ELSEVIER
Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Number Theory 116 (2006) 102–139
JOURNAL OF Number Theory
www.elsevier.com/locate/jnt

Mahler measure of some n-variable polynomial families

Matilde N. Lalín

Department of Mathematics, University of Texas at Austin, 1 University Station C1200, Austin, TX 78712, USA

Received 20 September 2004; revised 13 January 2005

Available online 30 April 2005

Communicated by F. Rodriguez Villegas

Abstract

The Mahler measures of some n-variable polynomial families are given in terms of special values of the Riemann zeta function and a Dirichlet L-series, generalizing the results of Lalín (J. Number Theory 103 (2003) 85–108). The technique introduced in this work also motivates certain identities among Bernoulli numbers and symmetric functions.

Keywords: Mahler measure; Riemann zeta function; L-functions; Polynomials; Bernoulli numbers; Symmetric functions

1. Introduction

The goal of this work is to exhibit three families of multivariable polynomials whose Mahler measure depend (in most of the cases) on special values of the Riemann zeta function and the L-series on the Dirichlet character of conductor four.

E-mail address: mlalin@math.utexas.edu.

0022-314X/$ - see front matter
doi:10.1016/j.jnt.2005.03.002

The Mahler measure of a polynomial $P \in \mathbb{C}[x_1, \dots x_n]$ is defined as

$$m(P) = \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{\mathrm{d}x_1}{x_1} \cdots \frac{\mathrm{d}x_n}{x_n}.$$

Here $\mathbb{T}^n = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_1| = \cdots = |z_n| = 1\}$ is the unit torus.

For one-variable polynomials, Jensen's formula gives a simple expression for the Mahler measure as a function on the roots of the polynomial. However, it is in general a very hard problem to give an explicit closed formula for the Mahler measure of a polynomial in two or more variables.

For up to five variables, several examples have been produced by Bertin [3], Boyd [5–7], Boyd and Rodriguez-Villegas [8,9], Cassaigne and Maillot [14], Condon [10], Smyth [16,17], Vandervelde [19], the author [13], among others.

Smyth [18] gave an example of an n-variable family of polynomials whose Mahler measures are related to special values of hypergeometric series.

We have analyzed the n-variable versions of the polynomials studied in [13] and found closed formulas for their Mahler measures, which in most of the cases depend on special values of the Riemann zeta function and Dirichlet L-series. More precisely, we have proved that

Theorem 1. *We have the following identities*:[1]
(i) *For* $n \geqslant 1$:

$$\pi^{2n} m\left(1 + \left(\frac{1-x_1}{1+x_1}\right) \cdots \left(\frac{1-x_{2n}}{1+x_{2n}}\right) z\right)$$
$$= \frac{1}{(2n-1)!} \sum_{h=1}^{n} s_{n-h}\left(2^2, \dots, (2n-2)^2\right) \frac{(2h)!(2^{2h+1}-1)}{2} \pi^{2n-2h} \zeta(2h+1). \tag{1}$$

For $n \geqslant 0$:

$$\pi^{2n+1} m\left(1 + \left(\frac{1-x_1}{1+x_1}\right) \cdots \left(\frac{1-x_{2n+1}}{1+x_{2n+1}}\right) z\right)$$
$$= \frac{1}{(2n)!} \sum_{h=0}^{n} s_{n-h}\left(1^2, \dots, (2n-1)^2\right)(2h+1)! 2^{2h+1} \pi^{2n-2h} \mathrm{L}(\chi_{-4}, 2h+2). \tag{2}$$

[1] In order to simplify notation, we describe the polynomials as rational functions, writing $1 + \frac{1-x}{1+x} z$ instead of $1 + x + (1-x)z$, and so on. The Mahler measure does not change since the denominators are products of cyclotomic polynomials.

(ii) *For* $n \geqslant 1$:

$$\pi^{2n+2} m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_{2n}}{1+x_{2n}}\right)(1+y)z\right)$$
$$= \frac{1}{(2n-1)!}\sum_{h=1}^{n}\frac{(2h+2)!(2^{2h+3}-1)}{8}$$
$$\times\left(\sum_{l=0}^{n-h} s_{n-h-l}(2^2,\ldots,(2n-2)^2)\binom{2(l+h)}{2h}(-1)^l\frac{2^{2l}}{l+h}B_{2l}\right)\pi^{2n-2h}\zeta(2h+3). \tag{3}$$

For $n \geqslant 0$:

$$\pi^{2n+3} m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_{2n+1}}{1+x_{2n+1}}\right)(1+y)z\right)$$
$$= \frac{1}{(2n)!}\sum_{h=0}^{n} s_{n-h}(1^2,\ldots,(2n-1)^2)$$
$$\times 2^{2h+1}\pi^{2n-2h}\left(\mathrm{i}(2h)!\mathcal{L}_{3,2h+1}(\mathrm{i},\mathrm{i})+(2h+1)!\pi^2\mathrm{L}(\chi_{-4},2h+2)\right). \tag{4}$$

(iii) *For* $n \geqslant 1$:

$$\pi^{2n+1} m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_{2n}}{1+x_{2n}}\right)x+\left(1-\left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_{2n}}{1+x_{2n}}\right)\right)y\right)$$
$$= \frac{\pi^{2n+1}}{2}\log 2$$
$$+\frac{1}{(2n-1)!}\sum_{h=1}^{n} s_{n-h}(2^2,\ldots,(2n-2)^2)\frac{(2h)!(2^{2h+1}-1)}{4}\pi^{2n-2h+1}\zeta(2h+1)$$
$$+\frac{1}{(2n-1)!}\sum_{h=1}^{n}\frac{(2h)!(2^{2h+1}-1)}{4}$$
$$\times\left(\sum_{l=0}^{n-h} s_{n-h-l}(2^2,\ldots,(2n-2)^2)\binom{2(l+h)}{2l}(-1)^{l+1}\frac{2^{2l}(2^{2l-1}-1)}{l+h}B_{2l}\right)$$
$$\times\pi^{2n-2h+1}\zeta(2h+1). \tag{5}$$

For $n \geqslant 0$:

$$\pi^{2n+2} m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_{2n+1}}{1+x_{2n+1}}\right)x+\left(1-\left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_{2n+1}}{1+x_{2n+1}}\right)\right)y\right)$$
$$= \frac{\pi^{2n+2}}{2}\log 2$$
$$+\frac{1}{(2n+1)!}\sum_{h=0}^{n} s_{n-h}(2^2,\ldots,(2n)^2)\frac{(2h+2)!(2^{2h+3}-1)}{4}\pi^{2n-2h}\zeta(2h+3)$$
$$+\frac{1}{(2n-1)!}\sum_{h=1}^{n}\frac{(2h)!(2^{2h+1}-1)}{4}$$
$$\times\left(\sum_{l=0}^{n-h} s_{n-h-l}(2^2,\ldots,(2n-2)^2)\binom{2(l+h)}{2l}(-1)^{l+1}\frac{2^{2l}(2^{2l-1}-1)}{l+h}B_{2l}\right)$$
$$\times\pi^{2n-2h+2}\zeta(2h+1). \tag{6}$$

where B_h *is the h-Bernoulli number*, $\frac{x}{\mathrm{e}^x-1}=\sum_{n=0}^{\infty}\frac{B_n x^n}{n!}$.

ζ is the Riemann zeta function,

$$\mathrm{L}(\chi_{-4},s) := \sum_{n=1}^{\infty}\frac{\chi_{-4}(n)}{n^s},$$

$$\chi_{-4}(n) = \begin{cases} \left(\frac{-1}{n}\right) & \text{if } n \text{ odd,} \\ 0 & \text{if } n \text{ even} \end{cases}$$

and $\mathcal{L}_{r,s}(\alpha,\alpha)$ are linear combinations of multiple polylogarithms (they will be defined later).

Also,

$$s_l(a_1,\ldots,a_k) = \begin{cases} 1 & \text{if} \quad l=0, \\ \sum_{i_1<\cdots<i_l} a_{i_1}\cdots a_{i_l} & \text{if} \quad 0<l\leqslant k, \\ 0 & \text{if} \quad k<l \end{cases} \tag{7}$$

are the elementary symmetric polynomials, i.e.,

$$\prod_{i=1}^{k}(x+a_i) = \sum_{l=0}^{k} s_l(a_1,\ldots,a_k)x^{k-l}. \tag{8}$$

For concreteness, we list the first values for each family in the following tables:

$\pi^2 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)z\right)$	$7\,\zeta(3)$
$\pi^4 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_4}{1+x_4}\right)z\right)$	$62\zeta(5)+\frac{14\pi^2}{3}\zeta(3)$
$\pi^6 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_6}{1+x_6}\right)z\right)$	$381\zeta(7)+62\pi^2\zeta(5)+\frac{56\pi^4}{15}\zeta(3)$
$\pi^8 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_8}{1+x_8}\right)z\right)$	$2044\zeta(9)+508\pi^2\zeta(7)+\frac{868\pi^4}{15}\zeta(5)+\frac{16\pi^6}{5}\zeta(3)$
$\pi m\left(1+\left(\frac{1-x_1}{1+x_1}\right)z\right)$	$2\mathrm{L}(\chi_{-4},2)$
$\pi^3 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_3}{1+x_3}\right)z\right)$	$24\mathrm{L}(\chi_{-4},4)+\pi^2\mathrm{L}(\chi_{-4},2)$
$\pi^5 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_5}{1+x_5}\right)z\right)$	$160\mathrm{L}(\chi_{-4},6)+20\pi^2\mathrm{L}(\chi_{-4},4)+\frac{3\pi^4}{4}\mathrm{L}(\chi_{-4},2)$
$\pi^7 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_7}{1+x_7}\right)z\right)$	$896\mathrm{L}(\chi_{-4},8)+\frac{560}{3}\pi^2\mathrm{L}(\chi_{-4},6)+\frac{259}{15}\pi^4\mathrm{L}(\chi_{-4},4)+\frac{5}{8}\pi^6\mathrm{L}(\chi_{-4},2)$

$\pi^2 m\left(1+x+(1+y)z\right)$	$\frac{7}{2}\zeta(3)$
$\pi^4 m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)(1+y)z\right)$	$93\zeta(5)$
$\pi^6 m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_4}{1+x_4}\right)(1+y)z\right)$	$\frac{1905}{2}\zeta(7)+31\pi^2\zeta(5)$
$\pi^8 m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_6}{1+x_6}\right)(1+y)z\right)$	$7154\zeta(9)+635\pi^2\zeta(7)+\frac{248\pi^4}{15}\zeta(5)$
$\pi^3 m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)(1+y)z\right)$	$2\pi^2\mathrm{L}(\chi_{-4},2)+2\mathrm{i}\mathcal{L}_{3,1}(\mathrm{i},\mathrm{i})$
$\pi^5 m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_3}{1+x_3}\right)(1+y)z\right)$	$24\pi^2\mathrm{L}(\chi_{-4},4)+\pi^4\mathrm{L}(\chi_{-4},2)+16\mathrm{i}\mathcal{L}_{3,3}(\mathrm{i},\mathrm{i})+4\pi\mathrm{i}\mathcal{L}_{3,1}(\mathrm{i},\mathrm{i})$

$\pi^3 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)x+\left(1-\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)\right)y\right)$	$\frac{21\pi}{4}\zeta(3)+\frac{\pi^3}{2}\log 2$
$\pi^5 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_4}{1+x_4}\right)x+\left(1-\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_4}{1+x_4}\right)\right)y\right)$	$\frac{155\pi}{4}\zeta(5)+\frac{14\pi^3}{3}\zeta(3)+\frac{\pi^5}{2}\log 2$
$\pi^2 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)x+\left(1-\left(\frac{1-x_1}{1+x_1}\right)\right)y\right)$	$\frac{7}{2}\zeta(3)+\frac{\pi^2}{2}\log 2$
$\pi^4 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_3}{1+x_3}\right)x+\left(1-\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_3}{1+x_3}\right)\right)y\right)$	$31\zeta(5)+\frac{7\pi^2}{3}\zeta(3)+\frac{\pi^4}{2}\log 2$

2. An important integral

Before proving our main result, we will need to prove some auxiliary statements.

We will need to compute the integral $\int_0^\infty \frac{x \log^k x \, \mathrm{d}x}{(x^2+a^2)(x^2+b^2)}$. The following lemma will help:

Lemma 2. *We have the following integral*:

$$\int_0^\infty \frac{x^\alpha \, \mathrm{d}x}{(x^2+a^2)(x^2+b^2)} = \frac{\pi(a^{\alpha-1} - b^{\alpha-1})}{2\cos\frac{\pi\alpha}{2}(b^2-a^2)} \qquad \text{for} \quad 0 < \alpha < 1. \tag{9}$$

Proof. We write the integral as a difference of two integrals:

$$\int_0^\infty \frac{x^\alpha \, \mathrm{d}x}{(x^2+a^2)(x^2+b^2)} = \int_0^\infty \left(\frac{1}{x^2+a^2} - \frac{1}{x^2+b^2}\right) \frac{x^\alpha \, \mathrm{d}x}{b^2-a^2}. \tag{10}$$

Now, when $0 < \alpha < 1$,

$$\int_0^\infty \frac{x^\alpha \, \mathrm{d}x}{x^2+a^2} = \frac{1}{1-\mathrm{e}^{2\pi\mathrm{i}\alpha}} 2\pi\mathrm{i} \sum_{x \neq 0} \operatorname{Res}\left\{\frac{x^\alpha}{x^2+a^2}\right\}$$

(see, for instance, Section 5.3 in chapter 4 of the Complex Analysis book by Ahlfors [2]). Then,

$$\int_0^\infty \frac{x^\alpha \, \mathrm{d}x}{x^2+a^2} = \frac{\pi a^{\alpha-1}}{2\cos\frac{\pi\alpha}{2}}.$$

Thus, we get the result. □

By continuity, the formula in the statement is true for $\alpha = 1$, in fact the integral converges for $\alpha < 3$.

Next, we will define some polynomials that will be used in the formula for $\int_0^\infty \frac{x \log^k x \, \mathrm{d}x}{(x^2+a^2)(x^2+b^2)}$.

Definition 3. Let $P_k(x) \in \mathbb{Q}[x]$, $k \geqslant 0$, be defined recursively as follows:

$$P_k(x) = \frac{x^{k+1}}{k+1} + \frac{1}{k+1} \sum_{j>1 \,(\text{odd})}^{k+1} (-1)^{\frac{j+1}{2}} \binom{k+1}{j} P_{k+1-j}(x). \tag{11}$$

For instance, the first $P_k(x)$ are:

$$P_0(x) = x,$$
$$P_1(x) = \frac{x^2}{2},$$
$$P_2(x) = \frac{x^3}{3} + \frac{x}{3},$$
$$P_3(x) = \frac{x^4}{4} + \frac{x^2}{2},$$
$$P_4(x) = \frac{x^5}{5} + \frac{2x^3}{3} + \frac{7x}{15},$$
$$P_5(x) = \frac{x^6}{6} + \frac{5x^4}{6} + \frac{7x^2}{6}.$$

Lemma 4. *The following properties are true*

(1) $\deg P_k = k + 1$.
(2) *Every monomial of* $P_k(x)$ *has degree odd* (*even*) *for* k *even* (*odd*).
(3) $P_k(0) = 0$.
(4) $P_{2l}(\mathrm{i}) = 0$ *for* $l > 0$.
(5) $(2l + 1)P_{2l}(x) = \frac{\partial}{\partial x} P_{2l+1}(x)$.
(6) $2l P_{2l-1}(x) \equiv \frac{\partial}{\partial x} P_{2l}(x) \bmod x$.

The above properties can be easily proved by induction. These properties, together with P_0, determine the whole family of polynomials P_k because of the recursive nature of the definition. At this point, it should be noted that this family is closely related to Bernoulli polynomials. We postpone the discussion of this topic for the appendix, since the explicit form of the polynomials P_k is barely needed in order to perform the computation of the Mahler measures.

We are now ready to prove the key Proposition for the main Theorem:

Proposition 5. *We have*:

$$\int_0^\infty \frac{x \log^k x \, \mathrm{d}x}{(x^2 + a^2)(x^2 + b^2)} = \left(\frac{\pi}{2}\right)^{k+1} \frac{P_k\left(\frac{2\log a}{\pi}\right) - P_k\left(\frac{2\log b}{\pi}\right)}{a^2 - b^2}. \tag{12}$$

Proof. The idea, suggested by Rodriguez-Villegas, is to obtain the value of the integral in the statement by differentiating k times the integral of Lemma 2 and then evaluating at $\alpha = 1$. Let

$$f(\alpha) = \frac{\pi(a^{\alpha-1} - b^{\alpha-1})}{2\cos\frac{\pi\alpha}{2}(b^2 - a^2)}$$

which is the value of the integral in the Lemma 2. In other words, we have

$$f^{(k)}(1) = \int_0^\infty \frac{x \log^k x \,\mathrm{d}x}{(x^2 + a^2)(x^2 + b^2)}.$$

By developing in power series around $\alpha = 1$, we obtain

$$f(\alpha)\cos\frac{\pi\alpha}{2} = \frac{\pi}{2(b^2 - a^2)} \sum_{n=0}^{\infty} \frac{\log^n a - \log^n b}{n!}(\alpha - 1)^n.$$

By differentiating k times,

$$\sum_{j=0}^{k} \binom{k}{j} f^{(k-j)}(\alpha) \left(\cos\frac{\pi\alpha}{2}\right)^{(j)} = \frac{\pi}{2(b^2 - a^2)} \sum_{n=0}^{\infty} \frac{\log^{n+k} a - \log^{n+k} b}{n!}(\alpha - 1)^n.$$

We evaluate in $\alpha = 1$,

$$\sum_{j=0\,(\mathrm{odd})}^{k} (-1)^{\frac{j+1}{2}} \binom{k}{j} f^{(k-j)}(1) \left(\frac{\pi}{2}\right)^j = \frac{\pi(\log^k a - \log^k b)}{2(b^2 - a^2)}.$$

As a consequence, we obtain

$$f^{(k)}(1) = \frac{1}{k+1} \sum_{j>1\,(\mathrm{odd})}^{k+1} (-1)^{\frac{j+1}{2}} \binom{k+1}{j} f^{(k+1-j)}(1) \left(\frac{\pi}{2}\right)^{j-1} + \frac{\log^{k+1} a - \log^{k+1} b}{(k+1)(a^2 - b^2)}.$$

When $k = 0$,

$$f^{(0)}(1) = f(1) = \frac{\log a - \log b}{a^2 - b^2} = \frac{\pi}{2} \frac{P_0\left(\frac{2\log a}{\pi}\right) - P_0\left(\frac{2\log b}{\pi}\right)}{a^2 - b^2}.$$

The general result follows by induction on k and the definition of P_k. $\square$

3. Integrals and polylogarithms

In order to understand how special values of zeta functions and L-series arise in our formulas, we are going to need the definition of polylogarithms, which can be found, for instance, in Goncharov's works, [11,12]:

Definition 6. Multiple polylogarithms are defined as the power series

$$\mathrm{Li}_{n_1,\ldots,n_m}(x_1,\ldots,x_m) := \sum_{0<k_1<k_2<\cdots<k_m} \frac{x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}}{k_1^{n_1} k_2^{n_2} \cdots k_m^{n_m}}$$

which are convergent for $|x_i| < 1$. The length of a polylogarithm function is the number m and its weight is the number $w = n_1 + \cdots + n_m$.

Definition 7. Hyperlogarithms are defined as the iterated integrals

$$\begin{aligned}
&\mathrm{I}_{n_1,\ldots,n_m}(a_1 : \cdots : a_m : a_{m+1}) \\
&\quad := \int_0^{a_{m+1}} \underbrace{\frac{\mathrm{d}t}{t-a_1} \circ \frac{\mathrm{d}t}{t} \circ \cdots \circ \frac{\mathrm{d}t}{t}}_{n_1} \circ \underbrace{\frac{\mathrm{d}t}{t-a_2} \circ \frac{\mathrm{d}t}{t} \circ \cdots \circ \frac{\mathrm{d}t}{t}}_{n_2} \circ \cdots \\
&\qquad \circ \underbrace{\frac{\mathrm{d}t}{t-a_m} \circ \frac{\mathrm{d}t}{t} \circ \cdots \circ \frac{\mathrm{d}t}{t}}_{n_m}
\end{aligned}$$

where n_i are integers, a_i are complex numbers, and

$$\int_0^{b_{k+1}} \frac{\mathrm{d}t}{t-b_1} \circ \cdots \circ \frac{\mathrm{d}t}{t-b_k} = \int_{0 \leqslant t_1 \leqslant \cdots \leqslant t_k \leqslant b_{k+1}} \frac{\mathrm{d}t_1}{t_1-b_1} \cdots \frac{\mathrm{d}t_k}{t_k-b_k}.$$

The value of the integral above only depends on the homotopy class of the path connecting 0 and a_{m+1} on $\mathbb{C}\backslash\{a_1,\ldots,a_m\}$.

It is easy to see (for instance, in [12]) that,

$$\begin{aligned}
\mathrm{I}_{n_1,\ldots,n_m}(a_1 : \cdots : a_m : a_{m+1}) &= (-1)^m \mathrm{Li}_{n_1,\ldots,n_m}\left(\frac{a_2}{a_1}, \frac{a_3}{a_2}, \ldots, \frac{a_m}{a_{m-1}}, \frac{a_{m+1}}{a_m}\right), \\
\mathrm{Li}_{n_1,\ldots,n_m}(x_1,\ldots,x_m) &= (-1)^m \mathrm{I}_{n_1,\ldots,n_m}((x_1\ldots x_m)^{-1} : \cdots : x_m^{-1} : 1),
\end{aligned}$$

which gives an analytic continuation of multiple polylogarithms. Observe that we recover the special value of the Riemann zeta function $\zeta(k)$ for $k \geqslant 2$ as $\mathrm{Li}_k(1)$, as well as $\mathrm{L}(\chi_{-4}, k) = -\frac{\mathrm{i}}{2}(\mathrm{Li}_k(\mathrm{i}) - \mathrm{Li}_k(-\mathrm{i}))$.

In order to express the results more clearly, we will establish some notation.

Definition 8.

$$\mathcal{L}_r(\alpha) := \mathrm{Li}_r(\alpha) - \mathrm{Li}_r(-\alpha),$$
$$\mathcal{L}_{r,s}(\alpha,\alpha) := 2(\mathrm{Li}_{r,s}(\alpha,\alpha) - \mathrm{Li}_{r,s}(-\alpha,\alpha) + \mathrm{Li}_{r,s}(\alpha,-\alpha) - \mathrm{Li}_{r,s}(-\alpha,-\alpha)).$$

Note that the weight of any of the functions above is equal to the sum of its subindexes. This notation is the same as in [13].

Now we are ready to establish some technical results that will help us recognize special values of the Riemann zeta function and L-series out of integrals.

Lemma 9. *We have the following length-one identities*:

$$\int_0^1 \log^j x \frac{\mathrm{d}x}{x^2-1} = (-1)^{j+1} j! \left(1 - \frac{1}{2^{j+1}}\right)\zeta(j+1), \tag{13}$$

$$\int_0^1 \log^j x \frac{\mathrm{d}x}{x^2+1} = \frac{(-1)^{j+1} j!}{2} \mathrm{i}\mathcal{L}_{j+1}(\mathrm{i}) = (-1)^j j! \mathrm{L}(\chi_{-4}, j+1), \tag{14}$$

and the following length-two identities:

$$\int_0^1 \int_0^x \frac{\mathrm{d}s}{s^2-1} \circ \frac{\mathrm{d}s}{s} \circ \frac{\mathrm{d}s}{s} \log^j x \frac{\mathrm{d}x}{x^2-1} = \frac{(-1)^j j!}{8} \mathcal{L}_{3,j+1}(1,1), \tag{15}$$

$$\int_0^1 \int_0^x \frac{\mathrm{d}s}{s^2-1} \circ \frac{\mathrm{d}s}{s} \circ \frac{\mathrm{d}s}{s} \log^j x \frac{\mathrm{d}x}{x^2+1} = \frac{(-1)^{j+1} \mathrm{i} j!}{8} \mathcal{L}_{3,j+1}(\mathrm{i},\mathrm{i}). \tag{16}$$

Proof. The idea is to translate the integral into hyperlogarithms. We use the fact that $\int_x^1 \frac{\mathrm{d}s}{s} = -\log x$.

$$\int_0^1 \log^j x \frac{\mathrm{d}x}{x^2-1} = \frac{(-1)^j j!}{2} \int_0^1 \left(\frac{1}{x-1} - \frac{1}{x+1}\right) \mathrm{d}x \circ \underbrace{\frac{\mathrm{d}s}{s} \circ \cdots \circ \frac{\mathrm{d}s}{s}}_{j \text{ times}}$$

The $j!$ occurs as a way to count the possible permutations of the variables s, since they are ordered in the hyperlogarithm integral.

$$= \frac{(-1)^{j+1} j!}{2} (\mathrm{Li}_{j+1}(1) - \mathrm{Li}_{j+1}(-1)) = \frac{(-1)^{j+1} j!}{2} 2\left(1 - \frac{1}{2^{j+1}}\right)\zeta(j+1).$$

The last equality is a consequence of the Euler product decomposition for the zeta function. The second formula can be proved in a similar way.

Now, for the length-two identities, we do as before,

$$\begin{aligned}
&\int_0^1 \int_0^x \frac{\mathrm{d}s}{s^2-1} \circ \frac{\mathrm{d}s}{s} \circ \frac{\mathrm{d}s}{s} \log^j x \frac{\mathrm{d}x}{x^2-1} \\
&= \frac{(-1)^j j!}{4} \int_0^1 \left(\frac{1}{s-1} - \frac{1}{s+1}\right) \mathrm{d}s \circ \frac{\mathrm{d}s}{s} \circ \frac{\mathrm{d}s}{s} \circ \left(\frac{1}{x-1} - \frac{1}{x+1}\right) \mathrm{d}x \\
&\quad \underbrace{\circ \frac{\mathrm{d}t}{t} \circ \cdots \circ \frac{\mathrm{d}t}{t}}_{j \text{ times}} \\
&= \frac{(-1)^j j!}{4} (\mathrm{I}_{3,j+1}(1:1:1) - \mathrm{I}_{3,j+1}(-1:1:1) + \mathrm{I}_{3,j+1}(-1:-1:1) \\
&\quad - \mathrm{I}_{3,j+1}(1:-1:1)) \\
&= \frac{(-1)^j j!}{4} (\mathrm{Li}_{3,j+1}(1,1) - \mathrm{Li}_{3,j+1}(-1,1) + \mathrm{Li}_{3,j+1}(1,-1) - \mathrm{Li}_{3,j+1}(-1,-1)) \\
&= \frac{(-1)^j j!}{8} \mathcal{L}_{3,j+1}(1,1).
\end{aligned}$$

The other formula in the statement can be proved analogously. □

Now let us observe that the values $\mathcal{L}_{r,s}(1,1)$ for $r+s$ odd, can be expressed as combinations of values of $\zeta(k)$ for $2 \leqslant k \leqslant r+s$. This is possible because of the amazing formula (75) in [4], which claims:

$$\begin{aligned}
\mathrm{Li}_{r,s}(\rho,\sigma) &= \tfrac{1}{2}\left(-\mathrm{Li}_{r+s}(\rho\sigma) + (1+(-1)^s)\mathrm{Li}_r(\rho)\mathrm{Li}_s(\sigma)\right) \\
&\quad + \frac{(-1)^s}{2}\left(\binom{r+s-1}{r-1}\mathrm{Li}_{r+s}(\rho) + \binom{r+s-1}{s-1}\mathrm{Li}_{r+s}(\sigma)\right) \\
&\quad - \sum_{0<k<\frac{r+s}{2}} \mathrm{Li}_{2k}(\rho\sigma)(-1)^s \left(\binom{r+s-2k-1}{r-1}\mathrm{Li}_{r+s-2k}(\rho)\right. \\
&\quad \left. + \binom{r+s-2k-1}{s-1}\mathrm{Li}_{r+s-2k}(\sigma)\right) \qquad (17)
\end{aligned}$$

for $r+s$ odd, $\rho = \pm 1$, and $\sigma = \pm 1$.

We will only need $\mathcal{L}_{r,s}(1,1)$ for $r = 3$ and s even.

Proposition 10. *We have*:

$$\mathcal{L}_{3,2h}(1,1) = \frac{2^{2h+3}-1}{2^{2h+1}}(h+1)(2h+1)\zeta(2h+3) - \frac{2^{2h+1}-1}{2^{2h}}h(2h+5)\zeta(2)\zeta(2h+1) - \sum_{k=2}^{h-1}\binom{2h-2k+2}{2}\frac{2^{2h-2k+3}-1}{2^{2h}}\zeta(2k)\zeta(2h-2k+3)$$

for $h \geqslant 2$, *and*

$$\mathcal{L}_{3,2}(1,1) = \frac{93}{4}\zeta(5) - \frac{21}{2}\zeta(2)\zeta(3).$$

Expressing everything in terms of odd special values of ζ *and powers of* π:

$$\mathcal{L}_{3,2h}(1,1) = \frac{2^{2h+3}-1}{2^{2h+1}}(h+1)(2h+1)\zeta(2h+3) - \frac{2^{2h+1}-1}{2^{2h}}h(2h+5)\frac{\pi^2}{6}\zeta(2h+1) - \sum_{k=2}^{h-1}\binom{2h-2k+2}{2}\frac{2^{2h-2k+3}-1}{2^{2h}}\frac{(-1)^{k-1}B_{2k}(2\pi)^{2k}}{2(2k)!}\zeta(2h-2k+3)$$

for $h \geqslant 2$, *and*

$$\mathcal{L}_{3,2}(1,1) = \frac{93}{4}\zeta(5) - \frac{7}{4}\pi^2\zeta(3).$$

The proof of this statement is an easy application of formula (17) together with the well-known formula

$$\zeta(2k) = \frac{(-1)^{k-1}B_{2k}(2\pi)^{2k}}{2(2k)!}.$$

By applying formula (17) we can also obtain the following result:

Proposition 11. *We have*

$$\int_0^1 \log(1+x)\log^j x\,\frac{\mathrm{d}x}{x^2-1} = \frac{(-1)^j j!}{2}\left(\mathrm{Li}_{1,j+1}(-1,1) - \mathrm{Li}_{1,j+1}(1,-1)\right)$$

and

$$\begin{aligned}&\mathrm{Li}_{1,2h}(-1,1) - \mathrm{Li}_{1,2h}(1,-1)\\ &\quad= (2h-1)\frac{2^{2h+1}-1}{2^{2h+1}}\zeta(2h+1) - \log 2\,\frac{2^{2h}-1}{2^{2h-1}}\zeta(2h)\\ &\qquad - \sum_{k=1}^{h-1}\frac{(2^{2h+1-2k}-1)(2^{2k-1}-1)}{2^{2h-1}}\zeta(2k)\zeta(2h+1-2k).\end{aligned}$$

In other words,

$$\int_0^1 \log(1+x)\log^{2h-1} x \frac{\mathrm{d}x}{x^2-1}$$

$$= -(2h-1)!(2h-1)\frac{2^{2h+1}-1}{2^{2h+2}}\zeta(2h+1)$$

$$+(-1)^{h-1}\frac{\log 2(2^{2h}-1)B_{2h}\pi^{2h}}{4h}$$

$$+\frac{(2h-1)!}{2}\sum_{k=1}^{h-1}\frac{(2^{2h+1-2k}-1)(2^{2k-1}-1)}{2^{2h-2k}}\frac{(-1)^{k-1}B_{2k}}{(2k)!}\pi^{2k}\zeta(2h+1-2k).$$

Proposition 12. *We have*

$$\int_0^\infty \log(1+x^2)\log^{2h} x \frac{\mathrm{d}x}{x^2+1}$$

$$= (-1)^h 2E_{2h}\left(\frac{\pi}{2}\right)^{2h+1}\log 2$$

$$+2\sum_{l=1}^{h}\frac{(2h)!}{(2h-2l)!}\left(1-\frac{1}{2^{2l+1}}\right)(-1)^{h-l}E_{2h-2l}\left(\frac{\pi}{2}\right)^{2h-2l+1}\zeta(2l+1),$$

where E_h *is the h-th Euler number*, $\frac{2\mathrm{e}^x}{\mathrm{e}^{2x}+1} = \sum_{n=0}^\infty \frac{E_n x^n}{n!}$.

Proof. The proof of this Proposition is again a simple exercise in applying hyperlogarithms,

$$\int_0^\infty \log(1+x^2)\log^{2h} x \frac{\mathrm{d}x}{x^2+1} = \int_0^\infty\int_0^1 \frac{2x^2t\,\mathrm{d}t}{x^2t^2+1}\log^{2h} x\frac{\mathrm{d}x}{x^2+1}$$

$$= 2\int_0^1\int_0^\infty\left(\frac{1}{t^2x^2+1}-\frac{1}{x^2+1}\right)\log^{2h} x\,\mathrm{d}x\frac{t\,\mathrm{d}t}{1-t^2}.$$

Now we make the following change of variables: $y = tx$ in the first term but we let $y = x$ in the second term,

$$= 2\int_0^1\int_0^\infty \frac{\mathrm{d}y}{y^2+1}((\log y - \log t)^{2h} - t\log^{2h} y)\frac{\mathrm{d}t}{1-t^2}$$

$$= 2\int_0^\infty \frac{\log^{2h} y\,\mathrm{d}y}{y^2+1}\int_0^1\frac{\mathrm{d}t}{1+t} + 2\sum_{k=1}^{2h}\binom{2h}{k}(-1)^k\int_0^\infty\frac{\log^{2h-k} y\,\mathrm{d}y}{y^2+1}\int_0^1\frac{\log^k t\,\mathrm{d}t}{1-t^2}.$$

We will apply Lemma 9. But first, note that $\int_0^\infty \frac{\log^n y\,\mathrm{d}y}{y^2+1} = 0$ for n odd. Then we may write $k = 2l$

$$= 4(2h)!\mathrm{L}(\chi_{-4}, 2h+1)\log 2$$
$$+2\sum_{l=1}^{h}\binom{2h}{2l}2(2h-2l)!\mathrm{L}(\chi_{-4}, 2h-2l+1)(2l)!\left(1-\frac{1}{2^{2l+1}}\right)\zeta(2l+1)$$
$$= 4(2h)!\mathrm{L}(\chi_{-4}, 2h+1)\log 2 + 4\sum_{l=1}^{h}(2h)!\left(1-\frac{1}{2^{2l+1}}\right)\mathrm{L}(\chi_{-4}, 2h-2l+1)\zeta(2l+1).$$

The proof of the statement is now just an application of the well-known formula

$$\mathrm{L}(\chi_{-4}, 2k+1) = \frac{(-1)^k E_{2k}\left(\frac{\pi}{2}\right)^{2k+1}}{2(2k)!}. \qquad \square$$

Proposition 13. *We have*

$$\frac{1}{2\mathrm{i}}\int_0^\infty \left(\left(\mathrm{Li}_2(\mathrm{i}x) - \mathrm{Li}_2(-\mathrm{i}x)\right)\log^{2h} x\frac{\mathrm{d}x}{x^2+1}\right.$$
$$= \sum_{l=0}^{h} B_{2l}\frac{(2h)!}{(2l)!}(2^{2l-1}-1)(-1)^{l+1}\pi^{2l}(h-l+1)\left(\frac{2^{2h+3-2l}-1}{2^{2h+1}}\right)\zeta(2h+3-2l).$$

Proof. Note that

$$\frac{1}{2\mathrm{i}}\int_0^\infty \left(\left(\mathrm{Li}_2(\mathrm{i}x) - \mathrm{Li}_2(-\mathrm{i}x)\right)\log^{2h} x\frac{\mathrm{d}x}{x^2+1}\right.$$
$$= \frac{1}{2\mathrm{i}}\int_0^\infty\int_0^1 \frac{2\mathrm{i}x\,\mathrm{d}t}{t^2x^2+1}\circ\frac{\mathrm{d}t}{t}\log^{2h} x\frac{\mathrm{d}x}{x^2+1}$$
$$= -\int_0^1\int_0^\infty \frac{x\log^{2h} x\,\mathrm{d}x}{(t^2x^2+1)(x^2+1)}\log t\,\mathrm{d}t.$$

We are now ready to apply Proposition 5

$$= -\int_0^1\left(\frac{\pi}{2}\right)^{2h+1}\frac{P_{2h}\left(-\frac{2\log t}{\pi}\right)}{1-t^2}\log t\,\mathrm{d}t.$$

Applying Proposition A.1 from the appendix,

$$= \int_0^1 \left(\frac{\pi}{2}\right)^{2h+1} \frac{2}{2h+1} \sum_{l=0}^{h} B_{2l} \binom{2h+1}{2l} (2^{2l-1}-1)(-1)^l \frac{\left(-\frac{2\log t}{\pi}\right)^{2h+1-2l}}{1-t^2} \log t \, \mathrm{d}t$$

$$= \int_0^1 \frac{2}{2h+1} \sum_{l=0}^{h} B_{2l} \binom{2h+1}{2l} (2^{2l-1}-1)(-1)^{l+1} \left(\frac{\pi}{2}\right)^{2l} \frac{\log^{2h+2-2l} t}{1-t^2} \, \mathrm{d}t.$$

Then, apply Lemma 9

$$= 2 \sum_{l=0}^{h} B_{2l} \frac{(2h)!}{(2l)!} (2^{2l-1}-1)(-1)^{l+1} \left(\frac{\pi}{2}\right)^{2l} (2h-2l+2) \left(1 - \frac{1}{2^{2h+3-2l}}\right)$$
$$\times \zeta(2h+3-2l)$$
$$= \sum_{l=0}^{h} B_{2l} \frac{(2h)!}{(2l)!} (2^{2l-1}-1)(-1)^{l+1} \pi^{2l} (h-l+1) \left(\frac{2^{2h+3-2l}-1}{2^{2h+1}}\right) \zeta(2h+3-2l). \quad \square$$

4. An identity for symmetric polynomials

For dealing with the polynomials P_k, we will need to manage certain identities of symmetric polynomials. More specifically, we are going to use the following result:

Proposition 14.

$$2n(-1)^l s_{n-l}(2^2, \ldots, (2n-2)^2) = \sum_{h=l}^{n} (-1)^h \binom{2h}{2l-1} s_{n-h}(1^2, \ldots, (2n-1)^2),$$

$$(2n+1)(-1)^l s_{n-l}(1^2, \ldots, (2n-1)^2) = \sum_{h=l}^{n} (-1)^h \binom{2h+1}{2l} s_{n-h}(2^2, \ldots, (2n)^2).$$

Proof. These equalities are easier to prove if we think of the symmetric functions as coefficients of certain polynomials, as in Eq. (8).

In order to prove the first equality, multiply by x^{2l} on both sides and add for $l = 1, \ldots, n$:

$$2n \sum_{l=1}^{n} s_{n-l}(2^2, \ldots, (2n-2)^2)(-1)^l x^{2l}$$
$$= \sum_{l=1}^{n} \sum_{h=l}^{n} (-1)^h \binom{2h}{2l-1} s_{n-h}(1^2, \ldots, (2n-1)^2) x^{2l}.$$

The statement we have to prove becomes

$$2n\prod_{j=0}^{n-1}((2j)^2-x^2)=\sum_{h=1}^{n}(-1)^h s_{n-h}(1^2,\ldots,(2n-1)^2)\sum_{l=1}^{h}\binom{2h}{2l-1}x^{2l}. \tag{18}$$

The right-hand side of (18) is

$$=\sum_{h=0}^{n}(-1)^h s_{n-h}(1^2,\ldots,(2n-1)^2)\frac{x}{2}((x+1)^{2h}-(x-1)^{2h})$$

$$=\frac{x}{2}\left(\prod_{j=1}^{n}((2j-1)^2-(x+1)^2)-\prod_{j=1}^{n}((2j-1)^2-(x-1)^2)\right)$$

$$=\frac{x}{2}\left(\prod_{j=1}^{n}(2j+x)(2j-2-x)-\prod_{j=1}^{n}(2j-x)(2j-2+x)\right)$$

$$=((-x)(2n+x)-x(2n-x))\frac{x}{2}\prod_{j=1}^{n-1}((2j)^2-x^2)=2n\prod_{j=0}^{n-1}((2j)^2-x^2)$$

so Eq. (18) is true.

In order to prove the second equality, we apply a similar process. First multiply by x^{2l+1} on both sides and add for $l=1,\ldots,n$:

$$(2n+1)\sum_{l=1}^{n}s_{n-l}(1^2,\ldots,(2n-1)^2)(-1)^l x^{2l+1}$$

$$=\sum_{l=1}^{n}\sum_{h=l}^{n}(-1)^h\binom{2h+1}{2l}s_{n-h}(2^2,\ldots,(2n)^2)x^{2l+1}.$$

Hence, we have to prove

$$(2n+1)x\prod_{j=1}^{n}((2j-1)^2-x^2)=\sum_{h=1}^{n}(-1)^h s_{n-h}(2^2,\ldots,(2n)^2)\sum_{l=1}^{h}\binom{2h+1}{2l}x^{2l+1}. \tag{19}$$

The right-hand side is

$$= \sum_{h=0}^{n} (-1)^h s_{n-h}(2^2, \ldots, (2n)^2) \frac{x}{2}((x+1)^{2h+1} - (x-1)^{2h+1})$$

$$= \frac{x}{2} \left((x+1) \prod_{j=1}^{n} ((2j)^2 - (x+1)^2) - (x-1) \prod_{j=1}^{n} ((2j)^2 - (x-1)^2) \right)$$

$$= \frac{x}{2} \left((x+1) \prod_{j=1}^{n} (2j+1+x)(2j-1-x) - (x-1) \prod_{j=1}^{n} (2j-1+x)(2j+1-x) \right)$$

$$= ((2n+1+x) + (2n+1-x)) \frac{x}{2} \prod_{j=1}^{n} ((2j-1)^2 - x^2)$$

$$= (2n+1)x \prod_{j=1}^{n} ((2j-1)^2 - x^2)$$

thus proving Eq. (19). □

5. Description of the general method

We will prove our main result by first examining a general situation and then specializing to the particular families of the statement.

Let $P_\alpha \in \mathbb{C}[\mathbf{x}]$ such that its coefficients depend polynomially on a parameter $\alpha \in \mathbb{C}$. We replace α by $\left(\frac{x_1-1}{x_1+1}\right) \cdots \left(\frac{x_n-1}{x_n+1}\right)$ and obtain a new polynomial $\tilde{P} \in \mathbb{C}[\mathbf{x}, x_1, \ldots, x_n]$. By definition of Mahler measure, it is easy to see that

$$m(\tilde{P}) = \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} m\left(P_{\left(\frac{x_1-1}{x_1+1}\right) \cdots \left(\frac{x_n-1}{x_n+1}\right)}\right) \frac{\mathrm{d}x_1}{x_1} \cdots \frac{\mathrm{d}x_n}{x_n}.$$

We perform a change of variables to polar coordinates, $x_j = \mathrm{e}^{\mathrm{i}\theta_j}$:

$$= \frac{1}{(2\pi)^n} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \left(P_{\mathrm{i}^n \tan\left(\frac{\theta_1}{2}\right) \cdots \tan\left(\frac{\theta_n}{2}\right)}\right) \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n.$$

Set $x_i = \tan\left(\frac{\theta_i}{2}\right)$. We get

$$= \frac{1}{\pi^n}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} m\left(P_{\mathrm{i}^n x_1\cdots x_n}\right)\frac{\mathrm{d}x_1}{x_1^2+1}\cdots\frac{\mathrm{d}x_n}{x_n^2+1}$$

$$= \frac{2^n}{\pi^n}\int_0^{\infty}\cdots\int_0^{\infty} m\left(P_{\mathrm{i}^n x_1\cdots x_n}\right)\frac{\mathrm{d}x_1}{x_1^2+1}\cdots\frac{\mathrm{d}x_n}{x_n^2+1}.$$

Making one more change, $\hat{x}_1 = x_1, \ldots, \hat{x}_{n-1} = x_1\cdots x_{n-1}, \hat{x}_n = x_1\cdots x_n$:

$$= \frac{2^n}{\pi^n}\int_0^{\infty}\cdots\int_0^{\infty} m\left(P_{\mathrm{i}^n \hat{x}_n}\right)\frac{\hat{x}_1\,\mathrm{d}\hat{x}_1}{\hat{x}_1^2+1}\frac{\hat{x}_2\,\mathrm{d}\hat{x}_2}{\hat{x}_2^2+\hat{x}_1^2}\cdots\frac{\hat{x}_{n-1}\,\mathrm{d}\hat{x}_{n-1}}{\hat{x}_{n-1}^2+\hat{x}_{n-2}^2}\frac{\mathrm{d}\hat{x}_n}{\hat{x}_n^2+\hat{x}_{n-1}^2}.$$

We need to compute this integral. In most of our cases, the Mahler measure of P_α depends only on the absolute value of α. If not, for each n we may modify P, such that it absorbs the number i^n. From now on, we will write $m(P_x)$ instead of $m(P_{\mathrm{i}^n x})$ to simplify notation.

By iterating Proposition 5, the above integral can be written as a linear combination, with coefficients that are rational numbers and powers of π in such a way that the weights are homogeneous, of integrals of the form

$$\int_0^{\infty} m(P_x)\log^j x\,\frac{\mathrm{d}x}{x^2\pm 1}.$$

It is easy to see that j is even iff n is odd and the corresponding sign in that case is "+".

We are going to compute these coefficients.

Let us establish some convenient notation:

Definition 15. Let $a_{n,h} \in \mathbb{Q}$ be defined for $n \geqslant 1$ and $h = 0, \ldots, n-1$ by

$$\int_0^{\infty}\cdots\int_0^{\infty} m\left(P_{x_1}\right)\frac{x_{2n}\,\mathrm{d}x_{2n}}{x_{2n}^2+1}\frac{x_{2n-1}\,\mathrm{d}x_{2n-1}}{x_{2n-1}^2+x_{2n}^2}\cdots\frac{\mathrm{d}x_1}{x_1^2+x_2^2}$$

$$= \sum_{h=1}^{n} a_{n,h-1}\left(\frac{\pi}{2}\right)^{2n-2h}\int_0^{\infty} m(P_x)\log^{2h-1} x\,\frac{\mathrm{d}x}{x^2-1}. \tag{20}$$

Let $b_{n,h} \in \mathbb{Q}$ be defined for $n \geqslant 0$ and $h = 0, \ldots, n$ by

$$\int_0^{\infty}\cdots\int_0^{\infty} m\left(P_{x_1}\right)\frac{x_{2n+1}\,\mathrm{d}x_{2n+1}}{x_{2n+1}^2+1}\frac{x_{2n}\,\mathrm{d}x_{2n}}{x_{2n}^2+x_{2n+1}^2}\cdots\frac{\mathrm{d}x_1}{x_1^2+x_2^2}$$

$$= \sum_{h=0}^{n} b_{n,h}\left(\frac{\pi}{2}\right)^{2n-2h}\int_0^{\infty} m(P_x)\log^{2h} x\,\frac{\mathrm{d}x}{x^2+1}. \tag{21}$$

We claim:

Lemma 16.

$$\sum_{h=0}^{n} b_{n,h} x^{2h} = \sum_{h=1}^{n} a_{n,h-1} \left(P_{2h-1}(x) - P_{2h-1}(\mathrm{i}) \right) \tag{22}$$

$$\sum_{h=1}^{n+1} a_{n+1,h-1} x^{2h-1} = \sum_{h=0}^{n} b_{n,h} P_{2h}(x) \tag{23}$$

Proof. First observe that

$$\sum_{h=0}^{n} b_{n,h} \left(\frac{\pi}{2}\right)^{2n-2h} \int_0^\infty m(P_x) \log^{2h} x \frac{\mathrm{d}x}{x^2+1}$$

$$= \sum_{h=1}^{n} a_{n,h-1} \left(\frac{\pi}{2}\right)^{2n-2h} \int_0^\infty \int_0^\infty m(P_x)\, y \log^{2h-1} y \frac{\mathrm{d}y}{y^2-1} \frac{\mathrm{d}x}{x^2+y^2}. \tag{24}$$

But

$$\int_0^\infty \frac{y \log^{2h-1} y \,\mathrm{d}y}{(y^2+x^2)(y^2-1)} = \left(\frac{\pi}{2}\right)^{2h} \frac{P_{2h-1}\left(\frac{2\log x}{\pi}\right) - P_{2h-1}(\mathrm{i})}{x^2+1}$$

by applying Proposition 5 for $a = x$ and $b = \mathrm{i}$.

The right-hand side of Eq. (24) becomes

$$= \sum_{h=1}^{n} a_{n,h-1} \left(\frac{\pi}{2}\right)^{2n} \int_0^\infty m(P_x) \left(P_{2h-1}\left(\frac{2\log x}{\pi}\right) - P_{2h-1}(\mathrm{i}) \right) \frac{\mathrm{d}x}{x^2+1}.$$

As a consequence, Eq. (24) translates into the polynomial identity (22).

On the other hand,

$$\sum_{h=1}^{n+1} a_{n+1,h-1} \left(\frac{\pi}{2}\right)^{2n+2-2h} \int_0^\infty m(P_x) \log^{2h-1} x \frac{\mathrm{d}x}{x^2-1}$$

$$= \sum_{h=0}^{n} b_{n,h} \left(\frac{\pi}{2}\right)^{2n-2h} \int_0^\infty \int_0^\infty m(P_x)\, y \log^{2h} y \frac{\mathrm{d}y}{y^2+1} \frac{\mathrm{d}x}{x^2+y^2}. \tag{25}$$

But

$$\int_0^\infty \frac{y \log^{2h} y \, \mathrm{d}y}{(y^2+x^2)(y^2+1)} = \left(\frac{\pi}{2}\right)^{2h+1} \frac{P_{2h}\left(\frac{2\log x}{\pi}\right) - P_{2h}(0)}{x^2-1}$$

by applying Proposition 5 for $a = x$ and $b = 1$.

So the right-hand side of (25) becomes

$$= \sum_{h=0}^{n} b_{n,h} \left(\frac{\pi}{2}\right)^{2n+1} \int_0^\infty m(P_y)\, P_{2h}\left(\frac{2\log x}{\pi}\right) \frac{\mathrm{d}x}{x^2-1}$$

which translates into the identity (23). □

Theorem 17. *We have*

$$\sum_{h=0}^{n-1} a_{n,h} x^{2h} = \frac{(x^2+2^2)\cdots(x^2+(2n-2)^2)}{(2n-1)!} \tag{26}$$

for $n \geqslant 1$ *and* $h = 0, \ldots, n-1$, *and*

$$\sum_{h=0}^{n} b_{n,h} x^{2h} = \frac{(x^2+1^2)\cdots(x^2+(2n-1)^2)}{(2n)!} \tag{27}$$

for $n \geqslant 0$ *and* $h = 0, \ldots, n$.

In other words,

$$a_{n,h} = \frac{s_{n-1-h}(2^2, \ldots, (2n-2)^2)}{(2n-1)!}, \tag{28}$$

$$b_{n,h} = \frac{s_{n-h}(1^2, \ldots, (2n-1)^2)}{(2n)!}. \tag{29}$$

Proof. For $2n+1 = 1$, $n = 0$ and the integral becomes

$$\int_0^\infty m(P_x) \frac{\mathrm{d}x}{x^2+1}$$

so $b_{0,0} = 1$.

For $2n = 2$, $n = 1$ and we have

$$\int_0^\infty \int_0^\infty m(P_x) \frac{y\,\mathrm{d}y}{y^2+1} \frac{\mathrm{d}x}{x^2+y^2} = \int_0^\infty m(P_x) \frac{\log x \, \mathrm{d}x}{x^2-1}$$

so $a_{1,0} = 1$.

Then the statement is true for the first two cases.
We proceed by induction. Suppose that

$$a_{n,h} = \frac{s_{n-1-h}(2^2, \ldots, (2n-2)^2)}{(2n-1)!}.$$

We have to prove that

$$b_{n,h} = \frac{s_{n-h}(1^2, \ldots, (2n-1)^2)}{(2n)!}.$$

By Lemma 16, it is enough to prove that

$$\sum_{h=0}^{n} s_{n-h}(1^2, \ldots, (2n-1)^2) x^{2h}$$
$$= 2n \sum_{h=1}^{n} s_{n-h}(2^2, \ldots, (2n-2)^2)\left(P_{2h-1}(x) - P_{2h-1}(\mathrm{i})\right). \tag{30}$$

Recall Eq. (11) that defines the polynomials P_k, from which the following identity may be deduced:

$$x^{2h} = \sum_{k=0}^{h-1} (-1)^k \binom{2h}{2k+1} P_{2h-2k-1}(x). \tag{31}$$

Multiplying Eq. (31) by $s_{n-h}(1^2, \ldots, (2n-1)^2)$ and adding, we get

$$\sum_{h=0}^{n} s_{n-h}(1^2, \ldots, (2n-1)^2) x^{2h}$$
$$= \sum_{h=1}^{n} s_{n-h}(1^2, \ldots, (2n-1)^2) \sum_{k=0}^{h-1} (-1)^k \binom{2h}{2k+1} P_{2h-2k-1}(x)$$
$$+ s_n(1^2, \ldots, (2n-1)^2).$$

Now let us evaluate the above equality at $x = \mathrm{i}$, we obtain

$$\sum_{h=0}^{n} s_{n-h}(1^2, \ldots, (2n-1)^2)(-1)^h$$

$$= \sum_{h=1}^{n} s_{n-h}(1^2, \ldots, (2n-1)^2) \sum_{k=0}^{h-1} (-1)^k \binom{2h}{2k+1} P_{2h-2k-1}(\mathrm{i})$$
$$+ s_n(1^2, \ldots, (2n-1)^2).$$

But

$$\sum_{h=0}^{n} s_{n-h}(1^2, \ldots, (2n-1)^2)(-1)^h = (x+1^2)\cdots(x+(2n-1)^2)|_{x=-1} = 0,$$

from where

$$\sum_{h=0}^{n} s_{n-h}(1^2, \ldots, (2n-1)^2) x^{2h}$$
$$= \sum_{h=1}^{n} s_{n-h}(1^2, \ldots, (2n-1)^2) \sum_{k=0}^{h-1} (-1)^k \binom{2h}{2k+1} (P_{2h-2k-1}(x) - P_{2h-2k-1}(\mathrm{i})).$$

Let $l = h - k$, then this becomes

$$= \sum_{h=1}^{n} s_{n-h}(1^2, \ldots, (2n-1)^2) \sum_{l=1}^{h} (-1)^{h-l} \binom{2h}{2l-1} (P_{2l-1}(x) - P_{2l-1}(\mathrm{i}))$$
$$= \sum_{l=1}^{n} \left(\sum_{h=l}^{n} (-1)^h \binom{2h}{2l-1} s_{n-h}(1^2, \ldots, (2n-1)^2) \right) (-1)^l (P_{2l-1}(x) - P_{2l-1}(\mathrm{i})),$$

and equality (30) is proved by applying Proposition 14.

Now suppose that

$$b_{n,h} = \frac{s_{n-h}(1^2, \ldots, (2n-1)^2)}{(2n)!},$$

we want to see that

$$a_{n+1,h} = \frac{s_{n-h}(2^2, \ldots, (2n)^2)}{(2n+1)!}.$$

Then it is enough to prove that

$$\sum_{h=0}^{n} s_{n-h}(2^2, \ldots, (2n)^2) x^{2h+1} = (2n+1) \sum_{h=0}^{n} s_{n-h}(1^2, \ldots, (2n-1)^2) P_{2h}(x) \qquad (32)$$

by Lemma 16.

Eq. (11) implies

$$x^{2h+1} = \sum_{k=0}^{h} (-1)^k \binom{2h+1}{2k+1} P_{2h-2k}(x),$$

and so,

$$\begin{aligned}
&\sum_{h=0}^{n} s_{n-h}(2^2, \ldots, (2n)^2) x^{2h+1} \\
&\quad = \sum_{h=0}^{n} s_{n-h}(2^2, \ldots, (2n)^2) \sum_{k=0}^{h} (-1)^k \binom{2h+1}{2k+1} P_{2h-2k}(x).
\end{aligned}$$

Let $l = h - k$, then

$$\begin{aligned}
&= \sum_{h=0}^{n} s_{n-h}(2^2, \ldots, (2n)^2) \sum_{l=0}^{h} (-1)^{h-l} \binom{2h+1}{2l} P_{2l}(x) \\
&= \sum_{l=0}^{n} \left(\sum_{h=l}^{n} (-1)^h \binom{2h+1}{2l} s_{n-h}(2^2, \ldots, (2n)^2) \right) (-1)^l P_{2l}(x)
\end{aligned}$$

which proves (32) by Proposition 14. □

6. Proof of the main theorem

In the last section we managed to express the Mahler measure of $\tilde{P}$ as a linear combination of functions that depend on the Mahler measure of P_α. We are now ready to apply that machinery to the specific families of polynomials. At this point we need to strongly use the formulas for the Mahler measure of each particular polynomial P_α.

(i) $P_\alpha(z) = 1 + \alpha z$.

$$m(1 + \alpha z) = \log^+ |\alpha|$$

This is the simplest possible case. For the even case we get

$$\begin{aligned}
&\pi^{2n} m\left(1 + \left(\frac{1-x_1}{1+x_1}\right) \cdots \left(\frac{1-x_{2n}}{1+x_{2n}}\right) z\right) \\
&\quad = 2^{2n} \sum_{h=1}^{n} a_{n,h-1} \left(\frac{\pi}{2}\right)^{2n-2h} \int_0^\infty \log^+ x \log^{2h-1} x \frac{dx}{x^2-1}
\end{aligned}$$

$$= \sum_{h=1}^{n} \frac{s_{n-h}(2^2, \ldots, (2n-2)^2)}{(2n-1)!} 2^{2h} \pi^{2n-2h} \int_1^\infty \log^{2h} x \frac{\mathrm{d}x}{x^2-1}.$$

Now set $y = \frac{1}{x}$,

$$= \sum_{h=1}^{n} \frac{s_{n-h}(2^2, \ldots, (2n-2)^2)}{(2n-1)!} 2^{2h} \pi^{2n-2h} \int_0^1 \log^{2h} y \frac{\mathrm{d}y}{1-y^2}.$$

If we apply Lemma 9, we obtain

$$= \sum_{h=1}^{n} \frac{s_{n-h}(2^2, \ldots, (2n-2)^2)}{(2n-1)!} 2^{2h} \pi^{2n-2h} (2h)! \left(1 - \frac{1}{2^{2h+1}}\right) \zeta(2h+1)$$
$$= \sum_{h=1}^{n} \frac{s_{n-h}(2^2, \ldots, (2n-2)^2)}{(2n-1)!} \frac{(2h)!(2^{2h+1}-1)}{2} \pi^{2n-2h} \zeta(2h+1).$$

For the odd case we get

$$\pi^{2n+1} m\left(1 + \left(\frac{1-x_1}{1+x_1}\right) \cdots \left(\frac{1-x_{2n+1}}{1+x_{2n+1}}\right) z\right)$$
$$= 2^{2n+1} \sum_{h=0}^{n} b_{n,h} \left(\frac{\pi}{2}\right)^{2n-2h} \int_0^\infty \log^+ x \log^{2h} x \frac{\mathrm{d}x}{x^2+1}$$
$$= \sum_{h=0}^{n} \frac{s_{n-h}(1^2, \ldots, (2n-1)^2)}{(2n)!} 2^{2h+1} \pi^{2n-2h} \int_1^\infty \log^{2h+1} x \frac{\mathrm{d}x}{x^2+1}.$$

Now set $y = \frac{1}{x}$,

$$= -\sum_{h=0}^{n} \frac{s_{n-h}(1^2, \ldots, (2n-1)^2)}{(2n)!} 2^{2h+1} \pi^{2n-2h} \int_0^1 \log^{2h+1} y \frac{\mathrm{d}y}{y^2+1}.$$

Now apply Lemma 9,

$$= \sum_{h=0}^{n} \frac{s_{n-h}(1^2, \ldots, (2n-1)^2)}{(2n)!} (2h+1)! 2^{2h+1} \pi^{2n-2h} \mathrm{L}(\chi_{-4}, 2h+2).$$

(ii) $P_\alpha(x, y, z) = (1+x) + \alpha(1+y)z$.

This Mahler measure was computed by Smyth [5,17],

$$\pi^2 m((1+x)+\alpha(1+y)z) = \begin{cases} 2\mathcal{L}_3(|\alpha|) & \text{for} \quad |\alpha| \leqslant 1, \\ \pi^2 \log|\alpha| + 2\mathcal{L}_3\left(|\alpha|^{-1}\right) & \text{for} \quad |\alpha| > 1, \end{cases}$$

where

$$\mathcal{L}_3(\alpha) = -\frac{2}{\alpha}\int_0^1 \frac{\mathrm{d}t}{t^2 - \frac{1}{\alpha^2}} \circ \frac{\mathrm{d}t}{t} \circ \frac{\mathrm{d}t}{t}.$$

Now set $s = t\alpha$,

$$= -2\int_0^\alpha \frac{\mathrm{d}s}{s^2-1} \circ \frac{\mathrm{d}s}{s} \circ \frac{\mathrm{d}s}{s}.$$

We obtain

$$\pi^{2n+2} m\left(1 + x + \left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_{2n}}{1+x_{2n}}\right)(1+y)z\right)$$

$$= 2^{2n}\sum_{h=1}^{n} a_{n,h-1}\left(\frac{\pi}{2}\right)^{2n-2h}\left(-4\int_0^1\int_0^x \frac{\mathrm{d}s}{s^2-1} \circ \frac{\mathrm{d}s}{s} \circ \frac{\mathrm{d}s}{s} \log^{2h-1} x \frac{\mathrm{d}x}{x^2-1}\right.$$

$$\left. + \pi^2\int_1^\infty \log^{2h} x \frac{\mathrm{d}x}{x^2-1} - 4\int_1^\infty \int_0^{\frac{1}{x}} \frac{\mathrm{d}s}{s^2-1} \circ \frac{\mathrm{d}s}{s} \circ \frac{\mathrm{d}s}{s} \log^{2h-1} x \frac{\mathrm{d}x}{x^2-1}\right)$$

$$= \sum_{h=1}^{n} \frac{s_{n-h}(2^2, \ldots, (2n-2)^2)}{(2n-1)!} 2^{2h}\pi^{2n-2h}$$

$$\times \left(-8\int_0^1\int_0^x \frac{\mathrm{d}s}{s^2-1} \circ \frac{\mathrm{d}s}{s} \circ \frac{\mathrm{d}s}{s} \log^{2h-1} x \frac{\mathrm{d}x}{x^2-1} - \pi^2\int_0^1 \log^{2h} x \frac{\mathrm{d}x}{x^2-1}\right).$$

By Lemma 9,

$$= \sum_{h=1}^{n} \frac{s_{n-h}(2^2, \ldots, (2n-2)^2)}{(2n-1)!} 2^{2h}\pi^{2n-2h}$$

$$\times \left((2h-1)!\mathcal{L}_{3,2h}(1,1) + (2h)!\left(1 - \frac{1}{2^{2h+1}}\right)\pi^2\zeta(2h+1)\right).$$

But if we apply Proposition 10, we get just combinations of the Riemann zeta function:

$$= \sum_{h=1}^{n} \frac{s_{n-h}(2^2, \dots, (2n-2)^2)}{(2n-1)!} \pi^{2n-2h}$$
$$\times (2h-1)! \sum_{k=0}^{h-1} \binom{2h-2k+2}{2} (2^{2h-2k+3}-1) \frac{(-1)^k B_{2k}(2\pi)^{2k}}{2(2k)!} \zeta(2h-2k+3).$$

Now set $t = h - k$ and change the order of the sums

$$= \sum_{t=1}^{n} \frac{(2t+2)!(2^{2t+3}-1)}{8}$$
$$\times \left(\sum_{k=0}^{n-t} \frac{s_{n-t-k}(2^2, \dots, (2n-2)^2)}{(2n-1)!} \binom{2t+2k}{2t} (-1)^k \frac{2^{2k}}{t+k} B_{2k} \right) \pi^{2n-2t} \zeta(2t+3).$$

The odd case is

$$\pi^{2n+3} m \left(1 + x + \left(\frac{1-x_1}{1+x_1} \right) \cdots \left(\frac{1-x_{2n+1}}{1+x_{2n+1}} \right) (1+y)z \right)$$
$$= 2^{2n+1} \sum_{h=0}^{n} b_{n,h} \left(\frac{\pi}{2} \right)^{2n-2h} \left(-4 \int_0^1 \int_0^x \frac{\mathrm{d}s}{s^2-1} \circ \frac{\mathrm{d}s}{s} \circ \frac{\mathrm{d}s}{s} \log^{2h} x \frac{\mathrm{d}x}{x^2+1} \right.$$
$$\left. + \pi^2 \int_1^\infty \log^{2h+1} x \frac{\mathrm{d}x}{x^2+1} - 4 \int_1^\infty \int_0^{\frac{1}{x}} \frac{\mathrm{d}s}{s^2-1} \circ \frac{\mathrm{d}s}{s} \circ \frac{\mathrm{d}s}{s} \log^{2h} x \frac{\mathrm{d}x}{x^2+1} \right)$$
$$= \sum_{h=0}^{n} \frac{s_{n-h}(1^2, \dots, (2n-1)^2)}{(2n)!} 2^{2h+1} \pi^{2n-2h}$$
$$\times \left(-8 \int_0^1 \int_0^x \frac{\mathrm{d}s}{s^2-1} \circ \frac{\mathrm{d}s}{s} \circ \frac{\mathrm{d}s}{s} \log^{2h} x \frac{\mathrm{d}x}{x^2+1} - \pi^2 \int_0^1 \log^{2h+1} x \frac{\mathrm{d}x}{x^2+1} \right).$$

By Lemma 9,

$$= \sum_{h=0}^{n} \frac{s_{n-h}(1^2, \dots, (2n-1)^2)}{(2n)!} 2^{2h+1} \pi^{2n-2h}$$
$$\times \left(\mathrm{i}(2h)! \mathcal{L}_{3,2h+1}(\mathrm{i}, \mathrm{i}) + (2h+1)! \pi^2 \mathrm{L}(\chi_{-4}, 2h+2) \right).$$

We should observe that it would be nice to have a simpler expression for $\mathcal{L}_{3,2h+1}(\mathrm{i}, \mathrm{i})$. In fact, we believe that this number should be somehow related to $\mathrm{L}(\chi_{-4}, k)$ (in a result analogous to Eq. (17)), but we have been unable to find such a relation.

(iii) $P_\alpha(z) = 1 + \alpha x + (1-\alpha)y$.

This Mahler measure is a particular case of an example computed by Cassaigne and Maillot [14]. This case is different from the previously studied cases due to the fact that the Mahler measure of this polynomial does not just depend on the absolute value of the parameter α, it also depends on the argument of α. This fact makes the application of the general method a little bit more subtle. We will use

$$\begin{aligned}&\pi m(1+\alpha x+(1-\alpha)y)\\ &\quad= |\arg\alpha|\log|1-\alpha| + |\arg(1-\alpha)|\log|\alpha| + \begin{cases} D(\alpha) & \text{if } \operatorname{Im}(\alpha)\geqslant 0,\\ D(\bar\alpha) & \text{if } \operatorname{Im}(\alpha) < 0.\end{cases}\end{aligned}$$

The deduction of this formula can be found in [13]. For the even case we need to use the formula for the case in which the parameter α is real,

$$m(1+\alpha x+(1-\alpha)y) = \begin{cases} \log^+\alpha & \text{if } \alpha > 0,\\ \log(1-\alpha) & \text{if } \alpha < 0.\end{cases}$$

Then

$$\begin{aligned}&\pi^{2n+1} m\left(1 + \left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_{2n}}{1+x_{2n}}\right)x + \left(1 - \left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_{2n}}{1+x_{2n}}\right)\right)y\right)\\ &\quad= \pi 2^{2n}\sum_{h=1}^{n} a_{n,h-1}\left(\frac{\pi}{2}\right)^{2n-2h}\frac{1}{2}\int_{-\infty}^{\infty} m(P_{(-1)^n x})\log^{2h-1}|x|\frac{\mathrm{d}x}{x^2-1}.\end{aligned}$$

Note that we have taken into account that the formula depends on the sign of the parameter.

$$\begin{aligned}&= \sum_{h=1}^{n}\frac{s_{n-h}(2^2,\ldots,(2n-2)^2)}{(2n-1)!}2^{2h}\pi^{2n+1-2h}\int_0^{\infty}\frac{1}{2}(\log^+ x + \log(1+x))\log^{2h-1}x\\ &\quad\times\frac{\mathrm{d}x}{x^2-1}.\end{aligned}$$

But setting $y = \frac{1}{x}$,

$$\int_0^{\infty}\log(1+x)\log^{2h-1}x\frac{\mathrm{d}x}{x^2-1} = \int_0^{1}\log(1+x)\log^{2h-1}x\frac{\mathrm{d}x}{x^2-1}$$

$$+\int_0^1 \log\left(\frac{1+y}{y}\right)\log^{2h-1} y \frac{\mathrm{d}y}{y^2-1}$$
$$= 2\int_0^1 \log(1+y)\log^{2h-1} y \frac{\mathrm{d}y}{y^2-1}$$
$$+\int_0^1 \log^{2h} y \frac{\mathrm{d}y}{1-y^2}.$$

Then the Mahler measure is

$$= \sum_{h=1}^{n} \frac{s_{n-h}(2^2, \ldots, (2n-2)^2)}{(2n-1)!} 2^{2h}\pi^{2n+1-2h}$$
$$\times \left(\int_0^1 \log(1+y)\log^{2h-1} y \frac{\mathrm{d}y}{y^2-1} + \int_0^1 \log^{2h} y \frac{\mathrm{d}y}{1-y^2}\right).$$

If we apply Lemma 9 and Proposition 11, we obtain

$$= \sum_{h=1}^{n} \frac{s_{n-h}(2^2, \ldots, (2n-2)^2)}{(2n-1)!} 2^{2h}\pi^{2n+1-2h} \left((2h-1)!(2h+1)\frac{2^{2h+1}-1}{2^{2h+2}}\zeta(2h+1)\right.$$
$$+(-1)^{h-1}\frac{\log 2(2^{2h}-1)B_{2h}\pi^{2h}}{4h}$$
$$\left.+\frac{(2h-1)!}{2}\sum_{k=1}^{h-1}\frac{(2^{2h+1-2k}-1)(2^{2k-1}-1)}{2^{2h-2k}}\frac{(-1)^{k-1}B_{2k}}{(2k)!}\pi^{2k}\zeta(2h+1-2k)\right).$$

Finally, by applying equality (41) from the Appendix and changing the order of the sums (and setting $t = h - k$):

$$= \frac{\pi^{2n+1}}{2}\log 2 + \sum_{h=1}^{n}\frac{s_{n-h}(2^2, \ldots, (2n-2)^2)}{(2n-1)!}(2h)!\frac{2^{2h+1}-1}{4}\pi^{2n+1-2h}\zeta(2h+1)$$
$$+\sum_{t=1}^{n}\frac{(2t)!(2^{2t+1}-1)}{4(2n-1)!}$$
$$\times\left(\sum_{k=0}^{n-t} s_{n-t-k}(2^2, \ldots, (2n-2)^2)\binom{2(k+t)}{2l}(-1)^{k-1}\frac{2^{2k}(2^{2k-1}-1)}{k+t}B_{2k}\right)$$
$$\times\pi^{2n+1-2t}\zeta(2t+1).$$

For the odd case we need the formula when the parameter α is purely imaginary,

$$\pi m(1+\mathrm{i}\alpha x+(1-\mathrm{i}\alpha)y)=\frac{\pi}{4}\log\left|\alpha^2+1\right|+\mathrm{Im}\left(\mathrm{Li}_2\left(\mathrm{i}\left|\alpha\right|\right)\right),$$

where $\alpha\in\mathbb{R}$.

$$\pi^{2n+2}m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_{2n+1}}{1+x_{2n+1}}\right)x+\left(1-\left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_{2n+1}}{1+x_{2n+1}}\right)\right)y\right)$$
$$=\pi 2^{2n+1}\sum_{h=0}^{n}b_{n,h}\left(\frac{\pi}{2}\right)^{2n-2h}\frac{1}{2}\int_{-\infty}^{\infty}m(P_{(-1)^n\mathrm{i}x})\log^{2h}|x|\frac{\mathrm{d}x}{x^2+1}$$
$$=2^{2n+1}\sum_{h=0}^{n}b_{n,h}\left(\frac{\pi}{2}\right)^{2n-2h}\int_{0}^{\infty}\left(\frac{\pi}{4}\log(1+x^2)+\mathrm{Im}\left(\mathrm{Li}_2\left(\mathrm{i}x\right)\right)\right)\log^{2h}x\,\frac{\mathrm{d}x}{x^2+1}.$$

We will now apply Propositions 12 and 13,

$$=\sum_{h=0}^{n}\frac{s_{n-h}(1^2,\ldots,(2n-1)^2)}{(2n)!}2^{2h+1}\pi^{2n-2h}\left((-1)^h E_{2h}\left(\frac{\pi}{2}\right)^{2h+2}\log 2\right.$$
$$+\sum_{l=1}^{h}\frac{(2h)!}{(2h-2l)!}(-1)^{h-l}E_{2h-2l}\left(\frac{\pi}{2}\right)^{2h-2l+2}\left(\frac{2^{2l+1}-1}{2^{2l+1}}\right)\zeta(2l+1)$$
$$\left.+\sum_{l=0}^{h}B_{2l}\frac{(2h)!}{(2l)!}(2^{2l-1}-1)(-1)^{l+1}\pi^{2l}(h-l+1)\left(\frac{2^{2h+3-2l}-1}{2^{2h+1}}\right)\zeta(2h+3-2l)\right).$$

Applying Eq. (39) from the Appendix

$$=\frac{\pi^{2n+2}}{2}\log 2+\sum_{h=0}^{n}\frac{s_{n-h}(1^2,\ldots,(2n-1)^2)}{(2n)!}2^{2h+1}\pi^{2n-2h}$$
$$\times\left(\sum_{l=1}^{h}\frac{(2h)!}{(2h-2l)!}(-1)^{h-l}E_{2h-2l}\left(\frac{\pi}{2}\right)^{2h-2l+2}\left(\frac{2^{2l+1}-1}{2^{2l+1}}\right)\zeta(2l+1)\right.$$
$$+\sum_{l=0}^{h}B_{2l}\frac{(2h)!}{(2l)!}(2^{2l-1}-1)(-1)^{l+1}\pi^{2l}(h-l+1)\left(\frac{2^{2h+3-2l}-1}{2^{2h+1}}\right)$$
$$\left.\times\,\zeta(2h+3-2l)\right).$$

Let us observe the following term carefully,

$$\sum_{h=0}^{n} \frac{s_{n-h}(1^2,\dots,(2n-1)^2)}{(2n)!} 2^{2h+1}\pi^{2n-2h}$$
$$\times \sum_{l=0}^{h} B_{2l}\frac{(2h)!}{(2l)!}(2^{2l-1}-1)(-1)^{l+1}\pi^{2l}(h-l+1)\left(\frac{2^{2h+3-2l}-1}{2^{2h+1}}\right)\zeta(2h+3-2l).$$

Set $s=h-l$,

$$= \frac{1}{(2n)!}\sum_{s=0}^{n} \pi^{2n-2s}\zeta(2s+3)(s+1)(2s)!(2^{2s+3}-1)$$
$$\times \sum_{l=0}^{n-s} s_{n-s-l}(1^2,\dots,(2n-1)^2)B_{2l}\binom{2(s+l)}{2l}(2^{2l-1}-1)(-1)^{l+1}.$$

By Theorem A.4 from the Appendix,

$$= \frac{1}{(2n)!}\sum_{s=0}^{n} \pi^{2n-2s}\zeta(2s+3)(s+1)(2s)!(2^{2s+3}-1)\frac{2s+1}{2(2n+1)}s_{n-s}(2^2,\dots,(2n)^2)$$
$$= \frac{1}{(2n+1)!}\sum_{s=0}^{n} \pi^{2n-2s}\zeta(2s+3)(2s+2)!\frac{2^{2s+3}-1}{4}s_{n-s}(2^2,\dots,(2n)^2).$$

Finally, the Mahler measure is

$$= \frac{\pi^{2n+2}}{2}\log 2 + \frac{1}{(2n+1)!}\sum_{s=0}^{n}\frac{(2s+2)!(2^{2s+3}-1)}{4}s_{n-s}(2^2,\dots,(2n)^2)\pi^{2n-2s}$$
$$\times\zeta(2s+3)$$
$$+\sum_{l=1}^{n}\frac{(2l)!(2^{2l+1}-1)}{4(2n)!}\left(\sum_{h=0}^{n-l}s_{n-l-h}(1^2,\dots,(2n-1)^2)\binom{2(h+l)}{2l}(-1)^h E_{2h}\right)$$
$$\times\pi^{2n-2l+2}\zeta(2l+1).$$

Let us also add, that with the help of Proposition A.5 the above equation may be written in terms of Bernoulli numbers instead of Euler numbers.

7. Concluding remarks

In conclusion, the Mahler measure of these three families of n-variable polynomials can be computed explicitly as some linear combination of special values of zeta functions, the L-series on the Dirichlet character of conductor 4, (and $\mathcal{L}_{3,2h+1}(\mathrm{i},\mathrm{i})$ for the second family). It remains to relate $\mathcal{L}_{3,2h+1}(\mathrm{i},\mathrm{i})$ to L-series and perhaps zeta functions, which would simplify formula (4).

In some cases the coefficients of these formulas are related to Bernoulli numbers. It should be remarked that the results of Theorem A.4 and Proposition A.6 suggest that there should be a simpler expression for formulas of the kind of Theorem A.5, and that might allow to find better expressions for the formulas of case (iii) (Eqs. (5) and (6)), for instance.

Finally and most importantly, it would be interesting to find different families, perhaps, by adding new variables by using other forms of fractional transformations or other rational functions.

Acknowledgments

It is my pleasure to thank Fernando Rodriguez-Villegas for several helpful discussions and his constant encouragement. I am also grateful to David Boyd for his useful comments, and to the Referee, whose suggestions have improved the exposition of this article.

This work was developed while I was receiving financial support from the Harrington fellowship and John Tate to whom I am deeply grateful.

Appendix A. Some identities involving Bernoulli and Euler numbers

The main result of this section is a collection of identities involving Bernoulli numbers and symmetric functions, which can be deduced from the explicit form of the polynomials P_k and their behavior as it was studied in Section 5. In addition to those identities, and for completeness, we also mention some other properties of Bernoulli and Euler numbers that have been used in order to simplify the final form of the equations of Theorem 1.

We begin by explicitly computing the polynomials P_k:

Proposition A.1. *We have the following*:

$$P_k(x) = -\frac{2}{k+1}\sum_{h=0}^{k} B_h \binom{k+1}{h}(2^{h-1}-1)\mathrm{i}^h x^{k+1-h}. \tag{33}$$

Proof. It is clear that the equation is true for $k = 0, 1$. We will prove that the properties of Lemma 4 hold. But these properties are straightforward except for (4). Then it is enough to verify property (4).

$$-\frac{2l+1}{2\,\mathrm{i}^{2l+1}}P_{2l}(\mathrm{i}) = \sum_{h=0}^{2l} B_h \binom{2l+1}{h}\left(2^{h-1}-1\right).$$

Thus, it suffices to prove that

$$0 \stackrel{?}{=} \sum_{h=0}^{2l} B_h \binom{2l+1}{h}\left(2^{h-1}-1\right) \qquad \text{for } l > 0.$$

Using the well-known identity:

$$\sum_{s=0}^{k} \binom{k+1}{s} B_s = 0 \tag{34}$$

for $k = 2l$, we conclude that we only need to prove,

$$0 \stackrel{?}{=} \sum_{h=0}^{2l+1} B_h \binom{2l+1}{h} 2^{h-1} \qquad \text{for } l > 0$$

since

$$B_{2j+1} = 0 \quad j = 1, 2, \ldots$$

but that is true, because of this other well known identity

$$\left(1-2^{k-1}\right)B_k = \sum_{s=0}^{k} 2^{s-1}\binom{k}{s} B_s \quad \text{for } n > 1. \qquad \square \tag{35}$$

Let us mention the following technical consequence that will be used later.

Corollary A.2. *We have the following special values*:

$$P_{2l-1}(\mathrm{i}) = (-1)^l \frac{2^{2l}-1}{l} B_{2l}. \tag{36}$$

Proof.

$$P_{2l-1}(\mathrm{i}) = -\frac{1}{l}\sum_{h=0}^{2l-1} B_h \binom{2l}{h}(2^{h-1}-1)(-1)^l = \frac{(-1)^{l+1}}{l}\sum_{h=0}^{2l-1} B_h\binom{2l}{h}2^{h-1}$$
$$= \frac{(-1)^{l+1}}{l}((1-2^{2l-1})B_{2l} - 2^{2l-1}B_{2l}) = \frac{(-1)^{l+1}}{l}(1-2^{2l})B_{2l}$$

because of Eq. (35). □

In fact,

Corollary A.3. *We have*

$$P_k(x) = \frac{2\mathrm{i}^{k+1}}{k+1}\left(B_{k+1}\left(\frac{x}{\mathrm{i}}\right) - 2^k B_{k+1}\left(\frac{x}{2\mathrm{i}}\right)\right) + \frac{(2^{k+1}-2)\mathrm{i}^{k+1}}{k+1}B_{k+1}, \tag{37}$$

where $B_k(x)$ is the Bernoulli polynomial.

We are now ready to prove the main Theorem of this section.

Theorem A.4. *We have the following identities*:
For $1 \leqslant l \leqslant n$:

$$s_{n-l}(1^2,\ldots,(2n-1)^2) = n\sum_{s=0}^{n-l} s_{n-l-s}(2^2,\ldots,(2n-2)^2)\frac{1}{l+s}B_{2s}\binom{2(l+s)}{2s}(2^{2s}-2)(-1)^{s+1}.$$

For $1 \leqslant n$:

$$\left(\frac{(2n)!}{2^n n!}\right)^2 = 2n\sum_{s=1}^{n} s_{n-s}(2^2,\ldots,(2n-2)^2)\frac{1}{s}B_{2s}(2^{2s}-1)(-1)^{s+1}.$$

For $0 \leqslant l \leqslant n$:

$$(2l+1)s_{n-l}(2^2,\ldots,(2n)^2) = (2n+1)\sum_{s=0}^{n-l} s_{n-l-s}(1^2,\ldots,(2n-1)^2)B_{2s}\binom{2(l+s)}{2s}(2^{2s}-2)(-1)^{s+1}.$$

Proof. By Lemma 16 and Theorem 17 we have

$$\frac{(x^2+1^2)\cdots(x^2+(2n-1)^2)}{(2n)!}$$
$$=\sum_{h=1}^{n}\frac{s_{n-h}(2^2,\dots,(2n-2)^2)}{(2n-1)!}\left(-\frac{2}{2h}\sum_{j=0}^{2h-1}B_j\binom{2h}{j}(2^{j-1}-1)\mathrm{i}^j x^{2h-j}\right.$$
$$\left.-(-1)^h\frac{2^{2h}-1}{h}B_{2h}\right).$$

Set $j=2s$, then the first term in the difference is

$$\sum_{h=1}^{n}\frac{s_{n-h}(2^2,\dots,(2n-2)^2)}{(2n-1)!}\left(-\frac{1}{h}\sum_{s=0}^{h-1}B_{2s}\binom{2h}{2s}(2^{2s-1}-1)(-1)^s x^{2h-2s}\right).$$

Now set $l=h-s$,

$$=\sum_{h=1}^{n}\frac{s_{n-h}(2^2,\dots,(2n-2)^2)}{(2n-1)!}\left(\frac{1}{h}\sum_{l=1}^{h}B_{2(h-l)}\binom{2h}{2l}(2^{2(h-l)-1}-1)(-1)^{h-l+1}x^{2l}\right)$$
$$=\sum_{l=1}^{n}\left(\sum_{h=l}^{n}\frac{s_{n-h}(2^2,\dots,(2n-2)^2)}{(2n-1)!}\frac{1}{h}B_{2(h-l)}\binom{2h}{2l}(2^{2(h-l)-1}-1)(-1)^{h-l+1}\right)x^{2l}.$$

Comparing coefficients we get

$$\frac{s_{n-l}(1^2,\dots,(2n-1)^2)}{(2n)!}$$
$$=\sum_{h=l}^{n}\frac{s_{n-h}(2^2,\dots,(2n-2)^2)}{(2n-1)!}\frac{1}{h}B_{2(h-l)}\binom{2h}{2l}(2^{2(h-l)-1}-1)(-1)^{h-l+1}.$$

Thus

$$s_{n-l}(1^2,\dots,(2n-1)^2)$$
$$=n\sum_{s=0}^{n-l}s_{n-l-s}(2^2,\dots,(2n-2)^2)\frac{1}{l+s}B_{2s}\binom{2(l+s)}{2s}(2^{2s}-2)(-1)^{s+1}.$$

The second equality is obtained by comparing the independent coefficients.

For the third equality, we do a similar process:

$$\frac{x(x^2+2^2)\cdots(x^2+(2n)^2)}{(2n+1)!}$$
$$=\sum_{h=0}^{n}\frac{s_{n-h}(1^2,\ldots,(2n-1)^2)}{(2n)!}\left(-\frac{2}{2h+1}\sum_{j=0}^{2h}B_j\binom{2h+1}{j}(2^{j-1}-1)\mathrm{i}^j x^{2h+1-j}\right)$$
$$=\sum_{h=0}^{n}\frac{s_{n-h}(1^2,\ldots,(2n-1)^2)}{(2n)!}\left(-\frac{2}{2h+1}\sum_{s=0}^{h}B_{2s}\binom{2h+1}{2s}\right.$$
$$\left.\times(2^{2s-1}-1)(-1)^s x^{2h+1-2s}\right).$$

Now set $l=h-s$,

$$=\sum_{h=0}^{n}\frac{s_{n-h}(1^2,\ldots,(2n-1)^2)}{(2n)!}\left(\frac{2}{2h+1}\sum_{l=0}^{h}B_{2(h-l)}\binom{2h+1}{2l+1}\right.$$
$$\left.\times(2^{2(h-l)-1}-1)(-1)^{h-l+1}x^{2l+1}\right)$$
$$=\sum_{l=0}^{n}\left(\sum_{h=l}^{n}\frac{s_{n-h}(1^2,\ldots,(2n-1)^2)}{(2n)!}\frac{2}{2h+1}B_{2(h-l)}\binom{2h+1}{2l+1}\right.$$
$$\left.\times(2^{2(h-l)-1}-1)(-1)^{h-l+1}\right)x^{2l+1}.$$

Comparing coefficients we get

$$\frac{s_{n-l}(2^2,\ldots,(2n)^2)}{(2n+1)!}$$
$$=\sum_{h=l}^{n}\frac{s_{n-h}(1^2,\ldots,(2n-1)^2)}{(2n)!}\frac{2}{2h+1}B_{2(h-l)}\binom{2h+1}{2l+1}(2^{2(h-l)-1}-1)(-1)^{h-l+1}.$$

Thus

$$(2l+1)s_{n-l}(2^2,\ldots,(2n)^2)$$
$$=(2n+1)\sum_{s=0}^{n-l}s_{n-l-s}(1^2,\ldots,(2n-1)^2)B_{2s}\binom{2(l+s)}{2s}(2^{2s}-2)(-1)^{s+1}.\qquad\Box$$

The next result illuminates the last formula of Theorem 1.

Proposition A.5. *We have*

$$n\sum_{s=0}^{n-l} s_{n-l-s}(2^2,\ldots,(2n-2)^2)\frac{1}{l+s}B_{2s}\binom{2(l+s)}{2s}2^{2s}(2^{2s}-2)(-1)^{s+1}$$
$$=\sum_{k=l}^{n}(-1)^{k+l}\binom{2k}{2l}s_{n-k}(1^2,\ldots,(2n-1)^2)E_{2(k-l)}. \tag{38}$$

Proof. By Proposition 14,

$$n\sum_{s=0}^{n-l} s_{n-l-s}(2^2,\ldots,(2n-2)^2)\frac{1}{l+s}B_{2s}\binom{2(l+s)}{2s}2^{2s}(2^{2s}-2)(-1)^{s+1}$$
$$=\sum_{s=0}^{n-l}\frac{(-1)^{l+s}}{2}$$
$$\times\sum_{k=l+s}^{n}(-1)^k\binom{2k}{2s+2l-1}s_{n-k}(1^2,\ldots,(2n-1)^2)\frac{1}{l+s}$$
$$\times B_{2s}\binom{2(l+s)}{2s}2^{2s}(2^{2s}-2)(-1)^{s+1}$$
$$=\sum_{s=0}^{n-l}\sum_{k=l+s}^{n}\frac{(-1)^{k+l+1}}{2k+1}\binom{2k+1}{2l}s_{n-k}(1^2,\ldots,(2n-1)^2)$$
$$\times B_{2s}\binom{2(k-l)+1}{2s}2^{2s}(2^{2s}-2).$$

Changing the order of the sums,

$$=\sum_{k=l}^{n}\frac{(-1)^{k+l+1}}{2k+1}\binom{2k+1}{2l}s_{n-k}(1^2,\ldots,(2n-1)^2)\sum_{s=0}^{k-l}B_{2s}\binom{2(k-l)+1}{2s}2^{2s}(2^{2s}-2).$$

Now observe that

$$\sum_{s=0}^{k-l}B_{2s}\binom{2(k-l)+1}{2s}2^{2s}(2^{2s}-2)=\sum_{m=0}^{2(k-l)+1}B_m\binom{2(k-l)+1}{m}2^m(2^m-2).$$

By Eq. (35),

$$= \sum_{m=0}^{2(k-l)+1} B_m \binom{2(k-l)+1}{m} 2^{2m} = 4^{2(k-l)+1} B_{2(k-l)+1}\left(\frac{1}{4}\right)$$

for $k - l > 0$ and $= -1$ otherwise.

Now use the following identity:

$$2^{2n} B_n\left(\frac{1}{4}\right) = (2 - 2^n) B_n - n E_{n-1},$$

which can be found, for instance, as Eq. (23.1.27) in [1].

Then, we get

$$\sum_{s=0}^{k-l} B_{2s} \binom{2(k-l)+1}{2s} 2^{2s}(2^{2s} - 2) = -(2(k-l)+1) E_{2(k-l)}.$$

Therefore,

$$-\sum_{k=l}^{n} \frac{(-1)^{k+l+1}}{2k+1} \binom{2k+1}{2l} s_{n-k}(1^2, \ldots, (2n-1)^2)(2(k-l)+1) E_{2(k-l)}$$

$$= \sum_{k=l}^{n} (-1)^{k+l} \binom{2k}{2l} s_{n-k}(1^2, \ldots, (2n-1)^2) E_{2(k-l)}. \qquad \square$$

We would like to finish by stating a few basic equalities that can be proved by induction:

Proposition A.6. *We have*

$$\sum_{h=0}^{n} s_{n-h}(1^2, \ldots, (2n-1)^2)(-1)^h E_{2h} = (2n)!, \tag{39}$$

$$\sum_{h=0}^{n} s_{n-h}(1^2, \ldots, (2n-1)^2)(-1)^{h+1} E_{2(h+1)} = (2n+1)!, \tag{40}$$

$$\sum_{h=1}^{n} s_{n-h}(2^2, \ldots, (2n-2)^2)(-1)^{h+1} \frac{2^{2h}(2^{2h}-1)}{h} B_{2h} = 2(2n-1)!. \tag{41}$$

References

[1] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Reprint of the 1972 edition, Dover Publications, Inc., New York, 1992, xiv+1046pp.
[2] L.V. Ahlfors, Complex Analysis, International Series in Pure and Applied Mathematics, McGraw-Hill, New York, 1979.
[3] M.J. Bertin, Une mesure de Mahler explicite, C. R. Acad. Sci. Paris Sér. I. Math. 333 (2001) 1–3.
[4] J.M. Borwein, D.M. Bradley, D.J. Broadhurst, Evaluations of k-fold Euler/Zagier sums: a compendium of results for arbitrary k, Electronic J. Combin. 4 (2) (1997) #R5.
[5] D.W. Boyd, Speculations concerning the range of Mahler's measure, Canad. Math. Bull. 24 (1981) 453–469.
[6] D.W. Boyd, Mahler's measure and special values of L-functions, Experiment. Math. 7 (1998) 37–82.
[7] D.W. Boyd, Mahler's measure and invariants of hyperbolic manifolds, in: M.A. Bennett et al. (Eds.), Number Theory for the Millennium, A K Peters, Boston, 2002, pp. 127–143.
[8] D.W. Boyd, F. Rodriguez-Villegas, Mahler's measure and the dilogarithm (I), Canad. J. Math. 54 (2002) 468–492.
[9] D.W. Boyd, F. Rodriguez-Villegas, With an appendix by N.M. Dunfield: Mahler's measure and the dilogarithm (II), (preprint, July 2003), math.NT/0308041.
[10] J.D. Condon, Calculation of the Mahler measure of a three variable polynomial, preprint, October 2003.
[11] A.B. Goncharov, Polylogarithms in arithmetic and geometry, in: Proceedings, ICM-94, Zurich, 1995, pp. 374–387.
[12] A.B. Goncharov, Multiple polylogarithms and mixed Tate motives, preprint, March 2001, math.AG/0103059.
[13] M.N. Lalín, Some examples of Mahler measures as multiple polylogarithms, J. Number Theory 103 (2003) 85–108.
[14] V. Maillot, Géométrie d'Arakelov des variétés toriques et fibrés en droites intégrables, Mém. Soc. Math. Fr. (N.S.) 80 (2000) 129.
[15] F. Rodriguez-Villegas, Modular Mahler measures I, in: Topics in Number Theory (University Park, PA, 1997), Mathematical Applications, vol. 467, Kluwer Academic Publishers, Dordrecht, 1999, pp. 17–48.
[16] C.J. Smyth, On measures of polynomials in several variables, Bull. Austral. Math. Soc. Ser. A 23 (1981) 49–63 Corrigendum (with G. Myerson): Bull. Austral. Math. Soc. 26 (1982) 317–319.
[17] C.J. Smyth, An explicit formula for the Mahler measure of a family of 3-variable polynomials, J. Theory Nombres Bordeaux 14 (2002) 683–700.
[18] C.J. Smyth, Explicit formulas for the Mahler measures of families of multivariable polynomials, preprint, April 2003.
[19] S. Vandervelde, A formula for the Mahler measure of $axy + bx + cy + d$, J. Number Theory 100 (2003) 184–202.