P-VARIETIES – A SIGNATURE INDEPENDENT CHARACTERIZATION OF VARIETIES OF ORDERED ALGEBRAS

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This paper is concerned mainly with classes (categories) of ordered algebras which in some signature are axiomatizable by a set of inequations between terms ('varieties' of ordered algebras) and also classes which are axiomatizable by implications between inequations ('quasi varieties' of ordered algebras). For example, if the signature contains a binary operation symbol (for the monoid operation) and a constant symbol (for the identity) the class of ordered monoids $M$ can be axiomatized by a set of inequations (i.e. expressions of the form $t \leq t'$). However, if the signature contains only the binary operation symbol, the same class $M$ cannot be so axiomatized (since it is not now closed under subalgebras). Thus, there is a need to find structural, signature independent conditions on a class of ordered algebras which are necessary and sufficient to guarantee the existence of a signature in which the class is axiomatizable by a set of inequations (between terms in this signature). In this paper such conditions are found by utilizing the notion of 'P-categories'. A P-category $C$ is a category such that each 'Hom-set' $C(a, b)$ is equipped with a distinguished partial order which is preserved by composition. Aside from proving the characterization theorem, it is also the purpose of the paper to begin the investigation of P-categories.

1. Introduction

Ever since Scott popularized their use in [12], ordered algebras have been used in many places in theoretical computer science. Most treatments of these ordered algebras has followed the Tarski style: i.e. one choses a 'signature' $\Sigma$ and then confines oneself to subclasses of the class of all $\Sigma$ algebras. The choice of the signature is important for certain purposes. For example, the class of (discretely ordered) groups is closed under subalgebras for one choice of a signature, but is not closed under subalgebras for another choice.

The work that is reported on here arose from the authors' desire to find a signa-

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ture independent, 'structural' characterization of certain important kinds of classes of ordered algebras: the 'varieties' and 'quasi varieties'. (Examples of some varieties of ordered algebras are: lattices, ordered monoids, ordered semirings, etc.; an example with an 'unbounded' signature is the collection of all complete posets (and sup preserving functions); examples of quasi-varieties of ordered algebras are: discretely ordered sets, discretely ordered torsion free abelian groups; a 'nonfinitary' example is the collection of posets such that every increasing omega chain has a least upper bound.) We believe that such a characterization will prove useful in applying the theory of ordered algebras to computer science and elsewhere. Just such a characterization of varieties and quasi varieties of unordered algebras has been obtained by Lawvere [8] (see also [11] and [7]) and his treatment of (unordered) universal algebra has been applied in theoretical science by several authors (see [13], [6], [5], [11]). Lawvere's result (i.e. the theorem on page 874 of [8]) characterizes up to equivalence categories of finitary quasi varieties and varieties of unordered algebras. We prefer to reformulate and slightly generalize his theorem as follows (see [3]):

**Theorem.** Let \( U: C \to \text{SET} \) be a functor. There is some signature \( \Sigma \) and a concrete quasi variety \( Q \) of (unordered) \( \Sigma \) algebras and an isomorphism \( I: C \to Q \) such that \( U = IU_Q \) (where \( U_Q \) is the underlying set functor) iff

(a) \( C \) has all coequalizers;
(b) \( U \) has a left adjoint;
(c) a morphism \( f \) in \( C \) is a coequalizer iff \( fU \) is a surjection;
(d) \( U \) creates isomorphisms (this notion is explained below).

If, in addition, \( U \) reflects congruences, \( Q \) will be a concrete variety.

Before giving the matter much thought, we had suspected that a characterization of the varieties and quasi varieties of ordered algebras could be easily obtained by retaining Lawvere's properties, only changing the target of the functor \( U \) to be the category \( \text{POS} \) of posets (particularly ordered sets) and order preserving functions. Of course, this naive conjecture immediately proved to be false. For example it is easy to see that there are surjections in \( \text{POS} \) itself which are not coequalizers. We thus began our investigation with the question: what category theoretic property characterizes the surjective order preserving homomorphisms?

After several false starts, the apparently 'correct' framework for the study of ordered algebras suggested itself, and it is partly the purpose of this paper to give an introduction to this framework. We call the appropriate categories 'P-categories', where the \( P \) stands for 'Poset'. In this framework, the statement of the characterization theorem sounds remarkably similar to the corresponding theorem for unordered algebras. Exactly why this is so is somewhat mysterious. (For example, one of our conditions is this: a morphism \( f \) in \( C \) is a 'P-coequalizer' iff \( fU \) is a surjection.) As a corollary of our arguments, we are able to derive the Lawvere theorem above.
We warn those readers who are familiar with the notion of ‘$V$-categories’ (see [4]) that even though our $P$-categories and $P$-functors are indeed $V$-categories and $V$-functors, where $V = \text{POS}$, not all of our notions are instances of $V$-category notions. For example, our ‘$P$-limits’ are not $V$-limits, and our $P$-epics, $P$-coequalizers and $P$-monics are not the same as the corresponding $V$ notions when $V = \text{POS}$.

The background necessary to read this paper is some knowledge of elementary categorical algebra (the first six chapters of [10] is more than sufficient) as well as familiarity with basic universal algebraic notions.

We use the notation ‘$g \circ f$’ to denote the composite of the morphisms $f: A \rightarrow B$, $g: B \rightarrow C$ and, if $f: A \rightarrow B$ is a function, the value of $f$ on an element $a$ in $A$ will be denoted variously as:

$$af, (a)f, \langle a, f \rangle \text{ and } f(a).$$

The set of morphisms $A \rightarrow B$ in a category $C$ is denoted $\text{SET}$ and $\text{POS}$ is the category whose objects are posets and whose morphisms are order preserving functions.

The paper is organized as follows. In Section 2, a description of ‘concrete’ varieties and quasi varieties of ordered algebras is given. In the following two sections are some results on ‘$P$-categories’. Several of these facts are used in the proof of the Main Theorem. The statement of the Main Theorem is in Section 5 and the proof of the theorem is given in Sections 6 and 7. Section 8 is devoted to some examples and remarks. In Section 9 we show how to extend any $P$-quasi variety to a $P$-variety in a canonical way.

2. Concrete $P$-quasi varieties and $P$-varieties

A signature for a class of unordered algebras is usually taken to be a mapping from an initial segment $\text{INIT}$ of the cardinal numbers into the class of sets. The most common case occurs when $\text{INIT}$ consists of the finite cardinals; in rare cases $\text{INIT}$ is not a proper initial segment. We will find it convenient to define a signature $\Sigma$ for a class of ordered algebras to be a mapping from an initial segment $\text{INIT}$ of the cardinal numbers (not necessarily proper) into the class of partially ordered sets (‘posets’). The value of $\Sigma$ on the cardinal $n$ is written $\Sigma_n$. Then an ordered $\Sigma$ algebra $A$ consists of a poset (also denoted $A$) and for each $\sigma$ in $\Sigma_n$ an order preserving function $\sigma_A: A^n \rightarrow A$

where the ordering on $A^n$ is componentwise. (It is convenient to consider an $n$-tuple of elements of $A$ to be an (order preserving) function from the discretely ordered set $n$ to $A$.) Further, if $\sigma \leq \sigma'$ in $\Sigma_n$, then for all $x$ in $A^n$,

$$x\sigma_A \leq x\sigma'_A \text{ in } A.$$
A signature $\Sigma$ for a class of ordered algebras is \textit{discrete} if $\Sigma_n$ is a discretely ordered poset, for all $n$ in INIT. (A poset is discretely ordered if $x=y$ whenever $x \leq y$.) $\Sigma$ is \textit{finitary} if INIT consists of the finite cardinals; $\Sigma$ is \textit{bounded} if INIT is a proper initial segment of the cardinals.

Let $\Sigma$ be a fixed signature for a class of ordered algebras. The class of all (ordered) $\Sigma$ algebras, $\Sigma\text{ALG}$, is a category whose objects are the algebras in $\Sigma\text{ALG}$ and whose morphisms are the order preserving homomorphisms. Explicitly, if $A$ and $B$ are $\Sigma$ algebras then $h$ is a morphism $A \rightarrow B$ in $\Sigma\text{ALG}$ iff

(i) for $x \leq y$ in $A$, $xh \leq yh$ in $B$;

(ii) for each $n$ in INIT and each $\sigma$ in $\Sigma_n$, each $x:n \rightarrow A$,

$$x\sigma_A h = xh\sigma_B,$$

where $xh$ is the composite

$$ n \xrightarrow{x} A \xrightarrow{h} B. $$

Any subclass of $\Sigma\text{ALG}$ determines a full subcategory of the category of all $\Sigma$ algebras. Each such category $C$ of ordered $\Sigma$ algebras has an ‘underlying poset functor’ $U:C \rightarrow \text{POS}$.

which forgets the algebraic structure on each algebra in $C$ and remembers only the order structure. When $C = \Sigma\text{ALG}$, we write $U$ as $U_\Sigma$.

Our main interest will be those categories of ordered algebras which are ‘$P$-quasi varieties’ or ‘$P$-varieties’. First we consider the finitary case.

Suppose that $\Sigma$ is a finitary signature.

\textbf{Definition.} A full subcategory $C$ of $\Sigma\text{ALG}$ is a (finitary) \textit{concrete $P$-quasi variety} if:

(i) $C$ is closed in $\Sigma\text{ALG}$ with respect to products; and

(ii) $C$ is closed under ‘strong’ subalgebras.

Condition (i) means that if $A_i$ is an ordered algebra in $C$, for each $i$ in the set $I$, then $\Pi A_i$ is also in $C$, where the ordering on the product is componentwise. A homomorphism $m:A \rightarrow B$ is ‘strong’ if $x \leq y$ in $A$ whenever $xm \leq ym$ in $B$. Condition (ii) means that if $B$ is in $C$ and $m:A \rightarrow B$ is a strong homomorphism, then $A$ is also in $C$. (A strong homomorphism is necessarily injective. Later we will call such homomorphisms ‘$P$-monics’.)

The following fact may be proved in the usual way.

\textbf{Proposition 1.} Suppose that $C$ is a finitary $P$-quasi variety. The $C$ has all ‘poset generated free algebras’; i.e.

\begin{itemize}
  \item[(*)] for each poset $X$ there is an algebra $XF$ in $C$ and an order preserving map $\eta:X \rightarrow XF\text{FL}$, where $\text{fl}:C \rightarrow \text{POS}$ is the underlying poset functor, such that for any algebra $A$ in $C$ and for any $f:X \rightarrow AU$ in $\text{POS}$ there is a unique homomorphism $f^\# :XF \rightarrow A$ in $C$ such that $\eta \cdot f^\# U = f$. Further, if $f$ and $g$ are morphisms $X \rightarrow AU$ with $f \leq g$ (i.e. $xf \leq xg$, for all $x$ in $X$), then $f^\# \leq g^\#$.
\end{itemize}
Definition. A (finitary) concrete P-variety $C$ of $\Sigma$ algebras is a full subcategory of $\Sigma \text{ALG}$ which is a concrete P-quasi variety which in addition satisfies the following condition:

$C$ is closed in $\Sigma \text{ALG}$ under surjective order preserving homomorphisms.

Thus, if $C$ is a P-variety and $A$ is an algebra in $C$ and $h : A \to B$ is a surjective (order preserving) homomorphism, then $B$ is in $C$ also.

We will state a Birkhoff style 'axiomatization theorem' for concrete P-quasi varieties and P-varieties in the case that the signature is finitary.

Let $\text{TM}(V)$ be the set of terms built in the usual way from a set $V$ of 'variables'.

Theorem. (See [2].) Suppose that $\Sigma$ is finitary. A full subcategory $C$ of $\Sigma$ algebras is a P-variety iff there is a set $AX$ of 'inequations' of the form

$t \leq t'$,

where $t$ and $t'$ are in $\text{TM}(V)$, for a countably infinite set $V$, such that an algebra $A$ is in $C$ iff $A$ is a model of $AX$.

(Each term $t$ in $\text{TM}(V)$ defines a function in each ordered $\Sigma$ algebra as usual, and $A$ is a model of the inequation $t \leq t'$ iff for all $x$ in $A^V$, $xt_A \leq xt'_A$.)

A full subcategory $C$ of $\Sigma$ algebra is a P-quasi variety iff there is a class (perhaps a proper class at that) of axioms $AX$ such that an algebra is in $C$ iff it is a model of $AX$. Each axiom $a$ is a 'generalized implication' of the following form: there is a set $Va$, which depends on the axiom $a$, and a set $H$ of inequations $t \leq t'$ in $\text{TM}(Va)$ and an inequation $s \leq s'$, with $s \cdot s'$ also in $\text{TM}(Va)$; the axiom is "$H \Rightarrow (s \leq s')$".

An algebra $A$ is a model of a generalized implication $H = (s \leq s')$ if whenever each implication in $H$ is satisfied in $A$, so is $(s \leq s')$.

We now consider the case that $\Sigma$ is a bounded signature. In this case as well as the finitary case, one may prove that any class $C$ of algebras closed under products and strong subalgebras has all poset generated free algebras, and an axiomatization theorem analogous to the preceding case holds. If $C$ is also closed under surjective order preserving homomorphisms, the stronger theorem holds, i.e. $C$ may be axiomatized by a set of inequations. (There is some work necessary to define the set of 'terms' and show it is a small set.)

In the case $\Sigma$ is unbounded (more precisely, when there is no cardinal $n$ such that $\Sigma_n$ is empty if $n < m$) then there is no 'concrete' notion of $\Sigma$ term. Thus no 'concrete' axiomatization theorem analogous to the one above exists. (Abstract axiomatization theorems do exist; see [1].) Furthermore, Proposition 1 fails. Indeed, in this case, the collection of all $\Sigma$ algebras is clearly closed under products and strong
subalgebras, but not all free algebras exist, indeed, there is no free algebra generated by a singleton poset, for example. (It is an easy exercise to show that for every cardinal \( n \) there is a \( 1 \)-generated \( \Sigma \) algebra \( A \) having at least \( n \) elements; thus the \( 1 \)-generated free algebra, if it were to exist, would also have at least \( n \) elements, for every \( n \).)

Thus we make the following definitions, one of whose conditions is redundant when \( \Sigma \) is bounded.

**Definition.** Let \( \Sigma \) be an arbitrary signature. A full subcategory \( C \) of the category \( \Sigma \text{ALG} \) is a **concrete \( P \)-quasi variety** if \( C \) is closed under products, strong subalgebras and \( C \) has all poset generated free algebras, i.e. (*) above holds. A concrete \( P \)-quasi variety is a **concrete \( P \)-variety** if \( C \) is closed under surjective order preserving homomorphisms.

Some examples of finitary \( P \)-varieties are: lattices, ordered monoids, ordered semi-rings, ordered rings, etc. An example of a bounded \( P \)-quasi variety is the class of all posets equipped with an \( \omega \)-ary operation \( \text{sup} \) which are models of

\[
\bigwedge_{n \in \omega} x_n \leq x_{n+1} \Rightarrow x_n \leq \text{sup}(x_1, x_2, \ldots);
\]

\[
\bigwedge_{n \in \omega} x_n \leq x_{n+1} \bigwedge \bigwedge_{n \in \omega} x_n \leq y \Rightarrow \text{sup}(x_1, x_2, \ldots) \leq y.
\]

An example of an unbounded \( P \)-variety is the class of all complete posets; i.e. posets equipped with a binary and \( n \)-ary \( \text{sup} \) operation, for every infinite cardinal number \( n \).

**Remark.** Note that the ‘Birkhoff style’ theorem characterizes concrete \( P \)-varieties in a relative manner – i.e. relative to the class of all \( \Sigma \) algebras. Secondly, the notion of ‘concrete \( P \)-variety’ is not invariant under category isomorphism. For example, let \( C \) be the category of all ordered groups, where ‘group’ means a \( \Sigma \) algebra (where \( \Sigma \) has one binary operation) satisfying the usual axioms. Let \( D \) be the category of all ordered groups, where now ‘group’ means a \( \Sigma' \) algebra, where \( \Sigma' \) has a binary operation, a unary operation (for inverse) and a nullary operation (for the identity) satisfying the usual equational axioms. Then the category \( C \) is isomorphic to the category \( D \), but \( C \) is not a concrete \( P \)-variety while \( D \) is. These observations should help explain why one might seek a signature independent characterization of \( P \)-varieties.

We end this section with an observation concerning discrete vs. arbitrary signatures.

For an arbitrary signature \( \Sigma \), let \( \Sigma' \) be the discrete signature such that for each \( n \) in \( \mathbb{N} \cap \). \( \Sigma' \) has the same underlying set as \( \Sigma_n \). Then clearly any \( \Sigma \) algebra is a \( \Sigma' \) algebra. Let \( I: \Sigma \text{ALG} \rightarrow \Sigma' \text{ALG} \) be the inclusion functor.
Proposition. For any full subcategory $C$ of $\Sigma \text{ALG}$ there is a full subcategory $C'$ of $\Sigma' \text{ALG}$ and an isomorphism $K : C \rightarrow C'$ such that the following diagram commutes:

\[
\begin{array}{ccc}
C & \xrightarrow{K} & C' \\
\downarrow{U} & & \downarrow{U'} \\
\text{POS} & & \\
\end{array}
\]

where $U$ and $U'$ are the underlying poset functors. In fact, $K$ is the restriction to $C$ of the inclusion functor $I$. Moreover, $C'$ is a concrete $P$-quasi variety (or $P$-variety) of $\Sigma'$ algebras whenever $C$ is a concrete $P$-quasi variety (or $P$-variety) of $\Sigma$ algebras.

Thus, there is no essential gain in generality if one allows poset valued signatures for ordered algebras. However there is a gain in convenience, as will be seen in our proof of the Main Theorem.

3. $P$-Categories

One of the more obvious features of concrete $P$-quasi varieties is that each ‘Hom set’ is a poset partially ordered by the relation: $f \leq g$ if $xf \leq xg$, for all $x$ in the source of $f$. Composition of functions preserves this ordering. In this section we examine some properties of such categories.

Definitions. A $P$-category is a category $C$ with the properties that each ‘Hom set’ $C(A, B)$ is equipped with a distinguished partial ordering $\leq$ and composition preserves this ordering; i.e. if $f \leq g$ in $C(A, B)$ then for all $h : X \rightarrow A$ and all $k : B \rightarrow Y$ both $h \cdot g$ in $C(X, B)$ and $f \cdot k \leq g \cdot k$ in $C(A, Y)$.

We should use special notation to indicate the set $C(A, B)$ on which $\leq$ is a partial ordering, but this omission should cause no confusion.

If $C$ and $C'$ are $P$-categories, a $P$-functor $F : C \rightarrow C'$ is a functor which preserves the partial orderings; i.e. if $f \leq g$ in $C(A, B)$ then $fF \leq gF$ in $C'(AF, BF)$.

When we say that $F : C \rightarrow C'$ is a $P$-functor”, we will assume that both $C$ and $C'$ are $P$-categories.

Note that any category may be considered to be a $P$-category all of whose partial orderings on the ‘Hom sets’ are discrete. A $P$-adjunction (or ‘$P$-adjoint situation’) $(U, I, \eta, \varepsilon)$ consists of $P$-functors $U : C \rightarrow A$, $F : C \rightarrow A$ and natural transformations

\[
\eta : 1 \Rightarrow FU, \quad \varepsilon : UF \Rightarrow 1
\]
satisfying the standard adjunction properties (cf. [10], p. 80).

We state the following facts without proof.
Proposition 1. Let \( C \) and \( C' \) be \( P \)-categories and suppose that
\[
(U: C \to C', F: C' \to C, \eta, \varepsilon)
\]
is an adjoint situation (not necessarily a \( P \)-adjoint situation). Let
\[
\varphi_{X,A}: C(XF, A) \to C'(X, AU)
\]
be the family of natural bijections associated with the adjunction in the usual way
\( \{ f: X \to AU \text{ then } \eta_f^{-1} \text{ is the unique morphism } f^*: XF \to A \text{ such that} \eta_A \}
\]
see [10], pp. 78–80). Then:
(a) \( U \) is a \( P \)-functor iff each \( \varphi_{X,A} \) is order-preserving; i.e. if \( f \leq g \) in \( C(XF, A) \) then \( f\varphi_{X,A} \leq g\varphi_{X,A} \) in \( C'(X, AU) \).
(b) \( F \) is a \( P \)-functor iff each \( \varphi_{X,A} \) is \( P \)-faithful; i.e. if \( f\varphi \leq g\varphi \), then \( f \leq g \).

Definition. Let \( C \) be a \( P \)-category. A \( P \)-monad over \( C \) is a monad
\[
(T, \eta, \mu)
\]
over \( C \) such that the endofunctor \( T \) is a \( P \)-functor.

Any \( P \)-adjoint situation
\[
(U: A \to C, F: C \to A, \eta, \varepsilon)
\]
determines a \( P \)-monad (in the usual way, where \( T = FU \)) and any \( P \)-monad is determined by some \( P \)-adjoint situation (viz. \( U^T \) and \( F^T \) [10], p. 135, which are \( P \)-functors if \( T \) is).

If \((T, \eta, \mu)\) is a monad or \( P \)-monad on \( \text{POS} \), we denote the corresponding category of Eilenberg–Moore algebras by \( \text{POS}^T \).

Before stating the next results, we need some terminology.

Definition. Let \( C \) be a \( P \)-category and suppose that \( f: A \to B \) is a morphism in \( C \).
(i) \( f \) is \( P \)-epic if for all \( g, h: B \to C \), \( g \leq h \) whenever \( f \cdot g \leq f \cdot h \).
(ii) \( f \) is \( P \)-monic if for all \( g, h: X \to A \), \( g \leq h \) whenever \( g \cdot f \leq h \cdot f \).
(iii) If \( U: C \to D \) is a \( P \)-functor, \( U \) is \( P \)-faithful if for all \( f, g: A \to B \) in \( C \), \( f \leq g \) whenever \( fU \leq gU \).

Proposition 2. Suppose that \((U: A \to C, F: C \to A, \eta, \varepsilon)\) is a \( P \)-adjoint situation. Then \( U \) is \( P \)-faithful iff each component of the counit is \( P \)-epic.

Proof. First suppose that \( U \) is \( P \)-faithful and that for some \( f, g: A \to B \) in \( A \)
\[
\varepsilon_1 \cdot f \leq \varepsilon_1 \cdot g;
\]
then \( \varepsilon_A \cdot fU = (\varepsilon_A \cdot f)U \leq \varepsilon_A \cdot gU \) in \( C \), since \( U \) is a \( P \)-functor, which in turn implies that

\[
\eta_{AU} \cdot \varepsilon_A \cdot fU \leq \eta_{AU} \cdot \varepsilon_A \cdot gU
\]

since \( h \) is preserved by composition. But by one of the triangle identities it now follows that \( fU \leq gU \). Thus, since \( U \) is \( P \)-faithful, \( f \leq g \).

Now suppose that each component of the counit is \( P \)-epic and assume that \( fU \leq gU \), where \( f, g : A \rightarrow B \). Then since \( F \) is also a \( P \)-functor

\[
(fUF) \leq (gUF)
\]

in \( A \),

and thus by the naturality of \( \varepsilon \),

\[
(\varepsilon_A \cdot f) \leq (\varepsilon_A \cdot g)
\]

Since \( \varepsilon_A \) is \( P \)-epic, \( f \leq g \), as claimed.

We state a similar fact without proof.

**Proposition 3.** With the same assumption as in Proposition 2, each component of the unit is \( 'U \ P \)-epic', i.e. if \( f, g : X \rightarrow A \) and if \( (\eta_X \cdot fU) \leq (\eta_X \cdot gU) \), then \( f \leq g \).

Note that a special case of Proposition 3 is that \( f = g \) whenever

\[
\eta_X \cdot fU = \eta_X \cdot gU.
\]

**Definition.** Let \( C \) be a \( P \)-category and suppose that \( f, g : A \rightarrow B \), \( h : B \rightarrow C \) are morphisms in \( C \). Then \( h \) is a \( P \)-coequalizer of the ordered pair \((f \cdot g)\) if \( f \cdot h \leq g \cdot h \) and whenever \( h' : B \rightarrow C' \) is a morphism such that \( f \cdot h' \leq g \cdot h' \), there is a unique morphism \( k : C \rightarrow C' \) such that \( h \cdot k = h' \).

Clearly, a \( P \)-cocqualizer of \( f \) and \( g \) is unique up to isomorphism and we will sometimes say 'the' \( P \)-coequalizer.

We will say that \( h \) is \( P \)-regular if \( h \) is a \( P \)-coequalizer of some parallel pair.

With \( f, g \) and \( h \) as above, we will call the ordered pair \((f, g)\) a \( P \)-kernel pair of \( h \) if \( f \cdot h \leq g \cdot h \) and for any other pair \( f', g' : A' \rightarrow B \) such that \( f' \cdot h \leq g' \cdot h \) there is a unique morphism \( k : A' \rightarrow A \) such that \( k \cdot f = f' \) and \( k \cdot g = g' \).

Clearly, the \( P \)-kernel of \( h \) is unique up to isomorphism.

We will say that the pair \((f, g)\) is a \( P \)-congruence if it is the \( P \)-kernel pair of some morphism \( h \).

**Remark.** Suppose that \( f : A \rightarrow B \) in the category \textbf{POS} (or any concrete \( P \)-quasi-variety):

(i) \( f \) is \( P \)-regular iff \( f \) is a surjection, i.e. for each \( b \) in \( B \) there is some \( a \) in \( A \) with \( af = b \);
(ii) \( f \) is P-monic iff the ordering on \( A \) is inherited from that in \( B \), i.e. \( a \leq a' \) in \( A \) iff \( af \leq a'f \) in \( B \), i.e. \( f \) is a ‘strong’ monic;

(iii) the P-kernel of \( f \) is the preorder on \( A \) induced by \( f \), i.e. if \( X \) is the set of ordered pairs \((x, y)\) of elements of \( A^2 \) such that \( xf \leq yf \) in \( B \), then the two projections \( X \rightarrow A \) form the P-kernel of \( f \).

(iv) examples of P-quasi varieties in which the P-epis do not coincide with the P-regular morphisms are given in Section 8.

We now state a result that has an analogue in standard categories ([7] give this result the strange name of ‘pulation lemma’; see 21.16). This fact will be used many times below.

(The ‘P-pulation’) **Lemma.** Suppose that \( C \) is a P-category and that \( f, g : A \rightarrow B, h : B \rightarrow C \) are C-morphisms. Then the following statements are equivalent:

1. \((f, g)\) is a P-kernel of \( h \) and \( h \) is a P-coequalizer of \((f, g)\);
2. \((f, g)\) is a P-congruence and \( h \) is a P-coequalizer of \((f, g)\);
3. \((f, g)\) is the P-kernel of \( h \) and \( h \) is P-regular.

**Remark.** Other than the fact that a partial ordering is a binary relation, nothing in the preceding definitions and propositions depends on the special properties of partial orderings. One might define the notion of an ‘\( R \)-category’ as a category equipped with some distinguished binary relation (say ‘\( R \)’) on each Hom-set; this relation must be preserved by composition. (i.e. \( fRg = (f * h)R(g * h) \) and \( (h * f)R(h * g) \), whenever these composites are defined.) Similarly, we may define \( R \)-functors, \( R \)-adjoint situations, \( R \)-monics, \( R \)-epics, etc. and prove theorems analogous to those above. We have chosen not to present this more general theory since at the moment we have no application for it.

Now we consider some properties more closely connected with orderings.

**Definition.** Suppose that \( C \) is a P-category and that \( A \) is an object in \( C \). A canonical ordering on \( A \) is an ordered pair of morphisms

\[(f, g) : X \rightarrow A\]

with the following three properties:

1. \( f \leq g \) (in \( \text{Hom}(X, A) \));
2. \((f, g)\) is a P-monocone, i.e if \( h, k : Y \rightarrow X \) and both \( h \cdot f \leq k \cdot f \) and \( h \cdot g \leq k \cdot g \), then \( h \leq k \);
3. if \( f' : X' \rightarrow A \) and \( f' \leq g' \), then there is a (necessarily unique, by (ii)) morphism \( k : X' \rightarrow X \) such that \( k \cdot f = f' \) and \( k \cdot g = g' \).

We say that “\( C \) has canonical orderings” if there is a canonical ordering on each object of \( C \).
Remarks. (i) Note that canonical orderings, when they exist, are unique up to isomorphism.

(ii) The category \textbf{POS} and indeed any concrete \( P \)-quasi-variety has canonical orderings. For each poset (or algebra) \( A \), let \( O_A \) be the set of pairs \((a, b)\) of elements of \( A \) such that \( a \leq b \). Then the two projections from \( O_A \) to \( A \) form a canonical ordering on \( A \).

Several times below we will make use of the following concept, which is of interest in its own right.

A ‘\( P \)-diagram scheme’ \((N, E)\) is an extension of the notion of diagram scheme, in [11] for example: \( N \) is a set (of ‘nodes’) and for each ordered pair \((i, j)\) of nodes there is a set \( E(i, j) \) of ‘signed edges from \( i \) to \( j \)’. A signed edge from \( i \) to \( j \) is an ordered pair \((a, r)\) where \( r \) is one of the three relation symbols \( \leq, \geq, = \), and \( a \) is a member of some set of labels. If \((a, r)\) is a signed edge from \( i \) to \( j \), \( a \) is called ‘an edge from \( i \) to \( j \)’. The sets \( E(i, j) \) need not be pairwise disjoint and some of them may be empty. A \( P \)-diagram \( D \) over the scheme \((N, E)\) in a \( P \)-category \( C \) is an assignment of objects and morphisms to the nodes and edges of the scheme, as usual. (The object assigned to the node \( i \) will usually be denoted \( D_i \).) The relation symbols are involved in the notion of a ‘\( D \)-cone’. A \( D \)-cone \((L, g_i)\) (or ‘cone over \( D \)’) consists of a \( C \)-object \( L \) and morphisms \( g_i : L \to D \) for each node \( i \) with the following property: if \((a, r) : i \to j \) is a ‘signed edge’ in the scheme \((N, E)\) and \( f : D_i \to D_j \) is assigned to \( a \), then

\[
g_i \cdot f \tilde{r} g_j
\]

where \( \tilde{r} \) is the appropriate relation on \( \text{Hom}(D_i, D_j) \), either \( \leq, \geq \) or \( = \). A \( P \)-limit of a \( P \)-diagram \( D \) is a \( D \)-cone \((L, g_i)\) which is universal with this property; i.e. if \((L', g_i')\) is any \( D \)-cone there is a unique morphism \( k : L' \to L \) such that for every node \( i \),

\[
k \cdot g_i = g_i'.
\]

(All \( P \)-limits considered in this paper are ‘small’ limits – i.e. the set of nodes in any diagram is a small set. Note that if all relation symbols in a \( P \)-diagram scheme are ‘\( = \)’, then one has the usual notion of diagram scheme. Thus limits, e.g. products, are a special case of \( P \)-limits.)

As an example consider the following \( P \)-diagram in a \( P \)-category \( C \):

\[
\begin{array}{ccc}
A & \xrightarrow{h. \leq} & B \\
\downarrow h. = & & \\
A & \xrightarrow{h. =} & B
\end{array}
\]

The reader may easily convince himself that a \( P \)-limit of this \( P \)-diagram (more precisely, the pair of morphisms in the limit cone with target \( A \)) is the \( P \)-kernel of the morphism \( h \).
A $D$-cocone (or 'cone over $D$') consists of an object $L$ and morphisms $g_i : D_i \to L$, such that for every signed edge $(f, r) : D_i \to D_j$ as above,

$$(f \circ g_i) \circ r = g_i.$$ 

A $P$-colimit of $D$ is a $D$-cocone which is universal with this property; i.e. if $(L', g')$ is any $D$-cocone, there is a unique morphism $k : L \to L'$ such that for each node $i$, $g_i \circ k = g_i'$.

For example, consider the following $P$-diagram in any $P$-category:

$$A \xrightarrow{f, \leq} B.$$ 

Then any $P$-colimit of this diagram, more precisely, the morphism with source $B$, is a $P$-coequalizer of $(f, g)$.

**Proposition.** Let $C$ be a $P$-category that has canonical orderings. If $(L, g_i)$ is a $P$-colimit of the $P$-diagram $D$, then $(L, g_i)$ is a $P$-epi cocone; i.e. if

$$g_i \circ h \leq g_i \circ k \text{ for all } i,$$

where $h, k : L \to X$, then $h \leq k$. In particular, any $P$-regular morphism in $C$ is $P$-epic.

**Proof.** Let $(s, r) : O \to X$ be a canonical ordering on $X$. Then, for each $i$, there is a unique morphism $c_i$ such that

$$c_i \circ s = g_i \circ h \quad \text{and} \quad c_i \circ r = g_i \circ k.$$ 

Using the fact that $(s, r)$ is a $P$-monocone, one can show that $(O, c_i)$ is a cocone over $D$. But then there is a unique morphism $d : L \to O$ such that for every $i$

$$g_i \circ d = c_i.$$ 

But $d \circ s = h$ and $d \circ r = k$, since $(L, g_i)$ is clearly an epi cocone. Since $s \leq r$, it follows that $h \leq k$. The proof is complete.

It is easy to find an example of a $P$-category in which there is a $P$-regular epi which is not $P$-epi, so that some added assumption is necessary to ensure that the $P$-regular epis are in fact $P$-epi. In this connection see Sublemma 1, Section 4.

The following theorem shows that in a $P$-adjoint situation the appropriate $P$-limits and $P$-colimits are preserved.

**Proposition 4.** Suppose that

$$(U : C \to A, F : A \to C, r \in)$$

is a $P$-adjoint situation. Then $U$ preserves $P$-limits and $f$ preserves $P$-colimits.
The straightforward proof is omitted.

**Definition.** The P-category \( C \) is **P-complete** (or **P-cocomplete**) if every P-diagram in \( C \) has a P-limit (or P-colimit).

We note the following simple fact.

**Proposition.** Suppose that \( 1_A : A \to A \) has a P-kernel \((s, r) : O \to A\). Then \((s, r)\) is a canonical ordering on \( A \).

Thus, if \( C \) is P-complete, \( C \) has canonical orderings.

**Proposition 5.** The category \( \text{POS} \) is both P-complete and P-cocomplete.

**Lemma.** Let \( C \) be any P-category. Then \( C \) is P-complete iff \( C \) has all small products and all ‘collective P-equalizers’.

Let \( I \) be a set and suppose that for each \( i \) in \( I \),

\[
(f_i, g_i) : A \to X_i
\]

is an ordered pair of morphisms in \( C \) and \( r_i \) is one of the relation symbols \( \leq, \geq \) or =. \( m : E \to A \) is a **collective P-equalizer** of \((f_i, g_i)\) if for all \( i \), \((m \cdot f_i) r_i (m \cdot g_i)\) (where \( r_i \) is the appropriate relation on \( \text{Hom}(A, X_i) \)) and further, \( m \) is universal with this property; i.e. if \( m' f_i r_i m' g_i \) for all \( i \), then \( m' = k \cdot m \), for some unique morphism \( k \).

The proof of this lemma is an easy modification of the proof that the existence of products and equalizers are sufficient to guarantee the existence of all limits.

Thus to prove that \( \text{POS} \) is P-complete, it is enough to show that \( \text{POS} \) has all collective P-equalizers, since clearly \( \text{POS} \) has all products. So assume that

\[
(f_i, g_i) : A \to X_i
\]

is a set of ordered pairs of morphisms in \( \text{POS} \) and \( r_i \) is one of the relation symbols \( \leq, \geq \) or =. Let

\[
E = \{ x \in A : x f_i r_i x g_i, \text{for all } i \}.
\]

If \( m : E \to A \) is the P-monic inclusion, i.e. the ordering on \( E \) is the restriction of the ordering on \( A \), it is easy to show that \( m \) is the collective P-equalizer of \((f_i, g_i)\).

Thus \( \text{POS} \) is P-complete. We omit the similar proof of P-completeness. But note that it follows from the construction in the last proposition that P-limit cones are P-monocones.
Corollary. Let \((L, m_i)\) be a \(P\)-limit cone in \(\text{POS}\). Then \((L, m_i)\) is a \(P\)-monocone; i.e. if \(f, g: A \to L\) and \(f \cdot m_i \leq g \cdot m_i\) for all \(i\), then \(f \leq g\).

Proof. This fact follows from the fact that products and collective \(P\)-equalizers are \(P\)-monocones.

Definition. A \(P\)-functor \(U: C \to C'\) creates \(P\)-limits if whenever \(D\) is a \(P\)-diagram in \(C\) and \((L, m_i)\) is a \(P\)-limit of \(DU\) in \(C'\), then there is a unique \(D\)-cone \((X, g_i)\) in \(C\) with

\[(X, g_i) U = (L, m_i);\]

furthermore, \((X, g_i)\) is a \(P\)-limit of \(D\).

Remark. The following observation will be used later. If \(U\) creates isomorphisms, and \(fU = 1: AU \to BU\) for some morphism \(f: A \to B\), then \(A = B\) and \(f = 1\).

We will make use of several facts about \(P\)-monadic functors.

Proposition 6. Suppose that \(U: C \to C'\) is a \(P\)-monadic functor. Then \(U\) creates \(P\)-limits.

The proof of this proposition is similar to the unordered version ([11] 3.1.19), and is omitted.

The next proposition is quite useful in the proof of the Main Theorem.

Proposition 7. Let \((T, \eta, \mu)\) be a monad (not necessarily a \(P\)-monad) on \(\text{POS}\). Then if \(T\) preserves surjections, every \(T\)-homomorphism

\[f: (A, s) \to (B, t)\]

in \(\text{POS}^T\) may be written as a composite

\[f = e \cdot m\]

of a surjective \(T\)-homomorphism \(e\) and a \(P\)-monic \(T\)-homomorphism \(m\).

Proof. Let \(f = e \cdot m\) be a factorization of \(f\) into a \(P\)-regular epi and \(P\)-monic in \(\text{POS}\), where \(e: A \to D\). Thus the outside of following diagram commutes:

\[
\begin{array}{ccc}
AT & \xrightarrow{vT} & DT & \xrightarrow{mT} & BI \\
\downarrow s & & \downarrow d & & \downarrow t \\
A & \xrightarrow{e} & D & \xrightarrow{m} & B
\end{array}
\]
Now $eT$ is a surjection, by assumption, so we may attempt to define $d : DT \to D$ by:

$$\langle x, eT \rangle d = \langle x, s \cdot e \rangle,$$

where $x$ is an AT. We show that $d$ is well defined and order preserving. If $\langle x, eT \rangle \leq \langle y, eT \rangle$, then $\langle x, eT \cdot mT \cdot t \rangle \leq \langle y, eT \cdot mT \cdot t \rangle$. Hence, $\langle x, s \cdot e \cdot m \rangle \leq \langle y, s \cdot e \cdot m \rangle$, and since $m$ is $P$-monic, $\langle x, s \cdot e \rangle \leq \langle y, s \cdot e \rangle$.

By definition then the left-hand square in the above diagram commutes, and since $eT$ is surjective, the right-hand square commutes also. It now must be shown that $(D, d)$ is a $T$-algebra.

**Lemma 8.** Suppose the following diagram commutes.

\[
\begin{array}{ccc}
DT & \xrightarrow{mT} & BT \\
\downarrow d & & \downarrow t \\
D & \xrightarrow{m} & B
\end{array}
\]

(a) If $(B, t)$ is a $T$-algebra and $m$ is $P$-monic, then $(D, d)$ is also a $T$-algebra.
(b) If $(D, d)$ is a $T$-algebra and $m$, $mT$ and $mTT$ are epi, then $(B, t)$ is a $T$-algebra.

The proof of these facts is in [11] (see the proof of 1.1.14 and 3.1.10).

Using part (a) of Lemma 8, the proof of Proposition 7 is complete.

An example of a $P$-monad $(T, \eta, \mu)$ on POS where $T$ does not preserve surjections is given in Section 8. A characterization of those monads which do preserve surjections on POS will be given in Section 7.

We end this section with a simple ‘Poset $P$-adjoint functor theorem’, which is an analogue of the set version, given in [7], 30.19 for example.

First we need some terminology. If $C$ is a $P$-category, $A$ is an object in $C$ and $I$ is a poset, an ‘$I$-th copower of $A$’ consists of an object, denoted $I \cdot A$, and an $I$-indexed family of morphisms

$$c_i : A \to I \cdot A$$

where if $i \leq i'$ then $c_i \leq c_{i'}$; further, if $d_i$ is any other $I$-indexed family of morphisms

$$d_i : A \to B$$

with $d_i \leq d_{i'}$ whenever $i \leq i'$, then there is a unique morphism

$$d : I \cdot A \to B$$

such that for each $i$ in $I$

$$c_i \cdot d = d_i.$$
If $U: C \to \text{POS}$, we say that $U$ is 'naturally $P$-isomorphic' to the Hom functor $C(A, -)$ if there is a natural isomorphism 
$$\varphi: U \to C(A, -)$$

such that for all objects $X$ in $C$, and all elements $u, v$ in $XU$,
$$u \leq v \iff u\varphi \leq v\varphi.$$ 

**Proposition 9.** Let $U: C \to \text{POS}$ be a $P$-functor. Then $U$ has a $P$-left adjoint iff there is an object $A$ in $C$ such that for each poset $I$, the $I$-th copower $I \cdot A$ of $A$ exists and such that $U$ is naturally $P$-isomorphic to the "Horn functor" $C(A, -)$.

The proof of this Proposition is easy and is omitted.

4. Some useful lemmas

This section contains the proof of two main lemmas, some of whose corollaries are useful in our proof of the Main Theorem, and are of interest in themselves. These two Lemmas are extensions of some arguments given in Section 32 of [7]. Their statements involve cones, and we review some cone terminology.

A 'cone' in a category consists of an object $X$ and a set $g_i : X \to Z_i$ of morphisms, indexed by some (small) set $I$, which will usually not be named explicitly. If $(g_i : X \to Z_i)$ and $(m_i : Y \to Z_i)$ are two $I$-indexed cones such that the target of $g_i$ is the same as the target of $m_i$ for all $i$ in $I$, then a morphism $h : X \to Y$ is a 'cone morphism' 
$$h : (X, g_i) \to (Y, m_i)$$

if 
$$h \cdot m_i = g_i \text{ for all } i.$$

Two cones are 'isomorphic' if there are cone morphisms from one to the other whose composites are the appropriate identities. If $(X, g)$ is a cone in the category $C$ and $U: C \to C'$ is a functor, then $(X, g)U$ is the cone $(XU, g_iU)$ in $C'$. Recall that a cone $(X, m_i)$ is a 'monocone' (or 'P-monocone') if for any $f, g : Z \to X$, $f = g$ whenever $f \cdot m_i = g \cdot m_i$ for all $i$ (respectively, if $f \leq g$ whenever $f \cdot m_i \leq g \cdot m_i$, for all $i$).

**The Monocone Lemma.** Suppose that the $P$-functor $U: C \to \text{POS}$ has a left adjoint $F$ and that $U$ reflects $P$-regular epis. Lastly assume that $U$ is $P$-faithful. Let $(g_i : X \to Z_i)$ and $(m_i : y \to Z_i)$ be two cones in $C$ such that 
$$(m_i : y \to Z_i)U$$

is a monocone POS. If $h : XU \to YU$ is a cone morphism in POS from $(y, g_i)U$ to $(X, m_i)U$ then there is a unique $h' : X \to Y$ in $C$ with $h'U = h$; it follows that $h'$ is a cone morphism $(y, g_i) \to (X, m_i)$ in $C$. 

Proof. There is a unique morphism $h^n : XF \to Y$ in $C$ such that the two triangles and the outside square in the following diagram commute:

First we show that $\varepsilon_X \cdot g_i = h^n \cdot m_i$ for all $i$.

Indeed

$$\eta_{XU} \cdot (\varepsilon_X \cdot g_i) U = g_i U - h \cdot m_i U = \eta_{XU} \cdot (h^n \cdot m_i) U,$$

and $\eta_{XU}$ is 'U epic' (see Proposition 3, Section 3). We now show that

$$\varepsilon_X U \cdot h = h^n U. \quad \text{(*)}$$

and $(X, m_i)U$ is a monocone.

Now we use the 'lifting lemma'.

The lifting lemma. Suppose that $U$ is $P$-faithful and reflects $P$-regular epis. If $gU \cdot k = hU$ in $POS$, and $gU$ is a surjection, then 'k lifts to C'; i.e. there is a (unique) $k'$ in $C$ with $g \cdot k' = h$ and further $k'U = k$.

Proof. Since $U$ reflects $P$ regular epis, $g$ is a $P$-coequalizer, of $(a, b)$, say. But then $a \cdot h \leq b \cdot h$, since $U$ is $P$-faithful and since

$$(a \cdot h) U = (a \cdot g) U \cdot k \leq (b \cdot g) U \cdot k = (b \cdot h) U.$$

Thus there is a unique $k'$ with $g \cdot k' = h$. Since $gU$ is epi, it follows that $k'U = k$.

Now by (*) and the lifting lemma, $h$ lifts to a unique morphism $h' : X \to Y$ in $C$ with $\varepsilon_X \cdot h' = h^n$ and $h'U = h$. The fact that $h'$ is a cone morphism follows since $U$ is $P$-faithful. The proof is complete.

Corollary. Under the same hypotheses as the Monocone Lemma, $U$ reflects limits and $P$-limits.
The proof follows from the Corollary to Proposition 5, Section 3 and the Monocone Lemma.

The next fact has some important corollaries.

**The P-limit cone Lemma.** Assume that the P-functor \( U : C \to \text{POS} \) is P-faithful, has a P-left adjoint \( F \), that \( C \) has all P-coequalizers and that \( U \) preserves P-regular epis. Then for any P-diagram \( D \) in \( C \) and any P-limit cone \( (X, m_i) \) over \( DU \) in \( \text{POS} \) there is a cone \( (Z, d_i) \) over \( D \) such that \( (Z, d_i)U \) is isomorphic to \( (X, m_i) \).

**Proof.** For each node \( i \), let \( m^*_i : XF \to D \) be \( C \)-morphisms such that

\[
\eta_X \cdot m^*_i U = m_i.
\]

**Claim.** \( (m^*_i : XF \to D_i) \) is a D-cone.

Indeed, let \( (f, r) \) be a signed edge from \( D_i \) to \( D_j \). Then

\[
(m_i \cdot f U) r m_j,
\]

so that

\[
(\eta_X \cdot m_i^* U \cdot f U) r (\eta_X \cdot m_j^* U).
\]

Since \( \eta_X \) is U P-epi, it follows that

\[
(m_i^* \cdot f) r m_j^*.
\]

The claim is proved.

Since \( (X, m_i) \) is a P-limit cone, and since \( (XF, m_i^*) \) is a cone over \( LU \), there is a unique \( \varphi : XFU \to X \) such that for all \( i \)

\[
\varphi \cdot m_i = m_i^* U. \quad (1)
\]

Thus

\[
\eta_X \cdot \varphi = 1_X.
\]

Hence \( \varphi \) is a surjection in \( \text{POS} \) and there is an ordered pair

\[
(g_1, g_2) : A \to XFU
\]

such that

\( \varphi \) is the P-coequalizer of \( g_1 \) and \( g_2 \).

Using the fact that \( U \) has a left adjoint, we obtain morphisms\n
\[
g_j^* : AF \to XF, \quad j = 1, 2,
\]

such that

\[
\eta_A \cdot (g_j^* U) = g_j, \quad j = 1, 2. \quad (2)
\]

Let \( q : XF \to Z \) be the P-coequalizer of \( g_1^* \) and \( g_2^* \) in \( C \).

**Claim.** For each node \( i \),
Indeed,
\[ g_1^\# \cdot m_i^\# \leq g_2^\# \cdot m_i^\# . \]

But because \( q \) is the \( P \)-coequalizer of \( g_1^\# \) and \( g_2^\# \), for each \( i \) there is a unique \( d_i : Z \to D_i \) such that
\[ q \cdot d_i = m_i^\# . \quad (3) \]

We now prove that
\[ (Z, d_i) \] is a \( D \)-cone. (4)

**Sublemma 1.** Suppose that \( U : C \to \textbf{POS} \) is a \( P \)-faithful \( P \)-functor which preserves \( P \)-regular epis. Then if \( q \) is \( P \)-regular in \( C \), \( q \) is also \( P \)-epic.

**Proof.** If \( q \cdot h \leq q \cdot k \) in \( C \), then \( qU \cdot hU \leq qU \cdot kU \) in \( \textbf{POS} \). But \( qU \) is \( P \)-regular and hence \( P \)-epi in \( \textbf{POS} \), so that \( hU \leq kU \). Since \( U \) is \( P \)-faithful, \( h \leq k \).

We now prove (4). Suppose
\[ (f, r) : D_i \to D_j \]
is a signed edge in \( D \). Then
\[ (m_i^\# \cdot f) \circ (m_j^\#) , \]
since \( (XF, m_i^\#) \) is a \( D \)-cone. Thus, by (3),
\[ (q \cdot d_i \cdot f) \circ (q \cdot d_j) . \]

Finally we prove that \( (Z, d) \) is isomorphic to \( (X, m) \). Since \( U \) preserves \( P \) regular epis, there is an ordered pair \( \langle s_1, s_2 \rangle : E \to XFU \) in \( \textbf{POS} \) such that
\[ qU \] is the \( P \)-coequalizer of \( (s_1, s_2) \).

\[ (5) \]

**Claim.** \( s_1 \cdot \varphi \leq s_2 \cdot \varphi . \)

Indeed, for each \( i \),
\[ s_1 \cdot \varphi \cdot m_i = s_1 \cdot m_i^\# U \quad \text{by (1)} \]
\[ = s_1 \cdot qU \cdot d_i U \quad \text{by (3)} \]
\[ \leq s_2 \cdot qU \cdot d_i U \quad \text{by (5)} \]
\[ = s_2 \cdot \varphi \cdot m_i . \]
But \((X, m)\) is a \(P\)-monocone, by the Corollary to Proposition 5, Section 3, so that the claim is proved.

Since \(qU\) is the \(P\)-coequalizer of \(s_1\) and \(s_2\), there is a unique morphism \(t: XU \to X\) such that
\[
qU \cdot t = \varphi.
\]
But also,
\[
g_1 \cdot qU \leq g_2 \cdot qU,
\]
since \(\eta_A \cdot (g_1 \ast q) U \leq \eta_A \cdot (g_2 \ast q) U\). Hence there is a unique \(u: X \to ZU\) such that
\[
\varphi \cdot u = qU.
\]
But then,
\[
u \cdot t = 1_X \quad \text{and} \quad t \cdot u = 1_{ZU},
\]
so that both \(t\) and \(u\) are isomorphisms; furthermore, for each \(i\),
\[
u \cdot d_iU = m_i,
\]
since
\[
\varphi \cdot u \cdot d_iU = qU \cdot d_iU = m_i \ast U = \varphi \cdot m_i,
\]
and \(\varphi\) is epi. The proof of the \(P\) limit cone lemma is complete.

In the following corollaries, we will assume hypotheses which include those of the lemmas above; we assume that:

\(C\) has all \(P\)-coequalizers and \(U: C \to \text{POS}\) is a \(P\)-faithful \(P\)-functor which preserves and reflects \(P\)-regular epis, and has a \(P\)-left adjoint.

**Corollary 1.** \(C\) is \(P\)-complete.

**Proof.** Let \(D\) be a \(P\)-diagram in \(C\) and let \((X, m_i)\) be a \(P\)-limit for \(DU\) in \(\text{POS}\) (recall that \(\text{POS}\) is \(P\)-complete). By the \(P\)-limit cone lemma, there is a \(D\)-cone \((Z, d_i)\) in \(C\) such that \((Z, d_i)U\) is isomorphic to \((X, m_i)\). Now if \((Y, g_i)\) is any \(D\)-cone in \(C\), there is a unique morphism \(h\) in \(\text{POS}\), say \(h: YU \to ZU\), from \((Y, g_i)U\) to \((Z, d_i)U\). By the \(P\)-monocone lemma, \(h\) 'lifts' uniquely to a \(C\)-morphism \(h\) from \((Y, g_i)\) to \((Z, d_i)\), proving that \((Z, d_i)\) is the \(P\)-limit of \(D\) in \(C\).

The argument for Corollary 1 may be easily extended to prove:

**Corollary 2.** If \(U\) creates isomorphisms, \(U\) creates all \(P\)-limits.

**Corollary 3.** \(C\) has canonical orderings.
Proof. This statement follows immediately from Corollary 1, since a canonical ordering on the object $A$ in $C$ is a $P$-kernel of $1_A : A \to A$.

**Corollary 4.** Every morphism $f : A \to B$ in $C$ may be written as a composite

$$f = e \cdot m,$$

where $e : A \to E$ is $P$-regular and $m : E \to B$ is $P$-monic. If $e \cdot m = e' \cdot m'$ are two such factorizations, there is an isomorphism $c$ with $e \cdot c = e'$ and $c \cdot m' = m$.

**Proof.** Let $e : A \to E$ be the $P$-coequalizer of the $P$-kernel $(g, h)$ of $f$. Then there is a unique morphism $m$ such that $f = e \cdot m$. We show that $m$ is $P$-monic.

Indeed, $(g U, h U)$ is the $P$-kernel of $e U$, since $U$ preserves $P$-limits. But since $U$ also preserves $P$-regular epis, by the ‘$P$-ulation lemma’, $e U$ is the $P$-coequalizer of $(g U, h U)$ in $\text{POS}$. Thus, $m U$ is $P$-monic. Since $U$ is $P$-faithful, it follows that $m$ is also $P$-monic.

The essential uniqueness of this factorization is obvious.

**Corollary 5.** Every $P$-limit cone in $C$ is a $P$-monocone.

**Proof.** This statement follows, since $P$-limit cones in $\text{POS}$ are $P$-monocones and $U$ preserves $P$-limit cones and is $P$-faithful.

**Corollary 6.** $C$ is $P$-regular co-well powered.

**Proof.** Let $(f_i : A \to X_i, i \in I)$ be a class of $P$-regular epis in $C$. Then there is a small subset $J$ of $I$ such that for each $i$ in $I$ there is some $j$ in $J$ with

$$f_i U \text{ isomorphic to } f_j U,$$

because $\text{POS}$ is $P$-regular co-well powered. Since each morphism $f_i U$ is a surjection, the isomorphism lifts to $C$, by the ‘lifting lemma’. Thus $f_i$ is isomorphic to $f_j$.

**Corollary 7.** $C$ has all coequalizers (note: not just $P$-coequalizers).

**Proof.** This will follow from Lemma 3, Section 6.

5. **Statement of the Main Theorem**

Suppose that $U : C \to \text{POS}$ and $V : D \to \text{POS}$ are $P$-functors. We will say that $(C, U)$ is isomorphic to $(D, V)$ if there is a $P$-isomorphism $K : C \to D$ such that

$$KV = U.$$
(K is a P-isomorphism if K is a P-functor and there is a P-functor $K': D \to C$ with $KK' = 1$ and $K'K = 1$.)

The main theorem has two parts.

**Main theorem.** *Let $U : C \to POS$ be a P-functor.*

(a) *There is some signature $\Sigma$ and a concrete P-quasi variety $D$ of $\Sigma$-algebras such that $(C, U)$ is isomorphic to $(D, V)$, where $V$ is the underlying poset functor, iff*

(i) $C$ has all P-coequalizers;
(ii) $U$ is P-faithful;
(iii) $U$ has a P-left adjoint;
(iv) $U$ preserves and reflects P-regular epis;
(v) $U$ creates isomorphisms.

(b) *There is some signature $\Sigma$ and a concrete P-variety $D$ of $\Sigma$-algebras such that $(C, U)$ is isomorphic to $(D, V)$, where $V$ is as above, iff conditions (i)-(v) above hold, as well as*

(vi) $U$ reflects P-congruences.

**Remark.** By removing the prefix 'P-' from the above theorem, one obtains a version of Lawvere’s characterization of varieties and quasi-varieties of (standard unordered) universal algebras (see [8], also [3] and the references there). We were surprised that such a similar looking theorem could be obtained in this setting.

We will explain somewhat more fully the meaning of the conditions occurring in the statement of the theorem.

Condition (i) means that any parallel pair of morphisms $f, g : A \to B$ in $C$ has a P-coequalizer.

Conditions (ii) and (iii) need no further comment.

Condition (iv), that $U$ preserves and reflects P-regular epis, means that for any morphism $f : A \to B$ in $C$, $fU$ is a surjection in $POS$ iff $f$ is the P-coequalizer of some parallel pair.

Condition (v), that $U$ creates isomorphisms, means the following: whenever $g : AU \to X$ is an isomorphism in $POS$ whose source is the $U$ image of some $C$-object $A$, then there is a unique morphism $f : A \to B$ in $C$ such that $fU = g$; moreover, $f$ is an isomorphism.

Condition (vi), that $U$ reflects P-congruences, means the following: whenever $f, g : A \to B$ is a parallel pair in $C$ such that $fU, gU$ is a P-congruence (i.e. a preorder) in $POS$, then $f, g$ is also a P-congruence in $C$.

It is useful to point out that for any signature $\Sigma$, the underlying poset functor $U_\Sigma : \Sigma \text{ALG} \to POS$ has the following properties:
(i) \( U_\Sigma \) preserves and reflects \( P \)-regular epis;
(ii) \( U_\Sigma \) reflects \( P \) congruences;
(iii) \( U_\Sigma \) creates isomorphisms.

In the next section we will show that the conditions given in the Main Theorem are necessary. After proving some preliminary results, we then prove the more difficult half: that these conditions are sufficient.

6. Proof of necessity

In this section we will sketch the proof that the conditions listed in the two parts of the main theorem are necessary.

Assume now that \( C \) is a concrete \( P \)-quasi variety of \( \Sigma \) algebras, for some signature \( \Sigma \).

**Lemma 1.** Each morphism \( f: A \to B \) in \( C \) may be written as a composite
\[
f = e \circ m
\]
where \( e: A \to D \) is a surjective homomorphism and \( m: D \to B \) is a \( P \)-monic homomorphism.

**Proof.** Let \( D \) be the set of elements in \( B \) of the form \( xf \), for \( x \) in \( A \). Then \( D \) is a subalgebra of \( B \) and if the order on \( D \) is the restriction of the order on \( B \), then the inclusion \( m: D \to B \) is a \( P \)-monic. Since \( C \) is closed under \( P \)-monics (= strong subalgebras), \( D \) is an algebra in \( C \). The proof is complete.

**Corollary 2.** If \( f: A \to B \) is \( P \)-regular, then \( f \) is a surjection; thus \( fU \) is also \( P \)-regular.

**Proof.** Suppose that \( f \) is the \( P \)-coequalizer of \((g, h)\). Using the lemma, write \( f \) as \( e \circ m \), with \( e \) a surjective homomorphism. Since \( m \) is \( P \)-monic,
\[
g \circ e \leq h \circ e.
\]
Thus there is a unique morphism \( k \) with
\[
f \circ k = e,
\]
since \( f \) is the \( P \)-coequalizer of \((g, h)\). But then it follows that \( k \circ m = 1 \) and since \( m \) is monic, \( m \) is an isomorphism, proving the Corollary.

We now prove the converse of the Corollary. Suppose that \( f: A \to B \) is a surjective homomorphism in \( C \). Let \( K \) be the set of all ordered pairs \((a, b)\) in \( A \) such that
\[
af \leq bf.
\]
\( K \) is a \( P \)-monic subalgebra of the product \( A \times A \) and is thus in \( C \). Let \( g, h: K \to A \) be the first and second projection maps.
Claim. \( f \) is the \( P \)-coequalizer of \((g,h)\).

The easy proof of the claim is omitted.

The above argument has established the fact that the functor \( U : C \rightarrow \text{POS} \) preserves and reflects \( P \)-regular epis.

We now sketch a proof of the fact that \( C \) has all \( P \)-coequalizers. Since this same argument will be used again, we state it as a Lemma.

**Lemma 3.** Suppose that \( C \) is a \( P \)-category with the following properties: there is a class \( E \) of epis (not necessarily \( P \)-epis) and a class \( M \) of \( P \)-monics such that

1. every morphism \( f \) factors as a composite \( e \ast m \), where \( e \) is in \( E \) and \( m \) is in \( M \);
2. \( C \) is \( E \) co-well powered;
3. \( C \) has all products, and all product cones are \( P \)-monocones. Then \( C \) has all \( P \)-coequalizers.

**Proof.** Let \((f,g) : A \rightarrow B\) be a pair of parallel morphisms in \( C \). Let \( e_i : B \rightarrow D_i \) be a representative set of \( E \) epis in \( C \) such that

\[ f \ast e_i \leq g \ast e_i . \]

Let \( D \) be the product \( \prod D_i \) and let \( e : B \rightarrow D \) be the target tupling of the \( e_i \). Then \( f \ast e \leq g \ast e \), since product cones are \( P \)-monocones. Now write \( e \) as \( h \ast m \), where \( h \) is in \( E \) and \( m \) is in \( M \). It is easy to show that \( h \) is the \( P \)-coequalizer of \((f,g)\).

Clearly, any concrete \( P \)-quasi variety satisfies the hypotheses of the lemma, where \( E \) is the class of surjective homomorphisms and \( M \) is the class of all \( P \)-monics.

**Theorem 4.** If \( C \) is a concrete \( P \)-quasi variety and \( U : C \rightarrow \text{POS} \) is the underlying poset functor, then:

1. \( C \) has all \( P \)-coequalizers;
2. \( U \) is \( P \)-faithful;
3. \( U \) has a \( P \)-left adjoint;
4. \( U \) preserves and reflects \( P \)-regular epis;
5. \( U \) creates isomorphisms.

**Proof.** Condition (iii) holds by definition. Conditions (ii) and (v) are obvious and the other conditions have already been proved above. The proof is complete.

We turn now to concrete \( P \)-varieties. Suppose that \((f,g) : A \rightarrow B\) is an ordered pair of morphisms in a concrete \( P \)-variety \( C \) of \( \Sigma \) algebras. Suppose also that the pair \( fU, gU \) is the \( P \)-kernel in \( \text{POS} \) of \( h : BU \rightarrow X \), where we may assume that \( h \) is surjective. In order to show that \( U \) reflects \( P \)-congruences, it is enough to show that we may impose a \( \Sigma \)-algebra structure on \( X \) in such a way that \( h \) becomes a homo-
morphism. Since $C$ is closed under surjective homomorphisms, $(f, g)$ will be the $P$-kernel of $h$. There is only one possibility for defining this structure. For $\sigma$ in $\Sigma_n$, define

$$\sigma_X : X^n \to X$$

as follows: for $x : n \to X$, find $b : n \to B$ such that $bh = x$ (this is possible, since $h$ is surjective). Then we define $\sigma_X$ by

$$x\sigma_X = b\sigma_B h.$$ 

The fact that $\sigma_X$ is well defined and order preserving follows since $(f, g)$ is the $P$-kernel of $h$. Indeed, suppose that for $b, b' : n \to B$, we have $bh \leq b'h$. Then we can find an $n$-tuple $a : n \to A$ in $A^n$ such that

$$(b, b') = (af, ag),$$

since $(f, g)$ is the $P$-kernel of $h$. Thus

$$(b\sigma_B, b'\sigma_B) = (a\sigma_A f, a\sigma_A g),$$

since $f$ and $g$ are $\Sigma$-homomorphisms. But then $b\sigma_B h \leq b'\sigma_B h$, showing $\sigma_X$ is well defined and order preserving. The remaining details are omitted. We have completed the proof of:

**Theorem 5.** Let $C$ be a concrete $P$-variety of $\Sigma$ algebras and let $U : C \to \text{POS}$ be the underlying poset functor. Then conditions (i)-(v) of the previous theorem hold as well as the following:

(vi) $U$ reflects $P$-congruences.

The proof of the necessity of the conditions in the Main Theorem is complete.

7. The sufficiency proof

In this section we will complete the proof of the Main Theorem by showing that the conditions listed there are sufficient. Anticipating this result, we introduce the following terminology.

**Definition.** An (abstract) $P$-quasi variety $(C, U)$ consists of a $P$-category $C$ having all $P$-coequalizers and a $P$-functor $U : C \to \text{POS}$ with the following properties:

(i) $U$ is $P$-faithful;

(ii) $U$ has a $P$-left adjoint (always denoted $F$);

(iii) $U$ preserves and reflects $P$-regular epis;

(iv) $U$ creates isomorphisms.

An (abstract) $P$-variety is an (abstract) $P$-quasi variety $(C, U)$ such that $U$ reflects $P$-congruences.
From now on we drop the prefix 'abstract' – which was used only to emphasize the difference between the concrete P-quasi varieties discussed earlier and the current notion.

We call a P-quasi variety \((C, U)\) a monadic P-quasi variety if the functor \(U\) is monadic. The following proposition is one of the major steps in the sufficiency proof.

**Proposition 1.** Let \((C, U)\) be a monadic P-quasi variety. Then we can find a signature \(\Sigma\) and a concrete P-quasi variety \(C'\) of \(\Sigma\)-algebras, such that
\[
(C, U) \text{ is isomorphic to } (C', U')
\]
where \(U': C' \to \text{POS}\) is the underlying poset functor. More precisely, there is an embedding
\[
I: (C, U) \to (\Sigma\text{ALG}, U_\Sigma)
\]
such that if \(C'\) is the image of \(I\), \((C', U')\) is a concrete P-quasi variety, where
\[
U' = C' \xrightarrow{U_\Sigma} \Sigma\text{ALG} \xrightarrow{U_\Sigma} \text{POS}.
\]

**Proof.** Since \(U\) is monadic, \((C, U)\) is isomorphic to \((C', U')\), where \((T = FU, \eta, \mu)\) is the corresponding monad and
\[
C' = \text{POS}^T \text{ and } U' = U^T.
\]
We henceforth assume that \((C, U) = (\text{POS}^T, U^T)\), and, after defining the signature \(\Sigma\) we show how to define \(I: \text{POS}^T \to \Sigma\text{ALG}\).

For each cardinal \(n\), define
\[
\Sigma_n = nT,
\]
where \(n\) is being considered as a discrete poset. (Here is precisely the place that it is convenient to allow poset valued signatures.) Now we put a \(\Sigma\) algebra structure on each \(T\)-algebra \((A, s)\), where \(s: A^T \to A\), in the same way as was done in [11], namely:

For \(\sigma\) in \(nT\), define \(\sigma_A: A^n \to A\) as follows. An element of \(A^n\) is a morphism \(x: n \to A\) in \(\text{POS}\) (since \(n\) is discretely ordered). Thus, for each such \(x\), we obtain a morphism
\[
xT: nT \to AT.
\]
Now we define \(x\sigma_A\) by
\[
x\sigma_A = (\sigma, xT\cdot s).
\]
the value of \(xT\cdot s\) on \(\sigma\).

Note that for each \(\sigma\) in \(\Sigma_n\) the function \(\sigma_A\) is order preserving because \(T\) is a P-functor and \(s\) is order preserving. (Here is the place we need the left adjoint of \(U\) to be a P-functor.)
Now \( I \) is defined as follows: for each \( T \)-algebra \((A, s)\), \((A, s)I\) is \( A \) equipped with the \( \Sigma \)-structure defined above. On morphisms, \( fI = f \). (Clearly, \( I \) is faithful and injective on objects.) In order to show \( I \) is well defined on morphisms and full, we must prove that any mapping between the underlying posets of two \( T \)-algebra homomorphism iff it is a \( \Sigma \)-algebra homomorphism. From this it will follow that

\[
\text{Im } I \text{ is a full subcategory of the category of all } \Sigma \text{-algebras.} \tag{2}
\]

Assume that \( f: (A, s) \to (B, t) \) is a \( T \)-algebra homomorphism. If \( \sigma \) is in \( \Sigma_n \) and \( x:n \to A \), then we must show that

\[
xsAf = xfB.
\]

But \( xsAf = \langle \sigma, xT \cdot s \rangle f = \langle \sigma, xT \cdot s \cdot f \rangle \), and

\[
s \cdot f = fT \cdot t,
\]

since \( f \) is a \( T \)-homomorphism. Thus,

\[
xsAf = \langle \sigma, xT \cdot fT \cdot t \rangle = \langle \sigma, (x \cdot f)T \rangle t = xfB.
\]

Conversely, assume that \((A, s)\) and \((B, t)\) are \( T \)-algebras and

\[
f: (A, s)I \to (B, t)I
\]

is a \( \Sigma \) algebra homomorphism. We must show that

\[
s \cdot f = fT \cdot t. \tag{3}
\]

**Lemma 1.** \( T: \text{POS} \to \text{POS} \) preserves surjections.

**Proof.** If \( q \) is a surjection in \( \text{POS} \), \( q \) is a \( P \)-coequalizer, say of \( u, v \). But then \( qF \) is the \( P \)-coequalizer of \( uF, vF \) in \( C \). Lastly, \( U \) preserves \( P \)-regular epis, so \( qFU = qT \) is \( P \)-regular in \( \text{POS} \); i.e. \( qT \) is a surjection.

We now prove (3). Let \( q:n \to A \) be a surjection, where \( n \) is a (discretely ordered) cardinal. If \( y \) is an element of \( AT \), there is some \( \sigma \) in \( nT \) such that

\[
y = \langle \sigma, qT \rangle,
\]

by Lemma 1. Since \( f \) is a \( \Sigma \) homomorphism,

\[
q \sigma_A f = (qf) \sigma_B.
\]

But

\[
q \sigma_A = \langle \sigma, qT \cdot s \rangle = ys,
\]

and

\[
(qf) \sigma_B - \langle \sigma, qT \cdot fT \rangle \cdot t - y(fT \cdot t).
\]

Hence

\[
y(s \cdot f) = y(fT \cdot t).
Since \( y \) was arbitrary, (3) and, thus (2) are proved.

Since by definition, \( U \) has a \( P \)-left adjoint, it remains to show that

(a) \( \text{Im} I \) is closed in the class of all \( \Sigma \) algebras under products, and
(b) \( \text{Im} I \) is closed under \( P \)-monics.

Statement (a) follows easily from [11], Theorem 3.1.19. We only prove (b).

Assume that \( A \) is a \( \Sigma \) algebra, that \( (B, t) \) is a \( T \)-algebra and that

\[
m : A \to (B, t)I
\]

is a \( P \)-monic \( \Sigma \)-homomorphism. We will show that there is a morphism \( s : AT \to A \) in \( \text{POS} \) such that

\[
s \cdot m = mT \cdot t.
\]

It will then follow from Lemma 8, Section 3 that \( (A, s) \) is a \( T \)-algebra, and hence \( A = (A, s)I \).

Let \( q : n \to A \) be a surjection in \( \text{POS} \), where \( n \) is the cardinal of \( A \) (considered to be a discretely ordered poset). If \( y \) is some member of \( AT \), there is some \( \sigma \) in \( nT \) with

\[
y = \langle \sigma, qT \rangle.
\]

by Lemma 1. Now

\[
\langle y, mT \cdot t \rangle = \langle \sigma, (qm)T \rangle \cdot t = (qm)\sigma_B,
\]

by definition, and, since \( m \) is a \( \Sigma \) homomorphism,

\[
q\sigma_A M = (qm)\sigma_B.
\]

Thus, for each \( y \) in \( AT \) there is at least one element \( a \) in \( A \) such that

\[
\langle y, mT \cdot t \rangle = \langle a, m \rangle.
\]

There is also at most one such element \( a \) since \( m \) is \( P \)-monic (and hence monic). Thus we define \( s \) by

\[
y s = a \quad \text{if (4) holds.}
\]

This defines \( s \) as a function only. We must show that \( s \) is order preserving. But if \( y \leq y' \), then

\[
\langle y, mT \cdot t \rangle \leq \langle y', mT \cdot t \rangle,
\]

since \( mT \) and \( t \) are order preserving. Then, by the definition of \( s \),

\[
ysm \leq y'sm.
\]

Since \( m \) is \( P \)-monic, \( ys \leq y's \), completing the proof.

Finally we must show that the \( \Sigma \) structure on \( A \) is that determined by the \( T \)-algebra \( (A, s) \), i.e. that for any \( \sigma \) in \( \Sigma_k \) and any \( x : k \to A \)

\[
x\sigma_A = \langle \sigma, xT \rangle s.
\]
But $x\sigma_A m = x m \sigma_B$, since $m$ is a $\Sigma$ homomorphism; thus
\[
\begin{align*}
x \sigma_A m &= (\sigma, (x m) T) \cdot t \\
&= (\sigma, x T) m T \cdot t = (\sigma, x T) s \cdot m,
\end{align*}
\]
since by construction, $s \cdot m = m T \cdot t$. Now (5) follows, since $m$ is monic. The proof of Proposition 1 is complete.

Remark. Our argument for Proposition 1 is a modification of that used by Manes [11] to prove a representation theorem for finitary monadic set valued functors. The argument here is complicated by the ordering; Lemma 1 takes the place of Manes' assumption that $T$ is finitary.

Corollary 1. Suppose that $(C, U)$ is a monadic $P$-variety. Then if $\Sigma$ the signature and $I : (C, U) \to (\Sigma \text{ALG}, U_{\Sigma})$ is the embedding of Proposition 1, $\text{Im } I$ is closed in $\Sigma \text{ALG}$ under surjective homomorphisms; i.e. $(\text{Im } I, U')$ is a concrete $P$-variety.

Proof. Let $\Sigma$ and $T = FU$ be as in Proposition 1. We may assume that $C = \text{POS}^T$ and $U = U^T$.

Let $(A, s)$ be a $T$-algebra (with the $\Sigma$ structure $(A, s)_I$ given in Lemma 1) and let $h : (A, s)_I \to B$ be a surjective $\Sigma$ homomorphism. We must show $B = (B, t)_I$, for some $T$-algebra structure $t : BT \to B$ on $B$. Let $(f, g) : D \to A$ be the $P$-kernel of $h$ in $\Sigma \text{ALG}$. Since $U_{\Sigma}$ preserves $P$-limits and $U$ creates $P$-limits by Proposition 6, Section 3, $D = (D, d)_U$, for some (unique) $T$-algebra structure $d$, and both $f$ and $g$ are $T$-algebra homomorphisms. Since $U$ reflects $P$-congruences because $(C, U)$ is a $P$-variety, there is a $T$-algebra homomorphism $h' : (A, s)_I \to (B', t')$ with $P$-kernel $(f, g)$. By Proposition 7, Section 3, we may assume that $h'$ is surjective. Thus both $h$ and $h'$ are $P$-coequalizers of $(f, g)$ in $\text{POS}$, by the $P$-ulation lemma, and hence $h$ is isomorphic to $h'$. Since $U$ creates isomorphisms, the proof is complete.

There are many examples of non-monadic $P$-quasi varieties (see Section 8). However, we can reduce the case that $(C, U)$ is an arbitrary $P$-quasi variety to the case that $U$ is monadic. But before doing so, we will prove the following useful fact.

Proposition 2. Suppose that $U$ is a $P$-functor and that
\[(U : C \to \text{POS}, F, \eta, \varepsilon)\]
is a $P$-adjoint situation. Let
\[(T = FU, \eta, \mu)\]
be the monad induced by this adjunction. Then $T$ preserves surjections iff $(\text{POS}^T, U^T)$ is a $P$-quasi variety.

Proof. First suppose that $(\text{POS}^T, U^T)$ is a $P$-quasi variety. If $f$ is a surjection,
$fF^T U^T$ is a surjection in POS, by Lemma 1. But $fF^T U^T = fT$. Thus $T$ preserves surjections.

Now assume that $T$ preserves surjections. First we show that POS has all P-coequalizers. From Proposition 7, Section 3 it follows that every morphism in POS has an 'E–M factorization', where $E$ is the class of surjections and $M$ is the class of all P-mronics. Thus, by Lemma 1, Section 6, POS has all P-coequalizers.

Since $U^T$ has a P-left adjoint by assumption, and since $U^T$ creates all limits, it remains to prove that $U^T$ preserves and reflects P-regular epis.

Again, it follows by Proposition 7, Section 3, that every P-regular epi in POS is necessarily a surjection. Hence $U^T$ preserves P-regular epis.

Now assume that $f$ is a surjective $T$-homomorphism. Let $(u, v)$ be the P-kernel of $f$ in POS. Since $U^T$ creates P-limits, $u$ and $v$ are $T$-homomorphisms. Let $g$ be a P-coequalizer of $(u, v)$ in POS. Since $U^T$ preserves P-regulars, $g$ is a P-coequalizer of $(u, v)$ in POS and is thus isomorphic to $f$, proving that $f$ is a P-coequalizer of $(u, v)$ in POS.

The proof of the proposition is complete.

Let $(C, U)$ be a fixed P-quasi variety, where as always, $F : POS \to C$ denotes the P-left adjoint to $U$. Let $K : C \to POS$ denote the ‘comparison functor’ (see [10], p. 138). Thus the following diagram commutes:

\[
\begin{array}{ccc}
C & \xrightarrow{K} & POS^T \\
\downarrow{U} & & \downarrow{U^T} \\
POS & & \\
\end{array}
\]

**Lemma 3.** (a) $K$ is a full, P-faithful P-functor which is injective on objects;
(b) $K$ has a left adjoint and every component of the unit is P-regular;
(c) the image of $K$ is closed in POS under products and P-mronics.

**Proof.** (a) The fact that $K$ is a P-faithful P-functor is due to the fact that both $U$ and $U^T$ are. We show $K$ is full. Recall that if $f : A \to B$ is a morphism in $C$, then

\[ fK = fU : (AU, \varepsilon_A U) \to (BU, \varepsilon_B U). \]

Assume that $g : (AU, \varepsilon_A U) \to (BU, \varepsilon_B U)$ is a $T$-homomorphism. Since $U$ reflects P-regular epis, $\varepsilon_A$ is the P-coequalizer of some pair, $(u, v)$, say. Then:

\[ u^* gF^* \varepsilon_B \leq v^* gF^* \varepsilon_B, \]

since $U$ is P-faithful and since $g$ is a $T$-homomorphism. But then there is a unique morphism $g'$ such that

\[ \varepsilon_A g' = gF^* \varepsilon_B. \]
Applying $U$ and the fact that $\varepsilon_A U$ is epi, we obtain
\[ g = g'U, \]
proving that $K$ is full.

In order to show $K$ is injective on objects, assume that $AK = BK$. Then
\[ \varepsilon_A U \cdot 1 = \varepsilon_B U. \]
By the lifting lemma in Section 4, there is a unique $k$ with $\varepsilon_A \cdot k = \varepsilon_B$ and $kU = 1$. But since $U$ creates isomorphisms, $k = 1_A$. Thus $A = B$. The proof of (a) is complete and thus $C$ is isomorphic to a full subcategory of $\text{POS}^T$.

(b) The image of $K$ is closed under products since $U$ preserves limits and $U^T$ creates limits. To prove that the image of $K$ is closed under $P$-monic, assume that $(A, s)$ is a $T$-algebra, that $m$ is a $P$-monic and the following diagram commutes:

\[
\begin{array}{ccc}
AT & \xrightarrow{mT} & BUT \\
\downarrow & & \downarrow \\
A & \xrightarrow{m} & BU
\end{array}
\]

Suppose that $(u, v)$ is the $P$-kernel of $mF \cdot \varepsilon_B$ in $C$ (which exists by Corollary 1, Section 4). Let $h$ be the $P$-coequalizer of $(u, v)$ in $C$. Then it is easy to show that
\[ (u, v) \text{ is the } P\text{-kernel of } mT \cdot \varepsilon_B U \]
and
\[ hU \text{ is the } P\text{-coequalizer of } (u, v)U. \]

But the $P$-kernel of $mT \cdot \varepsilon_B U$ is also the $P$-kernel of $s$, since $m$ is $P$-monic. Since $s$ is $P$-regular (indeed, being a $T$-algebra, $s$ is a split epi), $s$ is a $P$-coequalizer of $(u, v)U$ so that $hU$ is isomorphic to $s$. Since $U$ creates isomorphisms, $(A, s)$ is in the image of $K$, by the lifting lemma, as before.

(c) $K$ has a left adjoint, say $G$, by the sandwich theorem ([11] p. 182). If
\[ \eta_X^G : X \rightarrow XGK \]
is a component of the unit of this adjunction, $\eta_X^G$ factors as $e \cdot m$, where $e$ is $P$-regular and $m$ is $P$-monic, by Proposition 7, Section 3. By part (b), the source of $m$ is in the image of $K$, and hence $m$ is an isomorphism. The proof is complete.

We may now complete the proof of the Main Theorem for $P$-quasi varieties. If $(C, U)$ is an arbitrary $P$-quasi variety, then, by Lemma 1 and Proposition 2, $(\text{POS}^T, U^T)$ is a monadic $P$-quasi variety, (where as usual, $T = FU$) and hence is isomorphic to a concrete $P$-quasi variety of $\Sigma$ algebras, by Proposition 1. By Lemma 3, $C$ is isomorphic to a full subcategory of $\text{POS}^T$ closed (in $\text{POS}^T$) with respect to products and $P$-monics. It is easy to see that this means that $C$ is closed in $\Sigma\text{ALG}$.
with respect to products and $P$-monics. Since, by assumption, $U: C \to \text{POS}$ has a $P$-left adjoint, this shows that $(C, U)$ is also isomorphic to a concrete $P$-quasi variety. This part of the proof of the Main Theorem is complete.

It remains to consider $P$-varieties. By Corollary 1, the representation theorem for $P$-varieties will be proved once we prove that if $(C, U)$ is a $P$-variety, then $U$ is monadic. This is proved in the next lemma, one of the main results.

**Lemma 4.** If $(C, U)$ is a $P$-variety, $U$ is monadic.

**Proof.** By the Beck theorem ([10], p. 147), we must show that $U$ creates coequalizers of those pairs

$$f, g : X \to A$$

in $C$ such that

$$fU, gU : XU \to AU$$

has a split coequalizer. Thus we suppose that the following diagram in $\text{POS}$ commutes:

$$
\begin{array}{ccc}
A' & \xrightarrow{d'} & XU & \xrightarrow{gU} & AU \\
\downarrow{h} & & \downarrow{j} & & \downarrow{h} \\
B' & \xrightarrow{d} & AU & \xrightarrow{h} & B \\
\downarrow{1} & & \downarrow{1} & & \downarrow{1}
\end{array}
$$

(6)

**Sublemma.** There is an ordered pair $(u, v) : E \to A$ in $C$ such that $(u, v)U = (uU, vU)$ is a $P$-kernel of $h$.

Before proving it, we will show how the sublemma will prove Lemma 4. Let $k$ be the $P$-coequalizer of $(u, v)$ in $C$. Then, since $U$ reflects $P$-kernel pairs, $(u, v)$ is a $P$-congruence, and thus the $P$-kernel of $k$, by the pulication lemma. Also, $(u, v)U$ is the $P$-kernel of $kU$, and since $U$ preserves $P$-regular epis, $kU$ is a $P$-coequalizer of $(u, v)U$. But, by assumption, $(u, v)U$ is the $P$-kernel of $h$. Since $h$ is a split epi, $h$ is also $P$-regular and is thus a $P$-coequalizer of $(u, v)U$. Since $U$ creates isomorphisms, we may assume that

$$kU = h. \tag{7}$$

Now we show that $k$ is a coequalizer of $f, g$. Indeed, by (6)

$$fU \cdot kU = fU \cdot h = gU \cdot h = gU \cdot kU,$$
and since \( U \) is faithful, \( f \cdot k = g \cdot k \). Now suppose that \( k' \) is any \( C \)-morphism such that \( f \cdot k' = g \cdot k' \). Since \( kU = h \) is the coequalizer of \( fU, gU \) in \( \text{POS} \), there is a unique morphism \( t \) such that

\[
 kU \cdot t = k'U. 
\]

But since \( k \) is \( P \)-regular, it follows from the lifting lemma that there is a unique \( t' \) in \( C \) with \( t'U = t \) and \( k \cdot t' = k' \). Thus \( k \) is a coequalizer of \( f, g \) in \( C \), proving Lemma 4.

**Proof of the sublemma.** Let \( (a_1, a_2): Y \to X \) be the \( P \)-kernel of \( f \) in \( C \) (which exists by Corollary 1, Section 4).

**Claim.** \( ((a_1 \cdot g)U, (a_2 \cdot g)U) \) is a ‘weak’ \( P \)-kernel pair of \( h \).

Indeed,

\[
 (a_1 \cdot g)U \cdot h = a_1U \cdot gU \cdot h = a_1U \cdot fU \cdot h \\
 \leq a_2U \cdot fU \cdot h = (a_2 \cdot g)U \cdot h.
\]

Now suppose that \( w_1, w_2 \) is any parallel pair such that \( w_1 \cdot h \leq w_2 \cdot h \). Then, by (6),

\[
 w_1 \cdot d' \cdot fU = w_1 \cdot h \cdot d \\
 \leq w_2 \cdot h \cdot d = w_2 \cdot d' \cdot fU.
\]

Since \( (a_1, a_2)U \) is the \( P \)-kernel of \( fU \), there is a unique morphism \( c \) such that

\[
 w_i \cdot d' = c \cdot (a_i, U), \quad i = 1, 2.
\]

Then

\[
 w_1 = w_1 \cdot d' \cdot gU = c \cdot a_1U \cdot gU = c \cdot (a_1 \cdot g)U, \\
 w_2 = w_2 \cdot d' \cdot gU = c \cdot a_2U \cdot gU = c \cdot (a_2 \cdot g)U.
\]

Since we cannot show \( c \) is necessarily unique, the pair \( (a_i, g)U, i = 1, 2 \), is only a weak \( P \)-kernel of \( h \). The claim is proved.

Let \( p = (p_1, p_2) \) be a \( P \)-kernel of \( h \) in \( \text{POS} \). Then there are morphisms \( e, e' \) in \( \text{POS} \) such that for each \( i = 1, 2 \):

\[
 (a_i \cdot g)U = e \cdot p, \quad p_i = e' \cdot (a_i, g)U.
\]

Hence,

\[
 e' \cdot e = 1. \tag{9}
\]

since \( p \) is a \( P \)-kernel; and

\[
 [((a_1 \cdot g)U, (a_2 \cdot g)U] = e \cdot [p_1, p_2], \tag{10}
\]

where \([ , ]\) denotes target tupling.

Let

\[
 \bar{f} = a_1 \cdot g, \quad \bar{g} = a_2 \cdot g,
\]

Rewriting (10), we have

\[
 [\bar{f}, \bar{g}]U = e \cdot [p_1, p_2]. \tag{11}
\]
Let \((y_1, y_2)\) be a P-kernel in \(C\) of the target tupling

\[
[f, g]: Y \to A \times A,
\]

and let \(q\) be the P-coequalizer of \((y_1, y_2)\). Then there is a mediating morphism \(m = [u, v]\) such that

\[
q \cdot [u, v] = [f, g].
\]

We know that \((y_1, y_2)U\) is the P-kernel of \(qU\) and that \(qU\) is the P-coequalizer of \((y_1, y_2)U\), since \(U\) preserves P-regular epis. Now we show that

\[
(y_1, y_2)U \text{ is the P-kernel of } e.
\]

Indeed, by (11),

\[
y_1U \cdot e \leq y_2U \cdot e
\]

since \([p_1, p_2]\) is P-monic. Further, if \(b_1, b_2\) is any parallel pair such that \(b_1 \cdot e \leq b_2 \cdot e\), there is a unique morphism \(c\) with

\[
b_i = c \cdot y_iU, \quad i = 1, 2.
\]

since \((y_1, y_2)U\) is the P-kernel of \([fU, gU]\). The claim is proved.

But \(e\), being a split epi by (9), is also P-regular and hence \(e\) is also a P-coequalizer of \((y_1, y_2)U\). Thus \(e\) is isomorphic to \(qU\). Since \(U\) creates isomorphisms and since

\[
qU \cdot [u, v]U = e \cdot [p_1, p_2]
\]

we may assume that \(qU = e\) and thus

\[
[u, v]U = [p_1, p_2],
\]

completing the proof of the sublemma.

**Corollary 2.** If \((C, U)\) is a P-variety, \((C, U)\) is isomorphic to a concrete P-variety.

**Proof.** By Lemma 4, \(U\) is monadic, and by Corollary 1 therefore, \((C, U)\) is isomorphic to the concrete P-variety \((\text{Im} \: I, U')\), where \(I\) is the embedding of Proposition 1.

The proof of the Main Theorem is complete.

**Remark.** In [9], Lehmann gave a definition of a ‘semi variety’ of ordered algebras in a category theoretic setting. Without going into the details of his definition, we should remark that his definition is a relative one - i.e. a category is a ‘semi-variety’ if it is a subcategory of the category of \('T\text{-alg}\) (not the same as our \(T\text{-alg}\)) closed under products and (the equivalent of) P-monic. If one assumes that the endofunctor \(T\) is a P-functor (in addition to the other assumptions) then Lehmann’s semi-varieties are P-quasi varieties and conversely, each P-quasi variety is a Lehmann semi-variety.
We now give an argument to show that the proof of our main theorem also proves the Lawvere theorem mentioned in the Introduction. Let \( I: \text{SET} \to \text{POS} \) be the inclusion functor. Regarding the category of sets as a discrete \( P \)-category, \( I \) is a \( P \)-functor with the following properties:

(a) \( I \) has a \( P \)-left adjoint;
(b) \( I \) preserves and reflects \( P \)-regular epis;
(c) \( I \) creates isomorphisms.

If \( U: C \to \text{SET} \) is any functor, call the pair \((C, U)\) an abstract quasi variety if (see the Introduction)

- \( U \) has a left adjoint;
- \( C \) has all coequalizers;
- \( U \) preserves and reflects regular epis;
- \( U \) creates isomorphisms.

An abstract variety is an abstract quasi variety \((C, U)\) such that \( U \) reflects congruences.

Using the properties of \( I \) mentioned above, it is straightforward to prove the following fact.

**Proposition 5.** Suppose that \( C \) is a category, regarded as a discrete \( P \)-category. Let \( U: C \to \text{SET} \) be a functor. Then the pair \((C, U)\) is an abstract quasi variety iff \((C, UI)\) is a \( P \)-quasi variety.

We now sketch a proof of the Lawvere theorem.

**Corollary 3.** Suppose that \((C, U)\) is an abstract quasi variety. Then there is a (discrete) signature \( \Sigma \) and a concrete quasi variety \( D \) of (discretely ordered) \( \Sigma \) algebras (with underlying set functor \( V \)) such that \((C, U)\) is isomorphic to \((D, U)\). Furthermore, if \( U \) reflects congruences, \( D \) is a concrete variety.

**Proof.** By Proposition 5, \((C, UI)\) is a \( P \)-quasi variety and hence there is a signature \( \Sigma \) and a concrete \( P \)-quasi variety \( D \) such that \((C, UI)\) is isomorphic to \((D, W)\), where \( W: D \to \text{POS} \) is the underlying poset functor. We need only show that for each \( n \), \( \Sigma_n \) is discretely ordered as is each algebra in \( D \). But \( \Sigma_n = nT = nFUI \), where \( F \) is the \( P \)-left adjoint to \( UI \). Hence the signature is discrete. Lastly, each object \( A \) in \( C \) corresponds to the \( T \)-algebra \((AUI, \varepsilon_A UI)\), and is thus also discretely ordered. The proof for quasi varieties is complete. The remaining argument is easy and is omitted.

If \( \Sigma \) is a discrete signature, let \( \Sigma_d \text{ALG} \) denote the category of all discretely ordered \( \Sigma \) algebras. A full subcategory \( C \) of \( \Sigma_d \text{ALG} \) is a concrete quasi variety (resp. variety) if the underlying set functor \( U: C \to \text{SET} \) has a left adjoint; if \( C \) is closed in \( \Sigma_d \text{ALG} \) under monics, and products (and surjective homomorphic images). Using Proposition 5 and Theorem 4, Section 6, one may easily prove the following converse to Corollary 3.
Corollary 4. If $C$ is a concrete quasi variety (or variety) of discretely ordered $\Sigma$ algebras, with underlying set functor $U$, then $(C, U)$ is an abstract quasi variety (abstract variety).

The two previous corollaries prove our version of the Lawvere theorem stated in the introduction.

It is not true that $(C, U)$ is an abstract variety iff $(C, U)$ is a P-variety. The pair $(\text{SET}, \text{Id})$, where $\text{Id}$ is the identity functor on SET, is an abstract variety, but $I$ does not reflect P-congruences.

We now state without proof a characterization of those discrete P-quasi varieties which are varieties.

Corollary 5. Let $(C, U)$ be an abstract quasi variety as in Corollary 3. Then $(C, U)$ is an abstract variety iff the functor $UI: C \rightarrow \text{POS}$ is monadic.

8. Examples and remarks

In this section, we will give some examples to illustrate some of the concepts introduced earlier.

The first example will show that a P-functor which has a left adjoint need not have a P-left adjoint, even if it preserves and reflects P-regular epis.

Example 1. Let $C$ be the ‘op’ of the category of sets (so that a morphism $f: X \rightarrow Y$ in $C$ is a function $Y \rightarrow X$ in SET). $C$ is a P-category where the ordering on each Hom-set is discrete: $f \leq g$ iff $f = g$. Let 2 denote the two element poset with elements 0, 1. We define the functor $U: C \rightarrow \text{POS}$ as follows:

For each set $X$, $XU = \text{POS}(X, 2)$, the poset which is the Hom-set in the category POS; if $f: X \rightarrow Y$ in $C$, then

$fU: \text{POS}(X, 2) \rightarrow \text{POS}(Y, 2)$

is the function defined by

$\langle u, fU \rangle = f \cdot u,$

for any order preserving $u: X \rightarrow 2$ in POS (of course, since $X$ is discretely ordered, every function $X \rightarrow 2$ is order preserving). Note that the composition $f \cdot u$ is in POS, since $f$ is a function with source $Y$ and target $X$.

Claim. (a) $U$ is a P-faithful P-functor;
(b) $U$ has a left adjoint which is not a P-functor;
(c) $U$ preserves and reflects P-regular epis.

Proof. (a) Suppose that $f, g: X \rightarrow Y$ in $C$ and $fU \leq gU$ in POS. Then, for every
u: X → 2, ⟨u, fu⟩ ≤ ⟨u, gu⟩; i.e. by definition, f•u ≤ g•u. We must show that f = g (since the ordering in C(X, Y) is discrete. But if y is an element of Y such that yf is not equal to yg, define u: X → 2 to be any function such that

⟨yf, u⟩ = 1 and ⟨yg, u⟩ = 0.

But then, for this u, it is not the case that f•u ≤ g•u.

U is necessarily a P-functor, since the orderings on the Hom-sets of C are discrete.

(b) Define the functor F: POS → C as follows. On the object A of POS,

AF is the underlying set of POS(A, 2),

the set of all order preserving maps from A to 2. For each object A in POS, we define the morphism

ηA: A → AFU = POS(AF, 2)

by

anηA is evaluation at a;

i.e. for each g: A → 2,

⟨g, anηA⟩ = ⟨a, g⟩.

Now let X be a (discretely ordered) set and let f: A → POS(X, 2) be a morphism in POS. Define the morphism f#: AF → X in C (i.e. a function X → AF) as follows: For each x in X, xf# is the function A → 2 defined by:

⟨x, xf#⟩ = ⟨x, af⟩.

It is straightforward to check that the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\eta_A} & \text{POS}(AF, 2) \\
\downarrow f & & \downarrow f^*U \\
& \text{POS}(X, 2) &
\end{array}
\]

commutes and that f# is the unique morphism with this property. Thus F is a left adjoint of U. In order to see that F is not a P-functor, one needs to find two morphisms f, g in POS with f ≤ g but not f# ≤ g#, by Proposition 1 in Section 3. Let A be the singleton poset and let 2d be the discretely ordered two-element poset. Define f and g: A → POS(2d, 2) by:

⟨x, af⟩ = 0 and ⟨x, ag⟩ = 1,

for each x in 2d. It is clear by the above definition of the mapping f → f#, that f# is not the same morphism as g#, so that it is not the case that f# ≤ g#, completing the proof.

(c) Since C is a discretely ordered P-category, a P-regular epi in C is just a co-
equalizer; i.e. a morphism \( e: X \rightarrow Y \) in \( C \) is P-regular iff \( e: Y \rightarrow X \) is a monic in \( \text{SET} \). Now the statement that \( U \) preserves and reflects: P-regular epis is equivalent to the following easily proved fact:

Let \( f: Y \rightarrow X \) be a mapping between discrete posets. Then, for all \( w: Y \rightarrow 2 \) in \( \text{POS} \) there is some \( u: X \rightarrow 2 \) in \( \text{POS} \) such that

\[
f \cdot u = w
\]

iff \( f \) is a monic in \( \text{SET} \).

The proof of the claim is complete.

**Remark 1.** Since the category \( C \) in Example 1 has all P-coequalizers, \( (C, U) \) fails to be a P-quasi variety because \( U \) does not create isomorphisms and the left adjoint is not a P-functor. Hence these two properties are not implied by the other properties that define P-quasi varieties.

**Problem 1.** Is the property "U is P-faithful" implied by the remaining properties defining P-quasi varieties? It is easy to show that in the presence of the other axioms, \( U \) is P-faithful iff every P-regular epi in \( C \) is P-epic iff every object in \( C \) has a 'canonical ordering' (see Section 3).

The next two examples show that the concepts of epi, P-epi and P-regular epi do not coincide in P-quasi varieties.

**Example 2.** A P-variety containing an epi that is not P-epi.

Let \( C \) be the P-variety of ordered monoids (an 'ordered monoid' is a monoid whose underlying set is partially ordered and the monoid operation preserves the ordering). Let \( N \) and \( Z \) denote the additive monoids of the nonnegative integers and all integers, respectively, ordered as usual (i.e. \( n < n+1 \)). Let \( m: N \rightarrow Z \) be the inclusion map. It is well known that \( m \) is an epi. We show that \( m \) is not a P-epi. Indeed, if \( f: Z \rightarrow Z \) is the identity map and \( g: Z \rightarrow Z \) is 'multiplication by 2' (i.e. \( xg = 2x \), all \( x \in Z \) ) then clearly

\[
m \cdot f \leq m \cdot g.
\]

But it is not the case that \( f \leq g \), since e.g. \(-2 = (-1, g) < (-1, f) = -1 \).

**Example 3.** A P-quasi variety containing a P-epi that is not P-regular.

Let \( C \) be the P-quasi variety of discretely ordered monoids. Let \( N \) and \( Z \) be the same monoids as before only this time suppose that they are discretely ordered. Then the inclusion \( m: N \rightarrow Z \) is P-epi. But \( m \) is not P-regular since \( m \) is not surjective.

**Remark 2.** Let the category of sets, \( \text{SET} \), be identified with the P-category of discretely ordered posets. Then the inclusion functor

\[
f: \text{SET} \rightarrow \text{POS}
\]
is P-quasi varietal. Indeed, \( \text{SET} \) is axiomatizable by the implication

\[
x \leq y = y \leq x.
\]

Furthermore, if \((C, U)\) is any set quasi-variety (i.e. \((C, U)\) satisfies all the properties of a P-quasi variety with the prefix ‘P’ removed) then \((C, V)\) is a P-quasi variety, where \(V = UI\) and where \(C\) is considered to be a discretely ordered P-category.

Lastly, it can be shown that \(U: C \to \text{SET}\) is monadic iff \(UI: C \to \text{POS}\) is monadic. Thus, to obtain an example of a P-quasi variety \((C, V)\) which is not monadic, start with a non-monadic quasi variety over sets (for example torsion free abelian groups) and impose the condition that the order is discrete. So, for one example, discretely ordered torsion free abelian groups (with the underlying poset functor) form a non-monadic P-quasi variety.

**Example 4.** A P-variety \((C, U)\) such that \((C, V)\) is not a \(\text{SET}\) quasi variety (i.e. not definable by implications of the form “\(E = s = t\)”, where \(E\) is a set of equations), where \(V\) is the composition \(U \cdot S\) (\(S: \text{POS} \to \text{SET}\) is the underlying set functor).

Let \(C\) be the category \(\text{POS}\) itself and let \(U\) be the identity functor. Then \(V\) is the functor \(S\) which does not reflect regular epis.

**Example 5.** A monad \((T, \eta, \mu)\) on \(\text{POS}\) where \(T\) does not preserve surjections.

Let 2 denote the two-element poset \(\{0, 1\}\) with \(0 \prec 1\). Let \(2d\) denote the two-element discretely ordered poset \(\{0, 1\}\). We define \(U: \text{POS} \to \text{POS}\) to be the Horn-functor

\[
XU = \text{Hom}(2, X),
\]

for each poset \(X\); (we use the notation ‘\(\text{Hom}(2, X)\)’ instead of ‘\(\text{POS}(2, X)\)’. since there is only one category involved; this was not the case in Example 1); the value of \(U\) on the morphism \(f: X \to Y\) is the function

\[
fU: \text{Hom}(2, X) \to \text{Hom}(2, Y)
\]

defined by

\[
\langle u, fU \rangle = u \ast f,
\]

the composite of \(u\) and \(f\).

It is clear that \(U\) is a P-functor. Before showing that \(U\) has a P-left adjoint, we briefly recall the notion of P-copowers in a P-category (see the end cf Section 3).

For example, in the category \(\text{POS}\), \(2 \cdot 2\) is the four-element poset \(\{b, a, a', t\}\) whose order structure makes it into a Boolean algebra; i.e. \(b\) is the least element, \(t\) is the greatest element and \(a\) and \(a'\) are incomparable. The morphisms \(c_0\) and \(c_1: 2 \to 2 \cdot 2\) are defined as follows:

\[
\langle 0, c_0 \rangle = b; \quad \langle 1, c_0 \rangle = a; \quad \langle 0, c_1 \rangle = a'; \quad \langle 1, c_1 \rangle = t.
\]

As a second example, the copower \(2d \cdot 2\) consists of the four-element poset
u, v, u', v' which looks like two disjoint two-element chains: u < v and u' < v'. The copower morphisms here are denoted \( \bar{c} \) and are defined by

\[
\langle 0, \bar{c}_0 \rangle = u; \quad \langle 1, \bar{c}_0 \rangle = v; \quad \langle 0, \bar{c}_1 \rangle = u'; \quad \langle 1, \bar{c}_1 \rangle = v'.
\]

We claim that the left adjoint to \( U \) is the P-functor \( F \) defined on objects \( X \) by

\( XF = X \cdot 2 \).

On morphisms \( f : X \rightarrow Y \), \( fF : X \cdot 2 \rightarrow Y \cdot 2 \) is the unique morphism such that

\( c_i \cdot fF = c_j \), where \( j = if \),

for all \( i \) in \( X \). (Here \( c_i : 2 \rightarrow X \cdot 2 \), \( i \in X \), and \( c_j : 2 \rightarrow Y \cdot 2 \), \( j \in Y \) are the copower injections.) The proof of the claim is the last Proposition in Section 3.

Now define \( T \) to be the composition \( FU \). We show that \( T \) does not preserve surjections. Indeed, let \( f : 2d \rightarrow 2 \) be the surjection

\( if = i, \quad i = 0, 1 \).

Then \( fF : 2d \cdot 2 \rightarrow 2 \cdot 2 \) satisfies

\( \langle u, fF \rangle = b; \quad \langle v, fF \rangle = a; \quad \langle u', fF \rangle = a'; \quad \langle v, fF \rangle = t \).

Thus, if \( h : 2 \rightarrow 2 \cdot 2 \) is defined by

\( 0h = b; \quad 1h = t \),

\( h \) is not of the form \( \langle g, fT \rangle = g \cdot fF \), for any order preserving \( g : 2 \rightarrow 2d \cdot 2 \). Hence \( fT \) is not a surjection.

**Example 6.** A monadic P-quasi variety \( (C, U) \) which is not a P-variety.

Let \( C \) consist of the category of discretely ordered posets (so that \( C \) is isomorphic to \( \text{SET} \)) and let \( U : C \rightarrow \text{POS} \) be the inclusion functor. It is easy to see that \( (C, U) \) is a monadic P-quasi variety. In order to show that \( U \) does not reflect P-congruences, let \( (u, v) : A \rightarrow 2d \) be the P-kernel of the function

\( f : 2d \rightarrow 2 \)

taking \( i \) in \( 2d \) to \( i \) in \( 2 \), \( i = 0, 1 \). (Recall that \( 2 \) is the two element chain and \( 2d \) is the discretely ordered poset with elements 0 and 1.) It is easy to see that \( u \) and \( v \) are \( U \) images (since \( A \) is discretely ordered), but there is no morphism \( g : 2d \rightarrow X \) in \( C \) whose P-kernel is \( u, v \). Thus \( U \) does not reflect P-congruences.

**Problem 2.** Is there a 'sandwich theorem' for P-functors? Specifically, suppose that the diagram

\[
\begin{array}{ccc}
C & \xrightarrow{U} & D \\
\downarrow{U} & & \downarrow{V} \\
E & \xrightarrow{W} & D \\
\end{array}
\]
commutes, where $U, V$ and $W$ are $P$-functors. Suppose that $C$ has all $P$-coequalizers and that $(D, V)$ is $P$-monadic. If $W$ has a $P$-left adjoint, does $U$ necessarily have a $P$-left adjoint? Usually we can show that $U$ has a left adjoint using the sandwich theorem ([11], p. 182). Is this adjoint always a $P$-functor?

9. An extension theorem

In this section we will give a description of a 'canonical' extension of a $P$-quasi variety to a $P$-variety. This description is complicated by the following fact. In the setting of unordered algebras, if $C$ is a (concrete) quasi variety of $\Sigma$ algebras and $\hat{C}$ is the least variety of $\Sigma$ algebras containing $C$, then for each set $X$, the algebra $XF$ freely generated by $X$ in $C$ is also the algebra freely generated by $X$ in $\hat{C}$ (for example, the free torsion free abelian group generated by $X$ is also the free abelian group generated by $X$). In the context of ordered algebras, the situation is not so simple.

Suppose that $(C, U)$ is a $P$-quasi variety. By the Main Theorem, we may assume that $C$ is a concrete $P$-quasi variety of $\Sigma$ algebras. Let $\hat{C}$ be the full subcategory of $\Sigma\text{ALG}$ consisting of all $P$-regular (i.e. surjective, order preserving) homomorphic images of algebras in $C$ (and let $\hat{U}: \hat{C} \to \text{POS}$ denote the underlying poset functor). It is not surprising that $(\hat{C}, \hat{U})$ is the 'canonical' $P$-variety over $(C, U)$.

Lemma 1. Let $I: C \to \hat{C}$ be the inclusion functor. Then:

(i) $I$ is a functor $(C, U) \to (\hat{C}, \hat{U})$; i.e. $IU = U$;

(ii) $I$ is full, $P$-faithful, injective on objects; the image of $I$ is closed under $P$-monics; and

(iii) $I$ is 'P-regular reflective' (i.e. $I$ has a $P$-left adjoint, and each component of the unit of the reflection is a $P$-regular epi).

Proof. Parts (i) and (ii) are totally obvious. As for (iii), given an algebra $B$ in $\hat{C}$, let $J$ be a representative set of $P$-regular epis

$$e: B \to AI,$$

where $A$ is in $C$. Target tuple the set of morphisms in $J$, and write the target tuple as a composite $h \cdot m$, where $h$ is $P$-regular and $m$ is $P$-monic. Then it is easy to see that $h$ is the $B$-component of the reflection, since the image of $I$ is closed under $P$-monics. We omit the easy argument that the reflector is a $P$-functor.

Lemma 2. $\hat{U}: \hat{C} \to \text{POS}$ has a $P$-left adjoint.

Proof. Suppose that $F: \text{POS} \to C$ is the $P$-left adjoint of $U$ and $\eta$ is the unit of this adjunction. First we show that if $X$ is a discrete poset, then $XFI$ is the free algebra in $\hat{C}$, freely generated by $X$. Indeed, suppose that $B$ is an algebra in $\hat{C}$ and that $f: X \to B\hat{U}$ is a morphism in $\text{POS}$. There is some $P$-regular homomorphism $e: A \to B$,
with \( A \) in \( C \). Since \( X \) is discrete, there is some morphism \( g : X \to AU \) such that

\[
\begin{array}{ccc}
X & \xrightarrow{f} & B\hat{U} \\
\downarrow{g} & & \downarrow{e\hat{U}} \\
AU & & 
\end{array}
\]

commutes. Thus, there is a unique homomorphism \( g^* : XF \to A \) such that

\[
\eta_X \cdot g^* U = g.
\]

Hence,

\[
\eta_X \cdot (g^* \cdot e)\hat{U} = f.
\]

If \( \eta_X \cdot h\hat{U} \leq \eta_X \cdot g^* U \) then \( h \leq g \), since if \( E = \{ x \mid xh \leq xg \} \), then the inclusion \( m : E \to XF \) is \( P \)-monic and \( \eta_X \) may be written as a composite:

\[
\eta_X = f \cdot m U.
\]

Hence \( m \) is an isomorphism. Thus, this component of the unit is a \( U \) \( P \)-epic.

Now if \( Y \) is an arbitrary (not necessarily discrete poset) let \( \hat{Y} \) be the discrete poset with the same underlying set as \( Y \) and let \( a : \hat{Y} \to Y \) be the \( P \)-regular epi taking \( y \) in \( \hat{Y} \) to \( y \) in \( Y \). Let \( \sqsubseteq \) be the least preorder on \( \hat{Y} \) (respecting the \( \Sigma \) algebra structure) such that

1. \((2.1)\) for all \( x, y \) in \( \hat{Y} \), \( x \sqsubseteq y = x \sqsubseteq y \);
2. \((2.2)\) for all \( x, y \) in \( \hat{Y} \), if \( xa \leq ya \), then \( x\eta \sqsubseteq y\eta \) (where \( \eta = \eta_Y \)).

Let \( k : \hat{Y} \to \hat{Y} \) be the \( P \)-coequalizer of \( \sqsubseteq \). Note that there is a (unique) morphism \( \beta \) in \( POS \) such that the following diagram commutes:

\[
\begin{array}{ccc}
\hat{Y} & \xrightarrow{\beta} & Y \\
\downarrow{\eta} & & \downarrow{\beta} \\
\hat{Y}F\hat{U} & \xrightarrow{k\hat{U}} & Y\hat{F}\hat{U}
\end{array}
\]

We show that \( Y\hat{F} \) is the algebra freely generated by \( Y \) in \( \hat{C} \). First note that \( Y\hat{F} \) is an algebra in \( \hat{C} \), since \( \hat{Y}F \) is in \( C \) and \( k \) is \( P \)-regular. Now if \( f : Y \to B\hat{U} \) is any morphism in \( POS \), \( a \cdot f : \hat{Y} \to B\hat{U} \) has a discrete poset as its source. Hence, by the above, there is a unique homomorphism \( (a \cdot f)^* : \hat{Y}F \to B \) such that

\[
a \cdot f = \eta \cdot (af)^* \hat{U}.
\]

But clearly, the \( P \)-kernel of \( (a \cdot f)^* \) has the properties (2.1) and (2.2) above, so there is a unique morphism, say \( f^* : Y\hat{F} \to B \) such that

\[
k \cdot f^* = (a \cdot f)^* \hat{U}.
\]

From these facts it is easy to see that

\[
f = \beta \cdot (f^*) \hat{U}.
\]
The remaining part of the argument is easy and is omitted.

**Lemma 3.** \((C, \bar{U})\) is a (concrete) \(P\)-variety.

**Proof.** This follows from Lemma 2, since it is clear that \(C\) is closed under products, \(P\)-monics and \(P\)-regular epis.

The first three lemmas have established the following theorem:

**Theorem 1.** For any \(P\)-quasi variety \((C, U)\) there is a \(P\)-variety \((C, \bar{U})\) and a functor \(I: (C, U) \to (\bar{C}, \bar{U})\) such that

1. \(I\) is a \(P\)-embedding (i.e. \(I\) is full, \(P\)-faithful and injective on objects);
2. the image of \(I\) is closed under \(P\)-monics and binary products;
3. for any object in \(C\), there is some \(P\)-regular epi \(e: AI \to B\), whose source is an \(I\) image.

We now show that the properties Hi–H3 characterize the extension of \((C, U)\) to \((\bar{C}, \bar{U})\). Suppose now that \(I: (C, U) \to (\bar{C}, \bar{U})\) is a \(P\)-embedding of the \(P\)-quasi variety \((C, U)\) in the \(P\)-variety \((\bar{C}, \bar{U})\) which satisfies the hypotheses H1–H3 in Theorem 1.

**Theorem 2.** There is a unique \(P\)-functor \(\bar{K}: (\bar{C}, \bar{U}) \to (D, V)\) such that \(I \cdot \bar{K} = K\).

(Note that this theorem may be restated in the following way: The category of all \(P\)-varieties is a reflective subcategory of the category of all \(P\)-quasi varieties.) The proof of the theorem is divided into a number of lemmas.

**Lemma 4.** For each \(P\)-regular epi \(e: AI \to B\) in \(\bar{C}\), there is a unique \(e': AK \to B'\) in \(D\) such that

\[ e\bar{U} = e'V; \]

Further \(e'\) is \(P\)-regular. Lastly, the \(P\)-kernel of \(e\) is of the form \((u, v)I\), for some parallel pair \((u, v)\) in \(C\), and \((u, v)K\) is the \(P\)-kernel of \(e'\).

**Proof.** Let \((u', v'): X \to A\) be the \(P\)-kernel of \(e\) in \(\bar{C}\). Since \((u', v')\) is a \(P\)-monocone, \(X' = XI\) and \((u', v') = (u, v)I\), by H2. Since \((u, v)I\bar{U} = (u, v)KV\), and since \(\bar{U}\) preserves and \(V\) reflects \(P\)-congruences, \((u, v)K: XK \to AK\) is a \(P\)-congruence in \(D\); say \((u, v)K\) is the \(P\)-kernel of \(e'\). Since \(V\) creates isomorphisms, and since \(e\bar{U}\) is the \(P\)-coequalizer of \((u, v)I\bar{U}\) in \(POS\), we can assume that \(e'V = e\bar{U}\). If \(e''\) also has source \(AK\) and \(e''V = e'V\), then \(e'' = e''\) by the lifting lemma. Lemma 4 is proved.

We now show how to extend the functor \(K\) to a functor \(\bar{K}: \bar{C} \to D\). First we define \(\bar{K}\) on objects. For any object or morphism \(x\) in \(C\), we define \(xI\bar{K} = xK\). If \(B\) is an
object in \( \hat{C} \) but not in \( C \), there is some P-regular epi \( e: AI \rightarrow B \) in \( \hat{C} \), by H3. Let \( e': AK \rightarrow B' \) be the unique morphism in \( D \) with \( e'V = eU \).

**Definition.** \( B\hat{K} = B' \).

In order to show \( \hat{K} \) is well defined on the objects of \( \hat{C} \), we need:

**Lemma 5.** Let \( e_i: A_i, I \rightarrow B \) be P-regular epis in \( \hat{C} \) where \( A_i \) is in \( C \), \( i = 1, 2 \). Let \( e'_i: A_iK \rightarrow B_i \) be the unique morphisms in \( D \) such that \( e'_iV = e_i\hat{U} \), \( i = 1, 2 \). Then \( B_1 = B_2 \).

**Proof.** Let

\[
\begin{array}{ccc}
X & \xrightarrow{e_2} & A_1 \\
\downarrow{e_1} & & \downarrow{e_1} \\
A_2 & \xrightarrow{e_2} & B
\end{array}
\]

be a pullback in \( \hat{C} \). Then, since P-limits in \( \hat{C} \) are P-monocones, \( X, e_1 \) and \( e_2 \) are in the image of \( I \). Rewrite \( X \) as \( XI \) and \( e_i \), \( i = 1, 2 \). Let

\[
d_1 = \tilde{e}_2K \cdot e'_1: XK \rightarrow B_1; \\
d_2 = \tilde{e}_1K \cdot e'_2: XK \rightarrow B_2.
\]

**Claim.** \( d_1V = d_2V \) and \( d_1V \) is a surjection.

Indeed,

\[
d_1V = e_2KV \cdot e'_1V = e_2U \cdot e_1\hat{U} = (e_2I \cdot e_1)\hat{U} \\
= (e_1I \cdot e_2)\hat{U} = e_1KV \cdot e'_2V = d_2V.
\]

Also, since pullbacks of P-regular epis (in P-varieties) are also P-regular, \( e_1 \cdot e_2 \) and thus \( (\tilde{e}_1 \cdot e_2)\hat{U} \) are P-regular. The claim is proved.

Hence, by the lifting lemma applied to \((D, V), d_1 = d_2\), and thus \( B_1 = B_2 \).

We will define \( \hat{K} \) on morphisms (not in the image of \( I \)) in two stages. First suppose that

\[
h: B_1 \rightarrow B_2
\]

is P-regular in \( \hat{C} \). If \( B_1 = AI \), \( h\hat{K} \) is defined by Lemma 4 as the unique morphism \( h' \) in \( D \) such that \( h'V = hU \). If \( B_1 \) is not in the image of \( I \), let \( e: AI \rightarrow B_1 \) be some P-regular epi with \( A \) in \( C \). Then both \( e \) and \( e \cdot h \) are P-regular and there is a (unique) morphism \( j \) from the P-kernel of \( e \) to the P-kernel of \( e \cdot h \). Hence, by Lemma 4, there is a unique morphism in \( D \) from the P-kernel of \( e \hat{K} \) to that of \( (e \cdot h)\hat{K} \). Thus, there is a unique morphism \( h' \) from \( e\hat{K} \) to \((e \cdot h)\hat{K} \). We define \( h\hat{K} = h' \). It is easy to see that \( h\hat{K}V = h\hat{U} \).
Now assume that \( m : B_1 \rightarrow B_2 \) is a P-monic in \( \mathcal{C} \). Let \( e : A I \rightarrow B_2 \) be a P-regular with \( A \) in \( C \). Form the pullback of \( e \) and \( m \) in \( \mathcal{C} \):

\[
\begin{array}{ccc}
X & \xrightarrow{\bar{e}} & B_1 \\
\downarrow{m} & & \downarrow{m} \\
A I & \xrightarrow{e} & B_2
\end{array}
\]

In \( \mathcal{C} \), pullbacks of P-monics are P-monics and pullbacks of P-regular epis are P-regular. Thus \( X \) is an \( I \) image, by H2, and \( \bar{e} \) is a P-regular epi. Now \((\bar{m} \cdot e)\bar{K}\) and \(\bar{e}\bar{K}\) are already defined. By the same argument as above, there is a unique morphism \( m' \) in \( D \) with

\[
\bar{e}\bar{K} \cdot m' = (\bar{m} \cdot e)\bar{K}.
\]

We define \( \bar{m}\bar{K} = m' \). Then necessarily,

\[
\bar{m}\bar{K} = m U.
\]

Now to define \( \bar{K} \) on an arbitrary morphism \( g \) in \( \mathcal{C} \), first write \( g \) as a composite \( h \cdot m \), where \( h \) is P-regular and \( m \) is P-monic. Then we define

\[
g\bar{K} = h\bar{K} \cdot m\bar{K}.
\]

We now must show that with this definition \( \bar{K} \) is well defined and is a functor (i.e. preserves composition).

If \( h \cdot m = f \cdot n \) in \( \mathcal{C} \), where \( h \) and \( f \) are P-regular and \( m \) and \( n \) are P-monic, then

\[
(h \cdot m)\bar{U} = h\bar{U} \cdot m\bar{U} = h\bar{K} \cdot \bar{m}\bar{K} = (h\bar{K} \cdot m\bar{K})\bar{V},
\]

and similarly,

\[
(f \cdot n)\bar{U} = (f\bar{K} \cdot n\bar{K})\bar{V}.
\]

But since \( h \cdot m = f \cdot n \), and since \( V \) is P-faithful,

\[
h\bar{K} \cdot m\bar{K} = f\bar{K} \cdot n\bar{K},
\]

proving that \( \bar{K} \) is well defined.

In order to prove that \( \bar{K} \) preserves composition, we again use the fact that \( V \) is P-faithful: for any composable \( f \) and \( g \) in \( \mathcal{C} \),

\[
(f \cdot g)\bar{K} V = (f \cdot g)\bar{U} = f\bar{U} \cdot g\bar{U}
= f\bar{K} \cdot \bar{g}\bar{K} V = (f\bar{K} \cdot g\bar{K}) V,
\]

proving the claim. The above argument completes the proof of Theorem 2.

We end this section with a question.
Problem 3. If the functor $K : (C, U) \to (D, V)$ is full, faithful and injective on objects, is the functor $\hat{K}$ necessarily injective on objects?

Here is an example which shows that $\hat{K}$ is not always full. Let $C$ be the category of all ordered cancellation monoids (i.e. $y = z$ if $xy = xz$) and let $U$ be the underlying poset functor. Then it is easy to see that $\hat{C}$ is the category of all ordered monoids (and $\hat{U}$ is also the underlying poset functor). Let $(D, V)$ be the $P$-variety of all ordered semigroups and let

$$K : (C, U) \to (D, V)$$

be the inclusion functor. We omit the easy proof that $K$ is full, faithful and injective on objects. $\hat{K} : \hat{C} \to \hat{D}$ is the inclusion functor but $\hat{K}$ is not full. Indeed, let $(N, *)$ be the multiplicative semigroup of the nonnegative integers and let $({\{0\}, *})$ be the one-element subsemigroup of $(N, *)$. Then both of these semigroups are images of $\hat{K}$, but the inclusion

$$({\{0\}, *}) \to (N, *)$$

is not, since it is not a monoid homomorphism.

References