Orthogonal polynomials and coherent pairs: the classical case

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Communicated by Prof. J. Korevaar at the meeting of September 26, 1994

ABSTRACT

Let \( \{P_n(x)\}_{n \geq 0} \) and \( \{R_n(x)\}_{n \geq 0} \) be two sequences of simple monic polynomials such that

\[
P_n(x) = \frac{1}{n+1} R'_{n+1}(x) - \sigma_n R_n(x), \quad n = 0, 1, 2, \ldots
\]

where \( \{\sigma_n\}_{n \geq 0} \) is a sequence of complex numbers. Consider the two following problems: (i) if \( \{R_n\}_{n \geq 0} \) is a given system of orthogonal polynomials, to characterize all the sequences of orthogonal polynomials \( \{P_n\}_{n \geq 0} \) and all the sequences of compatible parameters \( \{\sigma_n\}_{n \geq 0} \) for which (*) holds; (ii) the analogous problem, with the assumption that \( \{P_n\}_{n \geq 0} \) is the given system of orthogonal polynomials. The first problem has been partially solved by Iserles et al. in [6], in the case in which \( \{R_n\}_{n \geq 0} \) is a classical family. Here, we characterize the solution for both problems in the case in which the given system is some classical one.

1. INTRODUCTION

This paper is motivated by the concept of coherence for a pair of orthogonal polynomials when one of them is classical.

The idea of coherent pairs of orthogonal polynomials was introduced by A. Iserles et al. (see [6]) in the framework of the study of orthogonal polynomials with respect to a Sobolev inner product

\[
\langle f, g \rangle = \int_R f g \, d\mu_0 + \lambda \int_R f' g' \, d\mu_1
\]

where \( \mu_0 \) and \( \mu_1 \) are positive Borel measures on the real line satisfying the conditions
\[ \int_{\mathbb{R}} x^k \, d\mu_i < \infty, \quad k = 0, 1, 2, \ldots, \quad i = 0, 1 \]

(finite moments) and

\[ \int_{\mathbb{R}} p(x) \, d\mu_i > 0, \quad i = 0, 1 \]

for each polynomial \( p \) that is non-negative for all real \( x \) and not identically zero on the supports of \( \mu_0 \) and \( \mu_1 \) (which must be sets with an infinite number of elements), respectively. In fact, coherence means that a relation between the OPS's (orthogonal polynomial sequences) \( \{P_n\}_{n \geq 0} \) and \( \{R_n\}_{n \geq 0} \), with respect to the measures \( \mu_0 \) and \( \mu_1 \), as

\[ P_n(x) = C_{n+1} R_{n+1}'(x) - C_n R_n'(x), \quad n = 1, 2, \ldots \]

holds, where \( C_1, C_2, \ldots \) are non-zero constants.

More generally, one has:

**Definition 1.** Let \( \{P_n\}_{n \geq 0} \) and \( \{R_n\}_{n \geq 0} \) be two OPS's with respect to the linear functionals \( u \) and \( v \), respectively (not necessarily positive measures). The pair \( (u, v) \) is said to be **coherent** if there exist non-zero complex constants \( C_1, C_2, \ldots \) such that (2) holds.

Properties of the set of polynomials orthogonal with respect to the Sobolev inner product (1), where \( \{d\mu_0, d\mu_1\} \) is a coherent pair, were studied by H.G. Meijer in [15].

Since we consider monic polynomials, our study on coherence will be centered on a relation

\[ P_n = \frac{1}{n+1} R_{n+1}' - \sigma_n R_n', \quad n = 1, 2, \ldots \]

where \( \{P_n\}_{n \geq 0} \) and \( \{R_n\}_{n \geq 0} \) are MOPS and \( \{\sigma_n\}_{n \geq 1} \) a sequence of non-zero complex parameters.

Consider the following problems:

**P1.** If \( \{R_n\}_{n \geq 0} \) is a fixed system of orthogonal polynomials, determine all the sequences of orthogonal polynomials \( \{P_n\}_{n \geq 0} \) and all the sequences of compatible parameters \( \{\sigma_n\}_{n \geq 1} \) for which (3) holds;

**P2.** The analogous problem, with the assumption that \( \{P_n\}_{n \geq 0} \) is the fixed system of orthogonal polynomials.

The first problem has been partially solved by Iserles et al. in [6], with the assumption that \( \{R_n\}_{n \geq 0} \) is a classical family, for positive measures. Here, in Section 3, we will give a complete characterization for both problems in the case in which one of the families \( \{P_n\}_{n \geq 0} \) or \( \{R_n\}_{n \geq 0} \) is given and some classical one. Furthermore, in Section 4, we situate this kind of problems in the
context of semiclassical OPS's, as particular cases of some more general problems. In Section 2, we introduce some basic results.

2. BASIC CONCEPTS

We denote by $\mathbb{P}$ the space of all polynomials in one variable with complex coefficients and by $\mathbb{P}'$ its topological dual (see [12], [14], e.g.). Let us recall the definitions of some useful operations on $\mathbb{P}'$.

**Definition 2.** Let $u \in \mathbb{P}'$, $\phi \in \mathbb{P}$ and $c \in \mathbb{C}$. We define:

- the *left multiplication* of the functional $u$ by the polynomial $\phi$, which is the functional in $\mathbb{P}'$, denoted by $\phi u$, such that

$$\langle \phi u, p \rangle := \langle u, \phi p \rangle, \quad p \in \mathbb{P};$$

- the *distributional derivative* of the functional $u$, denoted by $Du$, which is the element of $\mathbb{P}'$ such that

$$\langle Du, p \rangle := -\langle u, p' \rangle, \quad p \in \mathbb{P};$$

- and the *division* of the functional $u$ by $x - c$, denoted by $(x - c)^{-1}u$:

$$\langle (x - c)^{-1}u, p \rangle := \left\langle u, \frac{p(x) - p(c)}{x - c} \right\rangle, \quad p \in \mathbb{P}.$$

**Definition 3.** Given a continuous linear functional $u$ on $\mathbb{P}$ and a set of polynomials $\{P_n\}_{n \geq 0}$ such that

$$\deg P_n = n, \quad n = 0, 1, 2, \ldots$$

$$\langle u, P_n P_m \rangle = k_n \delta_{nm} (k_n \in \mathbb{C} \setminus \{0\}), \quad n, m = 0, 1, 2, \ldots$$

we say that $\{P_n\}_{n \geq 0}$ is an *orthogonal polynomial system* (OPS) with respect to $u$.

If $u$ is a linear functional on $\mathbb{P}$ and an OPS $\{P_n\}_{n \geq 0}$ for $u$ exists, then $u$ is called regular and $\{P_n\}_{n \geq 0}$ the corresponding OPS.

Throughout this paper, we will consider always systems of orthogonal polynomials such that the polynomials are monic (MOPS). The polynomials $\{P_n\}_{n \geq 0}$ of any MOPS satisfy a three-term recurrence relation (see [4])

$$\begin{cases} P_0 = 1, & P_1 = x - \beta_0 \\ P_{n+1} = (x - \beta_n) P_n - \gamma_n P_{n-1}, \quad n \geq 1 \end{cases} \tag{4}$$

where $\{\beta_n\}_{n \geq 0}$ and $\{\gamma_n\}_{n \geq 1}$ are two sequences of complex numbers with $\gamma_n \neq 0$ for $n \geq 1$. It is important to remark that such a relation characterize completely a given MOPS, according with Favard's theorem (see [4], pp. 21, Theorem 4.4).

One of the more important classes of OPS's are the so-called classic orthogonal polynomial systems (Hermite, Laguerre, Jacobi and Bessel). The classical
OPS's can be defined in terms of the corresponding linear (regular) functional, \( u \), as solutions of a differential distributional equation

\[ D(\phi u) = \psi u, \]

where \( \phi \) and \( \psi \) are polynomials such that

\[ \deg \phi \leq 2, \quad \deg \psi = 1. \]

In these conditions, \( u \) is called a classical functional. If we put

\[ \phi(x) = ax^2 + bx + c, \quad \psi(x) = px + q \]

then the regularity condition implies that

\[ na + p \neq 0, \quad \phi \left( -\frac{nb + q}{2na + p} \right) \neq 0, \quad n \geq 0. \]

In fact, it can be shown [10] that conditions (7) are necessary and sufficient for the regularity of a functional \( u \) which is a solution of an equation like (5). Moreover, the corresponding parameters \( \beta_n, \gamma_n \) in (4) of a classical MOPS are given explicitly in terms of the coefficients of \( \phi \) and \( \psi \) by

\[ \beta_n = -\frac{(-2a + p)q + 2bn[(n - 1)a + p]}{(2na + p)[(2n - 2)a + p]}, \quad n \geq 0 \]

and

\[ \gamma_n = \frac{-n[(n - 2)a + p]}{[(2n - 1)a + p][(2n - 3)a + p]} \phi \left( -\frac{(n - 1)b + q}{(2n - 2)a + p} \right), \quad n \geq 1 \]

(see [16]).

Up to a linear change in the variable, we have

- Hermite polynomials, \( \{H_n(x)\}_{n \geq 0} \), in the case \( \phi \equiv \text{const.}; \)
- Laguerre polynomials, \( \{L_n^{(\alpha)}(x)\}_{n \geq 0} \), in the case \( \deg \phi = 1; \)
- Jacobi polynomials, \( \{P_n^{(\alpha, \beta)}(x)\}_{n \geq 0} \), in the case \( \deg \phi = 2 \) and \( \phi \) with simple roots;
- Bessel polynomials, \( \{B_n^{(\alpha)}(x)\}_{n \geq 0} \), in the case \( \deg \phi = 2 \) with a double root,

and, in each case, we can take some canonical forms for these polynomials \( \phi \) and \( \psi \) (see [14], e.g.):

- Hermite: \( \phi(x) = 1, \psi(x) = -2x; \)
- Laguerre: \( \phi(x) = x, \psi(x) = -x + \alpha + 1, \text{ with } \alpha \neq -n, n \geq 1; \)
- Jacobi: \( \phi(x) = 1 - x^2, \psi(x) = -(\alpha + \beta + 2)x + \beta - \alpha, \text{ with } \alpha \neq -n, \beta \neq -n, \alpha + \beta + 1 \neq -n, n \geq 1; \)
- Bessel: \( \phi(x) = -x^2, \psi(x) = (\alpha + 2)x + 2, \text{ with } \alpha \neq -n, n \geq 2, \)

where the restrictions can be justified by conditions (7).

Many characterizations of the classical OPS's are known (see [7], e.g.). For our purpose, we just need the following one, due to Hahn [5]: a sequence of orthogonal polynomials \( \{P_n\}_{n \geq 0} \) is classical if and only if the sequence of (monic) derivatives \( \{Q_n\}_{n \geq 0} \), where
\[ Q_n(x) := \frac{P_{n+1}'(x)}{n+1}, \]
is also a MOPS.

**Remark.** If (8) and (9) denote the coefficients of the three-term recurrence relation for the sequence \( \{P_n\}_{n \geq 0} \), then the corresponding coefficients for \( \{Q_n\}_{n \geq 0} \) can be computed by the change \( p \to p + 2a \) and \( q \to q + b \) in both formulas (8) and (9), because \( \{Q_n\}_{n \geq 0} \) is a MOPS for the linear functional \( \nu := \phi w \), which fulfills the differential distributional equation

\[ D(\phi \nu) = (\psi + \phi')\nu \]

(see [14], [10], e.g.). In this way, one can see that the following relations hold:

\[ H'_n(x) = nH_{n-1}(x) \]
\[ I^{(\alpha)}_n(x) = nI^{(\alpha+1)}_{n-1}(x) \]
\[ P^{(\alpha,\beta)}_n(x) = nP^{(\alpha+1,\beta+1)}_{n-1}(x) \]
\[ B^{(\alpha)}_n(x) = nB^{(\alpha+2)}_{n-1}(x). \]

3. THE CLASSICAL CASE

The problem of coherence in the case in which one of the families \( \{P_n\}_{n \geq 0} \) or \( \{R_n\}_{n \geq 0} \) is given and some classical one can be solved by developing the ideas presented in [6] and [13]. First, we state the following lemma:

**Lemma 1.** Let \( \{P_n\}_{n \geq 0} \) and \( \{Q_n\}_{n \geq 0} \) be two MOPS's, \( \{a_n\}_{n \geq 0} \) and \( \{b_n\}_{n \geq 0} \) be two sequences of complex parameters and suppose that these two families of polynomials are connected by

\[ P_n(x) = Q_n(x) - a_0 \]
\[ P_{n+1}(x) = Q_{n+1}(x) - a_n Q_n(x) - b_n, \quad n \geq 1. \]

Denote by \( \{\beta_n, \gamma_{n+1}\}_{n \geq 0} \) and \( \{\tilde{\beta}_n, \tilde{\gamma}_{n+1}\}_{n \geq 0} \) the sets of parameters of the three-term recurrence relations of the sequences \( \{P_n\}_{n \geq 0} \) and \( \{Q_n\}_{n \geq 0} \), respectively. Then, the following relations hold:

\[ \begin{align*}
\text{(i)} & \quad b_n = 0, \quad n \geq 1; \\
\text{(ii)} & \quad \beta_n = \beta_{n-1} + a_n + a_{n-1}, \quad n \geq 0 \ (a_{-1} = 0); \\
\text{(iii)} & \quad \tilde{\gamma} = \gamma_n - a_{n-1}(\beta_n - \beta_{n-1}), \quad n \geq 1; \\
\text{(iv)} & \quad a_n \tilde{\gamma}_n = a_{n-1} \gamma_{n+1}, \quad n \geq 1.
\end{align*} \]

**Proof.** Consider the three-term recurrence relation for the sequence \( \{P_n\}_{n \geq 0} \):

\[ \begin{align*}
P_{n+1}(x) &= (x - \beta_n)P_n(x) - \gamma_n P_{n-1}(x), \quad n \geq 1 \\
P_0(x) &= 1, \quad P_1(x) = x - \beta_0
\end{align*} \]

\((\gamma_n \neq 0, n \geq 1)\). Substitution of (11) in (13) yields to

291
\[
\begin{align*}
Q_{n+1} - a_n Q_n - b_n &= (x - \beta_n)(Q_n - a_{n-1} Q_{n-1} - b_{n-1}) \\
&\quad - \gamma_n(Q_{n-1} - a_{n-2} Q_{n-2} - b_{n-2}), \quad n \geq 3 \\
Q_3 - a_2 Q_2 - b_2 &= (x - \beta_2)(Q_2 - a_1 Q_1 - b_1) - \gamma_2(Q_1 - a_0) \\
Q_2 - a_1 Q_1 - b_1 &= (x - \beta_1)(Q_1 - a_0) - \gamma_1 \\
Q_1 - a_0 &= x - \beta_0.
\end{align*}
\]

Now, consider the three-term recurrence relation for the sequence \(\{Q_n\}_{n \geq 0}\):
\[
\begin{align*}
\begin{cases}
x Q_n(x) &= Q_{n+1}(x) + \tilde{\beta}_n Q_n(x) + \tilde{\gamma}_n Q_{n-1}(x), \quad n \geq 1 \\
Q_0(x) &= 1, \quad x = Q_1(x) + \tilde{\beta}_0
\end{cases}
\end{align*}
\]
\((\tilde{\gamma} \neq 0, n \geq 1)\). According with (15), in the right-side of (14), we can expand the terms of the form \(x Q_i\) as linear combination of the polynomials \(\{Q_n\}_{n \geq 0}\), so that
\[
\begin{align*}
Q_{n+1} - a_n Q_n - b_n &= Q_{n+1} + \tilde{\beta}_n Q_n + \tilde{\gamma}_n Q_{n-1} - a_{n-1}(Q_n + \tilde{\beta}_n Q_{n-1} + \tilde{\gamma}_n Q_{n-2}) \\
&\quad - b_{n-1}(Q_1 + \tilde{\beta}_0) - \beta_n(Q_{n-1} - a_{n-1} Q_{n-2} - b_{n-2}), \quad n \geq 3 \\
Q_3 - a_2 Q_2 - b_2 &= Q_3 + \tilde{\beta}_2 Q_2 + \tilde{\gamma}_2 Q_1 - a_1(Q_2 + \tilde{\beta}_1 Q_1 + \tilde{\gamma}_1) - b_1(Q_1 + \tilde{\beta}_0) \\
&\quad - \beta_2(Q_2 - a_1 Q_1 - b_1) - \gamma_2(Q_1 - a_0) \\
Q_2 - a_1 Q_1 - b_1 &= Q_2 + \tilde{\beta}_1 Q_1 + \tilde{\gamma}_1 - a_0(Q_1 + \tilde{\beta}_0) - \beta_1(Q_1 - a_0) - \gamma_1 \\
Q_1 - a_0 &= Q_1 + \tilde{\beta}_0 - \beta_0,
\end{align*}
\]
and, after identification of coefficients in these relations – using the fact that \(\{Q_n\}_{n \geq 0}\) is a basis for \(\mathbb{P}\) – we find
\[
\begin{align*}
- a_n &= \tilde{\beta}_n - a_{n-1} - \beta_n, \quad n \geq 3 \\
- b_n &= - b_{n-1} \tilde{\beta}_0 + \beta_n b_{n-1} + \gamma_n b_{n-2}, \quad n \geq 3 \\
0 &= \tilde{\gamma}_n - a_{n-1} \tilde{\beta}_n - \beta_n a_{n-1} - \gamma_n, \quad n \geq 3 \\
0 &= a_{n-1} \tilde{\gamma}_n - \gamma_n a_{n-2}, \quad n \geq 4 \\
0 &= - a_2 \tilde{\gamma}_2 - b_2 + \gamma_3 a_1 \\
0 &= - b_{n-1}, \quad n \geq 4 \\
- a_2 &= \tilde{\beta}_2 - a_1 - \beta_2 \\
- b_2 &= - a_1 \tilde{\gamma}_1 - b_1 \tilde{\beta}_0 + \beta_2 b_1 + \gamma_2 a_0 \\
0 &= \tilde{\gamma}_2 - a_1 \tilde{\beta}_1 - b_1 + \beta_2 a_1 - \gamma_2 \\
- a_1 &= \tilde{\beta}_1 - a_0 - \beta_1 \\
- b_1 &= \tilde{\gamma}_1 - a_0 \tilde{\beta}_0 + \beta_1 a_0 - \gamma_1 \\
- a_0 &= \tilde{\beta}_0 - \beta_0.
\end{align*}
\]
From (16), (22), (25) and (27), it follows the relations (ii) of (12). From (21), we get

\[ b_n = 0, \quad n \geq 3; \]

hence, (17) can be reduced to

\[ \gamma_n b_{n-2} = (\beta_0 - \beta_n) b_{n-1}, \quad n \geq 3. \]

Since \( \gamma_n \neq 0 \) for all \( n \geq 1 \), from (28) and (29) we deduce that \( b_n = 0 \) for \( n \geq 1 \), which proves the relations (i) of (12). Therefore, from (18), (24) and (26), (iii) follows, and (iv) can be derived from (19), (20) and (23). \( \square \)

Now, we will characterize the solution of problem P2, in the hypothesis that the given system is some classical one. Thus, consider that \( \{R_n\}_{n \geq 0} \) is one of the classical MOPS's. From all the sequences \( \{R_n\}_{n \geq 0} \), of monic polynomials satisfying (3), our purpose is to characterize the ones which are MOPS's.

We recall that, according with (10), the polynomials of each (concrete) classical family can be written as the derivatives of polynomials of the same type. So, in each case, the polynomials \( P_n \) which appear in the left side of (3) can be replaced by derivatives of polynomials of the same type. Thus, relation (3) can be written as an equality involving only derivatives of polynomials and, consequently, after integration, it can be reduced to a direct relation of polynomials, involving, for each \( n \), a constant, which actually can be arbitrary. This leads to study the following more general problem:

**P3.** Fixed a MOPS \( \{P_n\}_{n \geq 0} \), to characterize all the possible systems of orthogonal polynomials \( \{Q_n\}_{n \geq 0} \) and all the sequences of compatible parameters \( \{a_n\}_{n \geq 0} \) and \( \{b_n\}_{n \geq 0} \), such that

\[ P_{n+1}(x) = Q_{n+1}(x) - a_n Q_n(x) - b_n, \quad n \geq 0. \]

Without loss of generality, we can consider \( b_0 = 0 \). Hence, the solution for this problem can be characterized by the following result:

**Theorem 1.** Let \( \{P_n\}_{n \geq 0} \) be a MOPS, \( \{a_n\}_{n \geq 0} \) and \( \{b_n\}_{n \geq 1} \) two sequences of complex parameters and \( \{Q_n\}_{n \geq 0} \) a simple set of monic polynomials, such that

\[ \begin{aligned}
P_1(x) &= Q_1(x) - a_0 \\
P_{n+1}(x) &= Q_{n+1}(x) - a_n Q_n(x) - b_n, \quad n \geq 1.
\end{aligned} \]

Let \( \{\beta_n, \gamma_{n+1}\}_{n \geq 0} \) be the set of parameters of the three-term recurrence relation of the sequence \( \{P_n\}_{n \geq 0} \).

Then, the orthogonality of the sequence \( \{Q_n\}_{n \geq 0} \) depends only on the choice of \( a_0 \).

More precisely, one has:

1. If \( a_0 = 0 \), then \( \{Q_n\}_{n \geq 0} \) is a MOPS if and only if
\[ b_n = 0, \quad a_n = 0, \quad n \geq 1. \]

In such conditions
\[ \{Q_n\}_{n \geq 0} = \{P_n\}_{n \geq 0}. \]

2. If \( a_0 \neq 0 \), then \( \{Q_n\}_{n \geq 0} \) is a MOPS if and only if
\[ b_n = 0, \quad a_n = \gamma_{n+1} \frac{P_n(c)}{P_{n+1}(c)}, \quad P_n(c) \neq 0, \quad n \geq 1 \]
where
\[ c : = \beta_0 + \gamma_1 / a_0. \]

In this case, the coefficients \( \tilde{\beta}_n \) and \( \tilde{\gamma}_n \) of the recurrence relation satisfied by \( \{Q_n\}_{n \geq 0} \) are given by
\[ \begin{cases} 
\tilde{\beta}_n = \beta_{n+1} + \frac{P_{n+2}(c)}{P_{n+1}(c)} - \frac{P_{n+1}(c)}{P_n(c)}, & n \geq 0 \\
\tilde{\gamma}_n = \frac{P_{n-1}(c)P_{n+1}(c)}{P_n^2(c)} \gamma_n, & n \geq 1.
\end{cases} \]

Moreover, if \( u \) denotes the linear functional such that \( \{P_n\}_{n \geq 0} \) is the corresponding MOPS, then \( \{Q_n\}_{n \geq 0} \) is orthogonal with respect to \( v \) defined as
\[ v = (x - c)u. \]

**Proof.** Suppose that \( \{Q_n\}_{n \geq 0} \) is a MOPS. Thus, it satisfies a three-term recurrence relation
\[ xQ_n(x) = Q_{n+1}(x) + \beta_n Q_n(x) + \tilde{\gamma}_n Q_{n-1}(x), \quad n \geq 1 \]
\[ Q_0(x) = 1, \quad Q_1(x) = x - \beta_0. \]

From Lemma 1, relations (12) hold, and, therefore, we need to discuss two situations:
1. \( a_0 = 0 \). Hence, it follows from (12)-(iv) that also \( a_n = 0 \) for all \( n \geq 1 \), and according with (12)-(i), we deduce immediately from (30) that
\[ Q_n \equiv P_n, \quad n \geq 0. \]

2. \( a_0 \neq 0 \). Then from (12)-(iv), also \( a_n \neq 0 \) for all \( n \) and
\[ \tilde{\gamma}_n = \gamma_{n+1} \frac{a_{n-1}}{a_n}, \quad n \geq 1. \]

Substitute (ii) and (iii) of (12) in (iv), and then divide both sides by \( a_n a_{n-1} \), to find
\[ a_{n-2} + \frac{\gamma_n}{a_{n-1}} + \beta_{n-1} = a_{n-1} + \frac{\gamma_{n+1}}{a_n} + \beta_n, \quad n \geq 1 \]
which shows that \( a_{n-2} + \gamma_n / a_{n-1} + \beta_{n-1} \) is independent of \( n \). Therefore, there exists a constant \( c \) such that
\[ a_{n-1} + \frac{\gamma_{n+1}}{a_n} + \beta_n = c, \quad n \geq 0. \]

The value of \( c \) can be obtained for \( n = 0 \), only in terms of \( a_0 \) and the data:

\[ c = \beta_0 + \frac{\gamma_1}{a_0}. \]

Now, define recurrently a sequence \( \{y_n\}_{n \geq 0} \) by

\[ y_0 = 1, \quad y_{n+1} = \gamma_{n+1} y_n / a_n, \quad n \geq 0. \]

Remark that

\[ y_n \neq 0, \quad n \geq 0. \]

Hence, \( a_n = \gamma_{n+1} y_n / y_{n+1} \) for \( n \geq 0 \); if we substitute in (37) we can see that

\[ a_n = \gamma_{n+1} \frac{P_n(c)}{P_{n+1}(c)}, \quad P_n(c) \neq 0, \quad n \geq 1. \]

So, we have proved that, for each \( a_0 \neq 0 \) fixed, in order that \( \{Q_n\}_{n \geq 0} \), defined by (30), be a MOPS it is necessary that conditions (31) hold, where \( c \) is defined as (32). Furthermore, from the expression of \( a_n \) given by (31) and from (36), we find the expression (33) of \( \hat{\gamma}_n \). Also, from the same expression for \( a_n \) and (ii), and using the three-term recurrence relation for the system \( \{P_n\}_{n \geq 0} \), we can deduce the expression for the coefficient \( \tilde{\beta}_n \) as in (33).

Conversely, it is easy to verify that conditions (31) are also sufficient to guarantee that \( \{Q_n\}_{n \geq 0} \), defined by (30), be a MOPS. For that, define complex numbers \( \tilde{\beta}_n \) and \( \hat{\gamma}_n \) by formulas (42) or – which is the same – by (33). These parameters are well defined, according with (31), and, in addition, \( \hat{\gamma}_n \neq 0 \) for all \( n \geq 1 \); one can verify directly (by induction) that \( \{Q_n\}_{n \geq 0} \) satisfies the three-term recurrence (35), so that, by Favard’s theorem, it is a MOPS.

Finally, since \( b_n = 0 \) for \( n \geq 1 \), it follows from (30) that

\[ \langle v, P_{n+1} \rangle = \langle v, Q_{n+1} - a_n Q_n \rangle = 0, \quad n \geq 1; \]

hence, \( v = (sx + t)u \), where \( s \) and \( t \) are constants, given explicitly by

\[ s = -\frac{a_0 v_0}{\gamma_1 u_0}, \quad t = \left(1 + \frac{\beta_0}{\gamma_1}\right) \frac{v_0}{u_0}, \]

where \( u_0 = \langle u, 1 \rangle \) and \( v_0 = \langle v, 1 \rangle \). Thus, \( s \neq 0 \), and if we consider the normalization \( s = 1 \) (i.e., \( v_0 = -\gamma_1 u_0 / a_0 \)), it is easy to obtain (34), which completes the proof.

Remarks. 1. It follows from the previous proof that, instead of (33), the coefficients \( \tilde{\beta}_n \) and \( \hat{\gamma}_n \) can also be computed from

\[ \hat{\gamma}_n = -\frac{a_n + \beta_n}{\gamma_{n+1}}, \quad n \geq 0. \]

\[ \tilde{\beta}_n = \gamma_{n+1} a_n / \gamma_{n+2}, \quad n \geq 0. \]
(42) \[
\begin{align*}
\beta_n &= \beta_{n-1} - a_n + a_{n-1}, & n \geq 0 \\
\gamma_n &= \gamma_n + a_{n-1}(\beta_{n-1} - \beta_n - a_{n-1} + a_{n-2}), & n \geq 1
\end{align*}
\]
with the convention $a_{-1} \equiv 0$.

2. The case 1 can be interpreted as a limit case of 2. In fact, if $a_0 = 0$, and if we interpret the relations of case 2 in the sense of the limit as $c \to +\infty$, then the conclusions of the case 1 are obtained.

3. Using the Chihara’s notation (see [4], pp. 35), from (34) we deduce that

$$Q_n(x) = P_n^*(c; x).$$

We return to the problem of coherence (which has motivated problem P3). In fact, now we can give easily a characterization for all the sequences $\{R_n\}_{n \geq 0}$ which are solutions of problem P2, when we assume that $\{P_n\}_{n \geq 0}$ is a given system of classical orthogonal polynomials.

(1) The Hermite case

Let $\{P_n\}_{n \geq 0}$ be the MOPS of Hermite, in relation (3). We want to characterize all the sequences of MOPS’s $\{R_n\}_{n \geq 0}$ and the corresponding parameters $\sigma_n$ ($n \geq 1$) such that

$$H_n(x) = \frac{1}{n + 1} R_{n+1}'(x) - \sigma_n R_n'(x), \quad n \geq 1. \tag{43}$$

Since, in view of (10), for each $n \geq 0$, $H_n = H_{n+1}'/(n + 1)$, we get

$$H_{n+1}'(x) = R_{n+1}'(x) - (n + 1)\sigma_n R_n'(x), \quad n \geq 1.$$

After integration,

$$H_{n+1}(x) = R_{n+1}(x) - (n + 1)\sigma_n R_n(x) - b_n, \quad n \geq 1$$

and, of course, $R_0$ and $R_1$ must be of the form

$$R_0(x) = 1, \quad R_1(x) = H_1(x) + \sigma,$$

with arbitrary parameters $\sigma$ and $b_n$. We recall that - cf. formulas (8) and (9) - the coefficients of the three-term recurrence relation for the sequence $\{H_n\}_{n \geq 0}$ are given by

$$\beta_n = 0, \quad \gamma_{n+1} = \frac{n + 1}{2}, \quad n \geq 0.$$

Thus, according with Theorem 1, for each choice of $\sigma$ (which plays here the role of $a_0$), there exists only one MOPS $\{R_n\}_{n \geq 0} \equiv \{R_n(\cdot, \sigma)\}_{n \geq 0}$ which is a solution of (43) and it can be characterized by the following conditions:

1. if $\sigma = 0$, then necessarily $\sigma_n = 0, n \geq 1$ and

$$R_n \equiv H_n, \quad n \geq 0;$$

2. if $\sigma \neq 0$, then, putting $c := \frac{1}{2} \sigma$, we must have $H_n(c) \neq 0$ for all $n \geq 1$, the compatible parameters $\sigma_n$ are given by
and \( \{ R_n \}_{n \geq 0} \) is the MOPS such that the coefficients of the corresponding three-term recurrence relation are given by
\[
\beta_n = \frac{H_{n+2}(c)}{H_{n+1}(c)} - \frac{H_{n+1}(c)}{H_n(c)}, \quad n \geq 0,
\]
\[
\gamma_n = \frac{n}{2} \frac{H_{n-1}(c)H_{n+1}(c)}{H_n^2(c)}, \quad n \geq 1.
\]

We recall (see [4], pp. 146) that the monic Hermite polynomial of degree \( n \) can be expressed as
\[
H_n(x) = \frac{n!}{2^n} \sum_{k=0}^{[n/2]} \frac{(-1)^k (2x)^{n-2k}}{(n-2k)!k!}
\]
([n/2] denotes the largest integer not exceeding \( n/2 \)), so that we can, in fact, to compute the expressions for \( \sigma_n, \beta_n \) and \( \tilde{\gamma}_n \).

(II) The Laguerre case

Now, let \( \{ P_n \}_{n \geq 0} \) be the Laguerre MOPS \( \{ L_n^{(\alpha)} \}_{n \geq 0} \), in relation (3). Again, our problem is to characterize all MOPS's \( \{ R_n \}_{n \geq 0} \) and the corresponding sequences of parameters \( \{ \alpha_n \} \) for which
\[
(44) \quad L_n^{(\alpha)}(x) = \frac{1}{n+1} R_{n+1}^{(\alpha)}(x) - \sigma_n R_n^{(\alpha)}(x), \quad n \geq 1.
\]

We know that the sequence \( \{ L_n^{(\alpha)} \}_{n \geq 0} \) is orthogonal for \( \alpha \neq -1, -2, -3, \ldots \). Therefore, in order to apply (10), we must distinguish the cases \( \alpha = 0 \) and \( \alpha \neq 0 \). If \( \alpha = 0 \), it is well known (see [17], pp. 113, formula (6) – after normalization) that
\[
L_n^{(0)}(x) = \frac{1}{n+1} L_{n+1}^{(0)}(x) - L_n^{(0)}(x), \quad n \geq 1
\]
so that the sequence \( \{ R_n \}_{n \geq 0} \equiv \{ L_n^{(0)} \}_{n \geq 0} \) fulfills the equation (44) for \( \alpha = 0 \), with \( \sigma_n = -1 \) for all \( n \). If \( \alpha \neq 0 \), that is \( \alpha \neq 0, -1, -2, \ldots \), according with (10), (44) is equivalent to
\[
L_n^{(\alpha)}(x) = R_{n+1}(x) - (n+1)\sigma_n R_n(x), \quad n \geq 1
\]
or
\[
L_n^{(\alpha)}(x) = R_{n+1}(x) - (n+1)\sigma_n R_n(x) - b_n, \quad n \geq 1
\]
and
\[
R_0(x) = 1, \quad L_1^{(\alpha)}(x) = R_1(x) - \sigma,
\]
for arbitrary parameters \( \sigma \) and \( b_n \). The coefficients of the three-term recurrence relation for the sequence \( \{ L_n^{(\alpha)} \}_{n \geq 0} \) are given by
\[
\beta_n = 2n + \alpha, \quad \gamma_n = (n+1)(n+\alpha), \quad n \geq 0
\]
and then, it follows that the only MOPS \( \{ R_n \}_{n \geq 0} \equiv \{ R_n(\cdot, \sigma) \}_{n \geq 0} \) which are solutions of (44) can be characterized by the following conditions
1. if $\sigma = 0$, then $\sigma_n = 0$, $n \geq 1$ and
   \[ R_n \equiv L_n^{(\alpha - 1)}, \quad n \geq 0; \]
2. if $\sigma \neq 0$,
   \[ L_n^{(\alpha - 1)}(c) \neq 0, \quad \sigma_n = (n + \alpha) \frac{L_n^{(\alpha - 1)}(c)}{L_{n+1}^{(\alpha - 1)}(c)}, \quad n \geq 1 \]
   where $c := \alpha(1 + 1/\sigma)$, and the corresponding coefficients of the three-term recurrence relation can be obtained from (42) or (33).

   Also, we remember that the monic Laguerre polynomials are given explicitly by ([41, pp. 145]
   \[ L_n^{(0)}(x) = (-1)^n n! \sum_{k=0}^{\infty} \binom{n + \alpha}{n - k} \frac{(-x)^k}{k!}. \]
   In particular, for the choice $\sigma = -1$, we have $c = 0$ and, since the $n$th monic Laguerre polynomial satisfies $L_n^{(\alpha)}(0) = (-1)^n(\alpha + 1)_n$, we find that
   \[ \sigma_n = -1, \quad n \geq 1. \]
   According with (42), in this case the coefficients of the three-term recurrence relation for the sequence \{\(R_n(x, -1)\)\} are given by
   \[ \tilde{\beta}_n = 2n + \alpha + 1, \quad \tilde{\gamma}_n = n(n + \alpha), \]
   which shows that
   \[ R_n(x, -1) \equiv L_n^{(\alpha)}(x), \quad n \geq 0. \]

(III) The Bessel case

If \{\(P_n\)\} is the Bessel MOPS \{\(B_n^{(\alpha)}\)\} we have to distinguish the three cases $\alpha = -1$, $\alpha = 0$ and $\alpha \neq 0, -1, -2, \ldots$. We don’t consider here the first two cases, because the relations (10) are not applicable to go from \(B_n^{(\alpha)}\) to \(B_n^{(\alpha - 2)}\). In the case $\alpha \neq 0, -1, -2, \ldots$, following the same technique as in the Hermite and Laguerre cases, one can see that the only MOPS’s \{\(R_n\)\} \(\equiv \{R_n(\cdot, \sigma)\}\) which are solutions of
   \[ B_n^{(\alpha)}(x) = \frac{1}{n + 1} R'_{n+1}(x) - \sigma_n R_n'(x), \quad n \geq 1 \]
   can be characterized by the conditions
1. if $\sigma = 0$, then $\sigma_n = 0$, $n \geq 1$ and
   \[ R_n \equiv B_n^{(\alpha - 2)}, \quad n \geq 0; \]
2. if $\sigma \neq 0$,
   \[ B_n^{(\alpha - 2)}(c) \neq 0, \quad \sigma_n = \frac{-4(n + \alpha - 1)}{(2n + \alpha - 1)(2n + \alpha)^2(2n + \alpha + 1)} \frac{B_n^{(\alpha - 2)}(c)}{B_{n+1}^{(\alpha - 2)}(c)}, \quad n \geq 1 \]
where \( c := -2/\alpha[1 + 2/(\alpha(\alpha + 1))], \) and the corresponding coefficients of the three-term recurrence relation can be obtained from (42) or (33).

The explicit representation for the monic Bessel polynomial of degree \( n \) is ([4], pp. 183)

\[
B_n^{(\alpha)}(x) = \frac{2^n}{(n + \alpha + 1)_n} \sum_{k=0}^{n} \binom{n}{k} (n + \alpha + 1)_k \left( \frac{x}{2} \right)^k.
\]

In particular the choice \( \sigma = -2/[(\alpha + 1)] \) yields to \( c = 0 \) and, since \( B_n^{(\alpha)}(0) = 2^n/(n + \alpha + 1)_n \), we find

\[
\sigma_n = \frac{-2}{(2n + \alpha)(2n + \alpha + 1)}, \quad n \geq 1
\]

so that

\[
\tilde{\beta}_n = \frac{-2(\alpha - 1)}{(2n + \alpha - 1)(2n + \alpha + 1)}, \quad n \geq 0
\]

and

\[
\tilde{\gamma}_n = \frac{-4n(n + \alpha - 1)}{(2n + \alpha - 2)(2n + \alpha - 1)^2(2n + \alpha)}, \quad n \geq 1,
\]

which means that

\[
R_n\left(x, \frac{-2}{\alpha(\alpha + 1)}\right) \equiv B_n^{(\alpha - 1)}(x), \quad n \geq 0.
\]

(IV) The Jacobi case

Finally, consider the case in which \( \{P_n\}_{n \geq 0} \) is the Jacobi system \( \{P_n^{(\alpha, \beta)}\}_{n \geq 0} \). We do not discuss here the cases \( \beta + \alpha = 0 \) and \( \beta + \alpha + 1 = 0 \), because in these cases relations (10) are not applicable to go from \( P_n^{(\alpha, \beta)} \) to \( P_n^{(\alpha - 1, \beta - 1)} \). If \( \beta + \alpha \neq 0 \) and \( \beta + \alpha + 1 \neq 0 \) (and, of course, \( \alpha \) and \( \beta \) subject to their general restrictions), we apply the same technique as before, to conclude that the MOPS's \( \{R_n(\cdot, \sigma)\}_{n \geq 0} \) solutions of

\[
P_n^{(\alpha, \beta)}(x) = \frac{1}{n + 1} \left( R_{n+1}'(x) - \sigma_n R_n'(x) \right), \quad n \geq 1
\]

are characterized in the following way:

1. if \( \sigma = 0 \), then \( \sigma_n = 0, n \geq 1 \) and

\[
R_n \equiv P_n^{(\alpha - 1, \beta - 1)}, \quad n \geq 0;
\]

2. if \( \sigma \neq 0 \), then \( P_n^{(\alpha - 1, \beta - 1)}(c) \neq 0 \) holds for \( n \geq 1 \) and

\[
\sigma_n = \frac{4(n + \alpha)(n + \beta)(n + \alpha + \beta - 1)}{(2n + \alpha + \beta - 1)(2n + \alpha + \beta)^2(2n + \alpha + \beta + 1)} \frac{P_n^{(\alpha - 1, \beta - 1)}(c)}{P_{n+1}^{(\alpha - 1, \beta - 1)}(c)}, \quad n \geq 1
\]

where
Here, for the monic Jacobi polynomial of degree $n$ ([4], pp. 144),

$$P_n^{(\alpha, \beta)}(x) = \frac{1}{\frac{2n+\alpha+\beta}{n}} \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \binom{n+\beta}{k} (x-1)^k (x+1)^{n-k}.$$ 

In particular, the choice $\sigma = -2\alpha/[(\alpha+\beta)(\alpha+\beta+1)]$, gives $c = -1$, and according with the relation $P_n^{(\alpha, \beta)}(-1) = (-1)^n 2^n (\beta+1)_n/(n+\alpha+\beta+1)_n$, we find

$$\sigma_n = \frac{-2(n+\alpha)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}, \quad n \geq 1$$

and

$$R_n \left( x, \frac{-2\alpha}{(\alpha+\beta)(\alpha+\beta+1)} \right) \equiv P_n^{(\alpha-1, \beta)}(x), \quad n \geq 0.$$ 

Next, we will give a characterization for problem $P_1$, also under the hypothesis that the given system, $\{R_n\}_{n \geq 0}$, is some classical one. This will be a corollary of the following property.

**Theorem 2.** Let $\{Q_n\}_{n \geq 0}$ be a MOPS, $\{a_n\}_{n \geq 0}$ a sequence of complex parameters and $\{P_n\}_{n \geq 0}$ a simple set of monic polynomials, such that

$$Q_n(x) = Q_n(x) - a_n Q_{n-1}(x), \quad n \geq 1.$$ 

Suppose also that $\{\bar{\beta}_n, \bar{\gamma}_n\}_{n \geq 0}$ is the set of parameters of the three-term recurrence relation of the sequence $\{Q_n\}_{n \geq 0}$.

Then, the orthogonality of the sequence $\{P_n\}_{n \geq 0}$ depends at most of the choice of the pair $(a_1, a_2)$.

1. If $a_1 = 0$, then $\{P_n\}_{n \geq 0}$ is a MOPS if and only if

$$a_n = 0, \quad n \geq 2$$

case in which

$$\{P_n\}_{n \geq 0} \equiv \{Q_n\}_{n \geq 0};$$

2. if $a_1 \neq 0$, then $\{P_n\}_{n \geq 0}$ is a MOPS if and only if

$$Q_n(\alpha) - \lambda Q_{n-1}^{(1)}(\alpha) \neq 0, \quad a_n = \frac{Q_n(\alpha) - \lambda Q_{n-1}^{(1)}(\alpha)}{Q_{n-1}(\alpha) - \lambda Q_{n-2}^{(1)}(\alpha)}, \quad n \geq 1$$

where $\alpha$ and $\lambda$ are determined in terms of $a_1$ and $a_2$ (and the data $\bar{\beta}_0, \bar{\beta}_1$ and $\bar{\gamma}_1$) by

$$\alpha := a_2 + \bar{\beta}_1 + \bar{\gamma}_1/a_1, \quad \lambda := \alpha - \bar{\beta}_0 - a_1.$$ 

Furthermore, the coefficients $\beta_n$ and $\gamma_n$ of the three-term recurrence relation satisfied by $\{P_n\}_{n \geq 0}$ are given by
\[ \beta_n = \tilde{\beta}_n + a_{n+1} - a_n, \quad n \geq 0 \quad (a_0 \equiv 0) \]
\[ \gamma_n = \tilde{\gamma}_n + a_n(\tilde{\beta}_n - \tilde{\beta}_{n-1} + a_{n+1} - a_n), \quad n \geq 1, \]

(this formula still remains valid in case 1), or, alternatively, by

\[ \begin{cases} 
\beta_0 = \alpha - \lambda \\
\beta_n = \alpha - a_n - \tilde{\gamma}_n/a_n, \quad n \geq 1 \\
\gamma_n = a_n(\alpha - \tilde{\beta}_{n-1} - a_n), \quad n \geq 1.
\end{cases} \]

Finally, \( \{P_n\}_{n \geq 0} \) is a MOPS with respect to the linear functional

\[ u = v_0 \delta_\alpha - \lambda(x - \alpha)^{-1}v, \]

where \( v \) denotes the functional corresponding to the given MOPS \( \{Q_n\}_{n \geq 0}, v_0 := (v, 1) \) and \( \delta_\alpha \) is the Dirac measure at the point \( \alpha \).

**Proof.** Assume first that \( \{P_n\}_{n \geq 0} \) is a MOPS, and let \( \{\beta_n, \gamma_{n+1}\}_{n \geq 0} \) be the corresponding set of coefficients of the three-term recurrence relation. We can apply Lemma 1 (with \( b_n \equiv 0 \) for all \( n \)) and we proceed as in the proof of Theorem 1. Hence, the conclusion of the theorem in the case \( a_1 = 0 \) is trivial (as in the proof of Theorem 1). In the case \( a_1 \neq 0 \), we deduce that there exists a constant \( \alpha \) such that

\[ a_{n+1} + \tilde{\beta}_n + \tilde{\gamma}_n/a_n = \alpha, \quad n \geq 1. \]

Now, define a sequence \( \{y_n\}_{n \geq 0} \) as in the proof of Theorem 1 by \( y_0 = 1 \) and \( y_{n+1} = a_{n+1} y_n \), and conclude that

\[ y_{n+1} = (\alpha - \tilde{\beta}_n) y_n - \tilde{\gamma}_n y_{n-1}, \quad n \geq 1 \]
\[ y_1 = \alpha - (\tilde{\beta}_0 + \lambda), \]

where \( \lambda = \alpha - (\tilde{\beta}_0 + a_1) \) (hence \( \gamma_1 = a_1 \lambda \), and then \( \lambda \neq 0 \)). Therefore, it follows that

\[ y_n = Q_n^*(\alpha; \lambda, 0), \quad n \geq 0 \]

where \( \{Q_n^*(x; \lambda, 0)\}_{n \geq 0} \) denotes the co-recursive sequence at level zero corresponding to the sequence \( \{Q_n\}_{n \geq 0} \) and to the modification \( \lambda \) (see [9]). It is known (see [9], pp. 205, formula (4)) that \( Q_n^*(x; \lambda, 0) = Q_n(x) - \lambda Q_{n-1}^{(1)}(x) \), so that

\[ y_n = Q_n(\alpha) - \lambda Q_{n-1}^{(1)}(\alpha), \quad n \geq 1. \]

The conclusion of the proof is now easy, following the same steps as in the proof of Theorem 1. \( \Box \)

**Remark.** This result is related to a result stated by P. Maroni in [13], and a similar result was founded in [3], with a different proof.

The previous theorem solve the problem of coherence when \( \{R_n\}_{n \geq 0} \) is a
classical MOPS. In fact, if \( \{R_n\}_{n \geq 0} \) is classical, then the sequence \( \{Q_n\}_{n \geq 0} \), where
\[
Q_n := \frac{R_{n+1}'}{n+1}, \quad n \geq 0
\]
is also a (classical) MOPS and, therefore, relation (3) can be written as
\[
P_n(x) = Q_n(x) - n\sigma_n Q_{n-1}(x), \quad n \geq 1.
\]
Hence, we apply Theorem 2 with \( a_n = n\sigma_n \) for \( n \geq 1 \) and then a characterization for the solution of problem P1, in the case in which the given MOPS is some classical one, is the following:

**Corollary 1.** Let \( \{R_n\}_{n \geq 0} \) be a classical MOPS, \( \{\sigma_n\}_{n \geq 1} \) a sequence of complex numbers and \( \{P_n\}_{n \geq 0} \) a set of monic polynomials such that
\[
(51) \quad P_n = \frac{1}{n+1} R_{n+1}' - \sigma_n R_n', \quad n = 1, 2, \ldots
\]
then, the orthogonality of the sequence \( \{P_n\}_{n \geq 0} \) depends at most of the choice of the pair \( (\sigma_1, \sigma_2) \). More precisely, one has:
1. If \( \sigma_1 = 0 \), then \( \{P_n\}_{n \geq 0} \) is a MOPS if and only if
\[
\sigma_n = 0, \quad n \geq 2;
\]
in such a case,
\[
\{P_n\}_{n \geq 0} = \left\{ \frac{R_{n+1}'}{n+1} \right\}_{n \geq 0}.
\]
2. If \( \sigma_1 \neq 0 \), then for each choice of \( \sigma_2 \) the set \( \{P_n\}_{n \geq 0} \) is a MOPS if and only if
\[
Q_n(\alpha) - \lambda Q_{n-1}'(\alpha) \neq 0, \quad \sigma_n = \frac{1}{n} \frac{Q_n(\alpha) - \lambda Q_{n-1}'(\alpha)}{Q_{n-1}(\alpha) - \lambda Q_{n-2}'(\alpha)}, \quad n \geq 1
\]
where
\[
Q_n(x) := \frac{1}{n+1} R_{n+1}'(x), \quad n \geq 0
\]
(which is also a MOPS - since \( \{R_n\}_{n \geq 0} \) is classical), \( \{Q_{n}'\}_{n \geq 0} \) means the first kind associated OPS corresponding to the OPS \( \{Q_n\}_{n \geq 0} \), and \( \alpha \) and \( \lambda \) are complex numbers which can be expressed explicitly in terms of \( \sigma_1 \) and \( \sigma_2 \) (and the data) as
\[
\alpha = 2\sigma_2 + \beta_1' + \frac{\gamma_1'}{\sigma_1}, \quad \lambda = \alpha - (\beta_0' + \sigma_1).
\]
In these expressions, \( \beta_0', \beta_1' \) and \( \gamma_1' \) belong to the set \( \{\beta_n', \gamma_{n+1}'\}_{n \geq 0} \) of the coefficients of the three-term recurrence relation for the monic (classical) orthogonal system \( \{Q_n\}_{n \geq 0} \), which can be obtained directly by the change \( p \rightarrow p + 2a \) and \( q \rightarrow q + b \) in both formulas (8) and (9), so that

302
where we consider that \( \phi \equiv ax^2 + bx + c \) and \( \psi \equiv px + q \) are the polynomials that appear in the distributional equation \( D(\phi v) = \psi v \) which the linear functional \( v \) corresponding to the given classical sequence \( \{R_n\}_{n \geq 0} \) satisfies.

Furthermore, the coefficients of the three-term recurrence relation for the sequence \( \{P_n\}_{n \geq 0} \) are given by

\[
\beta_0' = \frac{-q + b}{p + 2a}, \quad \beta_1' = \frac{-p(q + b) + 2b(p + 2a)}{(p + 2a)(p + 4a)},
\]

\[
\gamma_1' = \frac{1}{p + 3a} \phi \left( \frac{-q + b}{p + 2a} \right)
\]

Finally, \( \{P_n\}_{n \geq 0} \) is a MOPS with respect to the linear functional \( u \) defined as

\[
u = \langle v, \phi \rangle \delta_\alpha - \lambda(x - \alpha)^{-1}(\phi v),
\]

where \( \delta_\alpha \) is the Dirac measure at the point \( \alpha \).

4. FURTHER REMARKS

The problem of coherence for a pair of orthogonal polynomials is related with some more general models of problems in the theory of orthogonal polynomials. One of this kind of problems is the following:

Given two linear regular functionals \( u \) and \( v \), with relative MOPS \( \{P_n\}_{n \geq 0} \) and \( \{R_n\}_{n \geq 0} \), respectively, and assuming that it is known some relation between \( u \) and \( v \), let say \( f(u, v) = 0 \), to determine expressions between the elements of the sequences \( \{P_n\}_{n \geq 0} \) and \( \{R_n\}_{n \geq 0} \).

The study of this kind of problem – which we call ‘direct problem’ – leads to study the corresponding ‘inverse problem’:

If some relation involving the elements of the MOPS’s \( \{P_n\}_{n \geq 0} \) and \( \{R_n\}_{n \geq 0} \) is known, to characterize the corresponding regular functionals \( u \) and \( v \), and to give explicit relations between them.

One important problem of this kind is the following:

(P): Given two MOPS’s \( \{P_n\}_{n \geq 0} \) and \( \{R_n\}_{n \geq 0} \), to characterize the corresponding regular functionals \( u \) and \( v \) such that there exist non-negative integers \( s \) and \( p \) and a (fixed) polynomial \( \phi \) of degree \( p \) such that

\[
\phi(x)R_{n+1}'(x) = \sum_{k=n-s}^{n+p} \lambda_{nk} P_k(x), \quad n \geq s,
\]
with $\lambda_{n,n-s} \neq 0$ and the convention $\lambda_{ni} = 0$ for $i < 0$.

This problem was solved by S. Bonan et al. in [2] when $u$ and $v$ are positive measures, and the solution of the problem in the general case was provided by F. Marcellán et al. in [13]: $u$ and $v$ are necessarily semiclassical functionals, related by

$$\phi(x)u = h(x)v,$$

where $h(x)$ is a polynomial, given explicitly by

$$h(x) = \langle u, \phi \rangle [P_1(y)K^{(0,1)}_{s+2}(x,y) - P_1(x)K^{(0,1)}_{s+1}(x,y)],$$

and $K^{(r,s)}_n(x,y)$ is the generalized kernel

$$K^{(r,s)}_n(x,y) = \sum_{j=0}^{n} \frac{R_j^{(r)}(x)R_j^{(s)}(y)}{\langle v, R_j^2 \rangle}.$$  

(We consider the normalization $u_0 \equiv \langle u, 1 \rangle = 1$.)

Remark that, in the particular case when $s = 0$ in (52), $p \leq 2$ and $R_n \equiv P_n$, relation (53) leads, trivially, to a well-known characterization of the classical orthogonal polynomials due to W. Al-Salam and T.S. Chihara [1].

Another model of inverse problem is the following:

(P'): Given two MPS $\{P_n\}_{n \geq 0}$ and $\{R_n\}_{n \geq 0}$ orthogonal with respect to the linear functionals $u$ and $v$, to characterize $u$ and $v$ such that there exist non-negative integers $h$ and $t$ and a polynomial $p$ of degree $t$ such that

$$\pi(x)P_n(x) = \sum_{k=n-h}^{n+t} a_{nk} R_{k+1}(x), \quad n \geq h,$$

where $a_{nk}$ are complex parameters, with the convention $a_{ni} = 0$ if $i < 0$.

There are some particular 'trivial' cases for which the solution of this problem is well known:

1. if $\pi \equiv 1$ and $h = 0$, then (55) is reduced to

$$P_n(x) = \frac{1}{n+1} R_{n+1}(x), \quad n \geq 0;$$

since we assume that both sequences $\{P_n\}_{n \geq 0}$ and $\{R_n\}_{n \geq 0}$ are MOPS, the above expression implies that $\{R_{n+1}\}_{n \geq 0}$ is also a MOPS; consequently, from the Hahn characterization for the classical OPS's, $\{P_n\}_{n \geq 0}$ and $\{R_n\}_{n \geq 0}$ are classical MOPS of the same type.

2. if $\pi \equiv 1$, $h = 2$ and $R_n \equiv P_n$, (55) can be reduced to

$$P_n(x) = \frac{1}{n+1} P_{n+1}(x) + r_n P_n(x) + s_n P_{n-1}(x), \quad n \geq 2.$$

In this case, $\{P_n\}_{n \geq 0}$ is also a classical orthogonal polynomial system (see [7]).
In general, if the solution for problem (P') exists, then it is characterized in
the following way (this result was stated – in a preliminary version – by the
authors of [8], and the proof can be found in [16]):

(i) the functional \( u \) is semiclassical of class at most \( h + t \), and satisfies the
differential distributional equation

\[
D(\Phi u) = \Psi u,
\]

where \( \Phi \) and \( \Psi \) are polynomials of degrees \( \leq h + t + 2 \) and \( h + t + 1 \), defined,
respectively, by

\[
\Phi(x) = R_2(x)\theta_{00}(x) - R_1(x)\theta_{10}(x)
\]

and

\[
\Psi(x) = R_2(x)\theta_{01}(x) - R'_2(x)\theta_{00}(x) - R_1(x)\theta_{11}(x) + \theta_{10}(x),
\]

where, for each \( m \in \{0, 1\} \), \( \theta_{nm} \) is the polynomial defined as

\[
\theta_{nm}(x) = (-1)^{m+1}(m+1)\pi^{(m)}(x)\langle v, R_{n+1}^2 \rangle \sum_{j=n-t}^{n+h} \frac{a_{jn}}{\langle u, P_j^2 \rangle} P_j(x),
\]

(remark that \( \text{deg} \theta_{nm} \leq n + h + t - m \));

(ii) the functionals \( u \) and \( v \) are related by

\[
\pi^2(x)\Phi(x)v = \pi(x)\theta(x)u,
\]

where \( \theta(x) \) is a polynomial of degree \( \leq 2h + t \),

\[
\theta(x) = -\langle v, R_1^2 \rangle \sum_{j=0}^{h} \frac{a_{j0}}{\langle u, P_j^2 \rangle} P_j(x)(\theta_{11}(x) + \theta'_{10}(x)) + \langle v, R_2^2 \rangle \sum_{j=0}^{h+1} \frac{a_{j1}}{\langle u, P_j^2 \rangle} P_j(x)(\theta_{01}(x) + \theta'_{00}(x));
\]

(iii) the functional \( v \) is also semiclassical, of class at most \( 4h + 6t + 2 \), and
satisfies the equation

\[
D(\Phi \bar{v}) = \bar{\Psi} \bar{v},
\]

with \( \bar{\Phi} \) and \( \bar{\Psi} \) polynomials of degrees less than or equal to \( 4h + 6t + 4 \) and
\( 4h + 6t + 3 \),

\[
\bar{\Phi}(x) = (\pi^3\Phi^2\theta)(x), \quad \bar{\Psi}(x) = \pi^2\Phi[\pi\Psi\theta + 2\Phi(\pi'\theta + \pi\theta')](x).
\]

Remark that, in the formulations of problems (P) and (P'), we have assumed
regularity, that is, orthogonality of both sequences \( \{P_n\}_{n \geq 0} \) and \( \{R_n\}_{n \geq 0} \).

Of course, an interesting problem is the following: if we assume that one of
the sequences \( \{P_n\}_{n \geq 0} \) or \( \{R_n\}_{n \geq 0} \) is orthogonal in relation (55) of problem (P')
– for example – , to give necessary and sufficient conditions for the regularity of
the other sequence, including the determination of the compatible sequences of
parameters \( a_{nk} \). This was the question about what we were interested in
the previous section of this paper. In fact, for \( \pi \equiv 1 \) and \( h = 1 \), relation (55) reduces
to a relation of the form

305
\[ P_n = \frac{1}{n+1} R'_{n+1} - \sigma_n R'_n, \quad n \geq 1. \]

According with the results above, we can see that the framework of this kind of questions is the theory of semiclassical orthogonal polynomials (see [11], [12] and [14], e.g.).

ACKNOWLEDGEMENTS

This paper was finished during a stay of the second author in Departamento de Ingeniería, Universidad Carlos III de Madrid, with financial support of a Grant from Junta Nacional de Investigación Científica e Tecnológica (JNICT) – BD976 – and Centro de Matemática da Universidade de Coimbra (CMUC).

The work of the first author was supported by Dirección General de Ciencia y Tecnología (DGICYT) of Spain under Grant PB93-0228-C02-01.

REFERENCES