Journal of the Egyptian Mathematical Society (2016) 24, 522–525

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

Original Article

On some subclasses of bi-univalent functions associated with pseudo-starlike functions

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Received 5 January 2016; revised 26 March 2016; accepted 28 March 2016
Available online 20 May 2016

Keywords

Univalent function;
Coefficient estimates;
Bi-univalent functions;
λ-bi-pseudo-starlike functions

Abstract In this paper, we have established the Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ of the two new subclasses of bi-univalent functions using pseudo-starlike functions in the open unit disk $U = \{z : |z| < 1\}$.

2010 Mathematics Subject Classification: 30C45; 30C50; 30C80

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$. Let S denote the subclass of A, which consist of functions of the form (1.1) that are univalent and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$ in U.

A function $f \in S$ is said to be starlike of order $\alpha (0 \le \alpha < 1)$ if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha,\ z \in U$$

and convex of order $\alpha (0 \le \alpha < 1)$ if and only if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha,\ z \in U.$$

Denote these classes respectively by $S^*(\alpha)$ and $K(\alpha)$.

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Peer review under responsibility of Egyptian Mathematical Society.

Production and hosting by Elsevier

S1110-256X(16)30024-4
http://dx.doi.org/10.1016/j.joems.2016.03.007

The Koebe one quarter theorem [1] asserts that the image of U under every univalent function $f \in S$ contains a disk of radius $\frac{1}{4}$. Thus every univalent function $f \in S$ has an inverse f^{-1}, defined by $f^{-1}[f(z)] = z$, $(z \in U)$ and $f[f^{-1}(w)] = w$, $(|w| < r_0(f);\ r_0(f) \geq \frac{1}{4})$, where

$$f^{-1}(w) = w - a_2\, w^2 + (2a_2^2 - a_3)\, w^3 - (5a_2^3 - 5a_2a_3 + a_4)\, w^4 + \cdots . \tag{1.2}$$

A function $f \in S$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U. Let Σ denote the class of bi-univalent functions in U given by (1.1). Some functions in the class Σ are as below (see Srivastava et al. [2]):

$$\frac{z}{1-z}\,, \quad -log(1-z)\,, \quad \frac{1}{2}log\left(\frac{1+z}{1-z}\right).$$

However, the familiar Koebe function is not bi-univalent.

The class Σ of bi-univalent functions was first investigated by Lewin [3] and it was shown that $|a_2| < 1.51$. Brannan and Clunie [4] improved Lewin's result and conjectured that $|a_2| \leq \sqrt{2}$. Later, Netanyahu [5], showed that $\max_{f\in\Sigma} |a_2| = \frac{4}{3}$. Brannan and Taha [6] also introduced certain subclasses of the bi-univalent function class Σ and obtained estimates for their initial coefficients.

Recently, Srivastava et al. [2], Frasin and Aouf [7], Bansal and Sokol [8] and Srivastava and Bansal [9] are also introduced and investigated the various subclasses of bi-univalent functions and obtained bounds for the initial coefficients $|a_2|$ and $|a_3|$. The coefficient problem i.e. bound of $|a_n|$ $(n \in \mathbb{N}\backslash\{1, 2\})$ for each $f \in \Sigma$ given by (1.1) is still an open problem.

Babalola [10] defined the class $\mathscr{L}_\lambda(\beta)$ of λ-pseudo-starlike functions of order β as below:

Definition 1. [10] Let $f \in A$, suppose $0 \leq \beta < 1$ and $\lambda \geq 1$ is real. Then $f(z) \in \mathscr{L}_\lambda(\beta)$ of λ-pseudo-starlike functions of order β in the unit disk if and only if

$$Re\left(\frac{z\,[f'(z)]^\lambda}{f(z)}\right) > \beta\,, \qquad (z \in U).$$

Babalola [10] proved that, all pseudo-starlike functions are Bazilevič of type $(1 - \frac{1}{\lambda})$, order $\beta^{(\frac{1}{\lambda})}$ and univalent in open unit disk U.

Motivated by this, we propose to introduce two new subclasses of bi-univalent function class Σ and find estimates on the initial coefficients $|a_2|$ and $|a_3|$. The techniques used are same as Srivastava et al. [2] and also we note that similar type of techniques for $\exp(-\phi(\xi))$ -expansion method used by Abdelrehman et al. [11] and $(\frac{G'}{G})$ -expansion method, were used by Zayed and Gepreel [12]. These methods are used to find exact traveling wave solutions of nonlinear partial differential equations in mathematical physics.

In order to derive our main results, we have to recall here the following lemma.

Lemma 1. *[13] Let $h \in P$ the family of all functions h analytic in U for which $Re\{h(z)\} > 0$ and have the form*

$$h(z) = 1 + p_1\, z + p_2\, z^2 + p_3\, z^3 + \cdots$$

for $z \in U$. Then $|p_n| \leq 2$, for each n.

2. Main results

2.1. Coefficient bounds for the function class $\mathscr{L}B_\Sigma^\lambda(\alpha)$

Definition 2. A function $f(z)$ given by (1.1) is said to be in the class $\mathscr{L}B_\Sigma^\lambda(\alpha)$ if the following conditions are satisfied:

$$f \in \Sigma$$

$$\text{and} \quad \left|arg\left(\frac{z[f'(z)]^\lambda}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \quad (z \in U;\, 0 < \alpha \leq 1,\ \lambda \geq 1) \tag{2.1}$$

$$\text{and} \quad \left|arg\left(\frac{w[g'(w)]^\lambda}{g(w)}\right)\right| < \frac{\alpha\pi}{2} \quad (w \in U;\, 0 < \alpha \leq 1,\ \lambda \geq 1)\,, \tag{2.2}$$

where the function g is extension of f^{-1} to U, and is given by

$$g(w) = w - a_2\, w^2 + (2a_2^2 - a_3)\, w^3 - (5a_2^3 - 5a_2a_3 + a_4)\, w^4 + \cdots .$$

We call $\mathscr{L}B_\Sigma^\lambda(\alpha)$ be the class of strongly λ-bi-pseudo-starlike functions of order α.

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for the function class $\mathscr{L}B_\Sigma^\lambda(\alpha)$.

Theorem 1. *Let $f(z)$ given by ( ) be in the class $\mathscr{L}B_\Sigma^\lambda(\alpha)$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{(2\lambda-1)(2\lambda-1+\alpha)}} \tag{2.3}$$

$$|a_3| \leq \frac{4\alpha^2}{(2\lambda-1)^2} + \frac{2\alpha}{(3\lambda-1)} \tag{2.4}$$

Proof. Clearly, conditions (2.1) and (2.2) can be written as

$$\frac{z[f'(z)]^\lambda}{f(z)} = [p(z)]^\alpha \tag{2.5}$$

and

$$\frac{w[g'(w)]^\lambda}{g(w)} = [q(w)]^\alpha \tag{2.6}$$

respectively.

Where $p(z), q(w) \in P$ and have the forms

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$$
$$\text{and} \quad q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots .$$

Clearly,

$$[p(z)]^\alpha = 1 + \alpha p_1\, z + \left(\alpha\, p_2 + \frac{\alpha(\alpha-1)}{2}\, p_1^2\right) z^2 + \cdots$$
$$\text{and} \quad [q(w)]^\alpha = 1 + \alpha q_1 w + \left(\alpha q_2 + \frac{\alpha(\alpha-1)}{2} q_1^2\right) w^2 + \cdots .$$

Also

$$\frac{z[f'(z)]^{\lambda}}{f(z)} = 1 + (2\lambda - 1)a_2 z + [(3\lambda - 1)a_3 + (2\lambda^2 - 4\lambda + 1)a_2^2]\, z^2 + \cdots$$

and $$\frac{w[g'(w)]^{\lambda}}{g(w)} = 1 - (2\lambda - 1)a_2 w + [(2\lambda^2 + 2\lambda - 1)a_2^2 - (3\lambda - 1)a_3]\, w^2 + \cdots$$

Now, equating the coefficients in (2.5) and (2.6), we get

$$(2\lambda - 1)a_2 = \alpha p_1 \;, \tag{2.7}$$

$$(3\lambda - 1)a_3 + (2\lambda^2 - 4\lambda + 1)a_2^2 = \alpha\; p_2 + \frac{\alpha(\alpha - 1)}{2}\; p_1^2 \;, \tag{2.8}$$

$$-(2\lambda - 1)a_2 = \alpha q_1 \;, \tag{2.9}$$

$$(2\lambda^2 + 2\lambda - 1)a_2^2 - (3\lambda - 1)a_3 = \alpha q_2 + \frac{\alpha(\alpha - 1)}{2} q_1^2. \tag{2.10}$$

From (2.7) and (2.9), we get

$$p_1 = -q_1 \tag{2.11}$$

and $$2(2\lambda - 1)^2\, a_2^2 = \alpha^2\big(p_1^2 + q_1^2\big) \tag{2.12}$$

Now by adding equation (2.10) and equation (2.8) , we get

$$(4\lambda^2 - 2\lambda)\; a_2^2 = \alpha(p_2 + q_2) + \frac{\alpha(\alpha - 1)}{2}\big(p_1^2 + q_1^2\big)\;,$$

by using (2.12), we get

$$(4\lambda^2 - 2\lambda)\; a_2^2 = \alpha(p_2 + q_2) + \frac{\alpha(\alpha - 1)}{2}\left(\frac{2(2\lambda - 1)^2\; a_2^2}{\alpha^2}\right),$$

$$\Rightarrow a_2^2 = \frac{\alpha^2(p_2 + q_2)}{(2\lambda - 1)(2\lambda - 1 + \alpha)}.$$

Applying Lemma 1 for the coefficients p_2 and q_2 , we immediately have

$$|a_2| \le \frac{2\alpha}{\sqrt{(2\lambda - 1)(2\lambda - 1 + \alpha)}}.$$

This gives the bound on $|a_2|$ as asserted in (2.3).

Next, in order to find the bound on $|a_3|$, by subtracting Eq. (2.10) from Eq. (2.8), we get

$$2(3\lambda - 1)\; a_3 - 2(3\lambda - 1)\; a_2^2 = \alpha(p_2 - q_2) + \frac{\alpha(\alpha - 1)}{2}\big(p_1^2 - q_1^2\big).$$

From (2.11) we get $p_1^2 = q_1^2$ and also using (2.12), we have

$$2(3\lambda - 1)\; a_3 - 2(3\lambda - 1)\left(\frac{\alpha^2(p_1^2 + q_1^2)}{2(2\lambda - 1)^2}\right) = \alpha(p_2 - q_2)$$

$$(3\lambda - 1)\left\{2\; a_3 - \left(\frac{\alpha^2(2p_1^2)}{(2\lambda - 1)^2}\right)\right\} = \alpha(p_2 - q_2) \quad (\text{by using } p_1^2 = q_1^2)$$

$$\Rightarrow 2\; a_3 = \frac{2\alpha^2\; p_1^2}{(2\lambda - 1)^2} + \frac{\alpha}{(3\lambda - 1)}(p_2 - q_2).$$

Applying Lemma 1 once again for the coefficients p_1, q_1 p_2 and q_2 , we get

$$|a_3| \le \frac{4\alpha^2}{(2\lambda - 1)^2} + \frac{2\alpha}{(3\lambda - 1)}.$$

This completes the proof of Theorem 1. □

Putting $\lambda = 1$ in Theorem 1, we have

Corollary 1. *Let f(z) given by (1.1) be in the class $\mathscr{L}B^1_\Sigma(\alpha)$. Then*

$$|a_2| \le \frac{2\alpha}{\sqrt{(1 + \alpha)}}$$

$$|a_3| \le 4\alpha^2 + \alpha$$

3. Coefficient bounds for the function class $\mathscr{L}B_\Sigma(\lambda, \beta)$

Definition 3. A function *f*(*z*) given by (1.1) is said to be in the class $\mathscr{L}B_\Sigma(\lambda, \beta)$ if the following conditions are satisfied:

$$f \in \Sigma$$

and $$Re\left(\frac{z[f'(z)]^{\lambda}}{f(z)}\right) > \beta \quad (z \in U;\; 0 \le \beta < 1, \lambda \ge 1) \tag{3.1}$$

and $$Re\left(\frac{w[g'(w)]^{\lambda}}{g(w)}\right) > \beta \quad (w \in U;\; 0 \le \beta < 1, \lambda \ge 1)\;, \tag{3.2}$$

where the function g is extension of f^{-1} to U, and is given by

$$g(w) = w - a_2\; w^2 + (2a_2^2 - a_3)\; w^3 - (5a_2^3 - 5a_2a_3 + a_4)\; w^4 + \cdots.$$

We call $\mathscr{L}B_\Sigma(\lambda, \beta)$ be the class of λ-bi-pseudo-starlike functions of order β.

For functions in the class $\mathscr{L}B_\Sigma(\lambda, \beta)$, the following coefficient estimates hold.

Theorem 2. *Let f(z) given by (1.1) be in the class $\mathscr{L}B_\Sigma(\lambda, \beta)$. Then*

$$|a_2| \le \sqrt{\frac{2(1 - \beta)}{\lambda(2\lambda - 1)}} \tag{3.3}$$

$$|a_3| \le \frac{4(1 - \beta)^2}{(2\lambda - 1)^2} + \frac{2(1 - \beta)}{(3\lambda - 1)} \tag{3.4}$$

Proof. Clearly, conditions (3.1) and (3.2) can be written as

$$\frac{z[f'(z)]^{\lambda}}{f(z)} = \beta + (1 - \beta)\; p(z) \tag{3.5}$$

and

$$\frac{w[g'(w)]^{\lambda}}{g(w)} = \beta + (1 - \beta)\; q(w) \tag{3.6}$$

respectively.

Where $p(z), q(w) \in P$ and have the forms

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$$
$$\text{and } q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots.$$

Clearly,

$$\beta + (1-\beta)\, p(z) = 1 + (1-\beta)\, p_1\, z + (1-\beta)\, p_2\, z^2 + \cdots$$
$$\text{and } \beta + (1-\beta)\, q(w) = 1 + (1-\beta)\, q_1\, w + (1-\beta)\, q_2\, w^2 + \cdots.$$

Also

$$\frac{z[f'(z)]^\lambda}{f(z)} = 1 + (2\lambda-1)a_2\, z + [(3\lambda-1)a_3 + (2\lambda^2 - 4\lambda + 1)a_2^2]\, z^2 + \cdots$$

$$\text{and } \frac{w[g'(w)]^\lambda}{g(w)} = 1 - (2\lambda-1)a_2\, w + [(2\lambda^2 + 2\lambda - 1)a_2^2 - (3\lambda-1)a_3]\, w^2 + \cdots$$

Now, equating the coefficients in (3.5) and (3.6), we get

$$(2\lambda-1)a_2 = (1-\beta)\, p_1, \tag{3.7}$$

$$(3\lambda-1)a_3 + (2\lambda^2 - 4\lambda + 1)a_2^2 = (1-\beta)\, p_2, \tag{3.8}$$

$$-(2\lambda-1)a_2 = (1-\beta)\, q_1, \tag{3.9}$$

$$(2\lambda^2 + 2\lambda - 1)a_2^2 - (3\lambda-1)a_3 = (1-\beta)\, q_2. \tag{3.10}$$

From (3.7) and (3.9), we get

$$p_1 = -q_1 \tag{3.11}$$

$$\text{and } 2(2\lambda-1)^2 a_2^2 = (1-\beta)^2(p_1^2 + q_1^2) \tag{3.12}$$

Now by adding Eq. (3.10) and Eq. (3.8) , we get

$$(4\lambda^2 - 2\lambda)\, a_2^2 = (1-\beta)(p_2 + q_2)$$

$$a_2^2 = \frac{(1-\beta)}{2\lambda\,(2\lambda-1)}(p_2 + q_2)$$

Thus, we have

$$|a_2^2| \le \frac{(1-\beta)}{2\lambda\,(2\lambda-1)}\,(|p_2| + |q_2|)$$

Applying Lemma 1 for the coefficients p_2 and q_2 , we have

$$|a_2| \le \sqrt{\frac{2(1-\beta)}{\lambda\,(2\lambda-1)}}.$$

Which is the bound on $|a_2|$ as given in (3.3).

Next, in order to find the bound on $|a_3|$, by subtracting Eq. (3.10) from Eq. (3.8), we get

$$2(3\lambda-1)\, a_3 - 2(3\lambda-1)\, a_2^2 = (1-\beta)(p_2 - q_2)$$
$$2(3\lambda-1)\, a_3 = 2(3\lambda-1)\, a_2^2 + (1-\beta)(p_2 - q_2).$$

From (3.11) we get $p_1^2 = q_1^2$ and also using (3.12), we have

$$2(3\lambda-1)\, a_3 = 2(3\lambda-1)\, \frac{(1-\beta)^2}{2(2\lambda-1)^2}(p_1^2 + q_1^2) + (1-\beta)(p_2 - q_2)$$
$$a_3 = \frac{(1-\beta)^2}{2(2\lambda-1)^2}(p_1^2 + q_1^2) + \frac{(1-\beta)}{2(3\lambda-1)}(p_2 - q_2)$$
$$a_3 = \frac{(1-\beta)^2}{2(2\lambda-1)^2}(2p_1^2) + \frac{(1-\beta)}{2(3\lambda-1)}(p_2 - q_2) \quad (\text{by using } p_1^2 = q_1^2)$$
$$a_3 = \frac{(1-\beta)^2}{(2\lambda-1)^2}(p_1^2) + \frac{(1-\beta)}{2(3\lambda-1)}(p_2 - q_2)$$

Applying Lemma 1 for the coefficients p_1, q_1 p_2 and q_2 , we get

$$|a_3| \le \frac{(1-\beta)^2}{(2\lambda-1)^2}(4) + \frac{(1-\beta)}{2(3\lambda-1)}(4)$$
$$\Rightarrow |a_3| \le \frac{4(1-\beta)^2}{(2\lambda-1)^2} + \frac{2(1-\beta)}{(3\lambda-1)}.$$

This completes the proof of Theorem 2. □

Putting $\lambda = 1$ in Theorem 2, we have

Corollary 2. *Let f(z) given by* (1.1) *be in the class* $\mathscr{L}B_\Sigma(1,\beta)$. *Then*

$$|a_2| \le \sqrt{2(1-\beta)}$$
$$|a_3| \le 4(1-\beta)^2 + (1-\beta).$$

Acknowledgments

The authors wish to express their sincere thanks to the referee of this paper for several useful comments and suggestions.

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