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A new weighted metric: the relative metric I

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Abstract

The *M-relative distance*, denoted by ρ_M is a generalization of the p-relative distance introduced in [R.-C. Li, SIAM J. Matrix Anal. Appl. 19 (1998) 956–982]. We establish necessary and sufficient conditions under which ρ_M is a metric. In two special cases we derive complete characterizations of this metric. We also present a way of extending the results to metrics sensitive to the domain in which they are defined and find some connections to previously studied metrics. An auxiliary result of independent interest is an inequality related to Pittenger's inequality in Section 4.

Keywords: Relative metrics; Weighted metrics; Pittenger's inequality

1. Introduction and main results

In this section we introduce the problem and state two useful corollaries of the core results. The core results themselves are stated only in Section 3, since they require an additional notation. The topic of this paper is *M-relative distances*, which are functions of the form

$$\rho_M(x, y) := \frac{|x - y|}{M(|x|, |y|)},$$

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where $M : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a symmetric function satisfying $M(|x|, |y|) > 0$ if $|x||y| > 0$ for x and y in some normed space (note that $\mathbb{R}^+$ denotes $[0, \infty)$). We want to know when ρ_M is a metric, in which case it is called the *M-relative metric*.

The first special case that we consider is when M equals a power of the power mean, $M = A_p^q$, where

$$A_p(x, y) := \big((x^p + y^p)/2\big)^{1/p}, \qquad A_0(x, y) := (xy)^{1/2},$$
$$A_{-\infty}(x, y) := \min\{x, y\} \quad \text{and} \quad A_\infty(x, y) := \max\{x, y\},$$

for $p \in \mathbb{R} \setminus \{0\}$ and $x, y \in \mathbb{R}^+$, see also Definition 4.1. In this case we denote ρ_M by $\rho_{p,q}$ and call it the *(p, q)-relative distance*. The $(p, 1)$-relative distance was introduced by Ren-Cang Li [10], who proved that it is a metric in $\mathbb{R}$ for $p \geqslant 1$ and conjectured that it is such in $\mathbb{C}$ as well. Later, the $(p, 1)$-relative distance was shown to be a metric in $\mathbb{C}$ for $p = \infty$ by David Day [7] and for $p \in [1, \infty)$ by Anders Barrlund [4]. These investigations provided the starting point for the present paper. The following theorem contains the results from [4,7] as special cases.

Theorem 1.1. *Let $q \neq 0$. The (p, q)-relative distance*

$$\rho_{p,q}(x, y) = \frac{|x - y|}{A_p(|x|, |y|)^q},$$

is a metric in $\mathbb{R}^n$ if and only if $0 < q \leqslant 1$ and $p \geqslant \max\{1 - q, (2 - q)/3\}$.

Remark 1.1. As is done in [4,7], we define $\rho_{p,q}(0, 0) = 0$ even though the expression for $\rho_{p,q}$ equals $0/0$ in this case.

The second special case which we study in depth is $M(x, y) = f(x)f(y)$, where $f : \mathbb{R}^+ \to (0, \infty)$.

Theorem 1.2. *Let $f : \mathbb{R}^+ \to (0, \infty)$ and $M(x, y) = f(x)f(y)$. Then ρ_M is a metric in $\mathbb{R}^n$ if and only if*

(i) *f is increasing,*
(ii) *$f(x)/x$ is decreasing for $x > 0$, and*
(iii) *f is convex.*

(There are non-trivial functions which satisfy conditions (i)–(iii); for instance, the function $f(x) := (1 + x^p)^{1/p}$ for $p \geqslant 1$.)

In Section 4 we derive an inequality of the Stolarsky mean related to Pittenger's inequality which is of independent interest. In Section 6 we present a scheme for extending the results of this investigation to metrics sensitive to the domain in which they are defined. This provides connections with previously studied metrics.

This paper is the first of two papers dealing with the M-relative distance. In the second paper [8] we will consider various properties of the M-relative metric. In particular, isometries and quasiconvexity of ρ_M are studied there.

2. Preliminaries

2.1. Metric and normed spaces

By a *metric* on a set X we mean a function $\rho: X \times X \to \mathbb{R}^+$ which satisfies the following conditions:

(1) ρ is symmetric;
(2) $\rho(x, y) \geqslant 0$ for all $x, y \in X$ and $\rho(x, y) = 0$ if and only if $x = y$;
(3) $\rho(x, y) \leqslant \rho(x, z) + \rho(z, y)$ for all $x, y, z \in X$.

A function which satisfies Condition (2) is known as *positive definite*; the inequality in Condition (3) is known as the *triangle inequality*.

By a *normed space* we mean a vector space X with a function $|\cdot|: X \to \mathbb{R}^+$ which satisfies the following conditions:

(1) $|ax| = |a||x|$ for $x \in X$ and $a \in \mathbb{R}$;
(2) $|x| = 0$ if and only if $x = 0$; and
(3) $|x + y| \leqslant |x| + |y|$ for all $x, y \in X$.

2.2. Ptolemaic spaces

A metric space (X, d) is called *Ptolemaic* if

$$d(z, w)d(x, y) \leqslant d(y, w)d(x, z) + d(x, w)d(y, z) \tag{1}$$

for all $x, y, z, w \in X$ (for background information on Ptolemy's inequality, see e.g. [5, 10.9.2]). A normed space $(X, |\cdot|)$ is Ptolemaic if the metric space (X, d) is Ptolemaic, where $d(x, y) = |x - y|$. The following lemma provides a characterization of Ptolemaic normed spaces.

Lemma 2.1 [2, 6.14]. *A normed space is Ptolemaic if and only if it is an inner product space.*

Since the Ptolemaic inequality (1), with d equal to the Euclidean metric, can be expressed in terms of cross-ratios (see (13) in Section 6), it follows immediately that $(\overline{\mathbb{R}^n}, q)$ is a Ptolemaic metric space, where q denotes the chordal metric:

$$q(x, y) := \frac{|x - y|}{\sqrt{1 + |x|^2}\sqrt{1 + |y|^2}}, \qquad q(x, \infty) := \frac{1}{\sqrt{1 + |x|^2}}, \tag{2}$$

with $x, y \in \mathbb{R}^n$. The following lemma provides yet another example of a Ptolemaic space.

Lemma 2.2 [9]. *Hyperbolic space is Ptolemaic.*

Thus, in particular, the Poincaré model of the hyperbolic metric (B^n, ρ) is Ptolemaic. This metric will be considered in Section 5 of the sequel of this investigation, [8].

2.3. Real-valued functions

An increasing function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be *moderately increasing* (or shorter, to be MI) if $f(t)/t$ is decreasing on $(0, \infty)$. A function $P : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is MI if $P(x, \cdot)$ and $P(\cdot, x)$ are MI for every $x \in (0, \infty)$. Equivalently, if P is symmetric and $P \not\equiv 0$ then P is MI if and only if $P(x, y) > 0$ and

$$\frac{z}{x} \leqslant \frac{P(z, y)}{P(x, y)} \leqslant 1 \leqslant \frac{P(x, z)}{P(x, y)} \leqslant \frac{z}{y} \quad \text{for all } 0 < y \leqslant z \leqslant x.$$

The next lemma shows why we have assumed that $M(x, y) > 0$ for $xy > 0$.

Lemma 2.3. *Let $P : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be symmetric and MI. Then exactly one of the following conditions holds*:

(i) $P \equiv 0$.
(ii) $P(x, y) = 0$ *if and only if* $x = 0$ *or* $y = 0$.
(iii) $P(x, y) = 0$ *if and only if* $x = 0$ *and* $y = 0$.
(iv) $P(x, y) > 0$ *for every* $x, y \in \mathbb{R}^+$.

Proof. Suppose $P \not\equiv 0$. Let $x, y \in (0, \infty)$ be such that $P(x, y) > 0$. Then

$$P(z, w) \geqslant \min\{1, z/x\} \min\{1, w/y\} P(x, y) > 0$$

for every $z, w \in (0, \infty)$. Let then $x \in (0, \infty)$ be such that $P(x, 0) > 0$. Then $P(z, 0) \geqslant \min\{1, z/x\} P(x, 0) > 0$ for every $z \in (0, \infty)$. Finally, if $P(0, 0) > 0$ then $P(x, y) > 0$ for every $x, y \in \mathbb{R}^+$ since P is increasing. $\square$

Lemma 2.4. *Let $P : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be symmetric and MI. Then P is continuous in* $(0, \infty) \times (0, \infty)$.

Proof. Fix points $x, y \in (0, \infty)$. Since P is MI we have

$$\begin{aligned} \min\{1, z/x\} \min\{1, w/y\} P(z, w) &\leqslant P(x, y) \\ &\leqslant \max\{1, x/z\} \max\{1, y/w\} P(z, w), \end{aligned}$$

for $w, z > 0$, from which it follows that $|P(x, y) - P(z, w)|$ is bounded from above by

$$\max\bigl\{1 - \min\{1, z/x\}\min\{1, w/y\}, \max\{1, x/z\}\max\{1, y/w\} - 1\bigr\},$$

and so the continuity is obvious. $\square$

A function $P : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is said to be *α-homogeneous,* $\alpha > 0$, if $P(sx, sy) = s^\alpha P(x, y)$ for every $x, y, s \in \mathbb{R}^+$. A 1-homogeneous function is called *homogeneous*.

Lemma 2.5. *Let $P : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be symmetric, increasing, and α-homogeneous for some $0 < \alpha \leqslant 1$. Then P is MI.*

Proof. Let $x \geqslant z \geqslant y > 0$. The relations

$$xP(z, y) = xz^\alpha P(1, y/z) \geqslant zx^\alpha P(1, y/x) = zP(x, y)$$

and

$$yP(x, z) = yz^\alpha P(x/z, 1) \leqslant zy^\alpha P(x/y, 1) = zP(x, y)$$

imply that P is MI. $\square$

2.4. Conventions

Recall from the introduction that $M : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a symmetric function which satisfies $M(x, y) > 0$ if $xy > 0$. Throughout this paper we will use the short-hand notation $M(x, y) := M(|x|, |y|)$ in the case when $x, y \in \mathbb{X}$. We will denote by $\mathbb{X}$ a Ptolemaic normed space which is non-degenerate, i.e. $\mathbb{X}$ non-empty and $\mathbb{X} \neq \{0\}$. Moreover, if $M(0, 0) = 0$ then "ρ_M is a metric in $\mathbb{X}$" is understood to mean that ρ_M is a metric in $\mathbb{X} \setminus \{0\}$ (similarly for $\mathbb{R}$ or $\mathbb{R}^n$ in place of $\mathbb{X}$).

3. The M-relative metric

Theorem 3.1. *Let M be MI. Then ρ_M is a metric in $\mathbb{X}$ if and only if it is a metric in $\mathbb{R}$.*

Proof. Since in all cases it is clear that ρ_M is symmetric and positive definite, when we want to prove that ρ_M is a metric we need to be concerned only with the triangle inequality. The necessity of the condition is clear; just restrict the metric to a one-dimensional subspace of $\mathbb{X}$ which is isometric to $\mathbb{R}$.

We will consider a triangle inequality of the form $\rho_M(x, y) \leqslant \rho_M(x, z) + \rho_M(z, y)$. Let $x, y, z \in \mathbb{X}$ be such that that $M(x, y), M(x, z), M(z, y) > 0$. Since M is increasing, the case $z = 0$ is trivial, and we may thus assume $|z| > 0$.

For sufficiency we use the triangle inequality for the norm $|\cdot|$ and Ptolemy's inequality with $w=0$ to estimate $|x-y|$ in the left-hand side of $\rho_M(x,y) \leqslant \rho_M(x,z)+\rho_M(z,y)$.

We get the following two sufficient conditions for ρ_M being a metric:

$$|x-z|\big(1/M(x,z)-1/M(x,y)\big)+|z-y|\big(1/M(z,y)-1/M(x,y)\big) \geqslant 0,$$
$$|x-z|\left(\frac{1}{M(x,z)}-\frac{|y|}{|z|M(x,y)}\right)+|z-y|\left(\frac{1}{M(z,y)}-\frac{|x|}{|z|M(x,y)}\right) \geqslant 0.$$

If $|z| \leqslant \min\{|x|,|y|\}$, the first inequality holds since M is increasing. The second one holds if $|z| \geqslant \max\{|x|,|y|\}$ since f is MI. By symmetry we may therefore assume that $|x|>|z|>|y|$. Then $|x-z|$ has a negative coefficient in the first inequality, whereas $|z-y|$ has a positive one. The roles are interchanged in the second inequality. Thus we get two sufficient conditions:

$$\frac{|z-y|}{|x-z|} \geqslant \frac{1/M(x,y)-1/M(x,z)}{1/M(z,y)-1/M(x,y)}$$

and

$$\frac{|z-y|}{|x-z|} \leqslant \frac{1/M(x,z)-|y|/(|z|M(x,y))}{|x|/(|z|M(x,y))-1/M(z,y)}.$$

Now if

$$\frac{1/M(x,z)-|y|/(|z|M(x,y))}{|x|/(|z|M(x,y))-1/M(z,y)} \geqslant \frac{1/M(x,y)-1/M(x,z)}{1/M(z,y)-1/M(x,y)},$$

then certainly at least one of the above sufficient conditions holds. Rearranging the last inequality gives

$$\frac{|x|-|y|}{M(x,y)} \leqslant \frac{|x|-|z|}{M(x,z)}+\frac{|z|-|y|}{M(z,y)}, \tag{3}$$

the triangle inequality for ρ_M in $\mathbb{R}$. Thus if ρ_M is a metric in $\mathbb{R}$, it is a metric in $\mathbb{X}$, so the condition is also sufficient. □

Remark 3.1. In the proof of Theorem 3.1 we actually proved that the $\mathbb{R}$ in the statement of the theorem could be replaced by $\mathbb{R}^+$. Since the latter in not a vector space, we prefer the above statement. Nevertheless, in proofs it will actually suffice to show that ρ_M satisfies the triangle inequality for $0<y<z<x$, since the other cases follow from the MI condition as was seen in the proof.

We may define ρ_M in metric spaces as well. Let $a \in X$ be an arbitrary fixed point. Then we define

$$\rho_M(x,y) := \frac{d(x,y)}{M(d(x,a),d(y,a))}.$$

(As with $\mathbb{X}$, if $M(0,0)=0$ then we consider whether ρ_M is a metric in $X \setminus \{a\}$.)

Corollary 3.1. *Let M be MI and let X be a Ptolemaic metric space and let $a \in X$ be an arbitrary fixed point. Then ρ_M is a metric in X if it is a metric in $\mathbb{R}$.*

Proof. As in the previous proof we conclude that

$$\frac{|d(x,a)-d(y,a)|}{M(d(x,a),d(y,a))} \leqslant \frac{|d(x,a)-d(z,a)|}{M(d(x,a),d(z,a))} + \frac{|d(z,a)-d(y,a)|}{M(d(z,a),d(y,a))}$$

is a sufficient condition for ρ_M being a metric in X (it corresponds to (3)). However, since $d(x,a)$, $d(y,a)$, and $d(z,a)$ are all just real numbers, this inequality follows from the triangle inequality of ρ_M in $\mathbb{R}$. $\square$

Corollary 3.2. *Let M be MI. Then each of* $\log\{1+\rho_M(x,y)\}$, $\operatorname{arcsinh}\rho_M(x,y)$, *and* $\operatorname{arccosh}\{1+\rho_M(x,y)\}$ *is a metric in $\mathbb{X}$ if and only if it is a metric in $\mathbb{R}$.*

Proof. Denote by f one of the functions e^x-1, $\cosh\{x\}-1$, or $\sinh x$ so that the distance under consideration equals $f^{-1}(\rho_M)$. Applying f to both sides of the triangle inequality of $f^{-1}(\rho_M)$ gives

$$\begin{aligned}\rho_M(x,y) \leqslant{}& \rho_M(x,z)+\rho_M(z,y) \\ &+ g\bigl(f^{-1}\bigl(\rho_M(x,z)\bigr), f^{-1}\bigl(\rho_M(z,y)\bigr)\bigr),\end{aligned} \tag{4}$$

where $g(x,y) := f(x+y)-f(x)-f(y)$. Proceeding as in the proof of Theorem 3.1, we conclude that (4) follows from

$$\begin{aligned}\frac{|x|-|y|}{M(x,y)} \leqslant{}& \frac{|x|-|z|}{M(x,z)} + \frac{|z|-|y|}{M(z,y)} \\ &+ \frac{|x|-|z|}{|x-z|} g\left(f^{-1}\left(\frac{|x-z|}{M(x,z)}\right), f^{-1}\left(\frac{|z-y|}{M(z,y)}\right)\right).\end{aligned} \tag{5}$$

We may replace the term $(|x|-|z|)/|x-z|$ by $(|z|-|y|)/|z-y|$ by considering the ratio $|x-z|/|z-y|$ instead of $|x-z|/|x-z|$ in the proof of Theorem 3.1. Since both conditions are sufficient we may write it as one condition by using the constant

$$m := \max\left(\frac{|x|-|z|}{|x-z|}, \frac{|z|-|y|}{|z-y|}\right) \geqslant \sqrt{\frac{|x|-|z|}{|x-z|}\frac{|z|-|y|}{|z-y|}}. \tag{6}$$

Then (5) follows from the triangle inequality in $\mathbb{R}$ if

$$\begin{aligned}&g\left(f^{-1}\left(\frac{|x|-|z|}{M(x,z)}\right), f^{-1}\left(\frac{|z|-|y|}{M(z,y)}\right)\right) \\ &\quad\leqslant mg\left(f^{-1}\left(\frac{|x-z|}{M(x,z)}\right), f^{-1}\left(\frac{|z-y|}{M(z,y)}\right)\right).\end{aligned}$$

For f equal to one of $e^x - 1$, $\cosh\{x\} - 1$, and $\sinh x$, we find that $g(f^{-1}(a), f^{-1}(b))$ equals ab, $ab + \sqrt{a^2 + 2a}\sqrt{b^2 + 2b}$, and $a(\sqrt{1 + b^2} - 1) + b(\sqrt{1 + a^2} - 1)$, respectively. Now we see that each of these terms has either a factor a, b, or $\sqrt{ab}$, hence by choosing a suitable term in m or the lower bound from (6), using $|x - y| \geqslant |x| - |y|$, etc., the inequality follows. □

The reason for considering $\log\{1 + \rho_M(x, y)\}$, $\operatorname{arccosh}\{1 + \rho_M(x, y)\}$, and $\operatorname{arcsinh} \rho_M(x, y)$ is that these metric transformations (see the next remark) are well-known and have been applied in various other areas, notably in generalizing the hyperbolic metric (see [8, Section 5]).

Remark 3.2. (i) Let X be a set and $d : X \times X \to \mathbb{R}^+$ be a function. Denote conditions on d as follows:

A: d is a metric in X;
B: $\log\{1 + d\}$ is a metric in X;
C: $\operatorname{arcsinh}\{d\}$ is a metric in X; and
D: $\operatorname{arccosh}\{1 + d\}$ is a metric in X.

Then A $\Rightarrow$ B $\Rightarrow$ D and A $\Rightarrow$ C $\Rightarrow$ D, but B and C are not comparable, in the sense that there exists d such that B is satisfied but C is not, and analogously the other way round. These claims are easily proved by applying inverse functions (that is, e^x, $\sinh x$, and $\cosh x$) to the triangle inequality. For instance, to prove A $\Rightarrow$ B we satisfied that the triangle inequality for the $\log\{1 + d\}$ variant transforms into $1 + d(x, y) \leqslant (1 + d(x, z))(1 + d(z, y))$, which is equivalent to $d(x, y) \leqslant d(x, z) + d(z, y) + d(x, z)d(z, y)$.

(ii) Another relevant remark is that if f is subadditive and d is a metric then $f \circ d$ is a metric as well. Since any MI function is subadditive, as noted in [3, Remark 7.42], it follows that the composition of an MI function with a metric is again a metric.

Definition 3.1. A function $P : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ which satisfies

$$\max\{x^\alpha, y^\alpha\} \geqslant P(x, y) \geqslant \min\{x^\alpha, y^\alpha\}$$

is called an *α-quasimean*, $\alpha > 0$. A 1-quasimean is called a *mean*. We define the *trace* of P by $t_P(x) := P(x, 1)$ for $x \in [1, \infty)$. If P is an α-homogeneous symmetric quasimean then

$$P(x, y) = y^\alpha P(x/y, 1) = y^\alpha t_P(x/y)$$

for $x \geqslant y > 0$, so that t_P determines P uniquely in this case.

If we normalize an α-homogeneous increasing symmetric function P so that $P(1, 1) = 1$ then P is an α-quasimean.

Definition 3.2. We define a partial order on the set of α-quasimeans by $P \succeq N$ if $t_P(x)/t_N(x)$ is increasing.

Note that $P \succeq N$ implies that $t_P(x) \geqslant t_N(x)$, since $t_P(1) = t_N(1) = 1$ by definition.

We will need the following family of quasimeans, related to the Stolarsky mean (see Remark 3.3):

$$S_p(x, y) := (1 - p)\frac{x - y}{x^{1-p} - y^{1-p}}, \quad S_p(x, x) = x^p, \quad 0 < p < 1,$$

$$S_1(x, y) := L(x, y) := \frac{x - y}{\log x - \log y}, \qquad S_1(x, x) = x,$$

defined for $x, y \in \mathbb{R}^+$, $x \neq y$. Note that $S_1(x, y) = \lim_{p\to 1} S_p(x, y)$ equals the classical logarithmic mean, L, and that $S_1(x, 0) := 0$.

Lemma 3.1. *Let $0 < \alpha \leqslant 1$ and M be increasing and α-homogeneous.*

I. *If $M \succeq S_\alpha$ then ρ_M is a metric in $\mathbb{X}$.*
II. *If ρ_M is a metric in $\mathbb{X}$, then $M(x, y) \geqslant S_\alpha(x, y)$ for $x, y \in \mathbb{R}^+$ and*

$$\frac{M(x, 1)}{S_\alpha(x, 1)} \leqslant \frac{M(x^2, 1)}{S_\alpha(x^2, 1)} \qquad \text{for } x \geqslant 1.$$

Proof. By Lemma 2.5, M is MI. By Remark 3.1 it suffices to show that the triangle inequality holds in $\mathbb{R}^+$ with $y < z < x$. We will consider the cases $\alpha = 1$ and $\alpha < 1$ separately.

If $\alpha = 1$, set $g(x) := t_M(x)/t_L(x)$ for $x \in [1, \infty)$. Since $M(x, 0) = xM(1, 0)$ and $M(z, 0) = zM(1, 0)$, the triangle inequality is trivial if $y = 0$, so we may assume that $y > 0$. Then the triangle inequality for ρ_M becomes

$$\frac{\log st}{g(st)} \leqslant \frac{\log s}{g(s)} + \frac{\log t}{g(t)}, \tag{7}$$

where $s = x/z$ and $t = z/y$. Since $\log st = \log s + \log t$, it is clear that this inequality holds if g is increasing, hence $L \preceq M$ is a sufficient condition. Choosing $s = t$ shows that $g(s) \leqslant g(s^2)$ is a necessary condition.

Assume, conversely, that ρ_M is a metric. Let $0 < y = x_0 < x_1 < \cdots < x_{n+1} = x$ (note that $\mathbb{X}$ has a subspace isomorphic to $\mathbb{R}$). Using $M(x_i, x_{i+1}) \geqslant x_i$ we conclude that

$$\frac{x - y}{M(x, y)} \leqslant \sum_{i=0}^{n} \frac{x_{i+1} - x_i}{M(x_i, x_{i+1})} \leqslant \sum_{i=0}^{n} \frac{x_{i+1} - x_i}{x_i};$$

it follows by taking the limit that

$$\frac{x-y}{M(x,y)} \leqslant \int_y^x \frac{\mathrm{d}z}{z} = \log\frac{x}{y}$$

and hence $L(x,y) \leqslant M(x,y)$.

Assume now that $\alpha < 1$ and let $g(x) := t_M(x)/t_{S_\alpha}(x)$ for $x \in [1,\infty)$. If $y = 0$ then the triangle inequality takes the form

$$\frac{x^{1-\alpha} - z^{1-\alpha}}{M(0,1)} \leqslant \frac{x-z}{M(x,z)}.$$

This is equivalent to

$$g(x/z) \leqslant M(1,0)/(1-\alpha) = \lim_{s\to\infty} g(s)$$

and hence the lemma follows, since g is increasing. Assume then that $y > 0$. Then the triangle inequality becomes

$$\frac{x^{1-\alpha} - y^{1-\alpha}}{g(x/y)} \leqslant \frac{x^{1-\alpha} - z^{1-\alpha}}{g(x/z)} + \frac{z^{1-\alpha} - y^{1-\alpha}}{g(z/y)},$$

where $y < z < x$. Clearly this holds if g is increasing. The necessary conditions $g(x) \geqslant 1$ and $g(x) \leqslant g(x^2)$ follow as above. $\square$

Remark 3.3. For $p \in (0,1]$ and $x, y \in \mathbb{R}^+$, the quasimean S_p defined above is related to Stolarsky's mean St_{1-p} by

$$St_p(x,y) := \left(\frac{x^p - y^p}{p(x-y)}\right)^{1/(p-1)} = S_{1-p}(x,y)^{1/(1-p)},$$

for $0 < p < 1$ and $St_0(x,y) := L(x,y)$. Note that the Stolarsky mean can also be defined for $p \notin [0,1)$, however, we will not make use of this fact. The reader is referred to [13] for more information on the Stolarsky mean.

Remark 3.4. Strong inequalities, i.e. inequalities of the type $A \succeq B$, have been recently proved by Alzer for polygamma function [1]. Also, although not stating so, some people have proved strong inequalities when they actually wanted to obtain just ordinary inequalities. Thus, for instance, Vamanamurthy and Vuorinen proved that $AGM \succeq L$, where AGM denotes the arithmetic–geometric mean, see [14]. Thus there are potentially many other forms which can be shown to be metrics by means of Lemma 3.1.

4. Stolarsky mean inequalities

Definition 4.1. Let $x, y \geqslant 0$. We define the *power-mean of order* p by

$$A_p(x, y) := \left(\frac{x^p + y^p}{2}\right)^{1/p}$$

for $p \in \mathbb{R} \setminus \{0\}$ and, additionally,

$$A_{-\infty}(x, y) = \min\{x, y\}, \qquad A_0(x, y) := \sqrt{xy}, \quad \text{and}$$
$$A_\infty(x, y) = \max\{x, y\}.$$

Also follow the convention that $A_p(x, 0) = 0$ for $p \leqslant 0$.

In order to use the results of the previous section, we need to investigate the partial order "$\preceq$" from Definition 3.2. The next result is an improvement of a result of Tung-Po Lin in [11] which states that $L \leqslant A_p$ if and only if $p \geqslant 1/3$. Lin's result is implied by Lemma 4.1, since "$\preceq$" implies "$\leqslant$".

Lemma 4.1. *$L \preceq A_p$ if and only if $p \in [1/3, \infty]$.*

Proof. Denote t_{A_p} by t_p. Since $t_L, t_p \in C^1$, $L \preceq A_p$ is equivalent to

$$\frac{d \log t_L(x)}{dx} \leqslant \frac{\partial \log t_p(x)}{\partial x}. \tag{8}$$

Since

$$\frac{\partial^2 \log t_p(x)}{\partial p \partial x} = \frac{x^{p-1} \log x}{(x^p + 1)^2} > 0,$$

(8) holds for $p \geqslant 1/3$ if it holds for $p = 1/3$. Calculating (8) for $p = 1/3$ gives

$$\frac{1}{x-1} - \frac{1}{x \log x} \leqslant \frac{1}{x + x^{2/3}}.$$

Substituting $x = y^3$ and rearranging gives

$$3 \log y \leqslant (y^3 - 1)(1 + 1/y)\big/\big(y^2 + 1\big).$$

Note that equality holds for $y = 1$. It suffices to show that the derivative of the right-hand side is greater than that of the left-hand side. Differentiating and rearranging leads to

$$y^6 - 3y^5 + 3y^4 - 2y^3 + 3y^2 - 3y + 1 \geqslant 0,$$

which is equivalent to the tautology $(y-1)^4(y^2 + y + 1) \geqslant 0$.

Since "$\preceq$" implies "$\leqslant$", it follows from [11] that $L \not\preceq A_p$ for $p < 1/3$. $\square$

The previous lemma can be generalized to the quasimean case.

Lemma 4.2. *For $0 < q \leqslant 1$, $A_p^q \succeq S_q$ if and only if*

$$p \geqslant \max\{1-q, (2-q)/3\}.$$

Proof. The claim follows from the previous lemma for $q = 1$. For $0 < q < 1$, we need to show that $g(x) := (x^p + 1)^{q/p}(x^{1-q} - 1)/(x-1)$ is increasing for all $x \geqslant 1$ and $p \geqslant \max\{1-q, (2-q)/3\}$. This is equivalent to show that the logarithmic derivative of g is non-negative for $x \geqslant 1$, i.e. that $g'(x)/g(x) \geqslant 0$. Rearranging the terms, we see that this is equivalent to

$$q\bigl(x^p + x^{1-q}\bigr)(x-1) \leqslant \bigl(x - x^{1-q}\bigr)(x^p + 1). \tag{9}$$

Letting $x \to \infty$ and comparing exponents, we see that this can hold only if $p \geqslant 1-q$. The other bound on p comes from $x \to 1^+$, however, only after some work.

As $x \to 1^+$ (x tends to 1 from above), both sides of (9) tend to 0. Their first derivatives both tend to $2q$ and the second derivatives to $2q(p+1-q)$. Only in the third derivatives is there a difference, the left-hand side tending to

$$3q\bigl(p(p-1) + q(1-q)\bigr)$$

and the right-hand side to

$$3p(p-1)q + 2p(1-q)q - 2q\bigl(1-q^2\bigr) + p(1-q)q.$$

Thus the right-hand side of (9) is greater than or equal to the left at 1^+ only if $3p \geqslant 2-q$.

We still need to check the sufficiency of the condition on p. Since $A_p \succeq A_s$ for $p \geqslant s$, it is enough to check $p = \max\{1-q, (2-q)/3\}$. For $q \leqslant 1/2$ set $q = 1-p$ in (9). This gives $(2p-1)x^p(x-1) + x - x^{2p} \geqslant 0$. Since the second derivative of this function is positive, the inequality follows easily.

Now set $q = 2-3p$ in (9). Dividing both sides by x^p and rearranging terms gives

$$g_p(x) := (3p-1)\bigl(x - x^{2p-1}\bigr) + (2-3p)\bigl(1 - x^{2p}\bigr) - x^{3p-1} + x^{1-p} \geqslant 0.$$

Since

$$g_{1/3}(x) = 1 - x^{2/3} - 1 + x^{2/3} = 0 \quad \text{and}$$
$$g_{1/2}(x) = (x-1)/2 + (1-x)/2 - x^{1/2} + x^{1/2} = 0,$$

the previous inequality follows if we show that $\partial^2 g_p(x)/\partial p^2 \leqslant 0$ for every x. Now

$$\frac{\partial^2 g_p(x)}{\partial p^2} = 12\bigl(x^{2p} - x^{2p-1}\bigr)\log x$$
$$- \bigl(4(3p-1)x^{2p-1} + 4(2-3p)x^{2p} - x^{1-p} + 9x^{3p-1}\bigr)\log^2 x,$$

and hence $\partial^2 g_p(x)/\partial p^2 \leqslant 0$ is equivalent to (we divide by x^{2p})

$$12(1-1/x) \leqslant \big(9x^{p-1} - x^{1-3p} + 4(2-3p) + 4(3p-1)/x\big)\log x. \tag{10}$$

We will show that inequality holds for $p = 1/3$ and $p = 1/2$ and that the right-hand side is concave in p. Hence the inequality holds for $1/3 < p < 1/2$ as well.

For $p = 1/3$, (10) is equivalent to

$$x\big(3x^{-2/3} + 1\big)\log x \geqslant 4(x-1).$$

Since

$$\log x \geqslant 4\frac{x-1}{x+3x^{1/3}}$$

holds for $x = 1$, it suffices to show that the derivative of the left-hand side is greater than that of the right-hand side:

$$\frac{1}{x} \geqslant 4\frac{x+3x^{1/3} - (x-1)(1+x^{-2/3})}{(x+3x^{1/3})^2} = 4\frac{2x^{1/3} + 1 + x^{-2/3}}{x^2 + 6x^{4/3} + 9x^{2/3}}.$$

We set $x = y^3$ and rearrange to obtain the equivalent condition:

$$y^5 - 2y^3 - 4y^2 + 9y - 4 = (y-1)^3(y^2 + 3y + 4) \geqslant 0,$$

which obviously holds. Next let $p = 1/2$ in (10). We now need to show that

$$6(x-1) \leqslant \big(x + 4x^{1/2} + 1\big)\log x$$

holds for $x \geqslant 1$. This follows by the same procedure as for $p = 1/3$. We still have to show that the right-hand side of (10) is concave. However, after we differentiate twice with respect to p all that remains is

$$9\big(x^{p-1} - x^{1-3p}\big)\log^3 x.$$

Clearly this is negative for $x \geqslant 1$ and $p \leqslant 1/2$. □

As we noted in Remark 3.3, the Stolarsky mean was introduced in [13] as a generalization of the logarithmic mean. The previous lemma may be reformulated to a result of independent interest. This result is related to Pittenger's inequality, which gives the exact range of values of p for which the inequality $A_p^q \geqslant S_q$ holds (see [6, p.204]). Note that the bounds in Pittenger's inequality equal our bounds only for $q \in [0, 1/2] \cup \{1\}$. For $q \in (1/2, 1)$, there are p such that the ratio A_p^q/S_q is initially increasing but eventually decreases, however its values are never below 1.

Corollary 4.1. *Let $0 \leqslant q < 1$. For fixed $y > 0$ the ratio $A_p(x,y)/St_q(x,y)$ is increasing in $x \geqslant y$ if and only if $p \geqslant \max\{q, (1+q)/3\}$. In particular, $A_p(x,y) \geqslant St_q(x,y)$ for all $x, y \in \mathbb{R}^+$ for the same p and q.*

Proof. The claim follows directly from Lemma 4.2 and the relationship between S and St given in Remark 3.3. □

5. Applications

In this section we combine the results from the previous two sections to derive our main results as to when ρ_M is a metric.

Proof of Theorem 1.1. Assume that the triangle inequality holds for some pair (p,q) with $p>0$. Then

$$2=\rho_{p,q}(-1,1)\leqslant\rho_{p,q}(-1,0)+\rho_{p,q}(0,1)=2^{1+q/p},$$

hence $q\geqslant 0$.

Suppose next that $p<0$ and $q>0$. Consider the triangle inequality $\rho_{p,q}(\varepsilon,1)\leqslant\rho_{p,q}(\varepsilon,1/2)+\rho_{p,q}(1/2,1)$ as $\varepsilon\to 0$. Then the left-hand side tends to ∞ like $2^{-q/p}(1-\varepsilon)\varepsilon^{-q}$ and the right-hand side like $2^{-q/p}(1/2-\varepsilon)\varepsilon^{-q}$; a contradiction for sufficiently small ε.

Suppose then that $p,q<0$. Then $\rho_{p,q}(x,0)=0$ for every $x\in\mathbb{X}$, contrary to the assumption that $\rho_{p,q}$ is a metric. For $p=0$ we arrive at contradiction of the triangle inequality by letting z tend to 0 or ∞ according as q is greater or less than 0.

Hence only the case $p,q>0$ remains to be considered. When $q>1$, the triangle inequality $\rho_{p,q}(x,y)\leqslant\rho_{p,q}(x,z)+\rho_{p,q}(z,y)$ cannot hold, as we see by letting $z\to\infty$.

The non-trivial cases follow from Lemmas 3.1 and 4.2: if $p\geqslant\max\{1-q, 2/3-q/3\}$, $\rho_{p,q}$ is a metric by the lemmas. If $p<\max\{1-q,2/3-q/3\}$, the ratio in the definition of $\succeq$ is decreasing in a neighborhood of 1 or ∞ (this is seen in the proof of Lemma 4.2). In the first case, $A_p^q(x,1)<S_q(x,1)$ in $(1,a)$ for some $a>1$, contradicting the first condition in Lemma 3.1. In the second case, $A_p^q(x,1)/S_q(x,1)>A_p^q(x^2,1)/S_q(x^2,1)$ for sufficiently large x and $\rho_{p,q}$ is not a metric by the second condition in Lemma 3.1. $\square$

We will now consider an application of Corollary 3.2.

Lemma 5.1. *Let $\lambda_M:\mathbb{X}\times\mathbb{X}\to\mathbb{R}^+$ be defined by the formula*

$$\lambda_M(x,y):=\log\bigl\{1+\rho_M(x,y)\bigr\}.$$

Then $\lambda_{A_p/c}$ is a metric in $\mathbb{X}$ if $c\geqslant 1$ for $p\in[0,\infty]$ and $c\geqslant 2^{-1/p}$ for $p\in[-\infty,0)$. The latter bound for the constant c is sharp.

Proof. By Corollary 3.2, it suffices to prove the claims in $\mathbb{R}$ with $y<z<x$. We start by showing that $\lambda_{A_p/c}$ is a metric for $c\geqslant\max\{1,2^{-1/p}\}$. Since the case $y=0$ is trivial, we may assume that $y>0$. Denote $f(x):=t_{A_p}(x)$. The triangle inequality for λ_M,

$$\log\bigl\{1+\rho_M(x,y)\bigr\}\leqslant\log\bigl\{1+\rho_M(x,z)\bigr\}+\log\bigl\{1+\rho_M(z,y)\bigr\},$$

is equivalent to

$$\frac{st-1}{f(st)} \leqslant \frac{s-1}{f(s)} + \frac{t-1}{f(t)} + c\frac{s-1}{f(s)}\frac{t-1}{f(t)}, \tag{11}$$

where $s = x/z$ and $t = z/y$. Since $st - 1 = (s-1)(t-1) + (s-1) + (t-1)$ and since f is increasing and greater than 1, the triangle inequality surely holds if $f(st) \geqslant f(s)f(t)/c$. However, this follows directly from Chebyshev's inequality (see [6, p. 50]) for $p > 0$ and is trivial for $p = 0$. For $p < 0$ it follows from the inequality $(1+s^p)(1+t^p) \geqslant 1 + (st)^p$.

We will now show that we cannot choose $c < 2^{-1/p}$ for $p < 0$. Let $s = t$ in (11): $(s+1)/f(s^2) \leqslant 2/f(s) + c(s-1)/(f(s)^2)$. As $s \to \infty$, $f(s) \to 2^{1/p}$, hence at the limit $2^{1/p}(s+1) \leqslant 2^{1+1/p} + c2^{2/p}(s-1)$ which implies that $c \geqslant 2^{-1/p}$. □

We now consider the second special case, $M(x,y) = f(x)f(y)$.

Lemma 5.2. *Let $M(x,y) = f(x)f(y)$ and assume $f(x) > 0$ for $x \geqslant 0$. Then ρ_M is a metric in $\mathbb{R}$ if and only if f is MI and convex in $\mathbb{R}^+$.*

Proof. Assume that ρ_M is a metric in $\mathbb{R}$. Let $y = -x$ in the triangle inequality for $-x < z < x$:

$$\frac{2x}{f(x)^2} \leqslant \frac{x-z}{f(x)f(z)} + \frac{x+z}{f(x)f(z)} = \frac{2x}{f(x)f(z)}.$$

Hence $f(x) \geqslant f(z)$, i.e. f is increasing. If $z > x$, we get instead $zf(x) \leqslant xf(z)$, i.e. $f(x)/x$ is decreasing, so that f is MI. Let now $0 \leqslant y < z < x$. Then the triangle inequality multiplied by $f(y)f(z)f(x)$ becomes

$$(x-y)f(z) \leqslant (x-z)f(y) + (z-y)f(x). \tag{12}$$

But this means that f is convex [6, p. 61]. (Alternatively, setting $z =: ay + (1-a)x$ gives more standard form of the convexity condition, $f(ay + (1-a)x) \leqslant af(y) + (1-a)f(x)$.)

Assume then conversely that f is MI and convex in $\mathbb{R}^+$. Then convexity gives (12) for $0 \leqslant y < z < x$, and, dividing this inequality by $f(y)f(z)f(x)$, we get the triangle inequality for the same y, z, x. However, we know from Remark 3.1 that this is a sufficient condition for ρ_M to be a metric, provided M is MI. □

Proof of Theorem 1.2. If ρ_M is a metric in $\mathbb{R}^n$ it is trivially a metric in $\mathbb{R}$, since $\mathbb{R}^n$ includes a subspace isometrically isomorphic to $\mathbb{R}$. Hence the claims regarding f follow from Lemma 5.2. If $f: \mathbb{R}^+ \to (0,\infty)$ is MI and convex then ρ_M is a metric in $\mathbb{R}$ by Lemma 5.2 and hence in $\mathbb{R}^n$ by Theorem 3.1. □

We now give an example of a relative-metric family where M is not a mean. Note that this family includes the chordal metric, q, as a special case ($p = 2$).

Example 5.1. The distance

$$\frac{|x-y|}{\sqrt[p]{1+|x|^p}\,\sqrt[p]{1+|y|^p}}$$

is a metric in $\mathbb{R}^n$ if and only if $p \geqslant 1$.

6. Further developments

In this section, we show how the approach of this paper can be extended to construct metrics that depend on the domain in which they are defined. The method is based on interpreting ρ_M as $\rho_{M,\mathbb{R}^n\setminus\{0\}}$, where $\rho_{M,G}$ is a distance function (defined in the next lemma) that depends both on the function M and the domain G. The proof of the next lemma is similar to that that of [12, Theorem 3.3]. Note that the topological operations (closure, boundary etc.) are taken in the compact space $\overline{\mathbb{R}^n}$.

Lemma 6.1. *Let $G \subset \mathbb{R}^n$ with $G \neq \mathbb{R}^n$. If M is continuous in $(0,\infty)\times(0,\infty)$ and ρ_M is a metric then*

$$\rho_{M,G}(x,y) := \sup_{a\in\partial G} \frac{|x-y|}{M(|x-a|,|y-a|)}$$

is a metric in G.

Proof. Clearly only the triangle inequality needs to be considered. Fix two points x and y in G. Since M is continuous and ∂G is a closed set in the compact space $\overline{\mathbb{R}^n}$ there exists a point $a \in \partial G$ such that $\rho_{M,G}(x,y) = \rho_M(x-a, y-a)$. Since

$$\begin{aligned}\rho_M(x-a,y-a) &\leqslant \rho_M(x-a,z-a) + \rho_M(z-a,y-a)\\ &\leqslant \rho_{M,G}(x,z) + \rho_{M,G}(z,y),\end{aligned}$$

it follows that $\rho_{M,G}$ is a metric in G. □

Remark 6.1. Let $M(x,y) := \min\{x,y\}$. Then

$$\rho_{M,G}(x,y) = \sup_{a\in\partial G} \frac{|x-y|}{\min\{|x-a|,|y-a|\}} = \frac{|x-y|}{\min\{d(x),d(y)\}},$$

where $d(x) = d(x,\partial G)$. We then have

$$\log\bigl\{1+\rho_{M,G}(x,y)\bigr\} = j_G(x,y) := \log\left(1+\frac{|x-y|}{\min\{d(x),d(y)\}}\right),$$

which provides our first connection to a well-known metric (j_G occurs in, e.g., [3,12,15]).

The previous lemma provides only a sufficient condition for $\rho_{M,G}$ to be a metric. It is more difficult to derive necessary conditions, but with some restrictions on G, such as convexity, this might not be impossible.

If M is homogeneous, we have a particularly interesting special case, as we may set

$$\rho'_{M,G}(x,y) = \sup_{a,b\in\partial G} \frac{|y,a,x,b|}{t_M(|x,b,a,y|)} = \sup_{a,b\in\partial G} \frac{1}{M(|x,y,a,b|,|x,y,b,a|)},$$

where

$$|a,b,c,d| := \frac{q(a,c)q(b,d)}{q(a,b)q(c,d)} \tag{13}$$

denotes the cross-ratio of the points $a,b,c,d \in \overline{\mathbb{R}^n}$, $a \neq b$, $c \neq d$, and q denotes the chordal metric (defined in (2)). With this notation we have

Lemma 6.2. *Let $G \subset \overline{\mathbb{R}^n}$ with* $\operatorname{card} \partial G \geqslant 2$*. If M is increasing and homogeneous and ρ_M is a metric in $\mathbb{R}^n$ then $\rho'_{M,G}$ is a metric in G.*

Proof. Fix points x and y in G. There are a and b in the compact set ∂G (possibly $a=\infty$ or $b=\infty$) for which the supremum in $\rho'_M(x,y)$ is attained. By the Möbius invariance of the cross ratio, we may assume that $a=0$ and $b=\infty$. Then $\rho'_{M,G}(x,y)=\rho_M(x',y')$, where x' and y' are the points corresponding to x and y, and we may argue as in the proof of Lemma 6.1. □

Corollary 6.1. *Let $G \subset \overline{\mathbb{R}^n}$ with* $\operatorname{card} \partial G \geqslant 2$ *and let* $M(x,y) = \max\{1, 2^{1/p}\} \times A_{-p}(x,y)$*. Then*

$$\delta_G^p(x,y) := \log\bigl\{1 + \rho'_{M,G}(x,y)\bigr\},$$

is a metric in G.

Proof. Follows directly from Lemmas 5.1 and 6.2. □

With this notation we have $\delta_G = \delta_G^\infty$, where δ_G is Seittenranta's cross ratio metric [12]. Also note that

$$\delta_G^p(x,y) = \sup_{a,b\in\partial G} \log\Bigl\{1 + \bigl(|x,a,y,b|^p + |x,b,y,a|^p\bigr)^{1/p}\Bigr\}$$

actually receives a quite simple form.

Instead of taking the supremum over the boundary we could integrate over it:

$$\tilde{\rho}^p_{M,G}(x,y) := \left(\int_{\partial G} \rho_M(x-a, y-a)^p \, d\mu\right)^{1/p}$$

(defined for μ-measurable ∂G). This metric takes the boundary into account in a more global manner, but is difficult to evaluate for most G's.

Lemma 6.3. *Let ρ_M, G, and μ be such that $\tilde\rho^p_{M,G}(x,y)$ exists for all $x, y \in G$. If ρ_M is a metric then $\tilde\rho^p_{M,G}$ is a metric in G for $p \geqslant 1$.*

Proof. From Minkowski's inequality

$$\left(\int_{\partial G} (f+g)^p\,d\mu\right)^{1/p} \leqslant \left(\int_{\partial G} f^p\,d\mu\right)^{1/p} + \left(\int_{\partial G} g^p\,d\mu\right)^{1/p},$$

where $f, g \geqslant 0$ and $p \geqslant 1$, and the basic triangle inequality (take $f = \rho_M(x-a, z-a)$ and $g = \rho_M(z-a, y-a)$ above) $\rho_M(x,y) \leqslant \rho_M(x,z) + \rho_M(z,y)$ it follows that $\tilde\rho^p_{M,G}$ also satisfies the triangle inequality. □

The integral form is quite difficult to evaluate in general, however, we can calculate the following explicit formulae. Note that H^2 denotes the upper half-plane.

Lemma 6.4. *For some constants c_t,*

$$\tilde\rho^{1/(1-2t)}_{A_2,H^2}(x,y) = c_t \frac{|x-y|}{\sqrt[t]{|x-y|^2 + 4h^2}}$$

for $0 < t < 1/2$, where h is the distance from the mid-point of the segment $[x,y]$ to the boundary of H^2, $h := d((x+y)/2, \partial H^2)$. Hence

$$\frac{|x-y|}{\sqrt[t]{|x-y|^2 + 4h^2}}$$

is a metric in H^2 for $0 < t < 1/2$.

Proof. The formula is derived directly by integration as follows:

$$\begin{aligned}\tilde\rho^s_{A_2,H^2}(x,y) &= c\left(\int_{\partial H^2} \frac{dm_1(\xi)}{(|x-\xi|^2 + |y-\xi|^2)^{s/2}}\right)^{1/s} |x-y| \\ &= c\left(\int_{-\infty}^{\infty} \frac{dw}{(a^2+b^2+h^2+w^2)^{s/2}}\right)^{1/s} |x-y|,\end{aligned}$$

where $2a := x_1 - y_1$ and $2b := x_2 - y_2$ and h is as above (x_i refers to the ith coordinate of x, similarly for y). Let us use the variable substitution $w = \sqrt{a^2+b^2+h^2}\,z$. Then we have

$$\begin{aligned}\tilde\rho^s_{A_2,H^2}(x,y) &= c\left(\int_{-\infty}^{\infty} \frac{\sqrt{a^2+b^2+h^2}\,dz}{((a^2+b^2+h^2)(1+z^2))^{s/2}}\right)^{1/s} |x-y| \\ &= c\big(|x-y|^2 + 4h^2\big)^{(1/s-1)/2} |x-y| c_s,\end{aligned}$$

where

$$c_s := \left(\int_{-\infty}^{\infty} \frac{dz}{(1+z^2)^{s/2}} \right)^{1/s}.$$

Note that $c_s < \infty$ for $s > 1$.

The last claim follows directly from Lemma 6.3, since ρ_{A_2} is a metric, by Theorem 1.1. □

Remark 6.2. We saw that

$$\iota_s(x,y) := \frac{|x-y|}{(|x-y|^2 + 4h^2)^{(1-1/s)/2}}$$

is a metric for $s > 1$. We then conclude that $\lim_{s\to\infty} \iota_s$ exists and hence that

$$\iota_\infty(x,y) := \frac{2|x-y|}{\sqrt{|x-y|^2 + 4h^2}}$$

is a metric also. Note that this metric is a lower bound of the hyperbolic metric in the half-plane, as is seen by the path-length metric method in [8, Section 4].

We may define yet another distance by taking the supremum over two boundary points:

$$\rho''_{M,G}(x,y) := \sup_{a,b\in\partial G} \frac{|x-y|}{M(|x-a|,|y-b|)}.$$

If we assume that M is increasing and continuous, this amounts to taking

$$\rho''_{M,G}(x,y) = \frac{|x-y|}{M(d(x),d(y))}, \tag{14}$$

where $d(x) := d(x, \partial G)$.

One could ask whether we could construct a general theory for $\rho''_{M,G}$-type metrics. This would be a very interesting theory, since it would involve metrics taking the geometry of the domain into account which would not include a complicated supremum. However, this cannot, in general, be done by our techniques: the following lemma has the important consequence that the proof technique of Lemma 6.1 cannot be extended to metrics of the type $\rho''_{M,G}$. In the following two lemmas we will use the convention that se_1 is denoted by s, etc.

Lemma 6.5. *Let $G := \mathbb{R}^n \setminus \{-a, a\}$ $(a > 0)$ $n \geqslant 2$, and assume that M is increasing and continuous. Then $\rho''_{M,G}$ is a metric if and only if $M \equiv c > 0$.*

Proof. We assume that $\rho''_{M,G}$ is a metric. Consider first the points $-a-r$ and $a+r$ and let y be on the line joining. We may choose y so that $d(y)$ varies between 0 and r. Then, by the triangle inequality,

$$\begin{aligned}\frac{2(r+a)}{M(r,r)} &= \rho''_{M,G}(-r-a, r+a) \leqslant \rho''_{M,G}(-r-a, y) + \rho''_{M,G}(y, r+a) \\ &= \frac{2(r+a)}{M(r,d(y))}.\end{aligned}$$

Consider next the points $-a+re_2$ and $a+re_2$ and let y be on the line joining them. We have

$$\frac{2a}{M(r,r)} \leqslant \frac{2a}{M(r,d(y))}$$

but now $d(y)$ varies between r and $\sqrt{r^2+a^2}$. Hence we have $M(x,y) \leqslant M(x,x)$ for $y \in [0, \sqrt{x^2+a^2}]$.

Let us now consider the points $x_1 := -a-s+he_2$, $y := a-s+he_2$, and $x_2 := a+t+he_2$, for $t \geqslant 0$, and $s \leqslant a$. We have

$$\begin{aligned}\frac{2a+s+t}{M(r,d(x_2))} &= \rho''_{M,G}(x_1, x_2) \leqslant \rho''_{M,G}(x_1, y) + \rho''_{M,G}(y, x_2) \\ &= \frac{2a}{M(r,r)} + \frac{s+t}{M(r,d(x_2))},\end{aligned}$$

where $r = \sqrt{s^2+h^2}$. From this it follows that $M(r,r) \leqslant M(r,y)$ where $y = \sqrt{t^2+h^2} = \sqrt{r^2-s^2+t^2}$. Combining the upper and lower bounds, we conclude that $M(x,x) = M(x,y)$ for $y \in [\sqrt{b}, \sqrt{x^2+a^2}]$, where $b := \max\{0, x^2-a^2\}$. From this it follows easily that $M \equiv c$. $\square$

The next idea might be to build a theory of $\rho''_{M,G}$-type metrics for sufficiently regular, e.g. convex domains only. The following lemma shows that this approach does not show much promise, either. (Note that B^n denotes the unit ball.)

Lemma 6.6. *Let $P:(0,1]\times(0,1] \to (0,\infty)$ be symmetric, increasing, and continuous. Then ρ''_{P,B^n} is a metric if and only if $P \equiv c > 0$.*

Proof. According to (14),

$$\rho''_{P,B^n}(x,y) = \frac{|x-y|}{P(d(x),d(y))} = \frac{|x-y|}{P(1-|x|, 1-|y|)}.$$

Consider the triangle inequality of the points $-r$, 0, and r, $0<r<1$:

$$\frac{2r}{P(1-r,1-r)} \leqslant \frac{2r}{P(1,1-r)}.$$

This implies that $P(1,s) \leqslant P(s,s)$ for $0 < s \leqslant 1$, and, since P is increasing, $P(1,s) = P(t,s)$ for $0 < s \leqslant t \leqslant 1$.

It follows that there exists an increasing function $g:(0,1] \to (0,\infty)$ such that $P(x,y) =: g(\min\{x,y\})$. Take points $0 < y < z < x \leqslant 1$ on the e_1-axis. Then the triangle inequality

$$\frac{x-y}{g(y)} \leqslant \frac{x-z}{g(z)} + \frac{z-y}{g(y)}$$

implies that $g(z) \leqslant g(y)$ and since g is increasing, by assumption it follows that g, and hence P, is constant. □

Since the unit ball is in many respects as regular a domain as possible, we see that the prospects of generalizing the theory by restricting the domain are not good. A better approach seems to be to consider $\log\{1 + \rho''_{M,G}(x, y)\}$, since we know from Remark 3.2 that this can be a metric even though $\rho''_{M,G}(x, y)$ is not. The metric j_G is an example of such a metric. This line of research seems to be the most promising further extension.

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