On the Control of Certain Interacting Populations

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We study the time optimal control of the system

$$\dot{x}_1 = x_1 f_1(x_1, x_2) + u_1(t) g_1(x_1), \quad \dot{x}_2 = x_2 f_2(x_1, x_2) + u_2(t) g_2(x_2),$$

where $x_1$ is the size of the population of one species, $x_2$ is the population size of the second species, $f_1$ and $f_2$ are the fractional growth rates of the respective species, $g_1$ and $g_2$ are nowhere vanishing functions of class $C^1(0, +\infty)$, and the control $u(t) = (u_1(t), u_2(t))$ takes on values in a closed rectangle. The functions $f_1$ and $f_2$ are chosen to represent prey-predator, competitive, and symbiotic interactions.

We show, for the various interactions, that a time optimal control, if it exists, must be "bang-bang," and give sufficient conditions for the controllability, and for the existence, of time optimal controls of the above system.

1. Introduction

Interacting biological populations have been the subject of much investigation, both experimental and theoretical, for many years (see [1]-[4] for extensive bibliographies). Yet, despite the current interest in ecological problems, very few studies have appeared in which control theory has been used to treat the control of interacting populations. Thau [5], using an integral quadratic cost functional, has obtained a quasi-optimum feedback control law for two competing species governed by Volterra's competition equations. Vincent [6] applied optimal control theory, with an integral linear cost functional, to the control of a prey-predator system described by the Lotka-Volterra equations. Goh, Leitmann, and Vincent [7], and Vincent, Cliff, and Goh [8] also studied optimal control of prey-predator systems governed by the Lotka-Volterra equations.

Here we study a certain time optimal control problem for two interacting populations. We employ the system

$$\frac{dx_1}{dt} = x_1 f_1(x_1, x_2)$$
$$\frac{dx_2}{dt} = x_2 f_2(x_1, x_2)$$

(1)

10 to describe the interaction of the two uncontrolled species; $x_1$ is the size
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of the population of one species, \(x_2\) the population size of the second species and \(f_1, f_2\) are the fractional growth rates of the respective species.

In Section 2 we state conditions on \(f_1\) and \(f_2\) which have been interpreted [10] as describing situations where the uncontrolled species cooperate (symbiosis), where they compete, and where one population consists of prey, and the other of predators. We also summarize there some properties of the models which we have proved earlier [11].

We consider a control system, obtained from (1), which depends linearly on the control variables and formulate a class of time optimal control problems. We first show, in Section 3, that if a time optimal control exists then it must be “bang-bang,” whatever the mode of interaction of the species. We then study, in Section 4, the controllability of each of the three interactions, and investigate, in Section 5, the existence of time optimal controls. Finally, in Section 6, we apply our controllability results to two examples from biology.

2. THE UNCONTROLLED DYNAMICAL MODELS

Kolmogorov [9] used (1) to describe the dynamics of prey-predator interactions; Rescigno and Richardson [10] extended (1) to the cases of competitors and cooperators by altering the conditions on \(f_1, f_2\). We omit Kolmogorov's and Rescigno and Richardson's verbal interpretations, merely listing the mathematical assumptions.

We consider only the first quadrant

\[
Q^\text{def} = \{x_1 \geq 0, x_2 \geq 0\},
\]

with

\[
Q^0 = \{x_1 > 0, x_2 > 0\}
\]

in the \(x_1-x_2\) plane, and assume throughout that \(f_1, f_2 \in C^0\) in \(Q\) and \(f_1, f_2 \in C^1\) in \(Q^0\). Notice that every trajectory of (1) starting in \(Q\) at \(t = 0\) lies entirely either in \(Q^0\), on one of the positive semiaxes, or is the point \((0, 0)\) [11].

2.1. Prey–Predator Interactions

We let \(x_1\) denote the prey and \(x_2\) denote the predator population, and assume

P1. (a) There exists an \(x_1^* > 0\) such that

\[
(x_1 - x_1^*) f_1(x_1, 0) < 0 \quad \text{for all } x_1 \geq 0, \quad x_1 \neq x_1^*.
\]

(b) There exists an \(x_2^* > 0\) such that

\[
(x_2 - x_2^*) f_1(0, x_2) < 0 \quad \text{for all } x_2 \geq 0, \quad x_2 \neq x_2^*.
\]
(c) \( \frac{\partial f_1}{\partial x_2} < 0 \) in \( Q^0 \).

(d) For every \((\alpha, \beta) \in Q^0\)

\[
\frac{\partial f_1}{\partial x_1} (\alpha, \beta) x + \frac{\partial f_1}{\partial x_2} (\alpha, \beta) \beta < 0.
\]

P2. (a) There exists an \( \delta > 0 \) such that

\[(x_1 - \delta_1) f_2(x_1, 0) > 0 \quad \text{for all } x_1 \geq 0, \quad x_1 \neq \delta_1.\]

(b) \( \frac{\partial f_2}{\partial x_2} \leq 0 \) in \( Q^0 \).

(c) For every \((\alpha, \beta) \in Q^0\)

\[
\frac{\partial f_2}{\partial x_1} (\alpha, \beta) x + \frac{\partial f_2}{\partial x_2} (\alpha, \beta) \beta > 0.
\]

P3. \( \hat{x}_1 < x_1^* \).

If P1–P3 are satisfied, then [11]:

(a) The equation \( f_1(x_1, x_2) = 0 \) defines a unique continuous function \( x_2 = \phi_1(x_1) \) on the interval \([0, x_1^*]\), such that \( \phi_1(0) = x_2^*, \phi_1(x_1^*) = 0 \), and \( \phi_1 \) is strictly positive and differentiable on \((0, x_1^*)\) with

\[
\phi_1'(x_1) < \frac{\phi_1(x_1)}{x_1}.
\]

(b) In \( Q^0 \), \( \frac{\partial f_2}{\partial x_1} > 0 \). The equation \( f_2(x_1, x_2) = 0 \) defines a unique continuous function \( x_1 = \phi_2(x_2) \) on the interval \([0, +\infty)\), such that \( \phi_2(0) = \delta_1 \), and \( \phi_2 \) is differentiable on \((0, +\infty)\) with

\[
0 < \phi_2'(x_2) < \frac{\phi_2(x_2)}{x_2}.
\]

(c) There exists a unique singular point \((x_{10}, x_{20})\) of (1) in \( Q^0 \). If \((x_{10}, x_{20})\) is unstable then there exists at least one periodic orbit in \( Q^0 \). Moreover all the periodic orbits lie within a region bounded by an outermost periodic orbit, which is semistable from the outside, and an innermost periodic orbit, which is semistable from the inside. If there is just one periodic orbit it is stable. If there is no periodic orbit, then \((x_{10}, x_{20})\) is a global attractor.

2.2. Competitive Interactions

Again \( x_1 \) and \( x_2 \) are to denote the populations of the two distinct (and, in this case, competing) species. We now assume
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Cl. (a) \( \frac{\partial f_i}{\partial x_j} < 0 \) in \( Q^0 \), for all \( i, j = 1, 2 \).

(b) There exist \( x_{11} > 0 \) and \( x_{22} > 0 \) such that

\[
(x_1 - x_{11})f_1(x_1, 0) < 0 \quad \text{for all } x_1 \geq 0, \ x_1 \neq x_{11},
\]

\[
(x_2 - x_{22})f_2(0, x_2) < 0 \quad \text{for all } x_2 \geq 0, \ x_2 \neq x_{22}.
\]

(c) There exist \( x_{12} > 0 \) and \( x_{21} > 0 \) such that

\[
(x_2 - x_{21})f_1(0, x_2) < 0 \quad \text{for all } x_2 \geq 0, \ x_2 \neq x_{21},
\]

\[
(x_1 - x_{12})f_2(x_1, 0) < 0 \quad \text{for all } x_1 \geq 0, \ x_1 \neq x_{12}.
\]

If Cl is satisfied then [11]:

(a) For \( i = 1, 2 \), the equation \( f_i(x_1, x_2) = 0 \) defines a unique continuous function \( x_2 = \phi_i(x_1) \) on \( [0, x_{1i}] \), such that \( \phi_i(0) = x_{2i}, \phi_i(x_{1i}) = 0 \) and \( \phi_i \) is strictly positive and differentiable on \( (0, x_{1i}) \) with

\[
\phi_i'(x_1) < 0,
\]

hence \( \phi_i \) is strictly decreasing.

(b) The only singular points of (1) on the boundary of \( Q \) are \((0, 0), (x_{11}, 0), \) and \((0, x_{22})\), while the points of intersection of \( x_2 = \phi_i(x_1) \) and \( x_2 = \phi_2(x_1) \) are the only singular points in \( Q^0 \). The set of these intersection points, which is closed, may be empty, finite, countably or uncountably infinite.

(c) All trajectories of (1) approach some singular point in \( Q \), as \( t \to \infty \).

2.3. Symbiotic Interactions

We consider the case of perfect "cooperation": an increase in the size of either population stimulates the growth rate of the other, regardless of the size of the populations.

We assume

S1. (a) \( \frac{\partial f_1}{\partial x_2} > 0 \) and \( \frac{\partial f_2}{\partial x_1} > 0 \) in \( Q^0 \).

(b) For any \( (x, \beta) \in Q^0 \), there exists a \( \gamma > 0 \) such that

\[
\frac{\partial f_1}{\partial x_1} (x \xi, \beta \xi) + \frac{\partial f_1}{\partial x_2} (x \xi, \beta \xi) \beta < -\gamma
\]

and

\[
\frac{\partial f_2}{\partial x_1} (x \xi, \beta \xi) + \frac{\partial f_2}{\partial x_2} (x \xi, \beta \xi) \beta < -\gamma
\]

for \( \xi > 0 \).
(c) There exists an $x_{11} > 0$ and an $x_{22} > 0$ such that
\[
(x_1 - x_{11}) f_1(x_1, 0) < 0 \quad \text{for all } x_1 \geq 0, \quad x_1 \neq x_{11}
\]
\[
(x_2 - x_{22}) f_2(0, x_2) < 0 \quad \text{for all } x_2 \geq 0, \quad x_2 \neq x_{22}.
\]

If $S1$ is satisfied, then [11]:

(a) The equation $f_1(x_1, x_2) = 0$ defines a unique continuous function $x_1 = \phi_1(x_2)$ on $[0, +\infty)$, such that $\phi_1(0) = x_{11}$ and $\phi_1$ is differentiable on $(0, +\infty)$ with
\[
0 < \phi_1'(x_2) < \frac{\phi_1(x_2)}{x_2}.
\]

(b) The equation $f_2(x_1, x_2) = 0$ defines a unique continuous function $x_2 = \phi_2(x_1)$ on $[0, +\infty)$, such that $\phi_2(0) = x_{22}$ and $\phi_2$ is differentiable on $(0, +\infty)$ with
\[
0 < \phi_2'(x_1) < \frac{\phi_2(x_1)}{x_1}.
\]

(c) There exists a unique singular point of (1), $(x_{10}, x_{20}) \in Q^0$.

(d) Every trajectory of (1) in $Q^0$ approaches $(x_{10}, x_{20})$ as $t \to +\infty$.

3. **Time-Optimal Control: The Bang-Bang Property**

We consider now the time-optimal control of two interacting populations by applying certain controls to the Kolmogorov model (1):

\[
\begin{align*}
\frac{dx_1}{dt} &= x_1 f_1(x_1, x_2) + u_1(t) g_1(x_1) \\
\frac{dx_2}{dt} &= x_2 f_2(x_1, x_2) + u_2(t) g_2(x_2),
\end{align*}
\]

where $g_1$ and $g_2$ are nowhere vanishing functions of class $C^1$ on $(0, +\infty)$. We assume, for definiteness, that $g_1$ and $g_2$ are both positive.

We assume that each control $u(t) = (u_1(t), u_2(t))$ is defined on a compact interval $[0, T]$ (whose length depends on the chosen control) and takes values in the rectangle $[a_1, b_1] \times [a_2, b_2]$, $a_i \ll 0 < b_i$, $i = 1, 2$. The pair $(u_1(t), u_2(t))$ ranges over an admissible family $\mathcal{U}$ of controls in the sense of [12, p. 309] or [13] (e.g. $\mathcal{U}$ may consist of all measurable, or all piecewise continuous, or all piecewise constant controls).

System (2) can serve as a mathematical description of a number of different controlled interactions, depending on the choices of the $g$'s and the signs of the products $g_1 u_1, g_2 u_2$. If $g_1 = g_2 = 1$ and $u_1 > 0, u_2 > 0$, then this can be
interpreted as a situation where members of both species are being introduced at rates \( u_1(t) \) and \( u_2(t) \), respectively; if \( u_1 < 0, u_2 < 0 \), then the species are being depleted at those rates. Another interpretation for \( g_1 = 1, g_2 = 1, u_1 > 0, u_2 > 0 \) is that the two species are being fed (and hence multiply) differentially. If \( g_1(x_1) = x_1, g_2(x_2) = x_2 \), then (2) can be said to describe a rate control of species proportional to the size of the species population. If, further, \( u_1 \) and \( u_2 \) are negative, then one may consider this a case of harvesting or seining, the size of the harvest or catch (and therefore the decrease in growth rate of each species) being proportional to the population size. Specific examples of such control systems will be considered later.

In the absence of control the two interacting populations may approach one of possibly several equilibrium states, or may oscillate in size.

We are concerned here with steering (2) in "minimal time," from any initial \( x_0 \in \mathbb{Q}^0 \) to regions close enough to stable equilibrium points, or stable (possibly semistable) periodic orbits of (1), so that once the control is eliminated the populations will remain close to these stable solutions of (1).

(We give in Section 4 a formal definition of the region \( G \) to be reached in minimal time.)

More precisely, for \( u \in \mathcal{U} \) denote by \( x(t; u, x_0) \) the unique solution of (2) satisfying the initial condition \( x(0; u, x_0) = x_0 \). Then an admissible control \( \dot{u} \), defined on \([0, T]\), is said to be time-optimal, if:

M1. \( \dot{x}(t) = x(t; \dot{u}, x_0) \in \mathbb{Q}^0 \) is defined on the entire interval \([0, T]\) with \( x(T) \in G \), where \( G \) is the "target" set of points in \( \mathbb{Q}^0 \); that is, \( \dot{u} \) steers \( x_0 \) to \( G \).

M2. For every \( u \in \mathcal{U} \) defined on \([0, T]\) and steering \( x_0 \) to \( G \), we have \( T \leq \hat{T} \).

We first prove

**Theorem 1.** If \( \frac{\partial f_1}{\partial x_2} \) and \( \frac{\partial f_2}{\partial x_1} \) are nowhere zero on \( \mathbb{Q}^0 \) and if for \( x_0 \in \mathbb{Q}^0 \) and \( G \subset \mathbb{Q}^0 \) there exists a time-optimal control \( \dot{u}(t) = (\dot{a}_1(t), \dot{a}_2(t)) \) steering \( x_0 \) to \( G \), then for \( i = 1, 2, \) \( \dot{a}_i(t) \) is a piecewise constant function taking only the values \( a_i \) or \( b_i \). In particular, this is true if any of the groups of hypotheses (P), (C), or (S) hold.

**Proof.** Consider the functions \( h_i: (0, +\infty) \to \mathbb{R} \), defined by

\[
h_i(x_i) = \int_{x_i}^{\infty} \frac{d\tau}{g_i(\tau)}, \quad i = 1, 2.
\]

Clearly each of the functions \( h_i \) is a \( C^2 \)-diffeomorphism of \((0, +\infty)\) onto its image in \( \mathbb{R} \). Therefore, the mapping

\[
h = h_1 \times h_2: \mathbb{Q}^0 \to h(\mathbb{Q}^0) \subset \mathbb{R}^2
\]
is a $C^2$-diffeomorphism. Let
\[ y = (y_1, y_2) = h(x) = (h_1(x_1), h_2(x_2)) \]
for $x = (x_1, x_2) \in Q^0$. Now write (2) in vectorial form
\[ dx/dt = f(x) \cdot x + g(x) \cdot u(t) \]
where $f(x)$ and $g(x)$ are the diagonal $2 \times 2$ matrices $f(x) = \text{diag}[f_1(x), f_2(x)]$, $g(x) = \text{diag}[g_1(x), g_2(x)]$.

The substitution $y = h(x)$ then yields in $h(Q^0)$
\[ dy/dt = Dh(h^{-1}(y)) \cdot [f(h^{-1}(y)) \cdot h^{-1}(y) + g(h^{-1}(y)) \cdot u(t)], \]
where
\[ Dh(x) = \begin{bmatrix} h_1'(x_1) & 0 \\ 0 & h_2'(x_2) \end{bmatrix} = \begin{bmatrix} 1/g_1(x_1) & 0 \\ 0 & 1/g_2(x_2) \end{bmatrix} \]
is the Jacobian matrix of $h$ at $x \in Q^0$. Equivalently, we obtain in scalar notation
\[ dy_1/dt = F_1(y_1, y_2) + u_1(t) \]
\[ dy_2/dt = F_2(y_1, y_2) + u_2(t) \]
where the $C^1$-functions $F_i$ are given by
\[ F_i(y_1, y_2) = -\frac{1}{g_i(h_i^{-1}(y_i))} \cdot h_i^{-1}(y_i) \cdot f_i(h_i^{-1}(y_1), h_i^{-1}(y_2)). \]

It is a straightforward matter to show that for every $x_0 \in Q^0$, $u \in \mathcal{U}$, and every $t$ such that $x(t; u, x_0)$ is defined, we have
\[ h(x(t; u, x_0)) = y(t; u, h(x_0)), \]
i.e., $h$ maps trajectories of (2) onto trajectories of (3), preserving both the control and the parameterization of the solutions. It follows that an admissible control $\hat{u} \in \mathcal{U}$ is time-optimal in $Q^0$ with respect to $x_0 \in Q^0$, the set $G$, and system (2), if and only if it is time-optimal in $h(Q^0)$ with respect to $h(x_0) \in h(Q^0)$, the set $h(G)$, and system (3).

Thus, it is sufficient to prove our assertion for the simpler system (3), in which the control enters additively. We apply now the Pontryagin Maximum Principle (see e.g. [12]) to the optimal control $\hat{u}$. Since $x_0$ is fixed, we write $\hat{y}(t) = y(t; \hat{u}, h(x_0))$. There exists a nontrivial solution
\[ \Psi(t) = (\Psi_1(t), \Psi_2(t)) \]
of the adjoint system for $\dot{y}(t) = (\dot{y}_1(t), \dot{y}_2(t))$:

$$
\begin{align*}
\dot{\psi}_1 &= -\left(\frac{\partial F_1}{\partial y_1} \psi_1 + \frac{\partial F_2}{\partial y_2} \psi_2\right)_{y=\hat{y}(t)} \\
\dot{\psi}_2 &= -\left(\frac{\partial F_1}{\partial y_2} \psi_1 + \frac{\partial F_2}{\partial y_2} \psi_2\right)_{y=\hat{y}(t)}
\end{align*}
$$

(5)

(the dot denoting differentiation with respect to $t$) such that for almost all $t \in [0, \hat{T}]$

$$
(P, F_1, R_1, F_2, R_2, u) = \sup_{a_i \leq u_i \leq b_i} (\psi_1 F_1 + \psi_1 u_1 + \psi_2 F_2 + \psi_2 u_2)_{y=\hat{y}(t), u=\hat{u}(t)}
$$

(6)

Note that although the right-hand side in (3) may not even be continuous, the linear system (5) has continuous coefficients and hence its solutions are of class $C^1$. This fact, which will be used below, is a major reason for our replacing (2) by (3).

It follows, from (6), that for $i = 1, 2$:

$$
\dot{u}_i(t) = b_i , \text{ if } \psi_i(t) > 0 \\
= a_i , \text{ if } \psi_i(t) < 0.
$$

It remains to be shown that $\psi_1(t)$ and $\psi_2(t)$ have at most finitely many zeros in $[0, \hat{T}]$. Otherwise one of them, say $\psi_1(t)$, would have a convergent sequence of zeros whose limit $\hat{t}$ would again be a zero of $\psi_1(t)$. By the continuity of $\psi_1$ we would have also $\psi_1(\hat{t}) = 0$, hence, from (5)

$$
(\rho F_2/\dot{y}_1) (\dot{y}(t)) \psi_2(t) = 0.
$$

But (4) together with the assumptions of the theorem implies that for every $y \in h(Q^0)$

$$
\frac{\partial F_2}{\partial y_1}(y) = \frac{h_2^{-1}(y_2)}{g_2(h_2^{-1}(y_2))} \frac{\partial f_2}{\partial x_1}(h^{-1}(y)) g_1(h_2^{-1}(y_2)) \neq 0
$$

and therefore $\psi_2(t) = 0$, which is impossible since $\psi_1(t) = \psi_2(t) = 0$ would imply $\psi(t) \equiv 0$.

The same argument shows that $\psi_2(t)$ has at most finitely many zeros, since $\partial F_1/\dot{y}_2 \neq 0$ in $h(Q^0)$.

Remarks. 1. It may seem at first sight that the change of variables $y = h(x)$ in (2) plays only a simplifying role, since we could have applied the Pontryagin Maximum Principle directly to (2). However, in that case the adjoint system for $\dot{y}(t)$ would depend explicitly on the control $u(t)$ and we
would not be able to conclude that its solutions are of class $C^1$, nor would it be possible to see directly that $\dot{u}(t)$ takes values only on the vertices of the rectangle $[a_1, b_1] \times [a_2, b_2]$.

2. Theorem 1 states, essentially, that no matter what admissible class $\mathcal{M}$ of controls is initially considered, any time-optimal control for system (2) lies in the smallest admissible class with values in $[a_1, b_1] \times [a_2, b_2]$. However, it should be stressed that the theorem does not make any assertions concerning the existence of time-optimal controls.

3. Notice that Theorem 1 holds for an arbitrary target set $G$.

4. THE CONTROLLABILITY OF SYSTEM (2)

We consider next the controllability of system (2), i.e., the existence of admissible controls steering (2) from points $p_0 \in Q^0$ to a given target set $G$ with the properties described earlier. More precisely, for every stable configuration $L$ of (1) the corresponding set $G$ will be the intersection of a sufficiently small closed neighborhood of $L$ with the region of attraction of $L$ in $Q^0$. We shall refer to this situation as the controllability of (2) from $p_0$ to the stable neighborhood of $L$. Since the trajectories of (2) are continuous curves we can assume, without loss of generality, that $G$ is the closure or even only the boundary of the above set. Notice that this concept of controllability is weaker than the notion of controllability to the set $L$ itself, which will also be considered in certain cases. There are two reasons for this distinction: first, it is dynamically more meaningful to steer the point $p_0$ only to the region of attraction of a stable configuration and then cease to control the motion, rather than to steer to the configuration itself. Secondly, in some cases (cited in Remark 5) the stronger requirement cannot be met mathematically.

Thus, if $L = \{(x_{i_0}, x_{m})\}$, where $(x_{i_0}, x_{m})$ is a stable equilibrium point of (1) in $Q^0$, then we may choose $G$ to be the closed disc of sufficiently small radius $\rho$ centered at this point. We shall have occasion (when discussing the control of competitors) to modify this definition somewhat and consider the stable singular point to be on the boundary of $Q$, namely on one of the axes; $G$ then will be changed accordingly to a semidisc.

If $L$ is a stable periodic orbit of (1) of period $T$, then $G$ can be chosen as a closed strip of width $2\rho$ about $L$ if $L$ is stable, and of width $\rho$ (with $L$ forming one boundary curve) if $L$ is semistable.

Observe that in all three cases the sets $G$ could be replaced by their boundary curves.

Before turning to our results it may be appropriate to emphasize that the controllability of system (2) is sometimes "size dependent," as is shown in
the following propositions. Under some circumstances (2) will be "steerable" from any point in $Q^0$ to a desired target set $G$ by means of arbitrarily small controls, $u(t)$; at times, however, the controls will have to be sufficiently large to achieve controllability.

We first consider, in Proposition I, the simplest situation, where (1) has either no singular points, or a globally stable singular point, in $Q^0$. This situation always occurs for symbiotic, and can occur for prey–predator, and competitive, interactions. We next treat, in Propositions II–IV, the controllability of those prey–predator interactions not covered by Proposition I. Similarly, we give, in Propositions V and VI, conditions for the controllability of two competitive species.

We state, for completeness, the obvious

**PROPOSITION I.** If any of the groups of hypotheses (P), (C) or (S) hold for (1), and if (1) has either no singular points in $Q^0$ or else a globally stable singular point $(x_{10}, x_{20})$ in $Q^0$, then system (2) is controllable from any point in $Q^0$ to any neighborhood of the unique stable singular point in $Q$.

Remark 4. Aside from establishing the controllability of symbiotic interactions, there are two other cases where Proposition I applies: first, if hypotheses (P) hold, and there are no periodic orbits of (1) in $Q^0$, and secondly, if hypotheses (C) hold, and if there is just one intersection point of $x_{2} = \phi_2(x_1)$ and $x_{2} = \phi_2(x_1)$ in $Q^0$ which is stable.

**PROPOSITION II.** Assume that hypotheses (P) hold, that the singular point $(x_{10}, x_{20}) \in Q^0$ is unstable for (1), and that (1) has a unique periodic orbit $L \subset Q^0$ (which is consequently stable). Then (2) can be steered to $L$ from every point in $Q^0$.

Proof. We consider the controlled system with $b_1 > 0$; all other cases can be treated similarly. We choose $u_2(t) = 0$ and then have

$$dx_1/dt = x_1 f_1(x_1, x_2) + u_1(t) f_2(x_1)$$
$$dx_2/dt = x_2 f_2(x_1, x_2) \quad 0 \leq u_1(t) \leq b_1.$$ 

For $u_1(t) \equiv b_1$, $(x_{10}, x_{20})$ is not a singular point of this system. Thus we may steer away from $(x_{10}, x_{20})$. Furthermore, since the unique periodic orbit is globally stable in $Q^0 - \{(x_{10}, x_{20})\}$ for (1), all solutions of (1) spiral onto $L$ from either the interior or the exterior. Now choose two points on $L$ such that $f_2$ evaluated at one point is positive and $f_2$ evaluated at the other point is negative. Let $(x_1^*, x_2^*)$ denote either of these points. Then the solution of (2) with $u_1(t) \equiv b_1$, $u_2(t) = 0$ through $(x_1^*, x_2^*)$ crosses $L$ from the interior to the exterior if $f_2(x_1^*, x_2^*) > 0$ and from the exterior to the interior if $f_2(x_1^*, x_2^*) < 0$, as shown in Fig. 1. By switching from $u_1 = 0$ to $u_1 = b_1$
appropriately, until $L$ is reached, we can steer every ordinary point of (1) in $Q^0$ to $L$ in finite time.

**Remark 5.** Notice that in Proposition I we are steering (2) to a neighborhood of a singular point of (1), whereas in Proposition II we steer (2) to the periodic orbit itself. Clearly we could use a neighborhood of $L$ instead of $L$ itself. In some cases (for example, if $(x_{10}, x_{20})$ is a stable spiral point) the point itself can be reached in finite time, but this is not always true. If $(x_{10}, x_{20})$ is a stable improper node for (1), whose linearized equations at $(x_{10}, x_{20})$ have two distinct eigenvalues, then for $b_1$ sufficiently small, and $a_1 = a_2 = b_2 = 0$ there is an open set with $(x_{10}, x_{20})$ in its boundary, such that the points of this open set cannot be steered by (2) to $(x_{10}, x_{20})$ in finite time. (A stable improper node can actually occur; see the example, (11)).

**Proposition III.** If hypotheses (P) hold for (1) and if at least one of $|a_1|$, $|a_2|$, $b_1$, $b_2$ is sufficiently large, then system (2) can be controlled from any point in $Q^0$ to any desired target set $G$ with the aforementioned properties.

**Proof.** If there are no periodic solutions of (1) then Proposition I applies. Assume that (1) has one or more periodic solutions and let $L$ be the outermost periodic orbit. Recall that $L$ is semistable from the outside, hence all points in $Q^0$ exterior to $L$ can be steered to a neighborhood of $L$. Furthermore, by using the same arguments as in the proof of Proposition II, points in $Q^0$ exterior to $L$ can be steered to $L$ itself. In order to complete the proof of Proposition III we show that if at least one of $|a_1|$, $|a_2|$, $b_1$, $b_2$ is sufficiently large, then system (2) can be steered from $L$ to any point in the interior of $L$ (by which we mean the bounded component of the complement of $L$, henceforth denoted by $\text{Int } L$) and from any point in $\text{Int } L$ to $L$ itself. Thus, in order to
steer between two points in $\text{Int}L$ we can steer first to $L$, set $u = 0$ and coast around $L$ to the appropriate point, and then steer to the second point in $\text{Int}L$. Now suppose that

$$\sup \left\{ \frac{x_1 f_1(x_1, x_2)}{g_1(x_1)} : (x_1, x_2) \in \overline{\text{Int}L} \text{ and } f_2(x_1, x_2) = 0 \right\} < -a_1 = |a_1|$$

(the left-hand side is finite because $f_1$ is continuous and $g_1$ is continuous and positive). Then if we choose $u_1(t) = a_1$, $u_2(t) = 0$, system (2) is an autonomous system with no singular point in $\text{Int}L$. The Poincaré–Bendixson Theorem implies that the solution curve through any point of $\text{Int}L$ at $t = 0$ must intersect $L$ both for positive and for negative $t$. Thus, system (2) can be steered from $L$ to any point of $\text{Int}L$ and from any point of $\text{Int}L$ to $L$.

Similarly if

$$\sup \left\{ \frac{x_2 f_2(x_1, x_2)}{g_2(x_1)} : (x_1, x_2) \in \overline{\text{Int}L} \text{ and } f_1(x_1, x_2) = 0 \right\} < -a_2,$$

$$\sup \left\{ \frac{-x_1 f_1(x_1, x_2)}{g_1(x_1)} : (x_1, x_2) \in \overline{\text{Int}L} \text{ and } f_2(x_1, x_2) = 0 \right\} < b_1,$$

$$\sup \left\{ \frac{-x_2 f_2(x_1, x_2)}{g_2(x_2)} : (x_1, x_2) \in \overline{\text{Int}L} \text{ and } f_1(x_1, x_2) = 0 \right\} < b_1,$$

then we can choose $u(t)$ so that system (2) is autonomous and has no singular point in $\text{Int}L$, and the proposition is proved.

The next proposition shows that a condition on the “size” of the admissible controls is essential in order to guarantee total controllability.

**Proposition IV.** Suppose that hypotheses (P) hold for (1), that $R \subset Q^0$ is a region bounded by two periodic orbits of (1), and that in $R$ all solutions of (1) spiral from the inner (outer) to the outer (inner) boundary. Then there exists a $\delta > 0$ such that for $\max \{|a_1|, |a_2|, b_1, b_2| < \delta$ system (2) cannot be controlled from the outer (inner) to the inner (outer) boundary of $R$.

**Proof.** We assume, for definiteness, that the outer orbit is stable from the inside. Let $p$ be a point in the interior of $R$ and on the curve $f_0(x_1, x_2) = 0$ as shown in Fig. 2. The solution curve $x(t)$ of (1) with $x(0) = p$ intersects the curve $f_2(x_1, x_2) = 0$ again at a point $x(t_1) = p'$ which lies between $p$ and the outer boundary of $R$. Let $x^*(t)$ be the solution of (2) (for a given control $u$) with $x^*(0) = p^*$. We then have

$$x(t) = p + \int_0^t F(x(\tau)) \, d\tau$$
FIG. 2. Prey–predator. The region $R$ with points $p$ and $p'$ shown. $R$ is bounded by $L_1$, a periodic orbit of (1) which is semistable from the inside, and by $L_2$, a periodic orbit of (1) which is unstable from the outside.

and

$$x^*(t) = p^* + \int_0^t [F(x^*(\tau)) + g(x^*(\tau)) u(\tau)] d\tau,$$

where

$$F(x) = \begin{bmatrix} x_1 f_1(x_1, x_2) \\ x_2 f_2(x_1, x_2) \end{bmatrix} \quad \text{and} \quad g(x) = \begin{bmatrix} g_1(x_1) \\ 0 \\ g_2(x_2) \end{bmatrix}.$$

Let $|x - x^*| = \sup_{i=1,2} |x_i - x_i^*|$, let $K$ be a Lipschitz constant for $F(x)$ in $R$, and let $M = \sup_{x \in R} \{g_1(x_1), g_2(x_2)\}$. It then follows that

$$|x(t) - x^*(t)| \leq |p - p^*| + K \int_0^t |x(\tau) - x^*(\tau)| d\tau + M\delta t_1,$$

provided $x^*(t)$ is in $R$ for $0 \leq \tau \leq t \leq t_1$, and

$$\sup\{|a_1|, |a_2|, b_1, b_2\} < \delta.$$  \hspace{1cm} (7)

Gronwall's inequality yields

$$|x(t) - x^*(t)| \leq (|p - p^*| + M\delta t_1) e^{Kt}.$$

Since $x(t)$ remains in the interior of $R$ for $0 \leq t \leq t_1$, there is an $\epsilon_1 > 0$ and a $\delta_1 > 0$ such that when $|p - p^*| < \epsilon_1$ and (7) holds with $\delta = \delta_1$, then $x^*(t)$ does not reach the boundary of $R$ for $0 \leq t \leq t_1$ and therefore

$$|x^*(t_1) - x(t_1)| \leq (\epsilon_1 + M\delta_1 t_1) e^{Kt_1},$$

where $x(t_1) = p'$. Furthermore, $f_1$, $f_2$ being continuous, $f_1|_{p'} \neq 0$ and
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\[ f_2|_{p'} = 0 \] together imply that there is a neighborhood \( N = \{ x : | x - p' | < \epsilon \} \) of \( p' \) and a \( \delta - \delta_2 > 0 \) such that when (7) is satisfied any solution of (2) originating in \( N \) intersects the curve \( f_2(x_1, x_2) = 0 \) between \( p \) and the outer boundary of \( R \). If necessary, we can choose \( \epsilon_1 \) and \( \delta_1 \) smaller to guarantee that \( (\epsilon_1 + M \delta_1 t_1) e^{\kappa t_1} < \epsilon_2 \). Then if (7) is satisfied with \( \delta = \min(\delta_1, \delta_2) \) and \( | p - p^* | < \epsilon_1, x^*(t) \) intersects the curve \( f_2(x_1, x_2) = 0 \) between \( p \) and the outer boundary of \( R \). Moreover, if \( p'' \) is any point on our solution curve, \( x(t) \), of (1) between \( p \) and \( p' \), a solution of (2) through \( p'' \) must intersect the curve \( f_2(x_1, x_2) = 0 \) between \( p \) and the outer boundary of \( R \).

Now let \( \Omega \) be the closed curve consisting of \( \Omega_1 \), the segment of the trajectory of (1) from \( p \) to \( p' \), and \( \Omega_2 \), the arc of \( f_2(x_1, x_2) = 0 \) between \( p' \) and \( p \) (see Fig. 2). By choosing, in (7), \( \delta \) smaller if necessary, we can guarantee that at any point \( q \) of \( \Omega_2 \) such that \( | q - p | > \epsilon_2 \) all trajectories of (2) cross from the interior to the exterior of \( \Omega \). Suppose that with \( \delta \) so chosen in (7) there is an orbit \( \Gamma \) of (2) which starts on the outer boundary of \( R \) and reaches the inner boundary of \( R \). Then \( \Gamma \) necessarily intersects \( \Omega \) at a last point \( p'' \), where \( \Gamma \) enters Int \( \Omega \). Hence, \( p'' \) must be either on \( \Omega_1 \) or on \( \Omega_2 \) with \( | p'' - p | < \epsilon_1 \). Then \( \Gamma \) must subsequently cross \( f_2(x_1, x_2) = 0 \) between \( p \) and the outer boundary of \( R \), and we have a contradiction.

We next consider the controllability of those competitive interactions that are not covered by Proposition I, i.e. the controllability of system (2) when system (1) satisfies hypotheses (C) and has in \( Q'' \) at least one singular point which is not stable. The target set is a neighborhood of a singular point of (1) in \( Q \), which is stable and therefore isolated. Clearly, the set of points that can be steered by (2) to a given target set contains the region of attraction with respect to (1) of the corresponding singular point.

We first describe this region of attraction. It is easily seen that if the singular point \( p' = (x_1', x_2') \) in \( Q \) is stable for (1), then for all \( x_1 \neq x_1' \) sufficiently close to \( x_1' \) and such that \( \phi_1 \) and \( \phi_2 \) (see Section 2.2) are defined at \( x_1 \) we have

\[ (\phi_2(x_1') - \phi_1(x_1)) (x_1 - x_1') > 0. \]

Consider the maximal interval on the \( x_1 \)-axis containing \( x_1' \) for which (8) holds and let \( x_1'' \) be its right endpoint (the situation at the left endpoint is treated in a similar way). We have either \( \phi_1(x_1'') = \phi_2(x_1'') = x_2'' \geq 0 \) or \( \phi_1(x_1'') = 0 < \phi_2(x_1'') \). In the latter case there are no singular points of (1) in \( Q'' \) to the right of \( p' \) and the positive \( x_1 \)-axis is the "lower" boundary of its region of attraction. The same conclusion holds also when \( x_2'' = 0 \). Suppose that \( x_2'' > 0 \), i.e. the singular point \( p'' = (x_1'', x_2'') \) of (1) lies in \( Q'' \). Then it is easy to see that the region of attraction \( R \) of \( p' \) is bounded "below" by a curve \( \Gamma \) through \( p'' \) which consists of two trajectories of (1) tending toward \( p'' \) and of \( p'' \) itself (see Fig. 3). Notice that, in general, \( p'' \) need not be a saddle point; in fact, we do not even exclude here the case of \( p'' \) being a limit point of
singular points. Notice also that \( \Gamma \) separates \( Q^0 \) into two unbounded regions as shown in Fig. 3.

As observed previously, system (2) is controllable to the stable neighborhood of \( p' \) from every point in \( R \). In order to insure controllability from points on or below \( \Gamma \) we need additional conditions, as seen below. We recall our previous remark that similar results hold for the "upper" boundary of \( R \).

The next two propositions show, once again, that the controllability of (2) depends on the size of the controls.

**Proposition V.** Assume that hypotheses (C) hold for system (1). Let \( p' = (x_1', x_2') \in Q^0 \) and \( p'' = (x_1'', x_2'') \in Q^0 \) be two singular points of (1) with \( p' \) stable, \( x_1' < x_1'' \) and such that the function \( \phi_2 - \phi_1 \) does not vanish on the open interval \((x_1', x_1'')\). Let \( R \) be the region of attraction for \( p' \) and \( \Gamma \) its "lower" boundary.

1. If \( a_1 = 0 \) and \( b_2 = 0 \), then no trajectory of system (2) originating on \( \Gamma \) enters \( R \).
2. If \( p'' \) is a saddle point of (1) (i.e. the linearization of (1) at \( p'' \) has one positive and one negative eigenvalue) then there exists a \( \delta > 0 \) such that for \( 0 < |a_1| < \delta, 0 < b_2 < \delta \), there is a curve \( \Gamma'' \) separating \( Q^0 \) into two unbounded regions with \( \Gamma'' \) the lower boundary of the set of points controllable by system (2) to the stable neighborhood of \( p' \).

**Proof.** Part 1. Suppose to the contrary that \( a_1 = b_2 = 0 \) and that there is a measurable control \( u(t) \) which steers (2) from \( \Gamma \) into \( R \). First assume that the corresponding solution curve of (2) enters \( R \) from some point of \( \Gamma \) other than \( p'' \). By using Gronwall's inequality as in Proposition IV, one sees that

![Fig. 3. Competitors. \( \Gamma \), the "lower" boundary of the region of attraction for \( p' \), is shown.](image-url)
any piecewise constant control which is sufficiently close to \( u(t) \) in the \( L^1 \) norm also steers (2) from \( \Gamma \) into \( R \) and enters \( R \) from \( p_0 \neq p^* \). Thus we may assume that \( u(t) \) is piecewise constant. Since \( a_1 = b_2 = 0 \), one has that \( u_1(t) \geq 0 \), \( u_2(t) \leq 0 \), and at least one of these inequalities is strict on some interval \( (t_0 , t_0 + \epsilon) \), where \( t_0 \) is the time at which the solution curve enters \( R \) and \( \epsilon > 0 \). At \( p_0 \neq p^* \), \( f_1 \) and \( f_2 \) are either both positive or both negative, and it is easily seen that the tangent vector to our solution curve at this point is directed away from \( R \). Thus, except possibly at the point \( p^* \), \( \Gamma \) cannot be crossed by trajectories of (2) entering \( R \). But if a solution curve of (2) originating at \( p^* \) would enter \( R \), by continuity the corresponding control would steer a neighborhood of \( p^* \) in \( \Gamma \) into \( R \), which is impossible.

**Part 2.** For \( \mid a_1 \mid \) and \( b_2 \) sufficiently small, we first define \( \Gamma' \), then show that it separates \( Q^0 \) into two unbounded regions and that solution curves of (2) originating on it do not enter the region above it, and finally observe that all points between \( \Gamma \) and \( \Gamma'' \) can be steered by system (2) into \( R \).

Consider the mapping \( (\Phi_1, \Phi_2): Q^0 \times \mathbb{R}^2 \to \mathbb{R}^2 \) defined by

\[
\Phi_i(x_1, x_2, c_1, c_2) = x_i f_i(x_1, x_2) + c_i g_i(x_i), \quad i = 1, 2.
\]

Since \( p^* \) is a saddle point of (1), \( (\partial_1(F_1, F_2)(\partial(x_1, x_2)))(x_1^*, x_2^*, 0, 0) \) is a nonsingular matrix. It follows from the Implicit Function Theorem that for \( \mid c_1 \mid, \mid c_2 \mid \) sufficiently small, the system of equations \( \Phi_i(x_1, x_2, c_1, c_2) = 0, \ i = 1, 2, \) has a unique solution \( (x_1(c_1, c_2), x_2(c_1, c_2)) \) with values in a neighborhood of \( p^* \), and this solution is of class \( C^1 \). Thus

\[
\Phi_i(x_1, x_2, c_1, c_2) = x_i(c_1, c_2, x_2(c_1, c_2), c_1, c_2)
\]

depends continuously on \( c_1 \) and \( c_2 \). If \( \mid a_1 \mid \) and \( b_2 \) are sufficiently small, then system (2) with \( u_1(t) = a_1 \), \( u_2(t) = b_2 \), that is

\[
dx_i/dt = F_i(x_1, x_2, a_1, b_2), \quad i = 1, 2,
\]

has a singular point at \( (x_1(a_1, b_2), x_2(a_1, b_2)) = p_1 \) which, like the singular point \( p^* \) of (1), is a saddle point and has only negative coefficients in the linearization of the system at that point. We assume, in addition, that \( \mid a_1 \mid \) and \( b_2 \) are small enough to guarantee that, except at \( p_1 \), \( F_1(x_1, x_2, a_1, b_2) \) is negative along the horizontal segment joining \( p_1 \) and the curve \( f_1(x_1, x_2) = 0 \), that, except at \( p_1 \), \( F_2(x_1, x_2, a_1, b_2) \) is positive along the vertical segment joining \( p_1 \) and the curve \( f_2(x_1, x_2) = 0 \), and that system (9) has no singular point other than \( p_1 \) in the region bounded by the line \( x_1 = x_1(a_1, b_2) \), the line \( x_2 = x_2(a_1, b_2) \), and the curve \( \Gamma \) (see Fig. 4).
Let $\Gamma''$ be the union of the two solution curves of (9) that approach $p_1$ and $\{p_1\}$ itself (see Fig. 5). By examining the linearization of (9) at $p_1$, one finds that, near $p_1$, $F_i(x_1, x_2, a_1, b_2)$, $i = 1, 2$, are either both positive or both negative on $\Gamma'' - \{p_1\}$. We prove first that the same is true on all of $\Gamma''$ except $p_1$. Then, after showing that $\Gamma''$ separates $Q^0$ into two unbounded regions, it follows as in the proof of part 1 that no solution curve of system (2) (with $u_1(t) \geq a_1, u_2(t) \leq b_2$) originating on $\Gamma''$ enters the region above $\Gamma''$.

Let $(x_1(t), x_2(t))$ be a solution of (9) which approaches $p_1$ and let $(t_0, +\infty)$ be the maximal interval on which $F_i(x_1(t), x_2(t), a_1, b_2)$ (henceforth denoted by $F_i(t)$) are both negative (the case when both are positive is treated simi-
larly). Suppose that $t_0 > -\infty$ and $(x_1(t_0), x_2(t_0))$ is in $Q^0$. Then $F_1(t_0)$ and $F_2(t_0)$ cannot both be zero. If $F_1(t_0) = 0$, we have

$$\frac{dF_1}{dt}(t_0) = \frac{\partial F_1}{\partial x_1}(t_0) \frac{dx_1}{dt}(t_0) + \frac{\partial F_1}{\partial x_2}(t_0) \frac{dx_2}{dt}(t_0)$$

$$= \frac{\partial F_1}{\partial x_2}(t_0) F_2(t_0) > 0$$

since hypotheses (C) imply that $\partial F_1/\partial x_2 < 0$ in $Q^0$. Then $F_1(t)$ is positive on an interval to the right of $t_0$, and we have a contradiction. Similarly if $F_2(t_0) = 0$, $(dF_2/dt)(t_0) = (\partial F_2/\partial x_1)(t_0) F_1(t_0) > 0$, since hypotheses (C) imply that $\partial F_2/\partial x_1 < 0$ in $Q^0$. We then have the contradiction that $F_2(t)$ is positive on an interval to the right of $t_0$. Therefore $F_1(t)$ and $F_2(t)$ are both negative for all $t$ such that $(x_1(t), x_2(t))$ is in $Q^0$. Thus $\Gamma' - \{p_1\}$ lies in the interior of the two regions, $R_1$ and $R_2$, shown in Fig. 5. We show that system (9) has no singular point in the interior of either region, hence $\Gamma'$ separates $Q^0$ into two unbounded regions. Since $a_1 \leq 0$,

$$F_1(x_1, x_2, a_1, b_2) = x_1 f_1(x_1, x_2) + a_1 g_1(x_1)$$

can be zero in $R_1$ only if $f_1(x_1, x_2) \geq 0$. Together, $F_1(x_1, x_2, a_1, b_2) < 0$ on the horizontal segment joining $p_1$ and the curve $f_1(x_1, x_2) = 0$ and $\partial F_1/\partial x_2 < 0$ in $Q^0$ imply that, except at $p_1$, $F_1(x_1, x_2, a_1, b_2) < 0$ in $R_1$. Hence there is no singular point in $R_1$ other than $p_1$. Similarly there is no singular point in the interior of $R_2$. Since $b_2 \geq 0$, $F_2(x_1, x_2, a_1, b_2)$ can be zero in $Q^0$ only if $f_2(x_1, x_2) \leq 0$. But $F_2(x_1, x_2, a_1, b_2) > 0$ on the horizontal segment between $p_1$ and the curve $f_2(x_1, x_2) = 0$ and $\partial F_2/\partial x_1 < 0$ in $Q^0$ imply that $F_2(x_1, x_2, a_1, b_2) > 0$ in the interior of $R_2$.

It remains to be shown that points in the region between $\Gamma$ and $\Gamma'$ can be steered to $R$ (the region of attraction of $p'$ with respect to (1)) by system (2).

First consider a point, $p_2$, which is in the interior of the region between $p$ and $p'$, but not in the interior of $R_1$. Let $u_1(t) = a_1, u_2(t) = b_2$; then system (2) becomes system (9). It follows from the preceding results and the properties obtained by assuming $|a_1|$ and $b_2$ to be small enough that in $Q^0$ between $\Gamma$ and $\Gamma'$ system (9) has no singular point other than $p_1$. A solution curve $x(t)$ originating at $p_2$ can neither intersect $\Gamma'$, which consists of trajectories of (9), nor approach $p_1$, a saddle point of (9), nor approach any portion of the $x_1$-axis which is a boundary of this region since $b_2 \geq 0$ and $f_2(x_1, x_2) > 0$ there. Since, in addition, $F_1(x_1, x_2, a_1, b_2) < 0$ on the vertical segment between $p_1$ and $p'$ (except at $p_1$), solution curves of (9) cross this segment from right to left. Thus $x(t)$ must eventually cross $\Gamma$.  


Now consider the interior of $R_1$ between $\Gamma$ and $\Gamma'$. Let $u_1(t) \equiv a_1$ and

$$u_2(t) = \min \left\{ \frac{-x_2(t)f_2(x_1(t), x_2(t))}{g_2(x_2(t))}, b_2 \right\}$$

$(0 \leq u_2(t) \leq b_2$ since $f_2 \leq 0$ in $R_1)$; then system (2) is

$$\begin{align*}
\frac{dx_1}{dt} &= F_1(x_1, x_2, a_1, b_2) \\
\frac{dx_2}{dt} &= \min\{F_2(x_1, x_2, a_1, b_2), 0\}.
\end{align*}$$

Wherever $F_2 \leq 0$ in $R_1$, this system becomes system (9). Recall that on $\Gamma''$ in $R_1$, except at $p_1$, $F_2 < 0$. Since $\partial F_2/\partial x_1$ and $\partial F_2/\partial x_2$ are both negative at $p_1$, one concludes that $F_2 \leq 0$ in $R_1$ near $p_1$. Thus any solution curve $x(t)$ of the preceding system originating in the interior of $R_1$ between $\Gamma$ and $\Gamma''$ can neither intersect $\Gamma''$ nor approach the saddle point, $p_1$. Since $F_1 < 0$ in $R_1$ except at $p_1$, we have $dx_1/dt < 0$ and $dx_2/dt < 0$. Therefore $x(t)$ must intersect either $\Gamma$ or the boundary of $R_1$ between $\Gamma''$ and $\Gamma$. In the latter case, by then switching to $u_1(t) = a_1$, $u_2(t) = b_2$, we eventually cross $\Gamma$.

**PROPOSITION VI.** Suppose that system (1) satisfies hypotheses (C) and let $p'$, $p''$, $\Gamma$, and $R$ be as in Proposition V. Then all points in $Q^0$ that lie below $\Gamma$ can be steered to $R$ by system (2) (and hence to the stable neighborhood of $p'$) if either of the following holds:

1. $|a_1|$ is sufficiently large.
2. $b_2$ is sufficiently large and for some $\delta > 0$, $g_2(x_2)/x_2 \geq \delta$ when $0 < x_2 \leq x_2''$.

**Proof.** Let $Z$ be the subset of $Q$ defined by

$$Z = \{(x_1, x_2) : x_1 \geq x_1'' \text{ and } f_2(x_1, x_2) = 0\}$$

$$\cup \{(x_1, 0) : f_2(x_1, 0) < 0 \text{ and } f_1(x_1, 0) \geq 0\},$$

that is $Z$ consists of the segment of the curve $f_2(x_1, x_2) = 0$ to the right of $p''$ and a (possibly empty) segment of the $x_1$-axis. Now suppose that

$$\sup \left\{ \frac{x_1f_2(x_1, x_2)}{g_1(x_1)} : (x_1, x_2) \in Z \right\} \leq |a_1| = -a_1.$$ 

The left hand side is finite since the supremum is taken over a closed bounded set in $Q$ on which $g_1$ does not vanish. We let $u_1(t) = a_1$, $u_2(t) = 0$ and show that the corresponding solution curve of (2) originating at any point in $Q^0$ below $\Gamma$ must eventually enter $R$. Consider the two regions below $\Gamma$ shown in Fig. 6. With $u_1(t) = a_1$, $u_2(t) = 0$ in system (2) we have $dx_1/dt < 0$ and
FIG. 6. Competitors. The regions $R_1$, $R_2$, and $\Gamma$, the bounding trajectory of (1), are shown.

d$x_2/dt < 0$ in the interior of $R_1$. As shown in [11] a solution curve starting there does not reach the $x_1$-axis before leaving $R_1$. Thus it must either cross $\Gamma$, in which case we are done, or cross the curve $f_3(x_1, x_2) = 0$. In $R_2$ we have $dx_2/dt > 0$ along our trajectory. Since for $u_1(t) \equiv a_1$, $u_2(t) \equiv 0$ system (2) has singular points in $R_2$ only on the $x_1$-axis, the solution curve must eventually cross $\Gamma$. Thus, we have proved the first part of the proposition.

We outline the modifications in the preceding argument that yield a proof of the second part. Suppose that

$$
\sup \left\{ \frac{-x_2 f_3(x_1, x_2)}{g_2(x_2)} : x_1 \geqslant x_1^*, f_1(x_1, x_2) = 0 \right\} < b_2.
$$

The assumption that $g_2(x_2)/x_2$ is bounded away from zero as $x_2 \to 0$ guarantees that the left hand side is finite. By continuity there is a point $(x_1^*, x_2^*)$ on $\Gamma$ such that $f_1(x_1^*, x_2^*) < 0$ and $x_2^* f_3(x_1^*, x_2^*) / b_2 g_2(x_2^*) > 0$. Below $\Gamma$ and the line $x_2 = x_2^*$, choose $u_1(t) \equiv 0$, $u_2(t) \equiv b_2$. Below $\Gamma$ but above, or on, the line $x_2 = x_2^*$ choose $u_1(t) \equiv 0$, $u_2(t) = \min \left\{ \frac{-x_2 f_3(x_1(t), x_2(t))}{g_2(x_2(t))} , b_2 \right\}$.

Then every solution curve of (2) starting in the interior of region $R_2$, shown in Fig. 7, must cross $\Gamma$ or enter $R_1$. In the unbounded region $R_1'$ (above $x_2 = x_2^*$), $dx_1/dt < 0$ and $dx_2/dt \leqslant 0$. Solution curves originating in $R_1'$ must either cross $\Gamma$, enter $R_1$, or eventually move to the left on the boundary between $R_1$ and $R_1'$, ultimately crossing $\Gamma$. In $R_1$ we have $dx_2/dt < 0$ and solution curves originating there must eventually cross $\Gamma$. 

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5. THE EXISTENCE OF TIME-OPTIMAL CONTROL

We consider the existence of time-optimal controls steering system (2) from a given point \( p_0 \in \mathcal{Q}^0 \) to one of the closed sets \( G \) described earlier.

**Theorem 2.** Assume any one of the groups of hypotheses (P), (C), or (S) and let system (2) be controllable from the point \( p_0 \) to \( G \) in "time" \( T \). Then there is a \( K (0 < K \leq +\infty) \), which depends in general on \( p_0 \), such that if \( T < K \), there exists a time-optimal control steering (2) from \( p_0 \) to \( G \).

**Proof.** We find a \( K \) such that \( T < K \) implies the hypotheses of Corollary 2 in [12, p. 262], where sufficient conditions are given for the existence of optimal controls for processes depending linearly on the control variable. In our case this reduces to establishing the following property: there exists a closed bounded region \( R \) of the \((x_1, x_2)\)-plane in which the right-hand sides of (2) are \( C^1 \)-functions in \((x_1, x_2, u)\) and such that for every control \( u(t), 0 \leq t \leq T, p_0 \to G \) in \( \mathcal{Q}^0 \) the corresponding trajectory \( x(t; u, p_0) \) lies entirely in \( R \).

Notice that if (2) is the restriction to \( \mathcal{Q} \) of a \( C^1 \)-system defined on an open set containing \( \mathcal{Q} \), and if the constraints imposed on \( x(t; u, p_0) \) in \( M_1 \) and \( M_2 \) (of Sect. 3) are relaxed correspondingly, then we only have to verify that our trajectories are uniformly bounded.

First suppose that \( x_i \) represents a species which is a prey, a competitor, or a cooperator (i.e. any species type except predator). Hypotheses (P), (C) or (S) then imply that \( f_i(x_1, x_2) \leq f_i(0, 0) \) for all \( (x_1, x_2) \in \mathcal{Q} \) and \( f_i(0, 0) > 0 \). Thus
it follows from (2) that for all admissible controls \( u(t) \) and all \( 0 \leq t \leq T_u \):

\[ \frac{dx_i(t)}{dt} \leq f_i(0, 0) x_i(t) + b_i g_i(x_i(t)). \]  

(10)

Now let

\[ V_i(x_i) = \int_{x_i^0}^{x_i} \frac{d\eta}{f_i(0, 0) \eta + b_i g_i(\eta)}, \quad x_i \geq 0 \]

where \( (x_1^0, x_2^0) = p_0 \). The function \( V_i \) is strictly increasing and has a (strictly increasing) inverse defined on the interval

\[ \left( -\int_{x_i^0}^{x_i} \frac{d\eta}{f_i(0, 0) \eta + b_i g_i(\eta)}, \int_{x_i^0}^{+\infty} \frac{d\eta}{f_i(0, 0) \eta + b_i g_i(\eta)} \right). \]

From (10) we obtain

\[ V_i(x_i(t; u, p_0)) \leq t, \quad 0 \leq t \leq T_u \leq T \]

and hence, if

\[ T < \int_{x_i^0}^{+\infty} \frac{d\eta}{f_i(0, 0) \eta + b_i g_i(\eta)} = K_i, \]

then \( V_i^{-1}(T) = c_i \) is defined and is an upper bound for \( x_i(t; u, p_0) \).

Now suppose that \( x_2 \) represents a predator and that \( T < K_1 \). Then \( x_1(t; u, p_0) \leq c_1 \) for all admissible controls \( u(t) \) and all \( 0 \leq t \leq T_u \leq T \). Hypotheses \((P)\) imply that \( f_2(x_1, x_2) \leq f_2(c_1, 0) \) for all \((x_1, x_2)\) such that \( 0 \leq x_1 \leq c_1 \). By choosing \( c_1 \) sufficiently large we guarantee that \( f_2(c_1, 0) \) is positive and, proceeding as before, we obtain an upper bound for \( x_2(t; u, p_0) \), provided that

\[ T < \int_{x_2^0}^{+\infty} \frac{d\eta}{f_2(c_1, 0) \eta + b_2 g_2(\eta)} = K_2. \]

Thus for any situation \( T < K = \min\{K_1, K_2\} \) implies that \( x(t; u, p_0) \) is uniformly bounded by some \( c > 0 \) for all admissible \( u(t) \) and all \( 0 \leq t \leq T_u \leq T \).

If system (2) is the restriction to \( Q \) of a \( C^1 \)-system defined on an open set containing \( Q \) (and if the definition of time optimality is changed accordingly) then the proof is complete. Otherwise, it remains to be shown that (for a possibly smaller \( K \)) there exists a \( d > 0 \) such that \( x_i(t; u, p_0) \geq d, i = 1, 2 \), for all admissible controls \( u(t) \) and all \( 0 \leq t \leq T_u < K \). Let \( f_i(x_1, x_2) \geq -m \) for \( 0 \leq x_1, x_2 \leq c \), where \( m > 0 \). We then have for all admissible \( u(t) \) and all \( 0 \leq t \leq T_u \):
It follows that

\[ -t \leq W_i(x_i(t; u, p_0)) \]

where

\[ W_i(x_i) = \int_{x_i^0}^{x_i} \frac{d\eta}{m\eta - a_i g_i(\eta)} \]

is obviously a strictly increasing function for \( x_i \geq 0 \). Therefore, as before,

\[ 0 < W_i^{-1}(-T) \leq x_i(t; u, p_0) \]

if

\[ T < \int_{0}^{x_i^0} \frac{d\eta}{m\eta - a_i g_i(\eta)} = K_i'. \]

Choosing \( K = \min\{K_1, K_2, K_1', K_2'\} \) completes the proof of the theorem.

**Remark 6.** With \( p_0 \) fixed and \( K \) finite, when one of \( |a_1|, |a_2|, b_1, b_2 \) increases, \( K \) decreases. Propositions III–VI show that in order to insure controllability one or more of \( |a_1|, |a_2|, b_1, b_2 \) have to be sufficiently large. Thus for controllability we need, in general, large bounds on \( u(t) \), but then the condition \( T < K \) may not be satisfied. The difficulty of actually comparing \( T \) and \( K \) is avoided if \( K = +\infty \). One condition that guarantees \( K \) to be \( +\infty \) (for all \( p_0 \)) is that for \( i = 1, 2 \) the function \( g_i(x_i)/x_i \) be bounded as \( x_i \to +\infty \) if \( b_i > 0 \), or as \( x_i \to 0 \) if \( a_i < 0 \).

Notice that the second part of this condition becomes unnecessary if system (2) is the restriction to \( Q \) of a \( C^1 \) system defined on an open set containing \( Q \) and if time optimality is redefined as indicated above. However in this case a time-optimal trajectory might move along one or both of the axes. Therefore, Theorem 1 would not be applicable and the time-optimal control might not be bang-bang.

### 6. Two Examples

We use two examples, taken from the biological literature [14, 15], to illustrate our results.

Our first example is a prey–predator system whose uncontrolled dynamical equations are

\begin{align*}
\dot{x}_1 &= x_1 [r (1 - x_1/K) - k x_2 ((1 - e^{-\gamma x_1})/x_1)] = x_1 f_1(x_1, x_2) \\
\dot{x}_2 &= x_2 [-b + \beta (1 - e^{-\mu x_1})] = x_2 f_2(x_1, x_2),
\end{align*}

where \( x_1 \) and \( x_2 \) denote, as before, the sizes of the prey, and the predator populations, respectively; \( b, r, k, K, \beta, \gamma, \mu \) are positive constants.
It is a straightforward matter to show that \( f_1(x_1, x_2) \) and \( f_2(x_1, x_2) \) of (11) satisfy (P1)-(P3) of Sect. 2.1, when \( b/\beta < 1 \), \( K > (1/\mu) \log(\beta/(\beta - b)) \). One has \( x_1^* = K \), \( \dot{x}_1 = (1/\mu) \log(\beta/(\beta - b)) \), and \( x_2^* = r/(\gamma k) \). The unique equilibrium point \((x_{10}, x_{20})\) of (11) in \( Q^0 \), given by

\[
x_{10} = x_1^* , \quad x_{20} = \frac{r}{k} \left( 1 - \frac{x_{10}}{K} \right) \frac{x_{10}}{1 - e^{-r x_{20}}} ,
\]

is stable if

\[
K < x_{10} \left[ 1 + \frac{1}{1 - \frac{\gamma x_{10}}{e^{r x_{10}}}} \right] ,
\]

and unstable, if this last inequality is reversed.

We choose as our controlled system

\[
\begin{align*}
\dot{x}_1 &= x_1 [f_1(x_1, x_2) + u_1(t)] \\
\dot{x}_2 &= x_2 [f_2(x_1, x_2) + u_2(t)]
\end{align*}
\]

with the above \( f_1, f_2 \).

We have not been concerned with obtaining a complete phase portrait of the trajectories of (11), limiting ourselves just to a discussion of all the control possibilities.

If \((x_{10}, x_{20})\) is a global attractor for (11) then (12) can be "steered" to a neighborhood of \((x_{10}, x_{20})\) in finite time by setting \( u(t) = 0 \).

If \((x_{10}, x_{20})\) is unstable and (11) has a unique periodic orbit, \( L \), then one can steer (11) to \( L \) from every point in \( Q^0 \) in finite time as in the proof of Proposition 11.

Notice that proof of Proposition III shows how to steer from any point in \( Q^0 \) to any desired target set \( G \). Finally, we emphasize that the existence of a time optimal control for (12), steering any point in \( Q^0 \) to \( G \), is insured by Theorem 2 (see Remark 6).

The other example is a competitive system governed by

\[
\begin{align*}
\dot{x}_1 &= \frac{r_1}{K_1} x_1 [K_1 - x_1 - \alpha x_2] \\
\dot{x}_2 &= \frac{r_2}{K_2} x_2 [K_2 - x_2 - \beta x_1]
\end{align*}
\]

with \( r_1, r_2, K_1, K_2, \alpha, \beta \) all positive. Hypotheses (C1) of Section (2.2) are satisfied, where \( x_{11} = K_1 \), \( x_{22} = K_2 \), \( x_{21} = K_1/\alpha \), \( x_{12} = K_2/\beta \). The singular point \((x_{10}, x_{20})\), given by

\[
x_{10} = \frac{1}{1 - \alpha\beta} [K_1 - \alpha K_2] , \quad x_{20} = \frac{1}{1 - \alpha\beta} [K_2 - \beta K_1]
\]

will be stable if \( \alpha\beta < 1 \) and unstable (a saddle point) if \( \alpha\beta > 1 \).
We now choose as the controlled system

\[ \begin{align*}
    \dot{x}_1 &= x_1 f_1(x_1, x_2) + x_1 u_1(t) \\
    \dot{x}_2 &= x_2 f_2(x_1, x_2) + x_2 u_2(t)
\end{align*} \tag{14} \]

similar to the system used by Thau [5]; \(f_1\) and \(f_2\) are as in (13).

If \((x_{10}, x_{20})\) is stable then Proposition I applies, and the system (14) will be controllable to the neighborhood of \((x_{10}, x_{20})\) by choosing \(u(t) \equiv 0\).

If \((x_{10}, x_{20})\) is a saddle point, then either Proposition V or VI applies. If the controls are too small then (14) cannot be steered from every point in \(Q^0\) to the neighborhood of a stable singular point in \(Q\).

If \(|a_1|\) or \(b_1\) is sufficiently large then (13) can be steered from any point in \(Q^0\) to the stable neighborhood of the point \(p'\). Again the condition that \(g_2(x_2)/x_2\) be bounded away from zero is satisfied by our example; Remark 6 is pertinent.

**References**