Journal of the Egyptian Mathematical Society (2015) 23, 263–270

Egyptian Mathematical Society
Journal of the Egyptian Mathematical Society
www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

Legendre wavelet operational matrix method for solution of fractional order Riccati differential equation

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Received 22 February 2014; revised 10 April 2014; accepted 22 April 2014
Available online 24 May 2014

KEYWORDS

Riccati equation;
Non linear ODE;
Fractional order differential equation;
Legendre wavelet;
Convergence criteria;
Analytical approximation

Abstract A Legendre wavelet operational matrix method (LWM) presented for the solution of nonlinear fractional order Riccati differential equations, having variety of applications in engineering and applied science. The fractional order Riccati differential equations converted into a system of algebraic equations using Legendre wavelet operational matrix. Solutions given by the proposed scheme are more accurate and reliable and they are compared with recently developed numerical, analytical and stochastic approaches. Comparison shows that the proposed LWM approach has a greater performance and less computational effort for getting accurate solutions. Further existence and uniqueness of the proposed problem are given and moreover the condition of convergence is verified.

2010 MATHEMATICS SUBJECT CLASSIFICATION: 34A50; 34K37; 65T60

1. Introduction

In recent years use of fractional-order derivative going very strongly in engineering and life sciences and also in other area of sciences. One of the best advantages of use of fractional differential equation is modeling and control of many dynamic systems. Fractional-order derivatives are used in fruitful way to model many remarkable developments in those areas of science such as quantum chemistry, quantum mechanics, damping laws, rheology and diffusion processes [1–5] described through the models of fractional differential equations (FDEs). Modeling of a physical phenomenon depends on two parameters such as the time instant and the prior time history, because of this reason reasonable modeling through fractional calculus successfully achieved. The above mentioned advantages and applications of FDEs attracted researchers in developing efficient methods to solve FDEs in order to get accurate solutions to such problems and more active research is still going on in those areas. Most of the FDEs are complicated in its structure, hence finding exact solutions for them cannot be simple. Therefore one can approach the best accurate solution of FDEs through analytical and numerical

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Peer review under responsibility of Egyptian Mathematical Society.

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http://dx.doi.org/10.1016/j.joems.2014.04.007

methods. Designing accurate or best solution to FDEs, many methods are developed in recent years; each method has its own advantages and limitations. This paper aims to solve a FDE called fractional-order Riccati differential equation, one of the important equations in the family of FDEs. Solving fractional order Riccati differential equation, the most significant methods are Adomian decomposition method [6], homotopy perturbation method [7–10], homotopy analysis method [11,12], Taylor matrix method [13] and Haar wavelet method [14], combination of Laplace, Adomian decomposition and Padé approximation [15] methods, stochastic technique based on particle swarm optimization and simulated annealing [16], fractional variational iteration method [17] and a combination of finite difference and Padé-variational iteration numerical scheme [18]. However, the above mentioned methods have some restrictions and disadvantages in their performance. For example, very complicated and toughest Adomian polynomials are constructed in the Adomian decomposition method. Similarly we can find disadvantages in other methods. Moreover, the convergence region and implementation of these results are very small.

In recent years, wavelets theory is one of the growing and predominantly a new method in the area of mathematical and engineering research [19,20]. In this work, the nonlinear Riccati differential equations of fractional-order approached analytically by using Legendre wavelets method. The operational matrix of Legendre wavelet is generalized for fractional calculus in order to solve fractional and classical Riccati differential equations. The Legendre wavelet method (LWM) is illustrated by application, and obtained results are compared with recently proposed method for the fractional-order Riccati differential equation. We have adopted Legendre wavelet method to solve Riccati differential equations not only due to its emerging application of but also due to its greater convergence region.

The rest of the paper is as follows: In Section 2 definitions and mathematical preliminaries of fractional calculus are presented. In Section 3 Legendre wavelet, its properties, function approximations and generalized Legendre wavelet operational matrix fractional calculus are discussed. Section 4 establishes application of proposed method in the solution Riccati differential equations, existence and uniqueness solution of the proposed problem and convergence analyses of the proposed approach. Section 5 deals with the illustrative examples and their solutions by the proposed approach. Section 6 ends with our conclusion.

2. Preliminaries and notations

The notations, definitions and preliminary facts present in this section will be used in forthcoming sections of this work. As stated in [21], the Caputo fractional derivative uses initial and boundary conditions of integer order derivatives having some physical interpretations. Caputo fractional derivative D^α proposed by Caputo [22] in the theory of viscoelasticity.The Caputo fractional derivative of order $\alpha > 0, (\alpha \in \mathrm{R}, n-1 < \alpha \leqslant n, n \in \mathrm{N})$ and

$h : (0, \infty) \to \mathrm{R}$ is continuous is defined by

$$D^\alpha f(t) = I^{n-\alpha}\left(\frac{d^n}{dt^n} f(t)\right) \tag{1}$$

where

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \tag{2}$$

is the Riemann–Liouville fractional integral operator of order $\alpha > 0$ and Γ is the gamma function.

The fractional integral of t^β, $\beta > -1$ is given as

$$I^\alpha (t-\alpha)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} (t-a)^{\beta+\alpha}, \quad a \geqslant 0. \tag{3}$$

Properties of fractional integrals and derivatives are as follows [21], for $\alpha, \beta > 0$.

The fractional order integral satisfies the semi group property

$$I^\alpha(I^\beta f(t)) = I^\beta(I^\alpha f(t)) = I^{\alpha+\beta} f(t). \tag{4}$$

The integer order derivative D^n and fractional order derivative D^α commute with each other,

$$D^n(D^\alpha f(t)) = D^\alpha(D^n f(t)) = D^{n+\alpha} f(t). \tag{5}$$

The fractional integral operator and fractional derivative operator do not satisfy the commutative property. In general,

$$I^\alpha(D^\alpha f(t)) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0) \frac{t^k}{k!}. \tag{6}$$

But in the reverse way we have,

$$D^\alpha(I^\beta f(t)) = D^{\alpha-\beta} f(t). \tag{7}$$

3. Generalized Legendre wavelet operational matrix to fractional integration

A family of functions constituted by Wavelets, constructed from dilation and translation of a single function called mother wavelet. When the parameters a of dilation and b of translation vary continuously, following are the family of continuous wavelets [23]

$$\psi_{a,b}(t) = |a|^{-1/2} \psi\left(\frac{t-b}{a}\right), \quad a, b \in \Re, a \neq 0.$$

If the parameters a and b are restricted to discrete values as $a = a_0^{-k}, b = nb_0 a_0^{-k}, a_0 > 1, b_0 > 0$ and n, and k are positive integers, following are the family of discrete wavelets:

$$\psi_{k,n}(t) = |a_0|^{k/2} \psi(a_0^k t - nb_0),$$

where $\psi_{k,n}(t)$ form a wavelet basis for $L^2(R)$. In particular, when $a_0 = 2$, and $b_0 = 1$, $\psi_{k,n}(t)$ forms an orthonormal basis [23].

Legendre wavelets $\psi_{n,m}(t) = \psi(k, \hat{n}, m, t)$ have four arguments; $\hat{n} = 2n-1, n = 1, 2, 3, \ldots, 2^{k-1}, k$ can assume any positive integer, m is the order for Legendre polynomials and t is the normalized time. They are defined on the interval $[0, 1)$ as [24,25]

$$\psi_{nm}(t) = \begin{cases} \sqrt{m+\frac{1}{2}} 2^{k/2} P_m(2^k t - \hat{n}), & \text{for } \frac{\hat{n}-1}{2^k} \leqslant t < \frac{\hat{n}+1}{2^k}, \\ 0, & \text{otherwise}, \end{cases}$$

where $m = 0, 1, \ldots, M-1$ and $n = 1, 2, 3, \ldots, 2^{k-1}$. The coefficient $\sqrt{m+\frac{1}{2}}$ is for orthonormality, the dilation parameter is $a = 2^{-k}$ and translation parameter is $b = \hat{n} 2^{-k}$. $P_m(t)$ are the well-known Legendre polynomials of order m defined on the interval $[-1,1]$, and can be determined with the aid of the following recurrence formulae:

$$P_0(t)=1, \qquad P_1(t)=t,$$
$$P_{m+1}(t)=\left(\tfrac{2m+1}{m+1}\right)tP_m(t)-\left(\tfrac{m}{m+1}\right)P_{m-1}(t), \quad m=1,2,3,\ldots$$

The Legendre wavelet series representation of the function $f(t)$ defined over $[0,1)$ is given by

$$f(t)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}c_{nm}\psi_{nm}(t), \tag{8}$$

where $c_{nm}=\langle f(t),\psi_{nm}(t)\rangle$, in which $\langle .,.\rangle$ denotes the inner product. If the infinite series in Eq. (8) is truncated, then Eq. (8) can be written as

$$f(t)\cong\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{nm}\psi_{nm}(t)=C^{\mathrm{T}}\Psi(t)=\hat{f}(t). \tag{9}$$

where C and $\Psi(t)$ are $2^{k-1}M\times 1$ matrices given by

$$C=[c_{10},c_{11},\ldots,c_{1M-1},c_{20},c_{21},\ldots,c_{2M-1},\ldots,c_{2^{k-1}0},$$
$$c_{2^{k-1}1},\ldots,c_{2^{k-1}M-1}]^{\mathrm{T}},$$
$$\Psi(t)=[\psi_{10}(t),\psi_{11}(t),\ldots,\psi_{1M-1}(t),\psi_{20}(t),\psi_{21}(t),\ldots,$$
$$\psi_{2M-1}(t),\ldots,\psi_{2^{k-1}0}(t),\psi_{2^{k-1}1}(t),\ldots,\psi_{2^{k-1}M-1}(t)]^{\mathrm{T}} \tag{10}$$

Taking suitable collocation points as following

$$t_i=\cos\left(\frac{(2i+1)\pi}{2^kM}\right) \quad i=1,2,\ldots,2^{k-1}M,$$

we defined the $\dot{m}$ – square Legendre matrix

$$\phi_{\dot{m}\times\dot{m}}=\left[\Psi\left(\cos\left(\frac{3\pi}{2^kM}\right)\right)\ \Psi\left(\cos\left(\frac{5\pi}{2^kM}\right)\right)\cdots\Psi\left(\cos\left(\frac{(2^kM+1)\pi}{2^kM}\right)\right)\right]$$

where $\dot{m}=2^{k-1}M$, correspondingly we have

$$\hat{f}=\left[\hat{f}\left(\cos\left(\frac{3\pi}{2^kM}\right)\right)\ \hat{f}\left(\cos\left(\frac{5\pi}{2^kM}\right)\right)\cdots\hat{f}\left(\cos\left(\frac{(2^kM+1)\pi}{2^kM}\right)\right)\right]$$
$$=C^T\phi_{\dot{m}\times\dot{m}}$$

The Legendre matrix $\phi_{\dot{m}\times\dot{m}}$ is an invertible matrix, the coefficient vector C^{T} is obtained by $C^{\mathrm{T}}=\hat{f}\phi^{-1}_{\dot{m}\times\dot{m}}$.

The integration of the $\Psi(t)$ defined in Eq. (10) can be approximated by Legendre wavelet series with Legendre wavelet coefficient matrix P

$$\int_0^t\Psi(t)dt=P_{\dot{m}\times\dot{m}}\Psi(t)$$

where the $\dot{m}$ – square matrix P is called Legendre wavelet operational matrix of integration.

Then the Legendre wavelet operational matrix $P^{\alpha}_{\dot{m}\times\dot{m}}$ of fractional integration is given by

$$P^{\alpha}_{\dot{m}\times\dot{m}}=\phi_{\dot{m}\times\dot{m}}F^{\alpha}\phi^{-1}_{\dot{m}\times\dot{m}}$$

where $F^{\alpha}_{m\times m}$ is given in [21].

4. Existence, uniqueness and convergence

In this section, we will use the generalized Legendre wavelet operational matrix to solve nonlinear Riccati differential equation and we discuss the existence and uniqueness of solutions with initial conditions and convergence criteria of the proposed LWM approach.

Consider the fractional-order Riccati differential equation of the form

$$D^{\alpha}y(t)=P(t)y^2+Q(t)y+R(t), \quad t>0, \quad 0<\alpha\leqslant 1. \tag{11}$$

subject to the initial condition

$$y(0)=k. \tag{12}$$

Definition 4.1. Let $I=[0,l]$, $l<\infty$ and $C(I)$ be the class of all continuous function defined on I, with the norm

$$\|y\|=\sup_{t\in I}|e^{-ht}y(t)|, \quad h>0,$$

which is equivalent to the sup-norm of y. i.e., $\|y\|=\sup_{t\in I}|e^{-ht}y(t)|$.

Remark. Assume that solution $y(t)$ of fractional-order Riccati differential Eqs. (11) and (12) belongs to the space $S=\{y\in R:|y|\leqslant c, c \text{ is any constant}\}$, in order to study the existence and uniqueness of the initial value problem.

Definition 4.2. The space of integrable functions $L_1[0,l]$ in the interval $[0,l]$ is defined as

$$L_1[0,l]=\left\{u(t):\int_0^l|u(t)|dt<\infty\right\}.$$

Theorem 4.1. *The initial value problem given by Eqs. (11) and (12) has a unique solution*

$$y\in C(I), y'\in X=\left\{y\in L_1[0,l], \quad \|y\|=\|e^{-ht}y(t)\|_{L_1}\right\}.$$

Proof. By the Eq. (1), the fractional differential Eq. (11) can be written as

$$I^{1-\alpha}\frac{dy(t)}{dt}=P(t)y^2+Q(t)y+R(t) \tag{13}$$

becomes

$$y(t)=I^{\alpha}(P(t)y^2+Q(t)y+R(t)) \tag{14}$$

Now we define the operator $\Theta:C(I)\to C(I)$ by

$$\Theta y(t)=I^{\alpha}(P(t)y^2+Q(t)y+R(t)) \tag{15}$$

$$\begin{aligned}e^{-ht}(\Theta y-\Theta w)&=e^{-ht}I^{\alpha}\left[(Py^2(t)+Qy(t)+R)-(Pw^2(t)+Qw(t)+R)\right]\\&\leqslant\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}e^{-h(t-s)}[(y(s)-w(s))(y(s)\\&\quad+w(s))-k(y(s)-w(s))]e^{-hs}ds\\&\leqslant\|y-w\|\frac{1}{\Gamma(\alpha)}\int_0^t s^{\alpha-1}e^{-hs}ds\end{aligned}$$

hence, we have

$$\|\Theta y-\Theta w\|<\|y-w\|,$$

which implies the operator given by Eq. (15), has a unique fixed point and consequently the given integral equation has a unique solution $y(t)\in C(I)$. Also we can see that

$$I^{\alpha}(P(t)y^2 + Q(t)y + R(t))|_{t=0} = k \tag{16}$$

Now from Eq. (14), we have

$$y(t) = \left[\frac{t^{\alpha}}{\Gamma(\alpha+1)}(Py_0^2 + Qy_0 + R) + I^{\alpha+1}(P'y^2 + 2y'P + Q'y + Qy' + R')\right]$$

and

$$\frac{dy}{dt} = \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}(Py_0^2 + Qy_0 + R) + I^{\alpha}(P'y^2 + 2y'P + Q'y + Qy' + R')\right],$$

$$e^{-ht}y'(t) = e^{-ht}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}(Py_0^2 + Qy_0 + R) + I^{\alpha}(P'y^2 + 2y'P + Q'y + Qy' + R')\right], \tag{17}$$

from which we can deduce that $y^{'}(t) \in C(I)$ and $y^{'}(t) \in S$.Now again from Eqs. (14)–(16) we get

$$\frac{dy}{dt} = \frac{d}{dt}I^{\alpha}[Py^2(t) + Qy(t) + R],$$

$$I^{1-\alpha}\frac{dy}{dt} = I^{1-\alpha}\frac{d}{dt}I^{\alpha}[Py^2(t) + Qy(t) + R] = \frac{d}{dt}I^{1-\alpha}I^{\alpha}[Py^2(t) + Qy(t) + R]$$

$$D^{\alpha}y(t) = \frac{d}{dt}I[Py^2(t) + Qy(t) + R] = Py^2(t) + Qy(t) + R$$

and

$$y(0) = I^{\alpha}(Py^2(t) + Qy(t) + R)|_{t=0} = k$$

which implies that the integral Eq. (16) is equivalent to the initial value problem (12) and the theorem is proved. □

4.1. Convergence analyses

Let $\psi_{k,n}(t) = |a_0|^{k/2}\psi(a_0^k t - nb_0)$, where $\psi_{k,n}(t)$ form a wavelet basis for $L^2(R)$. In particular, when $a_0 = 2$, and $b_0 = 1$, $\psi_{k,n}(t)$ forms an orthonormal basis [23].

By Eq. (2.22), let $y(t) = \sum_{i=1}^{M-1} c_{1i}\psi_{1i}(t)$ be the solution of the Eq. (11) where $c_{1i} = \langle y(t), \psi_{1i}(t)\rangle$, for $k = 1$in which $\langle .,. \rangle$ denotes the inner product.

$$y(t) = \sum_{i=1}^{n}\langle y(t), \psi_{1i}(t)\rangle\psi_{1i}(t)$$

Let $\beta_j = \langle y(t), \psi(t)\rangle$ where $\psi(t) = \psi_{1i}(t)$

Let $x_n = \sum_{j=1}^{n}\beta_j\psi(t_j)$ be a sequence of partial sums. Then,

$$\langle y(t), x_n\rangle = \left\langle y(t), \sum_{j=1}^{n}\beta_j\psi(t_j)\right\rangle = \sum_{j=1}^{n}\bar{\beta}_j\langle y(t), \psi(t_j)\rangle$$
$$= \sum_{j=1}^{n}\bar{\beta}_j\beta_j = \sum_{j=1}^{n}|\beta_j|^2$$

Further

$$\|x_n - x_m\|^2 = \left\|\sum_{j=m+1}^{n}\beta_j\psi(t_j)\right\|^2 = \left\langle \sum_{i=m+1}^{n}\beta_i\psi(t_i), \sum_{j=m+1}^{n}\beta_j\psi(t_j)\right\rangle$$
$$= \sum_{i=m+1}^{n}\sum_{j=m+1}^{n}\beta_i\overline{\beta_j}\langle\psi(t_i), \psi(t_j)\rangle = \sum_{j=m+1}^{n}|\beta_j|^2$$

As $n \to \infty$, from Bessel's inequality, we have $\sum_{j=1}^{\infty}|\beta_j|^2$ is convergent.

$$\langle x - y(t), \psi(t_j)\rangle = \langle x, \psi(t_j)\rangle - \langle y(t), \psi(t_j)\rangle$$
$$= \left\langle \underset{n\to\infty}{Lt}\, x_n, \psi(t_j)\right\rangle - \beta_j = \underset{n\to\infty}{Lt}\,\langle x_n, \psi(t_j)\rangle - \beta_j$$
$$= \underset{n\to\infty}{Lt}\left\langle \sum_{j=1}^{n}\beta_j\psi(t_j), \psi(t_j)\right\rangle - \beta_j = \beta_j - \beta_j = 0.$$

It implies that $\{x_n\}$ is a Cauchy sequence and it converges to x (say), which is possible only if $y(t) = x$. i.e. both $y(t)$ and x_n converges to the same value, which indeed give the guarantee of convergence of LWM.

5. Numerical examples

In order to show the effectiveness of the Legendre wavelets method (LWM), we implement LWM to the nonlinear fractional Riccati differential equations. All the numerical experiments carried out on a personal computer with some MATLAB codes. The specification of PC is intel core i5 processor and with Turbo boost up to 3.1 GHz and 4 GB of DDR3 memory. The following problems of nonlinear Riccati differential equations are solved with real coefficients.

Example 5.1. Consider the following nonlinear fractional Riccati differential equation

$$D^{\alpha}y(t) = 1 + 2y(t) - y^2(t), \quad 0 < \alpha \leqslant 1 \tag{18}$$

with initial condition $\quad y(0) = 0.$ (19)

Exact solution for $\alpha = 1$was found to be $y(t)$

$$= 1 + \sqrt{2}\tanh\left(\sqrt{2}t + \frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right) \tag{20}$$

The integral representation of the Eqs. (18) and (19) is given by

$$I^{\alpha}(D^{\alpha}y(t)) = I^{\alpha}(1 + 2y(t) - y^2(t)) \tag{21}$$

$$y(t) = y(0) + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + 2I^{\alpha}y(t) - I^{\alpha}y^2(t) \tag{22}$$

$$\text{Let } y(t) = C^{T}\Psi(t) \tag{23}$$

then

$$I^{\alpha}y(t) = C^{T}I^{\alpha}\Psi(t) = C^{T}P^{\alpha}_{2^{k-1}M\times 2^{k-1}M}\Psi(t) \tag{24}$$

By substituting Eqs. (23) and (24) in (22), we get the following system of algebraic equations

$$C^{T}\Psi(t) = \frac{t^{\alpha}}{\Gamma(\alpha+1)} + 2C^{T}P^{\alpha}_{2^{k-1}M\times 2^{k-1}M}\Psi(t) - C^{T}P^{2\alpha}_{2^{k-1}M\times 2^{k-1}M}\Psi(t)$$

By solving the above system of linear equations, we can find the vector C. Numerical results are obtained for different values of k, M and α. Solution obtained by the proposed LWM approach for $\alpha = 1$, $k = 1$ and $M = 3$ is given in Fig. 1 and for different values of $\alpha = 0.6$, 0.7, 0.8 and 0.9 and for $k = 2$ and $M = 5$ are graphically given in Fig. 2. It can be seen from Fig. 1 that the solution obtained by the proposed LWM approach is more close to the exact solution.

In order to analyses the effectiveness of the proposed approach further, the obtained results of example 1 for $\alpha = 0.5$, 0.75 and for $k = 1$ and $M = 3$ are compared with reported results of other numerical, analytical and stochastic solver such as solution by PSO [16] based on swarm intelligence, analytical approximation solution obtained by fractional variational iteration method (FVI) [17] and a finite difference numerical iteration scheme by Padé-variational iteration method (PVI) [18] based on Riemann–Liouville derivative. The compared results are provided in Table 1 and

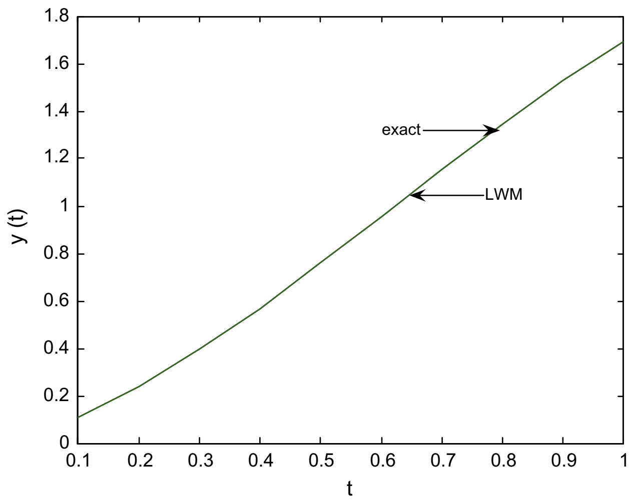

Figure 1 Numerical results of example 5.1 by LWM for $\alpha = 1$.

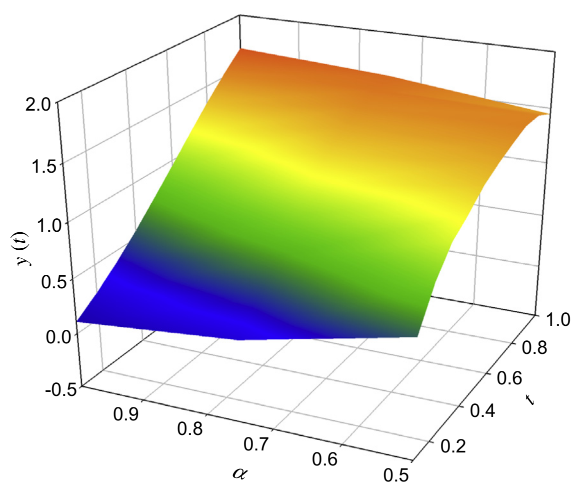

Figure 2 Numerical results of example 5.1 by LWM for different values of α.

it indicates that the results obtained by proposed LWM approach has good convergence than the other approaches in the given applicable domain.

Example 5.2. Consider another fractional order Riccati differential equation

$$D^{\alpha}y(t) = 1 - y^2(t), \quad 0 < \alpha \leqslant 1, \tag{25}$$

with initial condition $y(0) = 0$. (26)

Exact solution for the above equation was found to be

$$y(t) = \frac{e^{2t} - 1}{e^{2t} + 1}$$

The integral representation of Eqs. (25) and (26) is given by

$$y(t) = y(0) + \frac{t^{\alpha}}{\Gamma(\alpha + 1)} - I^{\alpha}y^2(t)$$

Let $y(t) = C^{\mathrm{T}}\Psi(t)$ (27)

then

$$I^{\alpha}y(t) = C^{\mathrm{T}}I^{\alpha}\Psi(t) = C^{\mathrm{T}}P^{\alpha}_{2^{k-1}M\times 2^{k-1}M}\Psi(t) \tag{28}$$

By substituting Eqs. (27) and (28) in (25), we get the following system of algebraic equations

$$C^{\mathrm{T}}\Psi(t) = \frac{t^{\alpha}}{\Gamma(\alpha + 1)} - C^{\mathrm{T}}P^{2\alpha}_{2^{k-1}M\times 2^{k-1}M}\Psi(t)$$

By solving the above system of linear equations, we can find the vector C. Numerical results are obtained for different values of k, M and α. Results obtained by LWM for $\alpha = 1$ $k = 2$ and $M = 3$ shown in Fig. 3 and it can be seen from the figure that solution given by the LWM merely coincide with the exact solution. Fig. 4 shows that the obtained results of Eqs. (25) and (26) by LWM for different values of α and for $k = 2$ and $M = 5$. Since exact solution for fractional order case is not available, like example 5.1, for the Eqs. (25) and (26) comparisons made with the approximate solution given by the proposed approach and reported approximate results of other approaches PSO [16], FVI [17], PVI [18]. Obtained results are provided in Table 2 and from these results we can identify that guarantee of convergence of the proposed LWM approach is very high.

Example 5.3. Let us consider another problem of nonlinear Riccati differential equation

$$D^{\alpha}y(t) = t^2 + y^2(t), \quad 0 < \alpha \leqslant 1, \quad t \geqslant 0. \tag{29}$$

Table 1 Numerical results of example 5.1.

t	$\alpha = 1/2$				$\alpha = 3/4$			
	LWM	PVI	PSO	FVI	LWM	PVI	PSO	FVI
0.1	0.574500	0.356803	0.574648	0.577431	0.244458	0.193401	0.283503	0.244460
0.2	0.890789	0.922865	0.890890	0.912654	0.469689	0.454602	0.539352	0.469709
0.3	1.100011	1.634139	1.090716	1.166253	0.698700	0.784032	0.768804	0.698718
0.4	1.205607	2.204441	1.230069	1.353549	0.924305	1.161986	0.971833	0.924319
0.5	1.334087	2.400447	1.334181	1.482633	1.137939	1.543881	1.147939	1.137952
0.6	1.415493	2.041435	1.415512	1.559656	1.296302	1.873658	1.296320	1.331462
0.7	1.480879	2.414889	1.480918	1.589984	1.416311	2.112944	1.416319	1.497600
0.8	1.534598	2.414248	1.534604	1.578559	1.506913	2.260134	1.506936	1.630234
0.9	1.530019	2.414246	1.579396	1.530028	1.569221	2.339134	1.569252	1.724439
1.0	1.448703	2.414231	1.617332	1.448805	1.605571	2.379356	1.605580	1.776542

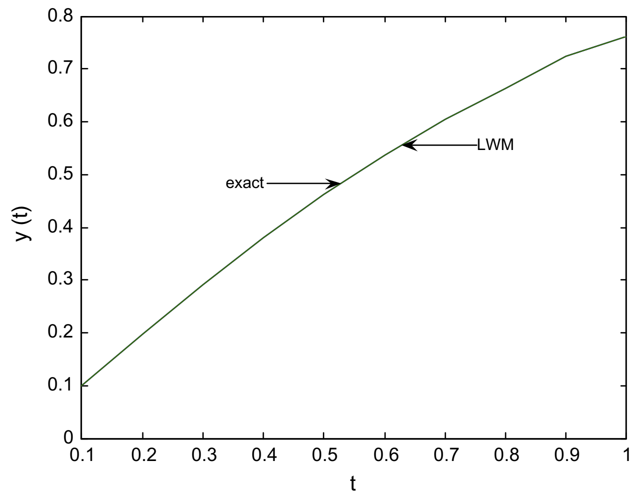

Figure 3 Numerical results of example 5.2 by LWM for $\alpha = 1$.

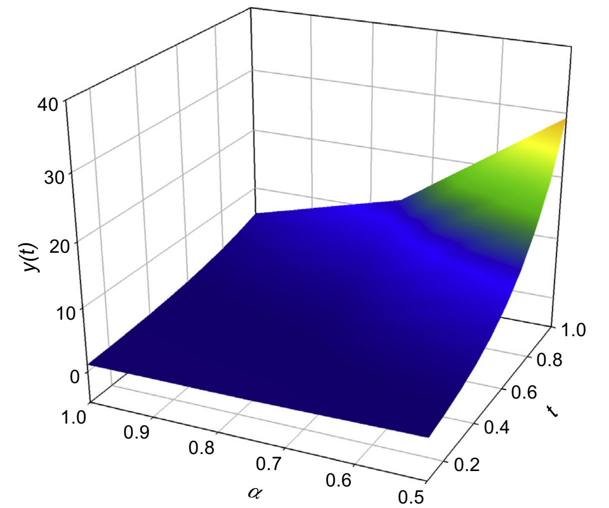

Figure 5 Numerical results of example 5.3 by LWM for different values of α.

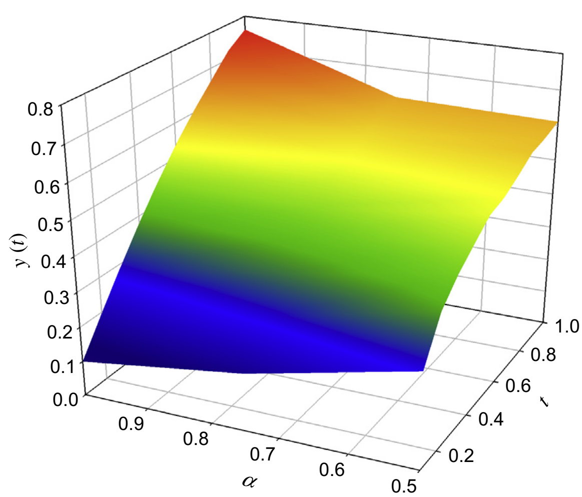

Figure 4 Numerical results of example 5.2 by LWM for different values of α.

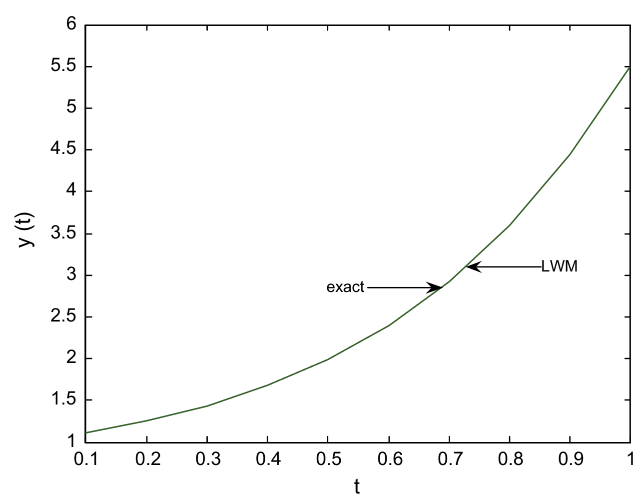

Figure 6 Numerical results of example 5.3 by LWM for $\alpha = 1$.

Table 2 Numerical results of example 5.2.

t	$\alpha = 1/2$				$\alpha = 3/4$			
	LWM	PVI	PSO	FVI	LWM	PVI	PSO	FVI
0.1	0.273600	0.313678	0.289667	0.273875	0.165056	0.187945	0.165087	0.184795
0.2	0.386358	0.468954	0.386489	0.454125	0.276332	0.324567	0.276350	0.313795
0.3	0.441104	0.593005	0.441120	0.573932	0.356115	0.456219	0.356196	0.414562
0.4	0.482304	0.650122	0.482348	0.644422	0.416817	0.667581	0.416916	0.492889
0.5	0.520664	0.898237	0.516379	0.674137	0.465480	0.900321	0.465520	0.462117
0.6	0.533287	1.000943	0.544872	0.671987	0.505894	1.110341	0.506004	0.597393
0.7	0.558743	1.512398	0.568545	0.648003	0.540606	1.516448	0.540629	0.631772
0.8	0.587812	1.816384	0.587895	0.613306	0.569998	1.916004	0.570632	0.660412
0.9	0.596234	2.005632	0.603344	0.579641	0.596600	2.012352	0.596636	0.687960
1.0	0.610642	2.006485	0.615268	0.558557	0.618824	2.034632	0.618873	0.718260

Table 3 Numerical results of example 5.3.

t	$\alpha = 1/2$				$\alpha = 3/4$			
	LWM	PSO	FVI	PVI	LWM	PSO	FVI	PVI
0.1	1.600345	1.671055	1.601055	1.671256	1.209113	1.249863	1.209863	1.249870
0.2	2.289652	2.342131	2.302131	2.342321	1.498342	1.513463	1.503463	1.513500
0.3	3.199986	3.241501	3.201501	3.241521	1.801287	1.852218	1.802218	1.852218
0.4	4.408328	4.468508	4.408508	4.468538	2.306457	2.306699	2.306699	2.306721
0.5	6.103679	6.144389	6.104389	6.144402	2.908663	2.928996	2.908996	2.929012
0.6	8.408745	8.429666	8.409666	8.429696	3.709612	3.789643	3.709643	3.789883
0.7	11.50010	11.54066	11.50066	11.54166	4.905128	4.985354	4.905354	4.985521
0.8	15.70017	15.77047	15.70047	15.77237	6.609123	6.649324	6.609324	6.649661
0.9	21.30823	21.51700	21.50700	21.51780	8.905289	8.965376	8.905376	8.965686
1.0	29.30017	29.32098	29.30098	29.32168	12.10721	12.18753	12.10753	12.18823

with initial condition $y(0) = 1$. (30)

When $\alpha = 1$, its exact solution is given by

$$y(t) = \frac{t(J_{-3/4}(t^2/2)\Gamma(1/4) + 2J_{3/4}(t^2/2)\Gamma(3/4))}{J_{1/4}(t^2/2)\Gamma(1/4) - 2J_{-1/4}(t^2/2)\Gamma(3/4)}$$

where $J_n(t)$ is the Bessel function of first kind.

The integral representation of the Eqs. (29) and (30) is given by

$$y(t) = 1 + \frac{2}{\Gamma(\alpha+1) + 2\Gamma(\alpha)} t^{\alpha+2} + I^{\alpha} y^2(t)$$

Let $y(t) = C^{\mathrm{T}}\Psi(t)$ (31)

then

$$I^{\alpha} y(t) = C^{\mathrm{T}} I^{\alpha}\Psi(t) = C^{\mathrm{T}} P^{\alpha}_{2^{k-1}M\times 2^{k-1}M}\Psi(t) \tag{32}$$

By substituting Eqs. (31) and (32) in (29), we get the following system of algebraic equations

$$C^{\mathrm{T}}\Psi(t) = 1 + \frac{2}{\Gamma(\alpha+1) + 2\Gamma(\alpha)} (C^{\mathrm{T}}\Psi(t))^{\alpha+2} + C^{\mathrm{T}} P^{2\alpha}_{2^{k-1}M\times 2^{k-1}M}\Psi(t)$$

By solving the above system of linear equations, we can find the vector C. Numerical results are obtained for different values of k, M and α. Obtained results for Eqs. (29) and (30) are shown in Figs. 5 and 6 and in Table 3. Fig. 5 shows the solutions obtained by LWM for different values of α and for $k = 2$, $M = 4$. Fig. 6 compares the solution obtained by LWM with the exact solution of Eqs. (29) and (30) when $\alpha = 1, k = 1, M = 2$ and Table 3 provides the obtained results of LWM and reported results of PSO [16], FVI [17], and PVI [18] for the values of $\alpha = 1/2$, $3/4$ with $k = 3$, $M = 5$ From these results we can see that the proposed LWM approach gives the solution which are very close to the exact solution and outperformed recently developed approaches.

6. Conclusions

In this work, a Legendre's wavelet operational matrix method called LWM, proposed for solving nonlinear fractional order Riccati differential equations. Comparison made for the solutions obtained by the proposed method and with the other recent approaches developed for same problem; obtained results show that the proposed LWM yields more accurate and reliable solutions with less computational effort. Further we have discussed the convergence criteria of proposed scheme, which indeed provides the guarantee of consistency and stability of the proposed LWM scheme for the solutions of nonlinear fractional Riccati differential equations.

Acknowledgment

The author thanks the editor and the referees for their comments and suggestions to improve the paper.

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