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On the Dependence of the Zeros of an Entire Function and Its Factorization

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Some criteria for testing whether certain types of entire functions are prime or not are given in this paper. Most of the studies are based upon the distribution of the zeros of the function or its derivative.

I. INTRODUCTION

According to [1], an entire function $F(z) = f(g)(z)$ is said to possess $f(z)$ as left factor and $g(z)$ as right factor, when they are both non-linear and entire. A prime (pseudo-prime) function $F(z)$ is one which cannot be factorized in the form $f(g)(z)$, where f and g are non-linear entire functions (polynomials).

Generally it is not easy to tell whether a given function is prime (pseudo-prime) or not. In [8], Rosenbloom stated that $e^z + z$ is prime and that the proof is complicated. Later on, Gross [6] proved it and generalized the previous result by showing that every entire function of the form $e^z + P(z)$ is prime, where $P(z)$ is a non-constant polynomial. The methods used in [1] and [6] depend heavily on the fact e^z is a periodic function.

In this paper we shall study some similar classes of functions from another viewpoint. Our emphasis will be put on the distribution of the zeros of a given function.

Here some new results which cannot be obtained by using the methods in [1] and [6] will be achieved. We shall denote by L and R the classes of all entire functions which possess no non-linear polynomials as their left and right factors respectively.

II. LEMMAS AND MAIN RESULTS

We begin with some of the lemmas in the proof of our theorems.

LEMMA 1 (Briot and Bouquet [3]). *If a solution of an algebraic differential equation of the first order,* $P(\omega, \omega') = 0$, *is uniform in the plane, then the* *solution is a rational function or a rational function of e^{bz} (b a constant), or an elliptic function.*

LEMMA 2. *Let $P(z)$ and $Q(z)$ ($\not\equiv$ constant) be polynomials, and $g(z)$ a non-constant entire function. Then $P(z)\, e^{g(z)} + Q(z) \in L$.*

Proof. Assume that the Lemma is false. Then there exist a non-linear polynomial $P_1(z) \equiv a_0 z^n + a_1 z^{n-1} + \cdots + a_n$ and a transcendental entire function $h(z)$ such that

$$P_1(h)(z) = P(z)\, e^{g(z)} + Q(z). \tag{1–1}$$

Then

$$a_0 h^n(z) + a_1 h^{n-1}(z) + \cdots + a_n = P(z)\, e^{g(z)} + Q(z). \tag{1–2}$$

or

$$a_0 h^n(z) + a_1 h^{n-1}(z) + \cdots + a_n - Q(z) = P(z)\, e^{g(z)}. \tag{1–3}$$

Now it is easy to see that Clunie's Theorem [4] is applicable to Eq. (1–3).

It follows that the leftside of (1–3) can be expressed in a binomial form.

Thus we have

$$a_0\{h(z) + c\}^n = P(z)\, e^{g(z)}, \tag{1–4}$$

where c is a constant.

From this and (2) we deduce

$$b_1 h^{n-1}(z) + b_2 h^{n-2}(z) + \cdots + b_{n-1} h(z) \equiv Q(z) - a_n - a_0 c^n, \tag{1–5}$$

where b_i $(i = 1, 2, \ldots, n-1)$ are constants.

This is impossible because the leftside of (1–5) is either transcendental entire or identically equal to zero, and, on the other hand, the rightside of (1–5) is a non-zero polynomial. Our Lemma is thus proved.

LEMMA 3 (Pólya [7]). *Suppose that $f(z)$, $g(z)$ are entire functions and that $\varphi(z) = g(f(z))$ has finite order. Then either f is a polynomial or $g(z)$ has zero order.*

The following lemma which will play a big role in the proofs of our theorems is due to Edrei [5].

LEMMA 4. *Let $f(z)$ be an entire function. Assume that there is an unbounded sequence $\{a_i\}_{i=1}^{\infty}$ such that all but a finite number of the roots of the equations $f(z) = a_i$ $(i = 1, 2, \ldots)$ lie on a straight line; then $f(z)$ is a polynomial of degree not greater than two.*

THEOREM 1. *Let F be an entire function of finite order with infinitely many zeros. Assume that all but a finite number of the zeros of F lie on a straight line.*

Then F is pseudo-prime. Furthermore the only possible right factors are polynomials of degree two.

Proof. Assume that $F = f(g)$. If f and g are not polynomials, then by virtue of Lemma 3 f has zero order. Therefore F has infinitely many zeros $\{a_i\}_{i=1}^{\infty}$. Hence all except a finite number of the roots of the equations $g(z) = a_i$ $(i = 1, 2,...)$ lie on a straight line. It follows from Lemma 4 that g is a polynomial of degree two. Thus the theorem is proved.

In a similar manner, we can obtain the following:

THEOREM 2. *Let F be an entire function of finite order with F′ has infinitely many zeros. Assume that all but a finite number of the zeros of F′ lie on a straight line. Then F is pseudo-prime.*

Furthermore the only possible right factors of F are polynomials of degree two.

As an illustration we prove the following:

THEOREM 3. $F(z) = e^z + z$ *is prime.*

Proof. It is trivial to check that F satisfies the assumptions of Theorem 2; therefore F is pseudo-prime. By Lemma 2 $F \in L$. It follows from this and Lemma 4 that $F = f(z^2 + c)$, an even function where f is entire and c is a constant, if F is not prime. This leads to a contradiction because $e^z + z$ is not an even function, and our proof is complete.

THEOREM 4. *Let $P(z)$ be a non-zero polynomial, and c a non-zero constant; then $P(z)\, e^z + cP(z)$ is prime.*

Proof. The argument will be the same as above. We omit the details.

THEOREM 5. *Let $P(z)$ be a polynomial, and let c be a non-zero constant; then*

$$F(z) = \int_0^z [P(z)\, e^z + cP(z)]\, dz$$

is prime

Proof. We note that $F(z)$ has the form $P_1(z)\, e^z + P_2(z)$, where P_i $(i = 1, 2)$ are polynomials. Then by Lemma 2 $F \in L$. This and Theorem 2 shows that the only possible factorizations of F will be of the form $F(z) = f(z^2 + c)$, where f is an entire function and c is a constant. That is

$$P_1(z) + P_2(z) = f(z^2 + c). \tag{1}$$

So

$$P_1(-z)\,e^{-z} + P_2(-z) = f(z^2 + c). \tag{2}$$

Combining (1) and (2) yields

$$P_1(z)\,e^{z} + P_2(z) = P_1(-z)\,e^{-z} + P_2(-z) \tag{3}$$

or

$$P_1(z)\,e^{z} + P_2(z) - P_2(-z) = P_1(-z)\,e^{-z}.$$

It is easily verified that (3) is impossible to hold by virtue of Lemma 2 or simply by considering the behavior of both sides when $z \to \infty$ (along positive real axis). Thus the proof is complete.

EXAMPLE. $(z - 1)\,e^{z} + z^2/2 + c$ is prime.

THEOREM 6. *Let m, n be two positive integers. Then $F(z) = e^{mz} \pm ie^{-nz}$ is pseudo-prime. Furthermore $F \in R$ and every possible right factor of F must be a rational function of e^{bz} (b is an integer).*

Proof. We now prove the case $F(z) = e^{mz} + ie^{-nz}$, that is

$$F(z) = e^{mz}(1 + ie^{-(m+n)z})$$

which will satisfy the hypothesis of Theorem 1. Therefore F is pseudo-prime. Since $e^{mz} + ie^{-nz}$ is not an even function, hence $F \in R$.

Now assume that there exists a polynomial $P(z)$ of degree $k \geq 2$ and an entire function $g(z)$ such that the following relation holds:

$$P(g) = e^{mz} + ie^{-nz}. \tag{4}$$

By differentiating (4),

$$g'P'(g) = me^{mz} - nie^{-nz}. \tag{5}$$

Solving for e^{mz} and e^{-nz} from (4) and (5) we obtain

$$e^{mz} = Q_1(g, g'), \tag{6}$$

$$e^{-nz} = Q_2(g, g'), \tag{7}$$

where $Q_1(x, y)$, $Q_2(x, y)$ are two polynomials in x and y. Then we have

$$[Q_1(g, g')]^n = [Q_2(g, g')]^{-m} \tag{8}$$

or

$$Q_3(g, g') = 0,$$

where $Q_3(x, y)$ is a non-constant polynomial in x and y. By Lemma 1, it follows that g must be a rational function of e^{bz} (b is a constant). Furthermore by a result of the impossibility of certain identity from Borel's [2] we can conclude that b must be an integer. This completes the proof.

COROLLARY. *With the above hypothesis let $n = 2km$, $k \geqslant 1$. Then F is a prime.*

Proof. We only need to show in the present case that $F \in L$.
Suppose that

$$e^{-mz} + ie^{nz} = P(g(z)) \tag{9}$$

for some polynomial $P(z)$ of degree $d \geqslant 2$. Then

$$e^{-m(z+\pi i/m)} + ie^{n(z+\pi i/m)} = P\left(g\left(z + \frac{\pi i}{m}\right)\right). \tag{10}$$

Thus we have

$$-e^{mz} + ie^{nz} = P\left(g\left(z + \frac{\pi i}{m}\right)\right), \tag{11}$$

(9) and (11) yield

$$(1 + i)\, e^{mz} = P\left(g\left(z + \frac{\pi i}{m}\right)\right). \tag{12}$$

This implies that $g(z) - g(z + \pi i/m)$ never vanishes. Therefore we get

$$g\left(z + \frac{\pi i}{m}\right) - g(z) = ce^{\beta z} \tag{13}$$

or

$$g\left(z + \frac{\pi i}{m}\right) = g(z) + ce^{\beta z},$$

where c is a constant and β is an integer (by Theorem 6).
We may write

$$g(z) = \sum_{k=0}^{n_0} A_k e^{\beta_k z}, \tag{14}$$

where A_k are non-zero constants and β_k are integers. We are going to show that $n_0 = 0$. If $n_0 > 0$, then we can assume β_k in the following order:

$$\beta_0 < \beta_1 < \cdots < \beta_{n_0}. \tag{15}$$

Also

$$P(g) = c_1(g - \alpha_1)(g - \alpha_2) \cdots (g - \alpha_d), \tag{16}$$

where c_1 is a constant, α_i are roots of $P(z) = 0$. From this and (13) we get

$$P\left(g\left(z + \frac{\pi i}{m}\right)\right) = c(g - ce^{\beta z} - \alpha_1)(g - ce^{\beta z} - \alpha_2) \cdots (g - ce^{\beta z} - \alpha_d). \tag{17}$$

It follows from (14), (16), and (17) that

$$c_1(g - ce^{\beta z} - \alpha_1) \cdots (g - ce^{\beta z} - \alpha_d) - c_1(g - \alpha_1)(g - \alpha_2) \cdots (g - \alpha_d) = -(1 + c)\, e^{mz}. \tag{18}$$

Substituting (14) into this and again using the impossibility of certain identity of Borel's to compare the types of both sides of (18) we find

$$\beta_i = \beta_j \tag{19}$$

for some i and j with $i \neq j$, which gives a contradiction to (15).

Thus we have to conclude that $n_0 = 0$.

Therefore

$$g(z) = A_0 e^{\beta 0 z}. \tag{20}$$

But then (9) will be impossible to hold.

Our assertion is thus proved.

THEOREM 7. *$F(z) = \sin z - z$ is prime.*

Proof. Suppose the theorem is false; then we have $F = f(g)$, where f and g are non-linear entire. If f is transcendental, then by Theorem 2, g is a polynomial of degree two. Therefore $F = h(z^2 + c)$, where h is an entire function and c is constant. Then F would be an even function but it is not the case. Thus $F \in R$.

Now we proceed to show that $F \in L$. If g is transcendental, then f is a polynomial of degree $k \geqslant 2$. Therefore $F' = f'(g)\, g'$ or

$$\cos z - 1 = g' f'(g) = c_0 g'(g - c_1)^{n_1} \cdots (g - c_k)^{n_k}, \tag{21}$$

where c_i $(i = 0, 1, 2, \ldots, k)$ are constants and n_i $(i = 1, 2, \ldots, k)$ are positive integers.

By observing the fact that g' in (21) must be a perfect square and comparing the power expansions of both sides of (21) one can conclude that (21) is impossible; hence $f \in L$.

Thus the Theorem is proved.

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