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A Class of Elliptic Systems Involving N-Functions

C. O. ALVES*
Departamento de Matemática, Universidade Federal de Campina Grande
58109-970 - Campina Grande (PB), Brazil
coalves@dme.ufcg.edu.br

P. C. CARRIÃO†
Departamento de Matemática, Universidade Federal de Minas Gerais
31270-010 - Belo Horizonte (MG), Brazil.
carrion@mat.ufmg.br

O. H. MIYAGAKI*†‡
Departamento de Matemática, Universidade Federal de Viçosa
36571-000 - Viçosa (MG), Brazil
olimpio@ufv.br

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Abstract—In this work, we state a result of compactness due to Lions in Orlicz spaces. We give an application proving an existence result for a gradient type elliptic systems in $\mathbb{R}^N$ involving N-functions. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

In this paper, we study the existence of solution for the following class of elliptic systems:

$$\begin{aligned} -\Delta u + Wu &= Q_u(u,v) + H_u(u,v), && \text{in } \mathbb{R}^N, \\ -\Delta v + Wv &= Q_v(u,v) + H_v(u,v), && \text{in } \mathbb{R}^N, \\ u, v &> 0, && \text{in } \mathbb{R}^N, \end{aligned} \tag{S}$$

where $W : \mathbb{R}^N \longrightarrow (0,\infty)$ is a continuous function, $H \in C^1(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R})$ is a 2^*-homogeneous function with gradient $\nabla H = (H_u, H_v)$, $2^* = 2N/(N-2)$ with $N \geq 3$, and $Q \in C^1(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R})$ is a positive function bounded from above by an N-function, (see [1,2] for definitions), that is, the function Q is assumed to verify the following conditions:

$$\begin{aligned} Q_u(u,v) \leq Ca(|u|) \quad \text{and} \quad Q_v(u,v) \leq Cb(|v|), \qquad u, v \in \mathbb{R}; \\ Q(u,v) \leq C(A(|u|) + B(|v|)), \qquad u, v \in \mathbb{R}, \end{aligned} \tag{Q_0}$$

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‡Author to whom all correspondence should be addressed.

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where C denotes a generic positive constant, $A(t) = \int_0^t a(s)\,ds$ and $B(t) = \int_0^t b(s)\,ds$ mean two N-functions, with a and b satisfying condition (G) below, namely, we say that a function g verifies condition (G) if

$$\begin{aligned} &g \text{ is odd}, \quad g(0) = 0, \quad g(t) > 0, \quad \text{if } t > 0, \qquad \lim_{t\to+\infty} g(t) = +\infty, \\ &\text{nondecreasing and right continuous on } t > 0, \quad |g(t)| \le |t|^{2^*-1}, \qquad \text{for all } t. \end{aligned} \tag{G}$$

Here F_z denotes the partial derivative of F with respect to variable z.

The main difficulty of establishing existence results to this kind of elliptic system is the fact that the function Q has critical growth involving N-functions. Because it is possible to find a N-function such that, at infinity, its growth is greater than any subcritical growth (see an example below). It is well known that, when we use the variational techniques, a lemma proved by Lions in [3] is one of the greatest tools to show the existence of solution to (S). In [4], it is given a different proof for this result. However, as far as we know, the proof of this result involving N-function has never appeared before in the literature. In this paper, we prove a version of this lemma for N-functions (see Main Lemma) and, as application, we state a result involving the above system which completes some results obtained by the authors in the papers [5,6]. Finally, we would like to cite papers [7,8] (and references therein) for some existence results for quasilinear problems in Orlicz-Sobolev space setting.

The example below, given in [9], shows a function whose primitive is a N-function with behavior mentioned above.

EXAMPLE. Let $a : [0,\infty) \to \mathbb{R}$ be given by

$$a(t) = \begin{cases} t^{2^*-1}, & \text{if } 0 \le t < 1, \\ t^{2^*-1-1/\log(\log 2)}, & \text{if } 1 \le t \le 3, \\ t^{2^*-1-1/\log(\log n)}, & \text{if } n \le t < n+1, \quad n = 3, 4, \dots. \end{cases}$$

DEFINITION. ORLICZ SPACE. *Let Ω be a domain in $\mathbb{R}^N$ and let A be an N-function. The Orlicz space $L_A(\Omega)$ is the set of all measurable functions u defined on Ω such that there exists $\lambda > 0$ satisfying $\int_\Omega A(|u(x)|/\lambda)\,dx < \infty$, endowed with the norm*

$$\|u\|_{L_A(\Omega)} = \inf\left\{\lambda : \int_\Omega A\left(\frac{|u(x)|}{\lambda}\right) dx \le 1\right\}.$$

LEMMA A. MAIN LEMMA. *Let $\{u_n\}$ be a bounded sequence in $H^1(\mathbb{R}^N)$, such that*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^2\,dx = 0, \qquad \textit{for some } R > 0.$$

Then

$$\|u_n\|_{L_A(\mathbb{R}^N)} \longrightarrow 0, \qquad \textit{as } n \to \infty,$$

where A is a N-function verifying (G).

The next result is related to the action of group in Sobolev spaces. For more details about this subject see [10, Theorem 1.24].

DEFINITION. *Let G be a subgroup of $O(N)$, $y \in \mathbb{R}^N$ and $r > 0$. We define*

$$m(y,r,G) = \sup\{n \in \mathbb{N} : \exists\, g_1,\dots,g_n \in G \text{ s.t. } B(g_j y, r) \cap B(g_k y, r) = \emptyset, \text{ for } j \ne k\},$$

where $B(y,r)$ denotes a ball centered in y with radius r. An open subset Ω of $\mathbb{R}^N$ is compatible with G if $g\Omega = \Omega$ for every $g \in G$ and for some $r > 0$

$$\lim_{|y|\to\infty,\,\mathrm{dist}\,(y,\Omega)\le r} m(y,r,G) = \infty.$$

COROLLARY B. *Let $\Omega \subset \mathbb{R}^N$ be a open set and G a subgroup of $O(N)$. If Ω is compatible with G, the following embedding is compact:*

$$H^1_{o,G}(\Omega) \hookrightarrow L_A(\Omega).$$

In order to give an application of above lemma, we shall impose the following hypothesis:

$$\begin{gathered} H \text{ is } 2^* - \text{homogeneous}, \ H(u,v) > 0 \text{ for every } u, v > 0, \\ H_u(0,1) = H_v(1,0) = 0. \end{gathered} \tag{H$_1$}$$

$$H_u(0,1) = H_v(1,0) = 0. \tag{H$_2$}$$

$$\begin{gathered} 0 < \theta, \qquad Q(u,v) \leq \nabla Q(u,v).(u,v), \qquad \theta > 2, \ u, v \in \mathbb{R}; \\ Q(u,v) \geq C\left(|u|^q + |v|^q\right), \qquad u, v \in \mathbb{R}, \\ \text{for some } q \in [2, 2^* - 1)\,, \qquad \text{if } N \geq 4 \text{ and } q \in (3,5), \quad \text{if } N = 3, \\ Q_u(0,1) \geq 0 \text{ and } Q_v(1,0) \geq 0, \\ \text{the 1-homogeneous function } G \text{ defined by} \\ G\left(s^{p^*}, t^{p^*}\right) = H(s,t), \ \forall\, s,t \geq 0, \ \text{is concave.} \\ \text{For } (u,v) \text{ fixed}, \ \frac{\nabla Q(t(u,v)).(u,v)}{t} \text{ is a strictly increasing function for } t \neq 0. \end{gathered} \tag{Q}$$

$$W \text{ is } 1 - \text{periodic.} \tag{P}$$

Thus, our application of Lemma A is the following.

THEOREM C. *Suppose that (Q), (P), (G), (H_1), and (H_2) hold. Then system (S) has a positive solution.*

2. PROOFS

PROOF OF LEMMA A. Observe that, without loss of generality, we can assume $u_n \geq 0$. On the contrary, for $u_n \in L_A(\mathbb{R}^N)$ writing $u_n = u_n^+ - u_n^-$ with $u_n^\pm = \max\{\pm u_n, 0)\}$ we have $u_n^\pm \in L_A(\mathbb{R}^N)$ and

$$\|u_n\|_{L_A(\mathbb{R}^N)} \leq \left\|u_n^+\right\|_{L_A(\mathbb{R}^N)} + \left\|u_n^-\right\|_{L_A(\mathbb{R}^N)}.$$

Let u_n^* be the Schwarz symmetrization of u_n (see e.g., [11]). Then

$$\int_{\mathbb{R}^N} A\left(\frac{u_n}{\lambda}\right) dx = \int_{\mathbb{R}^N} A\left(\frac{u_n^*}{\lambda}\right) dx \qquad \text{and} \qquad \|u_n\|_{L_A(\mathbb{R}^N)} = \|u_n^*\|_{L_A(\mathbb{R}^N)}, \tag{1}$$

so, it is sufficient to prove

$$\|u_n^*\|_{L_A(\mathbb{R}^N)} \longrightarrow 0, \qquad \text{as } n \to \infty. \tag{2}$$

In order to prove the convergence above, we remark that the embedding

$$H^1_0(\Omega) \hookrightarrow L_A(\Omega) \text{ is compact, when } \Omega \text{ is bounded.} \tag{3}$$

Indeed, observe that if Ω is bounded, there exists $u_0 \in H^1_0(\Omega)$, such that $u_n \longrightarrow u_0$ in $L^q(\Omega)$ as $n \to \infty$, for all $q \in (2, 2^*)$. Moreover, notice that from (G) we infer that for $q \in (2, 2^*)$ and each $\delta > 0$, there exists $C_\delta > 0$, such that

$$A(t) \leq \delta\left(t^2 + t^{2^*}\right) + C_\delta t^q, \qquad \forall\, t \in \mathbb{R}. \tag{4}$$

By (4), by choosing $\delta = \epsilon^{2^*+1}$, we have

$$\int_\Omega A\left(\frac{u_n - u_0}{\epsilon}\right) dx \leq 1.$$

Hence, $u_n \longrightarrow u_0$ in $L_A(\Omega)$ as $n \to \infty$. This proves (3).

Now we will prove (2). For each $R > 0$ fixed and since $u \in L_A(\mathbb{R}^N)$ we have

$$\int_{B_R} A\left(\frac{u}{\lambda}\right) dx = \int_{\mathbb{R}^N} A\left(\frac{\chi_{B_R} u}{\lambda}\right) dx \leq \int_{\mathbb{R}^N} A\left(\frac{u}{\lambda}\right) dx < \infty$$

and

$$\int_{\mathbb{R}^N} A\left(\frac{(1-\chi_{B_R})\, u}{\lambda}\right) dx < \infty,$$

where χ_{B_R} denotes the characteristic function on a ball, B_R, centered at the origin with radius R. Then

$$\|u\|_{L_A(B_R)} = \|\chi_{B_R} u\|_{L_A(\mathbb{R}^N)} \qquad \text{and} \qquad \|u\|_{L_A(\mathbb{R}^N \setminus B_R)} = \|(1-\chi_{B_R})u\|_{L_A(\mathbb{R}^N)}.$$

By hypothesis and applying Lions's Lemma (see [3]) we have

$$|u_n|_q, |u_n^*|_q \longrightarrow 0, \; u_n^* \longrightarrow 0, \qquad \text{a.e., in } \mathbb{R}^N \text{ as } n \to \infty,$$

where $|u|_s$ denotes the usual L^s-norm.

But, recalling that $\{u_n^*\}$ is bounded in $H^1(\mathbb{R}^N)$ and using (3), we get

$$\|u_n^*\|_{L_A(B_R)} \longrightarrow 0, \text{ as } n \to \infty.$$

By the inequality below due to Strauss (see [11])

$$|u_n^*(x)| \leq \frac{\tilde{C}}{|x|^{(N-1/2)}} \equiv g(x), \qquad \tilde{C} > 0, \quad \forall n \in \mathbb{N}, \quad \forall x \in \mathbb{R}^N \setminus B_R,$$

we conclude that $g \in L^s(\mathbb{R}^N \setminus B_R)$, $s > 2N/(N-1)$.

So, given $\epsilon > 0$, choose $R_0 > R$ such that

$$\int_{\mathbb{R}^N \setminus B_{R_0}} |g(x)|^s \, dx \leq \frac{\epsilon}{2}, \qquad \forall x \in \mathbb{R}^N \setminus B_{R_0}.$$

Now since the embedding $H^1_{\text{rad}}(R < |x| < R_0) \hookrightarrow L^s(R < |x| < R_0)$ is compact for all $s \in (2, \infty)$, where $H^1_{\text{rad}}(R < |x| < R_0) = \{u \in H^1(R < |x| < R_0) : u \text{ is radial}\}$, taking $s = 2^*$ we obtain

$$\int_{\mathbb{R}^N \setminus B_R} |u_n^*|^{2^*} \, dx = \int_{B_{R_0} \setminus B_R} |u_n^*|^{2^*} \, dx + \int_{\mathbb{R}^N \setminus B_{R_0}} |u_n^*|^{2^*} \, dx \longrightarrow 0, \qquad \text{as } n \to \infty. \tag{5}$$

From (4) and using (5), we get

$$\int_{\mathbb{R}^N \setminus B_R} A\left(\frac{u_n^*}{\epsilon/2}\right) dx < 1, \qquad \forall \epsilon > 0 \text{ and for all } n \text{ sufficiently large},$$

that is, $\|u_n^*\|_{L_A(\mathbb{R}^N \setminus B_R)} \longrightarrow 0$ as $n \to \infty$. This completes the proof of lemma.

PROOF OF COROLLARY B. It suffices to observe that the embedding $H^1_G(\Omega) \subset L^p(\Omega)$ is compact, for all $p \in (2, 2^*)$ (see [10, Theorem 1.24]).

PROOF OF THEOREM C. It is well known that weak solutions of (S) are the critical points of functional

$$I(u,v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) \, dx + \frac{1}{2} \int_{\mathbb{R}^N} W\,(u^2 + v^2) \, dx - \int_{\mathbb{R}^N} Q(u,v) \, dx - \int_{\mathbb{R}^N} H(u,v) \, dx,$$

which belong to $C^1(H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N), \mathbb{R})$. Hereafter, we denote by $E = (H^1(\mathbb{R}^N))^2$ the Hilbert space endowed with the norm

$$\|(u,v)\|^2 = \int_{\mathbb{R}^N} \left((|\nabla u|^2 + |\nabla v|^2 + W\left(|u|^2 + |v|^2\right)\right) dx.$$

As in [12], in the definition of functional I, we are considering the following extension for the functions Q and H in whole space $\mathbb{R}^2$:

$$Q(u,v) := \begin{cases} Q(u,v), & u,v \geq 0, \\ Q(0,v) + Q_u(0,v)u, & u \leq 0 \leq v, \\ Q(u,0) + Q_v(u,0)v, & v \leq 0 \leq u, \\ Q(u,0), & u,v \leq 0, \end{cases}$$

and

$$H(u,v) := H\left(u^+, v^+\right), \qquad w^{\pm} := \max\{\pm w, 0\}.$$

We also make use of the following constant:

$$S_H = \inf\left\{ \frac{\int_{\mathbb{R}^N} \left(|\nabla u|^2 + |\nabla v|^2\right) dx}{\left(\int_{\mathbb{R}^N} H(u,v)\, dx\right)^{2/2^*}} : (0,0) \neq (u,v) \in \left(H^1\left(\mathbb{R}^N\right)\right)^2 \right\}.$$

The constant S_H was defined by de Morais and Souto in [12] (see also [13]), which is related to the best constant of Sobolev obtained in [14]. As in the scalar case, the levels where the functional I fails to satisfy Palais-Smale condition (see below the definition) will involve the constant S_H.

Under our hypothesis, it is standard to prove that our functional verifies the mountain pass geometry (see [15]), then there exists a $(PS)_c$ sequence $\{(u_n, v_n)\} \subset E$, that is,

$$I(u_n, v_n) \longrightarrow c \quad \text{and} \quad I'(u_n, v_n) \longrightarrow 0, \qquad \text{in } E^*,$$

where $c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) > 0$, with $\Gamma = \{\gamma \in C(E, \mathbb{R}) : \gamma(0) = 0\ \gamma(1) = (e,e)\}$. We point out that, arguing as in [4], the above sequence is bounded in E. So, we can assume that there exists $(u,v) \in E$, such that $(u_n, v_n) \rightharpoonup (u,v)$ weakly in E, as $n \to \infty$. Passing to the limit in $I'(u_n, v_n) = o(1)$, as $n \to \infty$, we conclude that (u,v) is a weak solution of problem (S). Adapting the arguments used in [16], we consider the solution $w_\epsilon > 0$ of equation

$$-\Delta w_\epsilon = w_\epsilon^{2^*-1}, \qquad \text{in } \mathbb{R}^N,$$

and $\psi_\epsilon = \varphi w_\epsilon$, where φ is cut-off function, that is, $\varphi = 1$ in B_1, $\varphi = 0$ in $\mathbb{R}^N \setminus B_2$, and $0 \leq \varphi \leq 1$. Define

$$v_\epsilon = \frac{\psi_\epsilon}{\left(\int_{B_2} \psi_\epsilon^{2^*}\right)^{1/2^*}},$$

then

$$I(tBv_\epsilon, tDv_\epsilon) \leq \frac{t^2}{2}\left(B^2 + D^2\right) \int \left(|\nabla v_\epsilon|^2 + W v_\epsilon^2\right) - t^{2^*} H(B,D) - \lambda t^q \left(B^q + D^q\right) C \int v_\epsilon^{q+1},$$

which implies

$$I(tBv_\epsilon, tDv_\epsilon) < \frac{1}{N}\left(S_H\right)^{N/2} \left(\frac{1}{2^*}\right)^{(N-2)/N}, \qquad \text{for } \epsilon \text{ sufficiently small.}$$

Therefore,

$$c < \frac{1}{N} (S_H)^{N/2} \left(\frac{1}{2^*}\right)^{(N-2)/N} \equiv S_0. \tag{$*$}$$

Repeating the arguments explored in [5, Lemma 3.3] together Lemma A and $(*)$, we have that there exist ρ, $\eta > 0$ and $y_n \in \mathbb{R}^N$, such that

$$\lim_{n\to\infty} \left[\sup_{y\in\mathbb{R}^N} \int_{B_\rho(y)} |u_n|^2 + |v_n|^2 \, dx \right] \geq \eta.$$

Putting $\hat{u}_n(x) = u(x - y_n)$ and $\hat{v}_n(x) = v(x - y_n)$ then $\{(\hat{u}_n, \hat{v}_n)\}$ is a $(PS)_c$ sequence with $c > 0$, such that $(\hat{u}_n, \hat{v}_n) \rightharpoonup (\hat{u}, \hat{v})$ weakly as $n \to \infty$. Moreover, $(\hat{u}, \hat{v})$ is a nontrivial solution of (S) with $\hat{u} > 0$ and $\hat{v} > 0$. This completes the proof of Theorem C.

3. FINAL COMMENTS

Adapting the arguments explored by the authors in [5,6], it is possible to show a result of existence of a positive solution to (S) when the function W is a perturbation of a 1-periodic function, that is, W behaves at infinity like a 1-periodic function.

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