On the symmetric matrix word equation
\[ XBX^2B^3X^2BX = A \]

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Abstract

In this paper we present explicitly the unique positive definite solution of the symmetric word equation \( XBX^2B^3X^2BX = A \) over \( 2 \times 2 \) positive definite matrices. This word equation appeared as a counterexample to the uniqueness of solution conjecture for symmetric word equations: it has multiple positive definite solutions for certain \( 3 \times 3 \) positive definite matrices \( A \) and \( B \).

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1. Introduction

Symmetric word equations arise naturally in matrix theory as equations over the cone of positive definite matrices \([1,4,5,7]\). Such equations have recently been the topic of active investigation because of their relationship to the Bessis–Moussa–Villani trace conjecture, an open conjecture arising from statistical physics (see \([4]\)). The best-known symmetric matrix word equation is the simple Riccati matrix equation \( XAX = B \), which has a unique positive definite solution, the geometric mean \( A^{-1} \# B \) of \( A^{-1} \) and \( B \): \( A^{-1} \# B = (A^{1/2}BA^{1/2})^{1/2}A^{-1/2} \).

A symmetric word equation \( S(X, B) = A \) is called (uniquely) solvable if there exists (uniquely) a positive definite solution \( X \) of \( S(X, B) = A \) for every pair of \( n \times n \) positive definite matrices \( A \) and \( B \).
and $B$. In [5], Hillar and Johnson proved that every positive definite symmetric word equation is solvable and they left open the uniqueness of solution problem. Lawson and Lim [6] have verified the uniqueness conjecture in the case that the degree is not greater than 5 using the non-positive curvature property of the Riemannian symmetric space of positive definite convex cones. In [1], Armstrong and Hillar have recently shown via the Brouwer mapping degree that the symmetric word equation of degree 6

$$XBX^2B^3X^2BX = A$$

(1.1)

has multiple positive definite solutions for certain $3 \times 3$ positive definite matrices $A$ and $B$, providing a negative solution to the conjecture. However, it does have a unique positive definite solution over the $2 \times 2$ positive definite matrices. General uniqueness of solutions remains open for the case of $2 \times 2$ positive definite matrices.

**Theorem 1.1** [1]. *The symmetric word equation (1.1) is not uniquely solvable over $3 \times 3$ positive definite letters, but it is uniquely solvable over $2 \times 2$ positive definite matrices.*

However as mentioned in [1], finding the unique positive definite solution of (1.1) over $2 \times 2$ positive definite matrices is non-trivial and presents another problem. The main purpose of this paper is to present an explicit form of the solution. Making $B$ diagonal via unitary similarity, and then making $X$ entrywise real via a diagonal unitary similarity (this does not change $B$) reduces the problem to the consideration of the word equation in $2 \times 2$ real positive definite matrices. We show that the unique solution of (1.1) in $2 \times 2$ real positive definite matrices is of the form

$$X = \frac{1}{\sqrt{u(t)}}(\alpha(t)B^{-1} + B^{-3} + tI - A^{-1}),$$

for some uniquely determined positive reals $s$ and $t$. Under the natural restriction that $\det(A) = \det(B) = 1$, the solution simplifies to

$$X = \frac{1}{\sqrt{u(t)}}(\alpha(t)B^{-1} + B^{-3} + tI - A^{-1}),$$

where

$$\alpha(t) = \frac{1}{2} \left[ -\text{tr}(B^2) + \sqrt{\text{tr}^2(B^2) + 4t^2 - 4\text{tr}(A)t} \right],$$

$$u(t) = \det(\alpha(t)B^{-1} + B^{-3} + tI - A^{-1})$$

and $t \in (\text{tr}(A), \infty)$ is a unique positive real solution of

$$[2 + \alpha(t)\text{tr}(B^2) + t \text{tr}(B^3) - \text{tr}(A^{-1}B^3)] \times [\alpha(t)\text{tr}(B) + \text{tr}(B^3) + 2t - \text{tr}(A)] = u(t)[t - \text{tr}(A) + \text{tr}(B^3)].$$

**2. Geometric means of $2 \times 2$ matrices**

For positive definite Hermitian matrices $A$ and $B$, we let

$$A \# B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$ 

The positive definite matrix $A \# B$ is called the geometric mean of $A$ and $B$. The following properties of the geometric mean operation are well-known [2,3,6].
Proposition 2.1. The geometric mean $A\#B$ is the unique positive definite solution of the Riccati equation (Riccati Lemma)

$$XA^{-1}X = B.$$ 

It satisfies that

1. $A\#B = B\#A$;
2. $(A\#B)^{-1} = A^{-1}\#B^{-1}$;
3. $(MAM^*)\#(MBM^*) = M(A\#B)M^*$ for any invertible matrix $M$;
4. If $AB = BA$, then $A\#B = A^{1/2}B^{1/2} = (AB)^{1/2}$;
5. (Determinant) $\det(A\#B) = \sqrt{\det(A)\det(B)}$;
6. $(sA)\#(tB) = \sqrt{s^t(A\#B)}$ for $t, s > 0$.

The following result on the geometric mean of $2 \times 2$ positive definite matrices is crucial for our purpose.

Theorem 2.2 [2,3,8]. Let $A$ and $B$ be $2 \times 2$ positive definite Hermitian matrices with determinant $1$. Then

$$A\#B = \frac{A + B}{\sqrt{\det(A + B)}}.$$ 

In general,

$$A\#B = \frac{\sqrt{\alpha\beta}}{\sqrt{\det(\alpha^{-1}A + \beta^{-1}B)}}(\alpha^{-1}A + \beta^{-1}B), \quad \alpha^2 = \det(A), \quad \beta^2 = \det(B). \quad (2.2)$$

If $\det(A\#B) = 1$ (equivalently, $\det(A) = \det(B)^{-1}$), then

$$A\#B = \frac{1}{\sqrt{\det(A + \alpha^2B)}}(A + \alpha^2B), \quad \alpha^2 = \det(A). \quad (2.3)$$

Proof. Let $X = B^{-1/2}AB^{-1/2}$. Then from $\det(A + B) - 2 = \det(X + I) - 2 = \text{tr}(X)$ and the Cayley–Hamilton theorem, we have $X + (2 - \alpha^2)I + X^{-1} = 0$, where $\alpha^2 = \det(A + B)$. This equation is equivalent to $(A + B)A^{-1}(A + B) = \alpha^2B$. By Riccati Lemma, $\alpha(A\#B) = A + B$. ⊓⊔

Lemma 2.3. Let $A$ and $B$ be $2 \times 2$ positive definite Hermitian matrices of determinant $1$.

1. $\text{tr}(A) = \text{tr}(A^{-1})$.
2. $\det(tI - A) = t^2 - \text{tr}(A)t + 1$, $\det(tA - sI) = t^2 - \text{tr}(A)s + s^2$.
3. If $AB = BA$ and $A + B = tI$ for some $t \in \mathbb{R}$, then $A = B^{-1}$.
4. For $tI > A$ and $\alpha > 0$, $\det((\alpha I + B^3)\#(tI - A)^{-1}) = 1$ if and only if $\alpha^2 + \text{tr}(B^2)\alpha = t^2 - \text{tr}(A)t$. In this case, $t > \text{tr}(A)$.

Proof. (1)–(3) By simultaneous diagonalization.

(4) It follows from $\det(\alpha I + B^3) = \det(\alpha B + B^3)$ that $\det((\alpha I + B^3)\#(tI - A)^{-1}) = 1$ if and only if $\det(\alpha I + B^3) = \det(tI - A)$ if and only if $\alpha^2 + \text{tr}(B^2)\alpha = t^2 - \text{tr}(A)t$. From $t, \alpha > 0$, we have $t > \text{tr}(A)$. ⊓⊔
3. The word equation $XBX^2B^3X^2BX = A$

We consider symmetric word equations in $2 \times 2$ real positive definite matrices

$$XBX^2B^3X^2BX = A. \quad (3.4)$$

We may assume that $\det(A) = \det(B) = 1$. Indeed, setting $A' := \frac{A}{\sqrt{\det(A)}}$, $B' := \frac{B}{\sqrt{\det(B)}}$, and $Y = \left[\frac{\det(B)^{5/12}}{\det(A)^{1/12}}\right] X$, (3.4) is equivalent to $YB'Y^2B^2Y^2B'Y = A'$. Furthermore, we may assume that $B$ is a diagonal matrix, $B = \begin{pmatrix} b & 0 \\ 0 & 1/b \end{pmatrix}$ and $A = \begin{pmatrix} x & y \\ y & x \end{pmatrix}$ is a fixed positive definite real matrix with determinant 1.

**Definition 3.1.** For $t > \text{tr}(A)$, we define

$$\alpha(t) = \frac{1}{2} \left[-\text{tr}(B^2) + \sqrt{\text{tr}^2(B^2) + 4t^2 - 4\text{tr}(A)t}\right], \quad (3.5)$$

$$X(t) = (\alpha(t)B^{-1} + B^{-3})#(tI - A)^{-1}. \quad (3.6)$$

**Proposition 3.2.**

1. $\alpha(t) > 0$ and $X(t)$ is positive definite.
2. $\alpha(t)$ is monotone increasing.
3. The following statements hold true and they are equivalent:

   $$\alpha(t)^2 + \text{tr}(B^2)\alpha(t) = t^2 - \text{tr}(A)t, \quad (3.7)$$

   $$\det(X(t)) = 1. \quad (3.8)$$

In particular,

$$X(t) = \frac{\alpha(t)B^{-1} + B^{-3} + tI - A^{-1}}{[\det(\alpha(t)B^{-1} + B^{-3} + tI - A^{-1})]^{1/2}}, \quad (3.9)$$

4. The following statements hold:

   $$2t - \text{tr}(A) = \alpha(t)\text{tr}(X(t)^2B) + \text{tr}(X(t)^2B^3), \quad (3.10)$$

   $$\text{tr}(X(t)^2B^3) = \text{tr}(X(t)) \cdot \text{tr}(X(t)B^3) - \text{tr}(B^3). \quad (3.11)$$

5. The following are equivalent:

   $$\alpha(t) = \sqrt{\det(B^{-3} + X(t)AX(t))}, \quad (3.12)$$

   $$t = \alpha(t)\text{tr}(X(t)^2B), \quad (3.13)$$

   $$t = \text{tr}(A) + \text{tr}(X(t)^2B^3), \quad (3.14)$$

   $$t = \text{tr}(A) - \text{tr}(B^3) + \text{tr}(X(t)) \cdot \text{tr}(X(t)B^3). \quad (3.15)$$

6. $\alpha(t) = t - \text{tr}(A)$ if and only if $\text{tr}(A) = \text{tr}(B^2)$.

**Proof.**

1. Since $t > \text{tr}(A)$, $\alpha(t) > 0$ and $tI > A$.

   (2) Straightforward.

   (3) The equivalence between (3.7) and (3.8) follows by (3.5), Lemma 2.3 and (1). Since $\det(X(t)) = 1$, $\det(\alpha(t)B^{-1} + B^{-3}) = \det(tI - A)$ and hence

   $$[\det(\alpha(t)B^{-1} + B^{-3})](tI - A)^{-1} = [\det(tI - A)](tI - A)^{-1} = tI - A^{-1}.$$
Then (3.9) follows from the geometric mean formula (2.3).

(4) The Riccati Lemma implies that
\[ tI - A = [X(t)(\alpha(t)B^{-1} + B^{-3})^{-1}X(t)]^{-1} = X(t)^{-1}(\alpha(t)B^{-1} + B^{-3})X(t)^{-1} \]
and by Lemma 2.3(1),
\[
2t - \text{tr}(A) = \text{tr}(tI - A) = \alpha(t)\text{tr}(X(t)^{-1}B^{-1}X(t)^{-1}) + \text{tr}(X(t)^{-1}B^{-3}X(t)^{-1})
\]
\[
= \alpha(t)\text{tr}(X(t)BX(t)) + \text{tr}(X(t)B^3X(t))
\]
\[
= \alpha(t)\text{tr}(X(t)^2B) + \text{tr}(X(t)^2B^3).
\]

By the Cayley–Hamilton theorem, \( X(t)^2 - \text{tr}(X(t))X(t) + I = 0 \) and hence
\[ X(t)^2B^3 = \text{tr}(X(t))X(t)B^3 - B^3. \]

(5) By the Riccati Lemma,
\[ X(t)(tI - A)X(t) = \alpha(t)B^{-1} + B^{-3} \]
and hence
\[ B^{-3} + X(t)AX(t) = tX(t)^2 - \alpha(t)B^{-1} = B^{-1/2}[tB^{1/2}X(t)^2B^{1/2} - \alpha(t)I]B^{-1/2}. \]

By Lemma 2.3(2),
\[
\text{det}(B^{-3} + X(t)AX(t)) = \text{det}(tB^{1/2}X(t)^2B^{1/2} - \alpha(t)I)
\]
\[
= t^2 - \text{tr}(B^{1/2}X(t)^2B^{1/2})\alpha(t)t + \alpha(t)^2
\]
\[
= t^2 - \text{tr}(X(t)^2B)\alpha(t)t + \alpha(t)^2.
\]

Therefore \( \alpha(t)^2 = \text{det}(B^{-3} + X(t)AX(t)) \) if and only if \( t^2 = \text{tr}(X(t)^2B)\alpha(t)t \) if and only if \( t = \alpha(t)\text{tr}(X(t)^2B) \). This shows that (3.12) and (3.13) are equivalent.

Other equivalences follow from (4).

(6) By (3), \( \text{tr}(A) = \text{tr}(B^2) \) if and only if \( t^2 - \text{tr}(A)t = \alpha(t)^2 + \alpha(t)\text{tr}(A) \) if and only if \( (t - \alpha(t))(t + \alpha(t)) = (t + \alpha(t))\text{tr}(A) \) if and only if \( \alpha(t) = t - \text{tr}(A). \)

**Remark 3.3.** We note that \( X(t) \) can be defined for \( t = \text{tr}(A) \). Indeed, if \( t = \text{tr}(A) \) then \( tI - A = A^{-1} > 0 \). In this case, \( \alpha(t) = 0 \) and
\[ X(\text{tr}(A)) = B^{-3}A. \]

Therefore, \( t \mapsto X(t) \) is a differentiable curve in the space of \( 2 \times 2 \) positive definite matrices of determinant 1 starting at \( B^{-3}A \).

**Definition 3.4.** For \( t \geq \text{tr}(A) \), we define
\[
f(t) := t - \text{tr}(A) + \text{tr}(B^3) - \text{tr}(X(t))\cdot\text{tr}(X(t)B^3) \quad \text{and} \quad g(t) := t\text{tr}(X(t)^2B) - t,
\]
\[
u(t) := u(t)[t - \text{tr}(A) + \text{tr}(B^3)] - [2 + \alpha(t)\text{tr}(B^2) + t\text{tr}(B^3) - \text{tr}(A^{-1}B^3)]
\]
\[
\times[\alpha(t)\text{tr}(B) + \text{tr}(B^3) + 2t - \text{tr}(A)],
\]
where \( u(t) = \text{det}(\alpha(t)B^{-1} + B^{-3} + tI - A^{-1}) \).

**Proposition 3.5.** We have that \( u(t)^{f(t)} = g(t) \) and
\[
u(t) = [\text{tr}(B)t - \text{tr}(A^{-1}B)]\alpha(t) + 2t^2 + [\text{tr}(B^3) - 2\text{tr}(A)]t - \text{tr}(A^{-1}B^3) + 2. \]
In particular, the following equations are equivalent for $t > \text{tr}(A)$:

\[
\begin{align*}
  f(t) &= 0, \\
  g(t) &= 0.
\end{align*}
\]  

(3.17)

**Proof.** From $X(t) = u(t)^{-1/2}(A(t)B^{-1} + B^{-3} + tI - A^{-1})$ we have that $u(t)f(t) = g(t)$ for all $t \geq \text{tr}(A)$. Indeed,

\[
\begin{align*}
  \text{tr}(X(t)\text{tr}(X(t)B^3)) &= u(t)^{-1}\text{tr}(A(t)B^{-1} + B^{-3} + tI - A^{-1}) \\
  &\quad \times \text{tr}(A(t)B^2 + I + tB^3 - A^{-1}B^3) \\
  &= u(t)^{-1}[\alpha(t)\text{tr}(B) + \text{tr}(B^3) + 2t - \text{tr}(A)] \\
  &\quad \times [2 + \alpha(t)\text{tr}(B^2) + t\text{tr}(B^3) - \text{tr}(A^{-1}B^3)].
\end{align*}
\]

Using the fact that $B$ is a diagonal matrix and that $\alpha(t)^2 + \text{tr}(B^2)\alpha(t) = t^2 - \text{tr}(A)t$, one can directly show that

\[
\begin{align*}
  u(t) &= \det(\alpha(t)B^2 + I + tB^3 - A^{-1}B^3) \\
  &= \alpha(t)^2 + [\text{tr}(B^2) + \text{tr}(B)t - \text{tr}(A^{-1}B)]\alpha(t) \\
  &\quad + t^2 + [\text{tr}(B^3) - \text{tr}(A)]t - \text{tr}(A^{-1}B^3) + 2 \\
  &= [\text{tr}(B)t - \text{tr}(A^{-1}B)]\alpha(t) + 2t^2 + [\text{tr}(B^3) - 2\text{tr}(A)]t - \text{tr}(A^{-1}B^3) + 2.
\end{align*}
\]

From $u(t) > 0$ for all $t > \text{tr}(A)$, the equations $f(t) = 0$ and $g(t) = 0$ are equivalent. \hfill \Box

**Proposition 3.6.** We have $g'(\text{tr}(A)) < 0$. In particular if $\text{tr}(A) = \text{tr}(B^2)$, then $g(t) = 0$ (and hence $f(t) = 0$) has a unique solution in $(\text{tr}(A), \infty)$.

**Proof.** The derivatives of $g$ and $u$ are given by

\[
\begin{align*}
  g'(t) &= u'(t)[t - \text{tr}(A) + \text{tr}(B^3)] + u(t) \\
        &\quad - [\alpha'(t)\text{tr}(B^2) + \text{tr}(B^3)]\alpha(t)\text{tr}(B) + \text{tr}(B^3) + 2t - \text{tr}(A)] \\
        &\quad - [2 + \alpha(t)\text{tr}(B^2) + \text{tr}(B^3)t - \text{tr}(A^3B^3)]\alpha'(t)\text{tr}(B) + 2]. \\
  u'(t) &= \text{tr}(B)\alpha(t) + [\text{tr}(B)t - \text{tr}(A^{-1}B)]\alpha'(t) + 4t + \text{tr}(B^3) - 2\text{tr}(A).
\end{align*}
\]

For $t = \text{tr}(A)$, we have $\alpha(\text{tr}(A)) = 0$ and $\alpha'(\text{tr}(A)) = \frac{\text{tr}(A)}{u(B^2)}$. Therefore

\[
\begin{align*}
  u(\text{tr}(A)) &= \text{tr}(B^3)\text{tr}(A) - \text{tr}(A^{-1}B^3) + 2, \\
  u'(\text{tr}(A)) &= 2\text{tr}(A) + [\text{tr}(B)\text{tr}(A) - \text{tr}(A^{-1}B)]\frac{\text{tr}(A)}{\text{tr}(B^2)} + \text{tr}(B^3), \\
  \text{tr}(B^2)g'(\text{tr}(A)) &= \text{tr}(A^{-1}B^3)\text{tr}(A)\text{tr}(B) + \text{tr}(A^{-1}B^3)\text{tr}(B^2) \\
  &\quad - (\text{tr}(A^{-1}B)\text{tr}(A)\text{tr}(B^3) + 2\text{tr}(A)\text{tr}(B)) \\
  &\quad - (\text{tr}(A)^2 + \text{tr}(A)\text{tr}(B^3) + 2)\text{tr}(B^2).
\end{align*}
\]

By direct computation, $g'(\text{tr}(A)) < 0$.

Suppose that $\text{tr}(A) = \text{tr}(B^2)$ or equivalently, $\alpha(t) = t - \text{tr}(A)$ (Proposition 3.2(6)). Then it is easy to see that $g(t)$ is a cubic polynomial with the leading coefficient $\text{tr}(B) + 2 > 0$. In this case, the uniqueness of solution of $g(t) = 0$, $t > \text{tr}(A)$ is equivalent to $g'(\text{tr}(A)) < 0$. \hfill \Box
Example 3.7. If \(A = B^2\), then
\[
X(t) = (\alpha(t)B^{-1} + B^{-3})^{1/2}(tI - B^2)^{-1/2} = \left(\frac{1}{\sqrt{b}}\right) \begin{pmatrix}
\alpha(t)B^{-1} + B^{-3} \\
0 \\
\end{pmatrix}.
\]
Now, \(\alpha(t) = t - \text{tr}(B^2)\) (Proposition 3.2(4)) implies that \(\alpha(t) = \frac{tb^2 - b^4 - 1}{b^2}\) and
\[
X(t) = \left(\frac{1}{\sqrt{b}} \begin{pmatrix}
0 \\
\sqrt{b} \\
\end{pmatrix} \right) = B^{-1/2}.
\]
Furthermore, \(f(t) = 0\) has the unique solution \(t = 2\text{tr}(A)\).

We note that for \(A = B^2\), \(t \mapsto X(t) = B^{-1/2}\) is constant on \((\text{tr}(A), \infty)\) and realizes the unique solution of (1.1). Otherwise, we have the following.

Proposition 3.8. If \(A \neq B^2\), the function \(t \mapsto X(t)\) is injective on \((\text{tr}(A), \infty)\).

Proof. Suppose that \(X := X(t) = X(s)\) for some \(t > s > \text{tr}(A)\). Then by Riccati Lemma,
\[
X(tI - A)X = \alpha(t)B^{-1} + B^{-3},
\]
\[
X(sI - A)X = \alpha(s)B^{-1} + B^{-3}.
\]
Subtracting yields \((t - s)X^2 = (\alpha(t) - \alpha(s))B^{-1}\). Since \(t - s > 0\) and \(X^2, B^{-1}\) are positive definite, we have \(\alpha(t) - \alpha(s) > 0\). From \(\det(X) = \det(B) = 1\), we see that
\[
t - s = \alpha(t) - \alpha(s)
\]
and \(X = B^{-1/2}\). That is, \(B^{-1/2} = X = (\alpha(t)B^{-1} + B^{-3})#(tI - A)^{-1}\). By Riccati Lemma, \(B^{-1/2}(tI - A)B^{-1/2} = \alpha(t)B^{-1} + B^{-3}\) and hence
\[
tI - A = B^{1/2}(\alpha(t)B^{-1} + B^{-3})B^{1/2} = \alpha(t)I + B^{-2}.
\]
This implies that \(A\) and \(B\) commute from \(-A = (\alpha(t) - t)I + B^{-2}\), and \((t - \alpha(t))I = A + B^{-2}\). By Lemma 2.3(3), \(A = B^2\), which gives a contradiction. \(\square\)

The next theorem is the main result of this paper.

Theorem 3.9. There is a unique solution \(t_0 > \text{tr}(A)\) of \(f(t) = 0\). Furthermore,
\[
X(t_0) = \frac{\alpha(t_0)B^{-1} + B^{-3} + t_0I - A^{-1}}{[\det(\alpha(t_0)B^{-1} + B^{-3} + t_0I - A^{-1})]^{1/2}}
\]
is the unique positive definite solution of the symmetric word equation (3.4).

Proof. By Example 3.7, we may assume that \(A \neq B^2\). The symmetric word equation (3.4) is equivalent to the equation
\[
(X^2BX^2)B^3(X^2BX^2) = XAX.
\]
Let \(X\) be a positive definite solution of the symmetric word equation (3.4) or (3.21). By Riccati Lemma,
\[
X^2BX^2 = B^{-3}(XAX).
\]
By \( \det(B^{-3}) = \det(XAX) = 1 \) and by Lemma 2.2, \( X^2 B X^2 = \frac{1}{\alpha} (B^{-3} + XAX) \) where
\[
\alpha = \sqrt{\det(B^{-3} + XAX)}.
\] (3.22)
Again by the Riccati Lemma and Lemma 2.2,
\[
X^2 = B^{-1} \# \left[ \frac{1}{\alpha} (B^{-3} + XAX) \right]
\]
\[
= \frac{1}{\beta} \left( B^{-1} + \frac{B^{-3} + XAX}{\alpha} \right) = \frac{1}{\alpha \beta} [\alpha B^{-1} + B^{-3} + XAX],
\]
where
\[
\beta = \sqrt{\det\left( B^{-1} + \frac{1}{\alpha} (B^{-3} + XAX) \right)} = \frac{1}{\alpha} \sqrt{\det(\alpha B^{-1} + B^{-3} + XAX)}.
\]
Observe that \( \alpha \beta = \frac{\sqrt{\det(\alpha B^{-1} + B^{-3} + XAX)}}{(\alpha \beta)^{-1}} \). Therefore, \( \alpha \beta X^2 = \alpha B^{-1} + B^{-3} + XAX \) or \( X(\alpha \beta I - A)X = \alpha B^{-1} + B^{-3} \).

This implies that \( \alpha \beta I - A > 0 \) and by the Riccati Lemma,
\[
X = (\alpha B^{-1} + B^{-3}) \# (\alpha \beta I - A)^{-1}.
\] (3.23)
Since \( \det(X) = 1 = \det((\alpha B^{-1} + B^{-3}) \# (\alpha \beta I - A)^{-1}) \), we then have by Lemma 2.3 that \( \alpha^2 + \text{tr}(B^{-2})\alpha = t^2 - \text{tr}(A)t > 0 \), where \( t := \alpha \beta \). We note that
\[
t = \frac{1}{2} \left[ \text{tr}(A) + \sqrt{\text{tr}^2(A) + 4\alpha^2 + 4\alpha\text{tr}(B^{-2})} \right] > \text{tr}(A),
\]
\[
\alpha = \frac{1}{2} \left[ -\text{tr}(B^{-2}) + \sqrt{\text{tr}^2(B^{-2}) + 4t^2 - 4t \text{tr}(A)} \right] = \alpha(t),
\]
and
\[
X \overset{(3.23)}{=} (\alpha B^{-1} + B^{-3}) \# (tI - A)^{-1} = (\alpha(t) B^{-1} + B^{-3}) \# (tI - A)^{-1} = X(t).
\]

By Proposition 3.2(5),
\[
\alpha = \alpha(t) \overset{(3.22)}{=} \sqrt{\det(B^{-3} + X(t)AX(t))} = \frac{t}{\text{tr}(X(t)^2B)}
\]
and therefore \( f(t) = 0 \).

Conversely, let \( t > \text{tr}(A) \) be a solution of \( f(t) = 0 \). Then
\[
X(t) = [\alpha(t) B^{-1} + B^{-3}] \# [tI - A]^{-1}
\]
is positive definite (Proposition 3.2). By Proposition 3.2, \( \det(X(t)) = 1 \) and \( \alpha(t) = \sqrt{\det(B^{-3} + X(t)AX(t))} \). Furthermore,
\[
t X(t)^2 = \alpha(t) B^{-1} + B^{-3} + X(t)AX(t)
\]
by the Riccati Lemma. Setting \( \beta := \frac{t}{\alpha(t)} \), we have
\[
\beta X(t)^2 = \frac{t}{\alpha(t)} X(t)^2 = B^{-1} + \frac{B^{-3} + X(t)AX(t)}{\alpha(t)}
\] (3.24)
and from \( \det(X(t)) = 1 \),
\[ \beta = \sqrt{\det \left( B^{-1} + \frac{B^{-3} + X(t)AX(t)}{\alpha(t)} \right)}. \]

Therefore,
\[
X(t)^2 \overset{\text{3.24}}{=} \frac{B^{-1} + \frac{B^{-3} + X(t)AX(t)}{\alpha(t)}}{\beta} = B^{-1} \# \left[ \frac{B^{-3} + X(t)AX(t)}{\alpha(t)} \right]
\]
\[= B^{-1} \# \left[ B^{-3} \# (X(t)AX(t)) \right]. \]

where the equalities follow from Theorem 2.2. Again by Riccati Lemma,
\[
[X(t)^2BX(t)^2]B^{-3}[X(t)^2BX(t)^2] = X(t)AX(t)
\]
or \(X(t)BX(t)^2B^{-3}X(t)^2BX(t) = A\), that is, \(X(t)\) is a positive definite solution of the symmetric word equation (3.4).

Finally, we will show that the equation \(f(t) = 0\) has a unique positive real solution \(t > \text{tr}(A)\). Let \(t, s > \text{tr}(A)\) be two solutions. Then by the preceding result, \(X(t)\) and \(X(s)\) are positive definite solutions of the symmetric word equation (1.1). By Theorem 1.1, \(X(t) = X(s)\). By Proposition 3.8, \(s = t\). \(\square\)

**Remark 3.10.** In the proof of Theorem 3.9, we have established that the unique solvability of (3.4) is equivalent to that of the equation \(f(t) = 0, t > \text{tr}(A)\). When \(\text{tr}(A) = \text{tr}(B^2)\), the uniqueness of solution of (3.4) follows directly by Proposition 3.6. In general from \(f'(t)\) is differentiable, it suffices to show that \(f'(t) > 0\) (equivalently, \(u(t)g'(t) > u'(t)g(t)\)) for all \(t > \text{tr}(A)\). Computer simulations (programmed in MatLab) show that \(f\) is strictly increasing on \((\text{tr}(A), \infty)\). We do not have a proof for this.

We provide examples in closing which show how our method of searching for the solution of the one variable function \(g(t) = 0\) is more efficient than solving the word equation directly.

**Example 3.11.** Let \(A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\), \(B = \begin{pmatrix} \sqrt{5} + 1 & 0 \\ 0 & \sqrt{5} - 1 \end{pmatrix}\). Then \(\text{tr}(A) = \text{tr}(B^2) = 3\) and \(\det(A) = \det(B) = 1\). Thus, \(\alpha(t) = t - 3\) and
\[
X(t) = u(t)^{-1/2} \begin{pmatrix} \sqrt{5} + 1 - \sqrt{5} + 3 & 1 \\ 1 & \sqrt{5} + 3 - \sqrt{5} + 3 \end{pmatrix},
\]
\[u(t) = \left( 2 + \sqrt{5} \right) t^2 - \frac{1}{2} \left( 11 + 5 \sqrt{5} \right) t + \frac{1}{2} \left( 5 + 3 \sqrt{5} \right),
\]
\[g(t) = \left[ \left( 2 + \sqrt{5} \right) t^3 - \frac{1}{2} \left( 35 + 17 \sqrt{5} \right) t^2 + \left( 38 + 18 \sqrt{5} \right) t - \frac{1}{2} \left( 45 + 27 \sqrt{5} \right) \right].
\]

From \(g'(3) = -13 - 6 \sqrt{5}\), \(g(t) = 0\) has a unique solution in \((3, \infty)\). We obtain approximatively that \(t_0 = 5.8050325230\) and
\[X(t_0) = \begin{pmatrix} 0.7381972840 & 0.1089637381 \\ 0.1089637381 & 1.3707353279 \end{pmatrix}.\]
Example 3.12. Let \( A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \). Then \( \text{tr}(A) = 3, \text{tr}(B) = 5/2, \text{tr}(B^2) = 17/4, \text{tr}(B^3) = 65/8, \text{tr}(A^{-1}B^3) = 33/4 \), and
\[
\begin{align*}
\alpha(t) &= -17/8 + a, \quad a := \sqrt{t^2 - 3t + 289/64}, \\
u(t) &= 2t^2 + (5t/2 - 3)a - 51t/16 + 1/8, \\
X(t) &= u(t)^{-1/2} \begin{pmatrix} 1 + a/2 - 31/16 & 1 \\ 1 & t + 2a + 7/4 \end{pmatrix}, \\
g(t) &= 2t^3 - 317t^2/16 + 191t/4 - 3213/64 + \left(5t^2/2 - 19t + 189/8\right)a.
\end{align*}
\]
The equation \( g(t) = 0 \) has a unique solution in \((3, \infty)\), approximatively \( t_0 = 6.4800630156 \) and hence
\[
X(t_0) = \begin{pmatrix} 0.6214932125 & 0.0869970293 \\ 0.0869970293 & 1.6212059323 \end{pmatrix}.
\]

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References