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Convergence of General Iteration Schemes

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In this paper the convergence of general iteration algorithms defined by point-to-set maps is examined first. Special practical convergence conditions are then derived from the general theory.

INTRODUCTION

In recent years, the study of general iteration schemes has included a substantial effort to identify properties of iteration schemes that will guarantee their convergence in some sense. A number of these results have used an abstract iteration scheme that consists of the recursive application of a point-to-set mapping. In this paper we are concerned with these type of results. Iteration schemes of this form have great importance in optimization, input–output systems, stability analysis of dynamic systems, and in all fields of applied mathematics.

The paper is divided in two parts. Section 1 gives an outline of general iteration schemes, and the convergence of such schemes is examined. We also show that our conditions are very general: most clasical results can be obtained as special cases, and if the conditions are weakened slightly then our results may not hold. In Section 2 the discrete time scale Liapunov theory is extended to time dependent, higher order, nonlinear difference equations. In addition, the speed of convergence is estimated in most cases.

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1.1. *Algorithmic Models*

Let X denote an abstract set, and introduce the following notation:

$$X^1 = X, \qquad X^2 = X \times X, ..., \qquad X^k = X^{k-1} \times X, \qquad k \geqslant 2.$$

Assume that l is a positive integer, and for all $k \geqslant l-1$, the point-to-set mappings f_k are defined on X^{k+1}, furthermore, for all $(x^{(1)}, ..., x^{(k+1)}) \in X^{k+1}$ and $x \in f_k(x^{(1)}, ..., x^{(k+1)})$, $x \in X$. For the sake of brevity we will use the notation $f_k: X^{k+1} \to 2^X$, where 2^X denotes the set of all subsets of X.

DEFINITION 1.1. Select $x_0, x_1, ..., x_{1-1} \in X$ arbitrarily, and construct the sequence

$$x_{k+1} \in f_k(x_0, x_1, ..., x_k) \qquad (k \geqslant l-1), \tag{1.1}$$

where an arbitrary point from the set $f_k(x_0, x_1, ..., x_k)$ xan be accepted as the successor of x_k. Recursion (1.1) is called the *general algorithmic model.*

Remark. Since the domain of f_k is X^{k+1} and $f_k(x_0, x_1, ..., x_k) \subseteq X$, the recursion is well defined for all $k \geqslant l-1$. Points $x_0, ..., x_{l-1}$ are called the *initial approximations*, and the maps f_k are called the *iteration mappings.*

DEFINITION 1.2. The algorithmic model (1.1) is called an 1-*step process* if for all $k \geqslant l-1$, f_k does not depend explicitly on $x_0, x_1, ..., x_{k-1}$, that is, if algorithm (1.1) has the special form

$$x_{k+1} \in f_k(x_{k-l+1}, ..., x_{k-1}, x_k). \tag{1.2}$$

It is easy to show that any l-step process is equivalent to a certain single-step process defined on x^l. For $k \geqslant 0$, introduce vectors $\underline{z}_k = (z_k^{(1)}, z_k^{(2)}, ..., x_k^{(l)})$. Starting from the initial approximation

$$\underline{z}_0 = (x_0, x_1, ..., x_{l-1}),$$

consider the single step algorithmic model,

$$\begin{aligned} z_{k+1}^{(1)} &= z_k^{(2)} \\ z_{k+1}^{(2)} &= z_k^{(3)} \\ &\vdots \\ z_{k+1}^{(l-1)} &= z_k^{(l)} \\ z_{k+1}^{(l)} &\in f_k(z_k^{(1)}, ..., z_k^{(l)}). \end{aligned} \tag{1.3}$$

This iteration algorithm is a single-step process, and obviously it is equivalent to the algorithmic model (1.2), since for all $k \geqslant 0$,

$$z_k^{(1)} = x_k,\ z_k^{(2)} = x_{k+1},\ \ldots,\ z_k^{(l)} = x_{k+l-1}.$$

This equivalence is the main reason why only single-step iteration methods are discussed in most publications.

DEFINITION 1.3. A l-step process is called *stationary*, if mappings f_k do not depend on k. Otherwise the process is called *nonstationary*.

Iteration models in the most general form (1.1) have a great importance in certain optimization methods. For example, in using cutting plane algorithms very early cuts can still remain in the latter stages of the process by assuming that they are not dominated by later cuts [4]. Hence the optimization problem of each step may depend on the solutions of very early problems. Multistep processes are also used in many other fields of applied mathematics. As an example we mention that the secant method for solving nonlinear equations [9, 1] is a special two-step method. Nonstationary methods have a great practical importance in analysing the global asymptotical stability of dynamic economic systems, when the state transition relation is time-dependent [6].

In this paper the most general algorithmic model (1.1) will be first considered, and then, special cases will be derived from our general convergence theorem. In order to establish any kind of convergence, X should have some topology.

Assume now that X is a Hausdorff topological space that satisfies the first axiom of countability. (For definitions see, for example, [10]). Let $S \subset X$ be the set of desirable points, which are considered as the solutions to the problem being solved by the algorithm. For example, in the case of an optimization problem X can be selected a the feasible set, and S as the set of the optimal solutions. If a linear or nonlinear fixed point problem is solved, then X is the domain of the mapping and S is the set of all fixed points. In analysing the global asymptotic stability of a discrete dynamic system, set X is the state space and S is the set of equilibrium points.

DEFINITION 1.4. An *algorithmic model* is said to be *convergent*, if the accumulation points of any iteration sequence $\{x_k\}$ constructed by the algorithm are in S.

Note that the convergence of an algorithm model does not imply that the iteration sequence is convergent. This more general convergence principle was introduced and investigated by many authors (see, for example, [11, and Refs. therein]). They presented a comprehensive summary of convergence criteria for algorithmic models. They compared the known criteria and showed that they are special cases of a new result being published first in that article. In the next section this theorem will be further generalized.

1.2. *Convergence Criteria for Algorithmic Models*

Assume that for $k \geqslant 0$ there exist functions $c_k: X \to \mathbb{R}^1$ with the following properties:

(A_1) For large k, functions $\{c_k\}$ are *uniformly locally bounded below* on $X\setminus S$. That is, there is a nonnegative integer N_1 such that for all $x \in X\setminus S$ there is a neighborhood U of x and $ab \in \mathbb{R}^1$ (which may depend on x) such that for all $k \geqslant N_1$ and $x' \in U$,

$$c_k(x') \geqslant b; \tag{1.4}$$

(A_2) If $k \geqslant N_1$ and $x' \in f_k(z^{(1)}, ..., z^{(k)}, x)$ $(x, z^{(i)} \in X,\ i = 1, ..., k)$, then

$$c_{k+1}(x') \leqslant c_k(x); \tag{1.5}$$

(A_3) For each $z \in X\setminus S$ if $\{z_i\} \subseteq X$ is any sequence such that $z_i \to z$ and $\{k_i\}$ is any strictly increasing sequence of nonnegative integers such that $c_{k_i}(z_i) \to c^*$, then for all iteration sequences $\{x_i\}$ such that $x_{k_i} = z_i$ $(i \geqslant 0)$ there exists an integer N_2 such that $k_{N_2} \geqslant N_1 - 1$ and

$$c_{k_{N_2}+1}(y) < c^* \quad \text{for all } y \in f_{k_{N_2}}(x_0, x_1, ..., x_{k_{N_2}}). \tag{1.6}$$

THEOREM 1.1. *If conditions* (A_1), (A_2), *and* (A_3) *hold, then the algorithmic model* (1.1) *is convergent.*

Proof. Let x^* be an accumulation point of the iteration sequence $\{x_k\}$ constructed by the algorithmic model (1.1), and assume that $x^* \in X\setminus S$. Let $\{k_i\}$ denote the index set such that $\{x_{k_i}\}$ is a subsequence of $\{x_k\}$ converging to x^*. Assumption (A_2) implies that for large k, $\{c_k(x_k)\}$ is decreasing, and from assumption (A_1) we conclude that $\{c_{k_i}(x_{k_i})\}$ is convergent. Therefore the entire sequence $\{c_k(x_k)\}$ converges to a $c^* \in \mathbb{R}$. From (1.5) we know that for $k \geqslant N_1$,

$$c_k(x_k) \geqslant c^*. \tag{1.7}$$

Use subsequence $\{x_{k_i}\}$ as sequence $\{z_i\}$ in condition (A_3) to see that there exists an N_2 such that $k_{N_2} \geqslant N_1 - 1$ and with the notation $M = k_{N_2} + 1$,

$$c_M(x_M) < c^*,$$

which contradicts relation (1.7) and completes the proof.

Remark 1. Note first that in the special case when (1.1) is a single-step nonstationary process and c_k does not depend on k this theorem generalizes [11, Theorem 4.3]. If the process is stationary, then this result further specializes to [11, Theorem 3.5].

Remark 2. The conditions of the theorem do not imply that sequence $\{x_k\}$ has an accumulation point, as the next example shows:

EXAMPLE 1.1. Select $X=\mathbb{R}^1$, $S=\{0\}$, and consider the single-step process with $f_k(x)\equiv f(x)=x-1$, and choose $c_k(x)=x$ for all $x\in X$.

Since functions c_k are continuous and $f_k(x)<x$ for all x, condition (A_1) obviously holds, and since functions f_k are strictly decreasing and continuous, assumptions (A_2) and (A_3) also hold. However, for arbitrary, $x_0\in X$, the iteration sequence is strictly decreasing and divergent. (Infinite limit is not considered here as limit point from X.)

Remark 3. Even in cases when the iteration sequence has an accumulation point, the sequence does not need to converge as the following example shows.

EXAMPLE 1.2. Select $X=\mathbb{R}^1$, $S=\{0;1\}$, and consider the single step iteration algorithm with function

$$f_k(x)\equiv f(x)=\begin{cases}1 & \text{if } x=0\\ 0 & \text{if } x=1\\ x-1 & \text{if } x\notin S.\end{cases}$$

Choose

$$c_k(x)\equiv c(x)=\begin{cases}0 & \text{if } x\in S\\ x & \text{otherwise.}\end{cases}$$

On $X\backslash S$, function c is continuous, hence assumption (A_1) is satisfied. If $x\notin S$, then $f(x)<x$, which implies that $c(f(x))<c(x)$. If $x\in S$, then $f(x)\in S$. Therefore in this case $c(f(x))=c(x)$. Hence condition (A_2) also holds. Assumption (A_3) follows from the definition of functions c_k and from the fact that $f(x)<x$ on $X\backslash S$. If x_0 is selected as a nonnegative integer, then the iteration sequence has two accumulation points: 0 and 1. If x_0 is selected otherwise, then no accumulation point exists.

Note that Definition 1.4 is considered as the definition of *global* convergence on X, since the initial approximations $x_0, x_1, ..., x_{l-1}$ are arbitrary elements of X. *Local* convergence of algorithmic models can be defined in the following way:

DEFINITION 1.5. An *algorithmic model* is said to be *locally convergent*, if there is a subset X_1 of X such that the accumulation points of any iteration sequence $\{x_k\}$ constructed by the algorithm starting with initial approximations $x_0, x_1, ..., x_{l-1}$ from X_1 are in S.

Theorem 1.1 can be modified as a local convergence theorem by substituting X and S by X_1 and $X_1 \cap S$, respectively.

2.1. *The General Convergence Theorem*

Consider again the algorithmic model

$$x_{k+1} \in f_k(x_0, x_1, ..., x_k) \qquad (k \geqslant l-1), \tag{2.1}$$

where for $k \geqslant l-1$, $f_k: X^{k+1} \to 2^X$. Here we assume again that X is a Hausdorf topological space which satisfies the first axiom of countability, and l is a given positive integer, furthermore in relation (2.1), any point from the set $f_k(x_0, x_1, ..., x_k)$ can be accepted as the successor of x_k. Assume furthermore that the set S of desirable points has only one element s^*.

Assume that

(B_1) There is a compact set $C \subseteq X$ such that for all k, $x_k \in C$;

(B_2) Condition (A_1), (A_2), and (A_3) of Theorem 1.1 are satisfied.

The main result of this section is given as

THEREM 2.1. *Under assumptions* (B_1) *and* (B_2), $x_k \to s^*$ *as* $k \to \infty$ *with arbitrary points* $x_0, x_1, ..., x_{l-1} \in X$.

Proof. Since C is compact, sequence $\{x_k\}$ has a convergent subsequence. From Theorem 1.1 we also know that all the limit points of this iteration sequence belong to S, which has only one point s^*. Hence the iteration sequence has only one limit point s^*, which implies that it converges to s^*.

The speed of convergence of algorithm (2.1) can be estimated as follows. Assume that

(B_3) X is a metric space with distance $d: X \times X \to \mathbb{R}^1$;

(B_4) There exist nonnegative constants a_{ki} $(k \geqslant l-1, 0 \leqslant i \leqslant k)$ such that if $k \geqslant l-1$ and $x \in f_k(x^{(0)}, x^{(1)}, ..., x^{(k)})$, then

$$d(x, s^*) \leqslant \sum_{i=0}^{k} a_{ki} d(x^{(i)}, s^*).$$

From (2.1) we have

$$\varepsilon_{k+1} \leqslant \sum_{i=0}^{k} a_{ki}\varepsilon_i,$$

where $\varepsilon_i = d(x_i, s^*)$ for all $i \geqslant 0$.

Starting from initial values $\delta_i = \varepsilon_i (i = 0, 1, ..., l-1)$ consider the nonstationary difference equation

$$\delta_{k+1} = \sum_{i=0}^{k} a_{ki}\delta_i. \tag{2.2}$$

Obviously, for all $k \geqslant 0$, $\varepsilon_k \leqslant \delta_k$. In order to obtain a direct expression for δ_k, and therefore the same for the error bound of x_k $(k \geqslant l-1)$, introduce the following additional notation:

$$\underline{d}_k = (\delta_0, \delta_1, ..., \delta_k)^T,$$

$$\underline{\underline{A}}_k = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ a_{k0} & \cdots & a_{kk} & \end{bmatrix} \quad \text{and} \quad \underline{a}_k^T = (a_{k0}, a_{k1}, ..., a_{kk}).$$

Then from (2.2),

$$\underline{d}_{k+1} = \underline{\underline{A}}_k \underline{d}_k,$$

and hence, finite induction shows that for all $k \geqslant 1$,

$$\underline{d}_k = \underline{\underline{A}}_{k-1}\underline{\underline{A}}_{k-2} \cdots \underline{\underline{A}}_{l-1}\underline{d}_{l-1}.$$

Note that the components of $\underline{d}_{l-1}$ are the errors of the initial approximations $x_0, x_1, ..., x_{l-1}$. From (2.2) we have

$$\delta_{k+1} = \underline{a}_k^T \underline{d}_k = (\underline{a}_k^T \underline{\underline{A}}_{k-1}\underline{\underline{A}}_{k-2} \cdots \underline{\underline{A}}_{l-1})\, \underline{d}_{l-1} = \underline{b}_k^T \underline{d}_{l-1}$$

with

$$\underline{b}_k^T = \underline{a}_k^T \underline{\underline{A}}_{k-1}\underline{\underline{A}}_{k-2} \cdots \underline{\underline{A}}_{l-1}$$

being a l-dimensional row vector. Introducing finally the notation

$$\underline{b}_k^T = (b_{k0}, b_{k1}, ..., b_{k,l-1}),$$

the definition of the numbers δ_k and relation (2.2) imply the following result:

THEOREM 2.2. *Under assumptions* (B_3)–(B_4),

$$d(x_{k+1}, s^*) \leqslant \sum_{i=0}^{l-1} b_{ki} d(x_i, s^*) \qquad (k \geqslant l-1). \tag{2.3}$$

COROLLARY. *If for all* $i = 0, 1, ..., l-1$, $b_{ki} \to 0$ *as* $k \to \infty$, *then the iteration sequence* $\{x_k\}$ *generated by algorithm* (2.1) *convergences to* s^*. *Hence, in this case conditions* (B_1) *and* (B_2) *are not needed to establish convergence.*

The conditions of Theorem 1.1 are usually difficult to verify in practical cases. Therefore in the next section we will relax these conditions in order to derive sufficient convergence conditions which can be easily verified.

2.2. *Convergence of l-Step Methods*

In this section l-step iteration processes of the form

$$x_{k+1} \in f_k(x_{k-l+1}, x_{k-l+2}, ..., x_k) \tag{2.4}$$

are discussed, where $l \geqslant 1$ is a given integer, and for all k, $f_k: X^l \to 2^X$. Assume again that the set S of desirable points has only one element s^*.

DEFINITION 2.1. A function $V: X^l \to \mathbb{R}^1_+$ is called the *Liapunov-function* of process (2.4), if for arbitrary $x^{(i)} \in X$ $(i = 1, 2, ..., 1, x^{(l)} \neq s^*)$ and $y \in f_k(x^{(1)}, x^{(2)}, ..., x^{(l)})$ $(k \geqslant l-1)$,

$$V(x^{(2)}, ..., x^{(l)}, y) < V(x^{(1)}, x^{(2)}, ..., x^{(l)}). \tag{2.5}$$

DEFINITION 2.2. The Liapunov function V is called *closed*, if it is defined on $\bar{X}^l$, where $\bar{X}$ is the closure of X, furthermore if $k_i \to \infty$, $x_i^{(j)} \to x^{(j)*}$ as $i \to \infty$ $(x_i^{(j)} \in X$ for $i \geqslant 0$ and $j = 1, 2, ..., l$ such that $x^{(l)*} \neq s^*)$ and $y_i \in f_{k_i}(x_i^{(1)}, ..., x_i^{(l)})$ $(i \geqslant 0)$ such that $y_i \to y^*$ as $i \to \infty$, then

$$V(x^{(2)*}, ..., x^{(l)*}, y) < V(x^{(1)*}, ..., x^{(l)*}). \tag{2.6}$$

Assume now that the following conditions hold:

(C_1) For all $k \geqslant l-1$, $f_k(x^{(1)}, ..., x^{(l-1)}, s^*) = \{s^*\}$ with arbitrary $x^{(1)}, ..., x^{(l-1)} \in X$;

(C_2) Process (2.4) has a continuous, cosed Liapunov function;

(C_3) X is compact.

THEOREM 2.3. *Under assumptions* (C_1), (C_2), *and* (C_3), $x_k \to s^*$ *as* $k \to \infty$.

Proof. Note first that this process is equivalent to the single-step method (1.3), where set X is replaced by $\hat{X} = X^l$, and the new set of desirable points is now $\hat{S} = S^l$. Select function c as the Liapunov function V.

We can now easily verify that the conditions of Theorem 2.1 are satisfied, which implies the convergence of the iteration sequence $\{x_k\}$.

Assumption (A_1) follows from (C_3) and the continuity of V. Condition (C_1) and the monotonicity of V imply assumption (A_2). And finally, assumption (A_3) is the consequence of condition (C_2) and relation (2.6).

Remark 1. Assumption (C_3) can be weakened as follows:

(C_3') For all $x \in X \backslash S$, there is a compact neighborhood $U \subseteq X$ of x.

In this case we have to assume that $s^* \in \bar{X}$, and condition (C_1) is required only if $s^* \in X$.

Remark 2. Assumption $s^* \in \bar{X}$ is needed in order to obtain s^* as the limit of sequences from X. Assumption (C_1) guarantees that if at any iteration step the solution s^* is obtained, then the process remains at the solution. We may also show that the existence of the Liapunov function is not a too strong assumption. Assume that X is a metric space, and consider the special iteration process $x_{k+1} = f(x_k)$ and assume that starting from an arbitrary initial point, $\{x_k\}$ converges to the solution s^* of equation $x = f(x)$. Let $V: X \to \mathbb{R}^1$ be constructed as follows. With selecting $x_0 = x$, consider sequence $x_{k+1} = f(x_k)$, $(k \geqslant 0)$, and define

$$V(x) = \begin{cases} 0 & \text{if} \quad x = s^* \\ \max d(x_k, s^*), & k \geqslant 0, \end{cases}$$

where d is the distance. Obviously, $V(f(x)) \leqslant V(x)$ for all $x \in X$. The continuity-type assumptions in (C_2) are also natural, since without certain continuity assumptions no convergence can be established. Assumption (C_3) says that the entire sequence $\{x_k\}$ is contained in a compact set. This condition is necessarily satisfied, for example, if X is in a finite dimensional Euclidean space, and is bounded or if for every $K > 0$ there exists a $Q > 0$ such that $\underline{t}^{(1)}, ..., \underline{t}^{(l)} \in X$ and $\|\underline{t}^{(j)}\| > Q$ (for at least one index j) imply that

$$V(\underline{t}^{(1)}, ..., \underline{t}^{(l)}) > K.$$

In the case of one-step processes (that is, if $l = 1$) this last condition can be reformulated as $V(x) \to \infty$ as $\|x\| \to \infty$, $x \in X$.

Assume next that the iteration process is stationary, that is, mappings f_k do not depend on k. Replace condition (C_2) by the following pair of conditions:

(C_2') The process has a continuous Liapunov-function;

(C_2'') Mapping f is closed on X, that is, if $x_i^{(j)} \to x^{(j)*}$ as $i \to \infty$ ($j = 1, 2, ..., l$) and $y_i \in f(x_i^{(1)}, ..., x_i^{(l)})$ such that $y_i \to y^*$, then $y^* \in f(x^{(1)*}, ..., x^{(l)*})$.

THEOREM 2.4. *If process* (2.1) *is stationary and conditions* (C_1), (C_2'), (C_2''), *and* (C_3) *hold, then* $x_k \to s^*$ *as* $k \to \infty$.

Remark 1. This result in the special case of $l = 1$ can be considered as the discrete-time counterpart of the famous stability theorem of Uzawa [12].

Remark 2. Assume that for all $k \geq l-1$, mapping f_k is closed, and the iteration sequence converges to s^*. Then for all $k \geq l-1$, $s^* \in f_k(s^*, ..., s^*)$. Hence, s^* is a *common fixed point* of mappings f_k.

The speed of convergence of process (2.4) is next examined. Two results will be introduced. The first one is based on the general result presented in Section 2.1, and the second one is based on special properties of the Liapunov function.

Note first that in the case of a l-step process assumption (B_4) is modified as

(C_4) There exist nonnegative constants a_{ki} ($k \geq l-1$, $k-l+1 \leq i \leq k$), such that for all $k \geq l-1$ and $x \in f_k(x^{(1)}, ..., x^{(1)})$, ($x^{(1)}, ..., x^{(1)} \in X$ are arbitrary),

$$d(x, s^*) \leq \sum_{i=1}^{l} a_{k,k-l+i} d(x^{(i)}, s^*).$$

Then Theorem 2.2 remains valid with the specification that $a_{ki} = 0$ for all $i \leq k-l$.

In the case of a stationary process constants $a_{k,k-l+1}$ do not depend on k. If we introduce the notation $\bar{a}_i = a_{k,k-l+i}$, then (2.2) reduces to

$$\delta_{k+1} = \sum_{i=1}^{l} \bar{a}_i \delta_{k+1-i}. \tag{2.7}$$

Observe that sequence $\{\delta_k\}$ is the solution of this lth order linear difference equation. Note first that the characteristic polynomial of this equation is as follows:

$$\varphi(\lambda) = \lambda^l - \bar{a}_1 \lambda^{l-1} - \bar{a}_2 \lambda^{l-2} - \cdots - \bar{a}_{l-1} \lambda - \bar{a}_l.$$

Assume that the roots of φ are $\lambda_1, \lambda_2, ..., \lambda_R$ with multiplicities $m_1, m_2, ..., m_R$, then the general solution of Eq. (2.7) is given as

$$\delta_{k+1} = \sum_{r=1}^{R} \sum_{s=0}^{m_r - 1} c_{rs} k^s \lambda_r^k,$$

where the coefficients c_{rs} are obtained by solving the initial-value equations

$$\sum_{r=1}^{R} \sum_{s=0}^{m_r-1} c_{rs} i^s \lambda_r^i = d(x_{i-1}, s^*) \qquad (i = 1, 2, ..., l).$$

Hence, we have proved the following:

THEOREM 2.5. *Under assumption* (C_4),

$$d(x_{k+1}, s^*) \leqslant \sum_{r=1}^{R} \sum_{s=0}^{m_r-1} c_{rs} k^s \lambda_r^k \qquad (k \geqslant l-1). \tag{2.8}$$

COROLLARY. *If for all* r, $r = 1, 2, ..., R$, $|\lambda_r| < 1$, *then* $x_k \to s^*$ *as* $k \to \infty$. *Hence, in this case the conditions of Theorem* 2.4 *are not needed to establish convergence.*

In the rest of the section the speed of convergence of process (2.4) is estimated based on some properties of the Liapunov function.

Assume now that

(C_5) There exist constants a_i, b_i $(i = 1, 2, ..., l, a_1 > 0)$ such that

$$\sum_{i=1}^{l} a_i d(x^{(i)}, s^*) \leqslant V(x^{(1)}, ..., x^{(l)}) \leqslant \sum_{i=1}^{l} b_i d(x^{(i)}, s^*)$$

for all $x^{(i)} \in X$ $(i = 1, 2, ..., l)$.

The following result holds.

THEOREM 2.6. *Assume that process* (2.4) *has a Liapunov function* V, *which satisfies condition* (C_5). *Then for* $k \geqslant l - 1$,

$$d(x_{k+1}, s^*) \leqslant a_1^{-1} \sum_{i=1}^{l} (b_i = a_{i-1})\, d(x_{k-l+i}, s^*) \qquad (a_0 = 0). \tag{2.9}$$

Proof. If $x_k = s^*$, then $x_{k+1} = s^*$, and therefore (2.9) obviously holds, since the left hand side equals zero. If $x_k \neq s^*$, then condition (C_5) implies that

$$\sum_{i=1}^{l} a_i d(x_{k-l+i+1}, s^*) \leqslant V(x_{k+2-l}, ..., x_{k+1})$$

$$\leqslant V(x_{k+1-l}, ..., x_k) \leqslant \sum_{i=1}^{l} b_i d(x_{k-l+i}, s^*).$$

The assertion is a simple consequence of this inequality.

COROLLARY. *Introduce next the notation* $\bar{a}_i = (b_i - a_{i-1})/a_1$ $(i = , 2, ..., l)$, *and let the sequence* $\{\delta_k\}$ *denote now the solution of difference equation* (2.7) *with initial conditions* $\delta_i = d(x_{i-1}, s^*)$ $(i = 1, 2, ..., l)$. *Then obviously,* $d(x_k, s^*) \leqslant \delta_k$ *for all* $k \geqslant l - 1$, *and with the above coefficients* $\bar{a}_i$, *Theorem* 2.5 *remains true.*

2.3. *Convergence of Single-Step Methods*

In this section single-step processes generated by point-to-set mappings are first examined. For the sake of simplicity we assume that X is a subset of a Banach space B. The iteration process now has the form

$$x_{k+1} \in f_k(x_k) \qquad (k \geqslant 0), \tag{2.10}$$

where $f_k: X \to 2^X$. It is also assumed that O is in X and $s^* = O$. We may have this last assumption without loosing generality, since any solution s^* can be transformed into zero by introducing the transformed mappings

$$g_k(x) = \{y - s^* \backslash y \in f_k(x + s^*)\}.$$

It is also assumed that for all k, $f_k(O) = \{O\}$.

We start our analysis with the following useful result.

THEOREM 2.7. *Assume that* X *is compact, and there is a real valued continuous function* $\alpha: X \backslash \{O\} \to [0, 1)$ *such that*

$$\| y \| \leqslant \alpha(x) \| x \| \tag{2.11}$$

for all $k \geqslant 0$, $x \neq O$, *and* $y \in f_k(x)$. *Then the iteration sequence* (2.10) *converges to* O *as* $k \to \infty$.

Proof. We now verify that all conditions of Theorem 2.3 are satisfied with the Liapunov function $V(x) = \| x \|$ and $s^* = O$. Note that (C_1) and (C_3) obviously hold, and condition (C_2) is implied by the facts that α and the norm are continuous, and $\alpha(x) < 1$ for $x \neq O$.

Remark 1. If (2.11) is replaced by the weaker assumption that

$$\| y \| < \| x \|$$

for all $k \geqslant 0, x \neq O$, and $y \in f_k(x)$, then the result may not hold, as the following example shows.

EXAMPLE 2.1. Select $B = \mathbb{R}^1$, $X = [0, 2]$, and for $k \geqslant 0$,

$$f_k(x) = [(k+1)^2 - 1](k+1)^{-2} x.$$

If the initial point is chosen as $x_0 = 2$, then finite induction shows that

$$x_k = 1 + (k+1)^{-1} \to 1 \neq 0 \qquad \text{as} \quad k \to \infty.$$

Furthermore, for all $k \geqslant 0$ and $x \neq 0$,

$$|f_k(x)| < |x|.$$

COROLLARY. *Recursion* (2.10) *and inequality* (2.11) *imply that for* $k \geqslant 0$,

$$\|x_{k+1}\| \leqslant \alpha(x_k)\|x_k\|,$$

and therefore finite induction shows that

$$\|x_{k+1}\| \leqslant \alpha(x_k)\,\alpha(x_{k-1}) \cdots \alpha(x_0)\|x_0\|. \tag{2.12}$$

As a special case assume that $\alpha(x) \leqslant q < 1$ *for all* $O \neq x \in X$. *Then for all* $k \geqslant 0$,

$$\|x_{k+1}\| \leqslant q^{k+1}\,\|x_0\|, \tag{2.13}$$

which shows the linear convergence of the process in this special case.

Relation (2.12) serves as the error formula of the algorithm. In addition, it has the following consequence: Assume that (2.11) holds for all $O \neq x \in X$, furthermore $\alpha(x_k)\,\alpha(x_{k-1}) \cdots \alpha(x_0) \to O$ as $k \to \infty$. Then $x_k \to O$ for $k \to \infty$. Hence in this case we may drop the assumptions that $\alpha(x) \in [0, 1)$ $(O \neq x \in X)$ and X is compact.

An alternative approach to Theorem 2.7 is based on the assumption that there exists a function $h\colon (0, \infty) \to \mathbb{R}$ such that

$$\|y\| \leqslant h(r)\|x\| \tag{2.14}$$

for all $k \geqslant 0$, $r > 0$, $\|x\| \leqslant r$, $x \in X$ and $y \in f_k(x)$.

In this case it is easy to verify that for all k,

$$\|x_k\| \leqslant q_k,$$

where q_k is the solution of the nonlinear difference equation

$$q_{k+1} = h(q_k)\,q_k, \quad q_0 = \|x_0\|.$$

Hence, the convergence analysis of iteration algorithms defined in a Banach space is reduced to the examination of the solution of a special scalar nonlinear difference equation.

In deriving further practical convergence conditions we will use the following special result.

LEMMA 2.1. *Assume that X is convex, and function $h: X \to X$ satisfies the following condition*:

$$\| h(x) - h(x') \| \leqslant \alpha(\xi) \| x - x' \| \tag{2.15}$$

for all $x, x' \in X$, where ξ is a point on the linear segment between x and x', furthermore $\alpha: X \to \mathbb{R}^1$ is a real function such that for all fixed x and $x' \in X$, $\alpha(x' + t(x - x'))$ as the function of the parameter t is Riemann integrable on $[0, 1]$. *Then for all x and $x' \in X$,*

$$\| h(x) - h(x') \| \leqslant \int_0^1 \alpha(x' + t(x - x'))\, dt\, \| x - x' \|. \tag{2.16}$$

Proof. Let $x, x' \in X$ and define $t_i = i/N$ $(i = 0, 1, 2, ..., N)$, where N is a positive integer. Then from (2.15),

$$\begin{aligned} \| h(x) - h(x') \| &\leqslant \sum_{i=1}^{N} \| h(x' + t_i(x - x')) - h(x' + t_{i-1}(x - x')) \| \\ &\leqslant \sum_{i=1}^{N} \alpha(x' + \tau_i(x - x')) \| (t_i - t_{i-1})(x - x') \|, \end{aligned}$$

where $\tau_i \in [t_{i-1}, t_i]$, which implies that

$$\| h(x) - h(x') \| \leqslant \left\{ \sum_{i=1}^{N} \alpha(x' + \tau_i(x - x'))(t_i - t_{i-1}) \right\} \| x - x' \|.$$

Observe that the first factor is a Riemann-sum of the integral $\int_0^1 \alpha(x' + t(x - x'))\, dt$ which converges to the integral. Let $N \to \infty$ in the above inequality to obtain the result.

Remark. If function α is continuous, then $\alpha(x' + t(x - x'))$ is continuous in t, therefore it is Riemann integrable.

Assume next maps f_k are point-to-point and process (2.10) satisfies the following conditions:

(D_1) $f_k(O) = O$ for $k \geqslant 0$;

(D_2) For all $k \geqslant 0$,

$$\| f_k(x) - f_k(x') \| \leqslant \alpha(\xi_k) \| x - x' \| \tag{2.17}$$

for all $x, x' \in X$, where $\alpha: X \to \mathbb{R}^1$ is a continuous function, and ξ_k is a point on the linear segment connecting x and x'.

(D_3) $\alpha(x) \in [0, 1)$ for all $O \neq x \in X$;

(D_4) X is compact and convex.

THEOREM 2.8. *Under the above conditions, $x_k \to O$ as $k \to \infty$.*

Proof. Let $O \neq x \in X$, then relation (2.16) implies that for all k,

$$\| f_k(x) \| \leqslant \int_0^1 \alpha(tx)\, dt\, \| x \|, \tag{2.18}$$

where we have selected $x' = O$. Break the integral into two parts to obtain

$$\| f_k(x) \| \leqslant \left\{ \int_0^\delta \alpha(tx)\, dt + \int_\delta^1 \alpha(tx)\, dt \right\} \| x \|.$$

Since α is continuous, $\alpha(O) \leqslant 1$, and since the interval $[\delta, 1]$ is compact, $\alpha(tx) \leqslant \beta_\delta(x) < 1$ for all $\delta \leqslant t \leqslant 1$, where $\beta_\delta : X \setminus \{0\} \to \mathbb{R}^1$ is the real valued function defined as

$$\beta^\delta(x) = \max_{\delta \leqslant t \leqslant 1} \{\alpha(tx)\}.$$

Therefore,

$$\| f_k(x) \| \leqslant \{\delta + (1 - \delta)\, \beta_\delta(x)\} \| x \| = \gamma_\delta(x) \| x \|,$$

where $\gamma_\delta : X \setminus \{O\} \to \mathbb{R}^1$ is a continuous function such that for all $x \neq O$, $\gamma_\delta(x) \in [0, 1)$.

Hence the conditions of Theorem 2.7 are satisfied with $\alpha = \gamma_\delta$, which implies the assertion.

Remark. Replace (2.17) by the following weaker condition: Assume that for all $k \geqslant 0$ and $x, x' \in X$,

$$\| f_k(x) - f_k(x') \| \leqslant \alpha_k(\xi_k) \| x - x' \|, \tag{2.19}$$

where for $k \geqslant 0$, $\alpha_k : X \to R$ is a continuous function, ξ_k is a point on the linear segent connecting x and x', and $\alpha_k(x) \in [0, 1)$ for all $k \geqslant 0$ and $O \neq x \in X$.

Then the assertion of the theorem may not hold, as it is illustrated in the case of Example 2.1.

COROLLARY. *Recursion* (2.10) *and inequality* (2.18) *imply that for* $k \geqslant 0$,

$$\| x_{k+1} \| = \| f_k(x_k) \| \leqslant \bar{\alpha}(x_k) \| x_k \|,$$

where

$$\bar{\alpha}(x) = \int_0^1 \alpha(tx)\, dt.$$

Hence, by replacing $\alpha(x)$ by $\bar{\alpha}(x)$, Corollary of Theorem 2.7 remains valid.

In the previous results no differentiability of functions f_k is assumed. In the special case of Frechet differentiable functions f_k, the above theorems can be reduced to very practical convergence conditions. These results are presented in the next section.

2.4. *Convergence of Single-Step Methods with Differentiable Iteration Functions*

Assume now that B is a Banach space, $X \subseteq B$ and functions $f_k: X \to X$ are continuously differentiable on X. It is also assumed that X is compact and convex, $O \in X$, furthermore O is a common fixed point of functions f_k. In this special case the following result holds.

THEOREM 2.9. *Let $f_k'(x)$ denote the Frechet derivative of f_k at x. Assume that for all $k \geqslant 0$,*

$$\| f_k'(x) \| \leqslant \beta(x), \tag{2.20}$$

where $\beta: X \to \mathbb{R}^1_+$ is a continuous function such that for $x \neq O$, $\beta(x) \in [0, 1)$.

Then $x_k \to O$ as $k \to \infty$.

Proof. Select

$$X_0 = \{x / x \in X \text{ and } \|x\| \leqslant \|x_0\|\},$$

then X_0 is compact. Select furthermore $\alpha = \beta$. We can easily verify that all conditions of Theorem 2.8 are satisfied with X_0 replacing X, which implies the assertion. Assumptions (D_1) and (D_3) are obviously satisfied. Assumption (D_2) follows from the mean value theorem of derivatives and then from the fact that the linear segment between x and x' is compact and function α is continuous. In order to verify assumption (D_4) we have to show that $x_k \in X_0$ for all $k \geqslant 0$. From the beginning of the proof of Theorem 2.4 we conclude that for $O \neq x \in X$, $\| f_k(x) \| < \| x \|$. Then finite induction implies that for all $k \geqslant 0$, $\| x_k \| \leqslant \| x_0 \|$. Hence $x_k \in X_0$ for all $k \geqslant 0$, which completes the proof.

Remark 1. If (2.20) is replaced by the weaker assumption that for all $k \geqslant 0$ and $x \neq O$,

$$\| f_k'(x) \| < 1,$$

the result may not hold, as the case of Example 2.1 illustrates. However, if f_k does not depend on k, that is, when $f_k = f$, the condition

$$\|f'(x)\| < 1 \qquad \text{for all} \quad x \neq O$$

implies that $x_k \to O$ as $k \to \infty$. To see this assertion select $\beta(x) = \| f'(x) \|$. Note that this special result was first introduced by Wu and Brown [13].

COROLLARY. *Note that the Corollary of Theorem* 2.8 *remains valid with* $\alpha(x)=\beta(x)$.

Remark 2. Assume that no assumption is made on the derivatives at the fixed point O.

Consider next the special case, when $B=\mathbb{R}^N$. Obviously the above results are still valid. However, this further specialization enables us to derive even stronger conditions for the convergence of the iteration process

$$x_{k+1}=f_k(x_k),$$

where $f_k\colon B\to B$.

THEOREM 2.10. *Let U be an open neighborhood of O. Assume that for all k, f_k is differentiable, and there exists a continuous function $\alpha\colon \mathbb{R}^N\to\mathbb{R}^1$ such that $\alpha(x)\in[0,1)$ for $x\neq O$, furthermore*

(E_1) $\|f_k(x)\|\leqslant\alpha(x)\|x\|$ *for all k and $O\neq x\in U$;*

(E_2) *If $x\notin U$ and $\|f_k(x)\|=\alpha(x)\|x\|$ with some k, then* $\|f_k'(x)\,x\|\leqslant\alpha(x)\|x\|$.

Under these assumptions $x_k\to O$ as $k\to\infty$.

Proof. We will prove that for all $k\geqslant 0$ and $x\neq O$, relation (2.11) holds, which implies the assertion.

Assume that for some k, (2.11) does not hold in the entire set $\mathbb{R}^N\setminus\{O\}$, then

$$r^*=\inf\{\|x\|\setminus x\neq O \text{ and } (2.11) \text{ does not hold for } k\}$$

exist and is positive. If for all vectors satisfying $\|x\|=r^*$, $\|f_k(x)\|>\alpha(x)\|x\|$, then the continuity of functions f_k and α implies that r^* can be reduced, which contradicts the definition of r^*. Therefore there is at least one x^* such that

$$\|x^*\|=r^* \quad \text{and} \quad \|f_k(x^*)\|=\alpha(x)\|x^*\|. \tag{2.21}$$

Since f_k is differentiable we know that for any $\varepsilon>0$, and sufficiently large $\lambda\in(0,1)$,

$$\|f_k((1-\lambda)\,x^*)-f_k(x^*)-\lambda f_k'(x^*)\,x^*\|<\varepsilon\lambda\,\|x^*\|$$

which together with (E_2) implies that

$$\begin{aligned}\|f_k((1-\lambda)\,x^*)-f_k(x^*)\|&<\lambda[\|f_k'(x^*)\,x^*\|+\varepsilon\,\|x^*\|]\\&=\lambda[\beta(x^*)+\varepsilon]\|x^*\|,\end{aligned}$$

where

$$\beta(x^*) = \| f_k'(x^*)\, x^* \| \, \| x^* \|^{-1} < \alpha(x^*).$$

From this and equality (2.20) we conclude that

$$\begin{aligned} \| f_k((1-\lambda)\, x^*)\| &> \| f_k(x^*)\| - \lambda[\beta(x^*) + \varepsilon] \| x^* \| \\ &= (\alpha(x^*) - \lambda\beta(x^*) - \lambda\varepsilon) \| x^* \| \\ &\geqslant \| x^* \| \, \alpha(x^*)(1-\lambda) = \|(1-\lambda)\, x^* \| \, \alpha(x^*), \end{aligned}$$

when ε is selected small enough. Since α is continuous, with sufficiently large λ,

$$\| f_k(1-\lambda)\, x^* \| > \|(1-\lambda)\, x^* \| \, \alpha((1-\lambda)\, x^*),$$

which contradicts again the definition of r^*, and completes the proof.

COROLLARY 1. *Note that the Corollary of Theorem 2.7 can be applied for estimating the convergence speed under the assumption of the theorem.*

COROLARY 2. *Consider the special case, when $f_k = f$. The assertion of the theorem remains valid, if conditions* (E_1) *and* (E_2) *are substituted by the following assumptions*:

There exists an $\varepsilon > 0$ and a $0 \leqslant q < 1$ such that

(E_1') *For all $x \neq O$ and $\| x \| < \varepsilon$,*

$$\| f(x)\| \leqslant q \, \| x \|;$$

(E_2') *If $\| x \| \geqslant \varepsilon$ and $\| f(x)\| = \| x \|$, then*

$$\| f'(x)\, x \| < \| x \|.$$

Proof. Define

$$\begin{aligned} r_k = \max \{ &\| f'(x)\, x \| \, \| x \|^{-1} \backslash \| f(x)\| \\ &= \| x \|, k \in \leqslant \| x \| \leqslant (k+1)\, \varepsilon \} \qquad \text{for} \quad k = 1, 2, \ldots . \end{aligned}$$

Obviously $r_k < 1$. Introduce constants

$$R_k = \max \{ q; r_1; r_2; \ldots; r_k \},$$

and the piece-wise linear function $s(t)$ with vertices $(0, q)$, (ε, R_1), $(2\varepsilon, R_2)$, $(3\varepsilon, R_3)$, ..., . Then all conditions of the theorem are satisfied with $U = \{ x \backslash \| x \| < \varepsilon \}$ and $\alpha(x) = s(\| x \|)$.

Remark. The mean value theorem of derivatives implies that if $\| f'(0)\| < 1$, then there exist $\varepsilon > 0$ and $0 \leqslant q < 1$ which satisfy condition (E_1'). Assume furthermore that if $x \neq O$ and $\| f(x)\| = \| x \|$, then $\| f'(x)\, x \| < \| x \|$. In this case condition (E_2') is also satisfied. Hence the iteration sequence $\{x_k\}$ converges to O. This special result was first introduced by Fujimoto [2].

Assume again that $X \subseteq B$, where B is a Banach space, furthermore for all $k \geqslant 0$, f_k is Frechet differentiable at O, and $\| f_k'(O)\| < 1$. As the following example shows, these conditions do not imply even the local convergence of the algorithm.

EXAMPLE 2.2. Select $X = \mathbb{R}^1$, and for $k \geqslant 0$ let

$$f_k(x) = \frac{(k+1)(k+4)}{(k+2)(k+3)}\, x.$$

It is easy to verify that for all $k \geqslant 0$,

$$0 \leqslant f_k'(O) = \frac{(k+1)(k+4)}{(k+2)(k+3)} < 1.$$

If $x_0 \neq O$ is any initial approximation, then finite induction shows that

$$x_k = \frac{k+3}{3(k+1)}\, x_0 \to \frac{1}{3}\, x_0 \neq 0 \qquad \text{as} \quad k \to \infty.$$

However, if the process is stationary, then the following result holds:

THEOREM 2.11. *Assume $f_k = f$ $(k \geqslant 0)$, O is in the interior of X, and f is Frechet differentiable at O, furthermore $\| f(O)\| < 1$. Then there is a neighborhood U of O such that $x_0 \in U$ implies $x_k \to O$ as $k \to \infty$.*

Proof. Since f is differentiable at O, we can write $f(x) = L(x) + R(x)$, where L is a bounded linear mapping of X into itself and $\lim \| R(x)\| \, \| x \|^{-1} = O$ as $x \to O$. By assumption $\| L \| < 1$. Select a number $b > 0$ such that $\| L \| < b < 1$. There exists a $d > 0$ such that

$$\| R(x)\| < (1-b)\| x \| \qquad \text{if} \quad \| x \| < d.$$

Let $U = \{x \in X \backslash \| x \| < d\}$. We shall now prove that U has the required properties. Using the triangle inequality we can easily show that

$$\| f(x)\| < e \, \| x \|, \qquad \text{if} \quad x \in U,$$

where $e = \| L \| + 1 - b$. Since $0 < e < 1$, it follows that U is f-invariant.

Consequently, if $x_0 \in U$, then the entire sequence of iterates x_k is also contained in U, and using finite induction we obtain

$$\| x_k \leqslant e^k \| x \| .$$

Since $e^k \to O$, $x_k \to O$ as $k \to \infty$.

Remark 1. Assumption $\| f'(O) \| < 1$ can be weakened by assuming only that the spectral radius of $f'(O)$ is less than one. In this case $\| f'(O)^N \| < 1$ with some $N > 1$, and then apply the theorem for the function

$$f^N(x) = (f \circ f \circ \cdots \circ f)(x).$$

Remark 2. Note that no differentiability is assumed for $x \neq O$.

Remark 3. When $X = \mathbb{R}^N$, our results can be reduced to the ones obtained by Ostrowskii [7], Argyros [1], and Rheinboldt [8].

In the previous results the special Liapunov function $V(x) = \| x \|$ was used, where $\| \cdot \|$ is some vector norm. Now select the Liapunov function $V(x) = \| Px \|$, where P is an $n \times n$ constant nonsingular matrix. For the sake of simplicity we assume that $f_k = f$ for all $k \geqslant 0$. Then in Theorem 2.7 and 2.10 conditions (2.11) and (2.20) can be substituted by the modified relations

$$\| Pf(x) \| < \| Px \|$$

and

$$\| Pf'(x)\, u \| < \| Pu \| \qquad (\text{for all} \quad u \neq O).$$

If one selects the Euclidean norm $\| x \| = x^T x$, then these conditions are equivalent to the relations

$$f^T(x)\, P^T P f(x) < x^T P^T P x \tag{2.22}$$

and

$$u^T f'(x)^T\, P^T P f'(x)\, u < u^T P^T P u. \tag{2.23}$$

Note that (2.22) holds for all $u \neq O$ if and only if matrix $f'(x)^T\, P^T P f'(x) - P^T P$ is negative definite. This condition has been derived in [3] and is a generalization of [5, Theorem 1.3.2.3]. The case of other Liapunov functions can be discussed in an analogous manner, the details are omitted.

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