A solution to the hydrodynamic lubrication of a circular point contact sliding over a flat surface with cavitation

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Abstract This letter presents an analytical solution to the hydrodynamic lubrication of a circular point contact sliding over a flat surface with cavitation. The solution is found by solving the Reynolds equation with Reynolds boundary condition for cavitation. The cavitation boundary is shown to be straight lines directed 108.4˚ against the sliding direction. The result is experimentally verified in the limit of large values of viscosity, sliding velocity and radius of a spherical ball. The solution raises questions about the coupling between cavitation and film rupture and can be used as an independent check on the validity of numerical solutions. © 2012 The Chinese Society of Theoretical and Applied Mechanics. [doi:10.1063/2.1203204]

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A sliding lubricated point contact appears for example when a spherical ball is translated along a lubricated plane, see Fig. 1. This case is one of the most fundamental ones in the science of tribology. Close to the point of contact effects of elasto-hydrodynamics (EHD) is important due to the high pressures of the lubricant, capable of elastically deforming the ball. These high pressures also affect the viscosity of the lubricant, known as the piezoviscous phenomena.\(^1\) Outside of the EHD region, which normally is very small, the pressure of the lubricant quickly falls to levels which do not deform the ball and the viscosity can be considered constant. Lubrication with constant viscosity without elastic deformation is called hydrodynamic lubrication (HL).

The incompressible HL of a sliding circular point contact is governed by the Reynolds equation

\[
\nabla \cdot \left( h^3 \nabla p \right) = -6 \mu V \frac{\partial h}{\partial x},
\]

where \( p \) is the pressure, \( h \) is the film thickness, \( \mu \) is the dynamic viscosity and \( V \) is the sliding velocity of the plane in the negative \( x \) direction (keeping the ball fixed).

The only known solution to this problem is of Sommerfeld type, for which cavitation is excluded,\(^2\) it is given by

\[
p = \frac{6}{5} \frac{\mu V}{h^2} x.
\]

Lubricants like oil and grease cannot withstand large negative pressures and ultimately will cavitate, followed by film rupture in the shape of streamers. Rough estimates of cavitation is commonly introduced by utilizing the half-Sommerfeld approximation, i.e., cutting the Sommerfeld solution at the cavitation pressure \( p = 0 \). These approximations suffer from the drawback of not fulfilling the continuity of flow.\(^3\) The necessary conditions for continuity across the cavitation boundary is

\[
\nabla p = 0,
\]

also known as Reynolds boundary condition. Analytical solutions of HL with cavitation is very rare due to the unknown cavitation boundary and the non-linearity it implies.

In this letter we like to draw attention to a certain solution to the Reynolds equation valid for the hydrodynamic region of a sliding circular point contact that fulfills the Reynolds cavitation condition. Solutions of this type may bring valuable insight into thin film cavitation and the subsequent film rupture.

Let us rewrite the governing problem in polar coordinates, see Fig. 2. The film thickness is then approximately described by \( h \sim r^2/(2R) \) for \( r \ll R \).

Reynolds equation (1) then becomes

\[
\frac{\partial^2 p}{\partial r^2} + \frac{7}{r} \frac{\partial p}{\partial r} + \frac{1}{r^2} \frac{\partial^2 p}{\partial \theta^2} = -48 \mu VR^2 \cos \theta \frac{1}{r^5},
\]

and Reynolds cavitation condition is

\[
p = \frac{\partial p}{\partial r} = \frac{\partial p}{\partial \theta} = 0, \text{ at } \Gamma,
\]

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where $\Gamma$ is the unknown cavitation boundary. Furthermore, the solution needs to be bounded

$$p \to 0 \text{ as } r \to \infty,$$

(6)

and symmetric with respect to $\theta$.

To solve this problem, we assume a solution of the form

$$p = \frac{A}{r^3} f(\theta),$$

(7)

where the function $f$, by substitution into Eqs. (4) and (5), is governed by

$$f'' - 9f = -10 \cos \theta,$$

(8)

$$f = f' = 0, \text{ at } \Gamma,$$

(9)

and $A = 24\mu VR^2/5$ is a constant.

This has the solution

$$f = \cos \theta + c \cosh(3\theta),$$

(10)

with

$$\frac{c \cosh(3\theta_0)}{\cos \theta_0} = \frac{\tan \theta_0}{3 \tanh(3\theta_0)} = -1,$$

(11)

where the cavitation boundary $\Gamma$ is found to be straight lines at the angles $\theta_0 = \pm 108.4^\circ$ and $c = 0.002164$.

The hydrodynamic pressure distribution is then finally given by

$$p = \frac{24}{5} \mu VR^2 \left[\cos \theta + c \cosh(3\theta)\right],$$

(12)

which is depicted in Fig. 3 up to the line of cavitation. Note that the first term of Eq. (12) is the Sommerfeld solution Eq. (2) written in polar coordinates.

Experimental evidence of the cavitation lines is found in the limit of large values of $\mu VR^2$ (see Fig. 4). For smaller values of $\mu VR^2$, the onset of film rupture is postponed downstream and it is unclear whether Eq. (12) is able to predict this.

The other important scalar in 2D lubrication theory is the volumetric stream function $Q$ defined as

$$q = \nabla \times (Qe_z),$$

(13)

where

$$q = -\frac{h^3}{12\mu} \nabla p + V \frac{h}{2}$$

(14)

is the volume flow per unit width and $e_z$ is the unit vector perpendicular to the plane. In cylindrical coordinates using Eq. (12), we get by integration

$$Q = \frac{VR^3}{20R} \left[c \sinh(3\theta) - 2\sin \theta\right],$$

(15)

where we let the stream function be zero at $\theta = 0$. The stream function is depicted in Fig. 5.

By inspection of Fig. 5 of the volumetric stream function, we find an interesting flow pattern downstream of the cavitation line. A second line, “the flow asymptote”, is found, which ultimately blocks the film flow if it does not rupture into streamers. This line, which is the dividing stream line $\theta = 0$, appears at

$$\frac{\sin \theta_\infty}{\sinh(3\theta_\infty)} = \frac{c}{2},$$

(16)

giving the angle $\theta_\infty = \pm 136.5^\circ$. 

Fig. 2. Polar coordinates.

Fig. 3. A plot of the pressure distribution Eq. (12).

Fig. 4. The formation of streamers along the lines of cavitation in the limit of large $\mu VR^2$.

Fig. 5. The volumetric stream function.
Fig. 5. The dimensionless volumetric stream function \( \frac{Q}{VR^2} \).

In future work, the existence of the flow asymptote needs to be addressed. If it exists, we shall have to investigate whether the cavitation line can coexist, which would imply that film rupture is a phenomenon of its own, not governed by the Reynolds boundary condition Eq. (3).

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