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Numerical Bounds on Fluctuating Linear Processes

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Let A be an $n \times n$ primitive nonnegative matrix. The long-run behavior $x_k/\|x_k\|_1$ of the linear process $x_{k+1} = x_k A$ is determined by the stochastic eigenvector π of A. In this paper we consider the linear process $x_{k+1} = x_k A_k$, where each A_k fluctuates about A, and provide numerical bounds on the difference between $x_k/\|x_k\|$ and π, thus showing how well π describes the long-run behavior of this fluctuating behavior. © 2001 Academic Press

The behavior of a linear process such as $x_{k+1} = x_k A$, where A is an $n \times n$ primitive nonnegative matrix, depends on the stochastic eigenvector belonging to the largest eigenvalue of A [3]. If the transition matrix fluctuates (the more likely case), yielding the process $x_{k+1} = x_k A_k$, then little is known. If A is a stochastic matrix, bounds on the components of the x_k's can be found [1] and thus something of the long-run behavior can be established. If A is simply nonnegative, some conditions which ensure that the x_k's converge to some limiting set are shown in [4]. However, computable bounds which indicate numerically where the x_k's tend are not known. In this paper, such bounds are established. In particular, if the A_k's remain near A, we show how closely the stochastic eigenvector of A describes the behavior of the fluctuating system.

In this paper, all vectors are row vectors. Thus, when multiplied by a matrix they appear to the left of the matrix.

Results. Throughout the paper we let A denote an $n \times n$ primitive matrix. By the Perron–Frobenius Theorem, A has a positive eigenvalue λ and corresponding stochastic eigenvector π.

By a fluctuation of A we will mean a nonnegative matrix $A + E$ where the entries of E are small compared to the corresponding entries of A.

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We suppose that the entries of E are bounded, $|e_{ij}| \leq \mathscr{E}_{ij}$, and $a_{ij} - \mathscr{E}_{ij} > 0$ for all i, j.

Let ρ denote the projective pseudo-metric on positive vectors defined by

$$\rho(x, y) = \max_{i,j} \ln\left(\frac{x_i/y_i}{x_j/y_j}\right).$$

Using this, the coefficient of ergodicity is defined for any primitive matrix B as

$$\mathscr{T}(B) = \sup_{\substack{x,y \\ x \neq y}} \rho(xB, yB)/\rho(x, y).$$

It follows that

$$\rho(xB, yB) \leq \mathscr{T}(B)\rho(x, y), \qquad \text{for all positive vectors } x, y.$$

It is known [3] that $\mathscr{T}(B) \leq 1$ and that if B has all its entries positive, then $\mathscr{T}(B) < 1$.

Since A is primitive there is a positive integer r such that A^r has all entries positive. Thus, its coefficient of ergodicity is positive. Throughout the paper we define

$$\mathscr{T} = \mathscr{T}(A^r).$$

Another number used in our bound work follows. Using ρ, define

$$\delta = \sup \rho(xA, x(A + E)),$$

where the sup is over all fluctuations $A + E$ and all positive vectors x. We show that δ is finite by proving the following two lemmas.

LEMMA 1. *Let $A + E$ be a fluctuation of A. Define $RE_i = \sup_{x>0}(|(xE)_i|/(xA)_i)$ and $RE = \max_i RE_i$. Then $RE \leq \max_{i,j}(\mathscr{E}_{ij}/a_{ij})$.*

Proof. For each i

$$RE_i = \sup_{x>0} \frac{|(xE)_i|}{(xA)_i} = \sup_{x>0} \frac{|x_1 e_{1i} + x_2 e_{2i} + \cdots + x_n e_{ni}|}{x_1 a_{1i} + x_2 a_{2i} + \cdots + x_n a_{ni}}$$

$$\leq \sup_{x>0} \frac{x_1 \mathscr{E}_{1i} + x_2 \mathscr{E}_{2i} + \cdots + x_n \mathscr{E}_{ni}}{x_1 a_{1i} + x_2 a_{2i} + \cdots + x_n a_{ni}}$$

$$\leq \max_k \frac{\mathscr{E}_{ki}}{a_{ki}},$$

an easily established bound which can be found in [2, p. 79]. Thus, $RE \leq \max_{i,j}(\mathscr{E}_{ij}/a_{ij})$. ∎

LEMMA 2. *Let $A + E$ be a fluctuation of A. If $RE < 1$, then, for any positive vector x,*

$$\rho(xA, x(A+E)) \leq \frac{1+RE}{1-RE}.$$

Proof. Let $x > 0$. For simplicity, set $(xA)_k = z_k$ and $(xE)_k = e_k$ for all k. Then, for any i and j,

$$\frac{z_j + e_j}{z_j} = 1 + \frac{e_j}{z_j} \qquad \text{and} \qquad \frac{z_i}{z_i + e_i} = \frac{1}{1 + e_i/z_i}.$$

Thus,

$$\frac{z_i}{z_i + e_i} \frac{z_j + e_j}{z_j} = \frac{1 + e_j/z_j}{1 + e_i/z_i}.$$

From this we have

$$\rho(xA, x(A+E)) \leq \ln \frac{1+RE}{1-RE}.$$

From this, the result follows. ∎

We put the lemmas together.

THEOREM 1. *If* $\max(\mathscr{E}_{ij}/a_{ij}) < 1$ *for all i, j. Then*

$$\delta \leq \ln \frac{1 + \max_{i,j}(\mathscr{E}_{ij}/a_{ij})}{1 - \max_{i,j}(\mathscr{E}_{ij}/a_{ij})}$$

We now consider the fluctuating linear process

$$x_0 > 0$$
$$x_{k+1} = x_k A_k,$$

where A_k is a fluctuation of A for each k. Our goal is to obtain some information about the behavior of the iterates $x_1/\|x_1\|, x_2/\|x_2\|, \ldots$. To do this, let S^+ denote the set of positive stochastic vectors. Note that ρ is a metric on S^+. Let

$$\bar{x}_k = x_k/\|x_k\|_1 \qquad \text{for all } k.$$

Thus $\bar{x}_1, \bar{x}_2, \ldots$ are scaled from the iterates $x_1, x_2, \ldots$ to be stochastic vectors.

The following lemma on vectors in S^+ is useful throughout.

LEMMA 3. *Let x, y be in S^+ and let $\epsilon > 0$. If $\rho(x, y) \le \epsilon$, then* $\|x - y\|_1 \le e^\epsilon - 1$.

Proof. Since $\rho(x, y) \le \epsilon$ it follows that

$$\max_{i,j} \ln \frac{x_i}{y_i}\frac{y_j}{x_j} \le \epsilon \qquad \text{or} \qquad \max_{i,j} \frac{x_i}{y_i}\frac{y_j}{x_j} \le e^\epsilon.$$

Thus,

$$\frac{1}{\max_{i,j}(x_i/y_i)(y_j/x_j)} \ge \epsilon^{-\epsilon}.$$

And since $1/\max_{i,j}(x_i y_j / y_i x_j) = \min_{i,j}(y_i x_j / x_i y_j)$, it follows that

$$\min_{i,j} \frac{x_i}{y_i}\frac{y_j}{x_j} \ge e^{-\epsilon}.$$

Thus,

$$e^{-\epsilon} \le \frac{x_i}{y_i}\frac{y_j}{x_j} \le e^\epsilon \qquad \text{for all } i, j.$$

Since x, y are stochastic there is an r and an s such that

$$\frac{y_r}{x_r} \le 1 \qquad \text{and} \qquad \frac{y_s}{x_s} \ge 1.$$

Using this, we see that

$$e^{-\epsilon} \le \frac{x_i}{y_i} \le e^\epsilon \qquad \text{for all } i.$$

Thus,

$$e^{-\epsilon} y_i \le x_i \le e^\epsilon y_i \qquad \text{and} \qquad (e^{-\epsilon} - 1) y_i \le x_i - y_i \le (e^\epsilon - 1) y_i$$

for all i. Since $e^\epsilon - 1 \ge 1 - e^{-\epsilon}$ it follows that

$$|x_i - y_i| \le (e^\epsilon - 1) y_i \qquad \text{for all } i.$$

Summing over i yields

$$\|x - y\|_1 \le e^\epsilon - 1.$$

∎

Concerning the behavior of the iterates $\bar{x}_1, \bar{x}_2, \ldots$ we first give a result on the closeness of any $\bar{x}_k$ to π. This requires two lemmas.

LEMMA 4. *For all k and t < r,*

$$\rho\left(\bar{x}_{kr}A^t, \bar{x}_{kr}A_{kr}\ldots A_{kr+t-1}\right) \le t\delta.$$

Proof. Note that

$$\begin{aligned}
&\rho\left(\bar{x}_{kr}A^t, \bar{x}_{kr}A_{kr}\ldots A_{kr+t-1}\right)\\
&\quad\le \rho\left(\bar{x}_{kr}A^t, \bar{x}_{kr}A_{kr}\ldots A_{kr+t-2}A\right)\\
&\qquad + \rho\left(\bar{x}_{kr}A_{kr}\ldots A_{kr+t-2}A, \bar{x}_{kr}A_{kr}\ldots A_{kr+t-1}\right)\\
&\quad\le \rho\left(\bar{x}_{kr}A^{t-1}, \bar{x}_{kr}A_{kr}\ldots A_{kr+t-2}\right) + \delta
\end{aligned}$$

and continuing

$$\le (t-1)\delta + \delta = t\delta.$$

∎

LEMMA 5. *For all k,*

$$\rho(\pi, \bar{x}_{kr}) \le \mathscr{T}^{k-1}\rho(\pi, \bar{x}_r) + (1 + \mathscr{T} + \cdots + \mathscr{T}^{k-2})r\delta.$$

Proof. Using Lemma 4,

$$\begin{aligned}
\rho(\pi, \bar{x}_{kr}) &= \rho\left(\pi A^r, \bar{x}_{(k-1)r}A^r\right)\\
&\qquad + \rho\left(\bar{x}_{(k-1)r}A^r, \bar{x}_{(k-1)r}A_{(k-1)r}\ldots A_{kr-1}\right)\\
&\le \mathscr{T}\rho\left(\pi, \bar{x}_{(k-1)r}\right) + r\delta.
\end{aligned}$$

For simplicity set $d_k = \rho(\pi, \bar{x}_{kr})$. Thus,

$$\begin{aligned}
d_2 &\le \mathscr{T}d_1 + r\delta\\
d_3 &\le \mathscr{T}d_2 + r\delta.
\end{aligned}$$

Substitution leads to

$$d_k \le \mathscr{T}^{k-1}d_1 + \mathscr{T}^{k-2}(r\delta) + \mathscr{T}^{r-3}(r\delta) + \cdots + r\delta.$$

From this the result follows. ∎

Putting the lemmas together, we have the following theorem.

THEOREM 2. *For all k and t < r,*

$$\rho(\pi, \bar{x}_{kr+t}) \le \left[\mathscr{T}^{k-1}\rho(\pi, \bar{x}_r) + (1 + \mathscr{T} + \cdots + \mathscr{T}^{k-2})r\delta\right] + t\delta.$$

Proof. Using Lemma 4,

$$\begin{aligned}\rho(\pi, \bar{x}_{kr+t}) &= \rho\left(\pi A^t, \bar{x}_{kr} A_{kr} \dots A_{kr+t-1}\right) \\ &\le \rho\left(\pi A^t, \bar{x}_{kr} A^t\right) + \rho\left(\bar{x}_{kr} A^t, \bar{x}_{kr} A_{kr} \dots A_{kr+t-1}\right) \\ &\le \rho(\pi, \bar{x}_{kr}) + t\delta.\end{aligned}$$

Now using Lemma 5

$$\rho(\pi, \bar{x}_{kr+t}) \le \left[\mathscr{T}^{k-1}\rho(\pi, \bar{x}_r) + (1 + \mathscr{T} + \cdots + \mathscr{T}^{k-2}) r\delta\right] + t\delta.$$

∎

This theorem ensures that in the long run $\bar{x}_1, \bar{x}_2, \dots$ get within $\frac{r\delta}{1-\mathscr{T}} + (r-1)\delta$ of π. We now intend to show that $\bar{x}_1, \bar{x}_2, \dots$ actually tend to a set of stochastic vectors. To do this, we use the following lemma.

LEMMA 6. *Let x be a vector in S^+ with $\rho(\pi, x) \le a$. Then* $\min_i x_i \ge (1/n)/(\max_{i,j}(\pi_i/\pi_j)e^a)$.

Proof. Since $\rho(\pi, x) \le a$,

$$\ln \max_{imj} \frac{\pi_i}{x_i} \frac{x_j}{\pi_j} \le a \qquad \text{or} \qquad \max_{i,j} \frac{\pi_i}{x_i} \frac{x_i}{\pi_j} \le e^a.$$

Suppose $\max_i x_i = x_p$ and $\min_i x_i = x_q$. Then

$$\frac{\pi_q}{x_q} \frac{x_p}{\pi_p} \le e^a.$$

So

$$\frac{x_p}{x_q} \le \max_{i,j} \frac{\pi_i}{\pi_j} e^a.$$

Since x is stochastic, $x_\rho \ge \frac{1}{n}$, so

$$\frac{1/n}{x_q} \le \max_{i,j} \frac{\pi_i}{\pi_j} e^a \qquad \text{and} \qquad \frac{1/n}{\max_{i,j}(\pi_i/\pi_j) e^a} \le x_q.$$

∎

Let b be a positive number. Define

$$S_b = \{x\colon x \text{ is stochastic and } x_i \ge b \text{ for all } i\}.$$

We show that S_b is compact in S^+ with the ρ metric.

LEMMA 7. *S_b is compact.*

Proof. First note that, in the 1-norm, S_b is compact. Now, let $z_1, z_2, \dots$ be a sequence in S_b. Then, in the 1-norm, there is a subsequence $z_{k_1}, z_{k_2}, \dots$ which converges to say $z \in S_b$. Then, by the continuity of ρ, $z_{k_1}, z_{k_2}, \dots$ converges to z in the ρ metric. Thus, S_b is compact in S^+ with the ρ metric. ∎

Define

$$C = \left\{ x \in S^+ : \rho(\pi, x) \le \frac{r\delta}{1 - \mathscr{T}} + (r-1)\delta \right\}.$$

Using Lemma 6 and Lemma 7, and that closed sets inside compact sets are themselves compact, it follows that C is compact in the ρ metric. Define, for any $x \in S^+$,

$$\rho(x, C) = \min_{c \in C} \rho(x, c),$$

the distance of x from C.

We now show that the iterates $\bar{x}_1, \bar{x}_2, \dots$ tend to the subset C of stochastic vectors. The theorem requires three lemmas.

LEMMA 8. *Let x be a vector in S^+ and suppose, without loss of generality, that $\pi_1/x_1 \ge \pi_2/x_2 \ge \cdots \ge \pi_n/x_n$. Then, for any α, $0 \le \alpha \le 1$,*

$$\frac{\pi_1}{\alpha\pi_1 + (1-\alpha)x_1} \ge \frac{\pi_2}{\alpha\pi_2 + (1-\alpha)x_2} \ge \cdots \ge \frac{\pi_n}{\alpha\pi_n + (1-\alpha)x_n}.$$

Proof. We show that if $\pi_i/x_i \ge \pi_j/x_j$, then $\pi_i/(\alpha\pi_i + (1-\alpha)x_i) \ge \pi_j/(\alpha\pi_j + (1-\alpha)x_j)$, where $0 \le \alpha \le 1$. This follows from the following equivalent inequalities:

$$\frac{\pi_i}{x_i} \ge \frac{\pi_j}{x_j}$$

$$\pi_i x_j \ge \pi_j x_i$$

$$\alpha\pi_i\pi_j + (1-\alpha)\pi_i x_j \ge \alpha\pi_i\pi_j + (1-\alpha)\pi_j x_i$$

$$\pi_i\big(\alpha\pi_j + (1-\alpha)x_j\big) \ge \pi_j\big(\alpha\pi_i + (1-\alpha)x_i\big)$$

$$\frac{\pi_i}{\alpha\pi_i + (1-\alpha x_i)} \ge \frac{\pi_j}{\alpha\pi_j + (1-\alpha)x_j}.$$

∎

LEMMA 9. *Using the hypothesis of Lemma* 8,

(i) $\rho(\pi, x) = \ln(\pi_i/x_1)(x_n/\pi_n)$.

(ii) $\rho(\pi, \alpha\pi + (1-\alpha)x) = \ln(\pi_1/(\alpha\pi_1 + (1-\alpha)x_1))((\alpha\pi_n + (1-\alpha)x_n)/\pi_n)$.

(iii) $\rho(\pi, x) = \rho(\pi, \alpha\pi + (1-\alpha)x) + \rho(\alpha\pi + (1-\alpha)x, x)$.

Proof. Both (i) and (ii) are applications of Lemma 8. By reversing the roles of x and π in (ii)

$$\rho(\alpha\pi + (1-\alpha)x, x) = \ln\left(\frac{x_n}{\alpha\pi_n + (1-\alpha)x_n}\frac{\alpha\pi_1 + (1-\alpha)x_1}{x_1}\right).$$

Result (iii) follows by direct calculation. ∎

LEMMA 10. *If x is a vector in S^+ and*

$$\rho(\pi, x) \le \frac{r\delta}{1-\mathscr{T}} + (r-1)\delta + \epsilon,$$

then there is a vector $\bar{x} \in C$ such that $\rho(x, \bar{x}) \le \epsilon$.

Proof. If $\rho(\pi, x) \le \frac{r\delta}{1-\mathscr{T}} + (r-1)\delta$ take $\bar{x} = x$. Otherwise, consider $x(\alpha) = \alpha\pi + (1-\alpha)x$. Since $x(0) = x$ and $x(1) = \pi$, α can be chosen so that $\rho(\pi, x(\alpha)) = \frac{r\delta}{1-\mathscr{T}} + (r-1)\delta$. Set $\bar{x} = x(\alpha)$ for this α. Then, by Lemma 9,

$$\begin{aligned}\rho(\pi, x) &= \rho(\pi, \bar{x}) + \rho(\bar{x}, x)\\ &= \frac{r\delta}{1-\mathscr{T}} + (r-1)\delta + \rho(\bar{x}, x).\end{aligned}$$

Since we are given that

$$\rho(\pi, x) \le \frac{r\delta}{1-\mathscr{T}} + (r-1)\delta + \epsilon,$$

it follows that

$$\rho(\bar{x}, x) \le \epsilon.$$

∎

THEOREM 3. *For any k and $t < r$,*

$$\rho(C, \bar{x}_{kr+t}) \le \mathscr{T}^k\rho(\pi, \bar{x}_r).$$

Proof. By Theorem 2,

$$\rho(\pi, \bar{x}_{kr+t}) \leq \left[\mathscr{T}^{k-1}\rho(\pi, \bar{x}_r) + (1 + \mathscr{T} + \cdots + \mathscr{T}^{k-2}) r\delta\right] + t\delta,$$

$$\leq \mathscr{T}^{k-1}\rho(\pi, \bar{x}_r) + \frac{r\delta}{1 - \mathscr{T}} + t\delta.$$

By Lemma 10, with $\epsilon = \mathscr{T}^{k-1}\rho(\pi, \bar{x}_r)$ we have $\bar{x} \in C$ such that

$$\rho(\bar{x}, \bar{x}_{kr+t}) \leq \mathscr{T}^{k-1}\rho(\pi, \bar{x}_r).$$

∎

As a concluding result, we now show that small changes in the iterates $\bar{x}_1, \bar{x}_2, \ldots$ also indicate within or close to C. This requires a preliminary lemma.

LEMMA 11. *Let x be a positive vector. Then, for any i, if π_i denotes the stochastic eigenvector of A_i,*

$$\lim_{k \to \infty} \rho(\pi_i, xA_i^k) = 0.$$

Hence $\pi_i \in C$.

Proof. It is known that $\lim_{k \to \infty} \frac{xA_i^k}{\|xA_i^k\|_1} = \pi_i$. Thus

$$\lim_{k \to \infty} \rho(\pi_i, xA_i^k) = \lim_{k \to \infty} \rho\left(\pi_i, \frac{xA_i^k}{\|xA_i^k\|_1}\right) = \rho(\pi_i, \pi_i) = 0.$$

Now, if we have in the sequence $A_j = A_i$ for all $j \geq i$, the previous results still hold. Thus, $\pi_i \in C$. ∎

THEOREM 4. *For all i, if $\mathscr{T}(A_i) \leq \bar{\bar{\mathscr{T}}}$ and $\bar{\bar{\mathscr{T}}} < 1$, then*

$$\rho(C, \bar{x}_i) \leq \frac{r}{1 - \bar{\bar{\mathscr{T}}}}\rho(\bar{x}_i, \bar{x}_{i+1}).$$

Proof. By the triangular inequality

$$\rho(\bar{x}_i, \bar{x}_{i+1}A_i^{kr+t}) \leq \rho(\bar{x}_i, \bar{x}_{i+1}) + \rho(\bar{x}_i A_i, \bar{x}_{i+1} A_i) + \cdots$$

$$+ \rho(\bar{x}_i A_i^{kr+t}, \bar{x}_{i+1} A^{kr+t})$$

$$\leq r\rho(\bar{x}_i, \bar{x}_{i+1}) + r\bar{\bar{\mathscr{T}}}\rho(\bar{x}_i, \bar{x}_{i+1}) + \cdots$$

$$+ r\bar{\bar{\mathscr{T}}}^{k-1}\rho(\bar{x}_i, \bar{x}_{i+1}) + t\bar{\bar{\mathscr{T}}}^k\rho(\bar{x}_i, \bar{x}_{i+1})$$

$$\leq \frac{r}{1 - \bar{\bar{\mathscr{T}}}}\rho(\bar{x}_i, \bar{x}_{i+1}) + t\bar{\bar{\mathscr{T}}}^k\rho(\bar{x}_i, \bar{x}_{i+1}).$$

Now, letting $k \to \infty$,

$$\rho(\bar{x}_i, \pi_i) \leq \frac{r}{1 - \mathscr{T}} \rho(\bar{x}_i, \bar{x}_{i+1}).$$

Since by Lemma 11, $\pi_i \in C$,

$$\rho(C, \bar{x}_i) \leq \frac{r}{1 - \mathscr{T}} \rho(x_i, \bar{x}_{i+1}).$$

∎

An example putting the work together may be helpful.

EXAMPLE. Let $A = \begin{bmatrix} 5 & 6 \\ 6 & 5 \end{bmatrix}$ and $\mathscr{E} = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{bmatrix}$. Using the formulas in [2], $\mathscr{T} = 0.10909$. Direct calculation also shows that

$$RE \leq 0.02041 \qquad \text{and} \qquad \delta \leq 0.04082.$$

Thus, the radius of C is $\frac{\delta}{1 - \mathscr{T}} = 0.04582$. Using Theorem 3, this says that the normalized iterates $\bar{x}_k$ tend to the stochastic vectors in C or stay within $e^{0.04582} - 1 = 0.04689$ of π.

REFERENCES

1. D. J. Hartfiel, Component bounds on Markov set-chain limiting sets, *J. Statist. Comput. Simulation* **38** (1991), 15–24.
2. M. Marcus and H. Minc, "Modern University Algebra," MacMillan Co., New York, 1966.
3. Eugene Seneta, "Nonnegative Matrices and Markov Chains," 2nd ed., Springer–Verlag, Berlin, 1981.
4. Eugene Seneta, On the limiting set of nonnegative matrix products, *Statist. Probab. Lett.* **2** (1984), 159–163.