Available online at www.sciencedirect.com
ScienceDirect
Journal of Combinatorial Theory, Series A 115 (2008) 485–504
Journal of Combinatorial Theory Series A
www.elsevier.com/locate/jcta

Forbidden patterns and shift systems

José María Amigó [a,1], Sergi Elizalde [b], Matthew B. Kennel [c]

[a] *Centro de Investigación Operativa, Universidad Miguel Hernández, Avda. de la Universidad s/n, 03202 Elche, Spain*
[b] *Department of Mathematics, Dartmouth College, Hanover, NH 03755-3551, USA*
[c] *Institute for Nonlinear Science, University of California, San Diego, La Jolla, CA 92093-0402, USA*

Received 15 December 2006

Available online 4 September 2007

Abstract

The scope of this paper is two-fold. First, to present to the researchers in combinatorics an interesting implementation of permutations avoiding generalized patterns in the framework of discrete-time dynamical systems. Indeed, the orbits generated by piecewise monotone maps on one-dimensional intervals have forbidden order patterns, i.e., order patterns that do not occur in any orbit. The allowed patterns are then those patterns avoiding the so-called forbidden root patterns and their shifted patterns. The second scope is to study forbidden patterns in shift systems, which are universal models in information theory, dynamical systems and stochastic processes. Due to its simple structure, shift systems are accessible to a more detailed analysis and, at the same time, exhibit all important properties of low-dimensional chaotic dynamical systems (e.g., sensitivity to initial conditions, strong mixing and a dense set of periodic points), allowing to export the results to other dynamical systems via order-isomorphisms.

Keywords: Order patterns; Deterministic and random sequences; Permutations avoiding consecutive patterns; Time series analysis; Dynamical systems; Shift maps

1. Introduction

Order has some interesting consequences in discrete-time dynamical systems. Just as one can derive sequences of symbol patterns from such a dynamic via coarse-graining of the phase space, so it is also straightforward to obtain sequences of *order patterns* if the phase space is linearly ordered. It turns out that, under some mild mathematical assumptions, not all order patterns can

E-mail address: jm.amigo@umh.es (J.M. Amigó).
[1] Fax: +34 96 665 8715.

0097-3165/$ – see front matter
doi:10.1016/j.jcta.2007.07.004

be materialized by the orbits of a given, one-dimensional dynamic. Furthermore, if an order pattern of a given length is 'forbidden,' i.e., cannot occur, its absence pervades all longer patterns in form of more missing order patterns. This cascade of outgrowth forbidden patterns grows super-exponentially (in fact, factorially) with the length, all its patterns sharing a common structure. Of course, forbidden and allowed order patterns can be viewed as permutations; allowed patterns are then those permutations avoiding the so-called forbidden root patterns and their shifted patterns (see Section 4 for an exact formulation). Let us mention at this point that permutations avoiding generalized and consecutive patterns is a popular topic in combinatorics (see, e.g., [4,8,9]). It is in this light that we approach order patterns in the present paper. In fact, the measure-theoretical aspects of the underlying dynamical system play no role in the combinatorial properties of the order patterns defined by its orbits and hence will be only considered when necessary. Also for this reason we will not dwell on the dynamical properties of shift systems and their role as prototypes of chaotic maps once endowed with appropriate invariant measures; see [5,7] for readable accounts.

Order relations belong rather to algebra than to continuous mathematics because of their discrete nature. Only in the standard real line, order and metric are coupled, leading to such interesting results as Sarkovskii's theorem [10,11]. But even in this special though important framework, order fails to be preserved by isomorphisms, that consistently only address dynamical properties such as invariant measures, periodicity, mixing properties, etc., and this reduces its applicability. Yet, order relations have been successfully applied in discrete dynamical systems and information theory, e.g., to evaluate the measure-theoretic and topological entropies [1,6]. This paper is an extension of those investigations. Isomorphisms that preserve the possibly existing order relations of the dynamical systems they identify, are called order-isomorphisms. The order isomorphy in one-dimensional dynamical systems is the subject of *Kneading Theory* [10]. In this paper, we will go beyond the framework of Kneading Theory in two respects: (i) the maps need not be continuous (but piecewise continuous) and (ii) we will also consider more general phase spaces (like finite-alphabet sequence spaces and two-dimensional intervals).

Forbidden order patterns, the only ones we will consider in this paper, should not be mistaken for other sorts of forbidden patterns that may occur in dynamics with constraints. Forbidden patterns in symbol sequences occur, e.g., in Markov subshifts of finite type and, more generally, in random walks on oriented graphs. On the contrary, the existence of forbidden *order* patterns does not entail necessarily any restriction on the patterns of the corresponding symbolic dynamic: the variability of *symbol* patterns is given by the statistical properties of the dynamic. As a matter of fact, the symbolic dynamic of one-dimensional chaotic maps are used to generate pseudo-random sequences, although all such maps used in practice have forbidden order patterns. In general it is very difficult to work out the specifics of the forbidden patterns of a given map, but we will see that shifts on finite-symbol sequence spaces are an important exception: the detailed analysis of the forbidden patterns of this transformations is precisely the topic of this paper.

The existence of forbidden patterns is a hallmark of deterministic orbit generation and thus it can be used to discriminate deterministic from random time series. Indeed, thanks to the super-exponentially growing trail of outgrowth forbidden patterns, the probability of a false forbidden pattern in a truly stochastic process vanishes very fast with the pattern length and, consequently, a time series with missing order patterns of moderate length can be promoted to deterministic with virtually absolute confidence. The quantitative details depend, of course, on the specificities of the process (probability distribution, correlations, etc.). Only those chaotic maps with all forbidden patterns of exceedingly long length seem to be intractable from the practical point of view. Besides, applications need to address some key issues, such as the robustness of the forbidden patterns against observational noise, and the existence of false forbidden patterns in *finite*, random time series. We refer to [3] for these issues.

This paper is organized as follows. In Section 2 we briefly recall the basics of shift systems and symbolic dynamics. The concepts and notation introduced in this section (including the examples) will be used throughout. Order patterns and forbidden root patterns, together with the outgrowth forbidden patterns, are presented in Section 3. The structure of the outgrowth forbidden patterns and their asymptotic growth with the length are discussed in Section 4. Finally, Sections 5 and 6 are devoted to the structure of allowed patterns and the existence of root forbidden patterns in one-sided and two-sided shift systems, respectively. In the examples we present some interesting by-products of the theoretical results.

2. Shift systems and symbolic dynamics

Let us start by recalling some basics of shift systems and symbolic dynamics. We set $\mathbb{N}_0 = \{0\} \cup \mathbb{N} = \{0, 1, 2, \ldots\}$.

Fix $N \geqslant 2$ and consider the measurable space $(\Omega, \mathcal{P}(\Omega))$, where $\Omega = \{0, 1, \ldots, N-1\}$ and $\mathcal{P}(\Omega)$ is the family of all subsets of Ω. Let $(\Omega^{\mathbb{N}_0}, \mathcal{B})$ denote the product space $\Pi_0^\infty(\Omega, \mathcal{P}(\Omega))$, i.e., $\Omega^{\mathbb{N}_0}$ is the space of (*one-sided*) *sequences* taking values on the 'alphabet' Ω,

$$\Omega^{\mathbb{N}_0} = \left\{\omega = (\omega_n)_{n\in\mathbb{N}_0}\colon\ \omega_n \in \Omega\right\},$$

and $\mathcal{B}$ is the sigma-algebra generated by the *cylinder sets*

$$C_{a_0,\ldots,a_n} = \left\{\omega \in \Omega^{\mathbb{N}_0}\colon\ \omega_k = a_k,\ 0 \leqslant k \leqslant n\right\}.$$

The topology generated by the cylinder sets makes $\Omega^{\mathbb{N}_0}$ compact, perfect (i.e., it is closed and all its points are accumulation points) and totally disconnected. Such topological spaces are sometimes called Cantor sets. The elements of Ω are called *symbols* or *letters*. Segments of symbols of length L, like $\omega_k\omega_{k+1}\ldots\omega_{k+L-1}$, will be sometimes shortened ω_k^{k+L-1}.

Furthermore, let $\Sigma : \Omega^{\mathbb{N}_0} \to \Omega^{\mathbb{N}_0}$ denote the (one-sided) *shift transformation* defined as

$$\Sigma : (\omega_0, \omega_1, \omega_2, \ldots) \mapsto (\omega_1, \omega_2, \omega_3, \ldots). \tag{1}$$

All probability measures on $(\Omega^{\mathbb{N}_0}, \mathcal{B})$ which make Σ a measure-preserving transformation are obtained in the following way [12]. For any $n \geqslant 0$ and $a_i \in \Omega$, $0 \leqslant i \leqslant n$, let a real number $p_n(a_0, \ldots, a_n)$ be given such that (i) $p_n(a_0, \ldots, a_n) \geqslant 0$, (ii) $\sum_{a_0\in\Omega} p_0(a_0) = 1$, and (iii) $p_n(a_0, \ldots, a_n) = \sum_{a_{n+1}\in\Omega} p_{n+1}(a_0, \ldots, a_n, a_{n+1})$. If we define now

$$m(C_{a_0,\ldots,a_n}) = p_n(a_0, \ldots, a_n),$$

then m can be extended to a probability measure on $(\Omega^{\mathbb{N}_0}, \mathcal{B})$. The resulting dynamical system, $(\Omega^{\mathbb{N}_0}, \mathcal{B}, m, \Sigma)$ is called the *one-sided shift space*.

Example 1.

(a) Let $\mathbf{p} = (p_0, p_1, \ldots, p_{N-1})$, $N \geqslant 2$, be a probability vector with non-zero entries (i.e., $p_i > 0$ and $\sum_{i=0}^{N-1} p_i = 1$). Set $p_n(a_0, a_1, \ldots, a_n) = p_{a_0} p_{a_1} \ldots p_{a_n}$. The resulting measure-preserving shift transformation is called the one-sided $\mathbf{p}$-*Bernoulli shift*.

(b) Let $\mathbf{p} = (p_0, p_1, \ldots, p_{N-1})$ be a probability vector as in (a) and $P = (p_{ij})_{0\leqslant i,j\leqslant N-1}$ an $N \times N$ stochastic matrix (i.e., $p_{ij} \geqslant 0$ and $\sum_{i,j=0}^{N-1} p_{ij} = 1$) such that $\sum_{i=0}^{N-1} p_i p_{ij} = p_j$. Set then $p_n(a_0, a_1, \ldots, a_n) = p_{a_0} p_{a_0a_1} p_{a_1a_2} \ldots p_{a_{n-1}a_n}$. The resulting measure-preserving shift transformation is called the one-sided $(\mathbf{p}, P)$-*Markov shift*.

(c) Let $\mathbf{S} = (S_n)_{n=0}^{\infty}$ be a discrete-time stochastic process on a probability space $(X, \mathcal{F}, \mu)$ started at time $n = 0$ with finitely many outcomes $\{0, 1, \ldots, N-1\} = \Omega$. The realizations (or "sample paths") $\mathbf{S}(x) = (S_0(x), \ldots, S_n(x), \ldots)$ are viewed as elements of $\Omega^{\mathbb{N}_0}$ endowed with the induced measure $p_n(a_0, \ldots, a_n) = \mu(\{x \in X\colon S_0(x) = a_0, \ldots, S_n(x) = a_n\}) \equiv \Pr\{S_0 = a_0, \ldots, S_n = a_n\}$, the probability of the event $S_0 = a_0, \ldots, S_n = a_n$. The resulting measure on $\Omega^{\mathbb{N}_0}$ is shift invariant if the stochastic process $\mathbf{S}$ is stationary.

There are several metrics compatible with the topology of $\Omega^{\mathbb{N}_0}$, the most popular being

$$d_K(\omega, \omega') = \sum_{n=0}^{\infty} \frac{\delta(\omega_n, \omega'_n)}{K^n}, \tag{2}$$

where $\delta(\omega_n, \omega'_n) = 1$ if $\omega_n \neq \omega'_n$, $\delta(\omega_n, \omega_n) = 0$ and $K > 2$. Observe that given $\omega \in C_{a_0,\ldots,a_n}$, then $d_K(\omega, \omega') < \frac{1}{K^n}$ if $\omega' \in C_{a_0,\ldots,a_n}$ and $d_K(\omega, \omega') \geqslant \frac{1}{K^n}$ if $\omega' \notin C_{a_0,\ldots,a_n}$, so that $C_{a_0,\ldots,a_n} = B_{d_K}(\omega; \frac{1}{K^n})$, the open ball of radius K^{-n} and center ω in the metric space $(\Omega^{\mathbb{N}_0}, d_K)$. Since the base of the measurable sets are open balls, we conclude that $\mathcal{B}$ is the Borel sigma-algebra in the topology defined by the metric (2). Observe furthermore that every point in $B_{d_K}(\omega; \frac{1}{K^n})$ is a center, a property known from non-Archimedean normed spaces (e.g., the rational numbers with p-adic norms).

Continuity will play a role below. Since $\Sigma^{-1} C_{a_0,\ldots,a_n} = \bigcup_{a \in \Omega} C_{a,a_0,\ldots,a_n}$, Σ is continuous in $(\Omega^{\mathbb{N}_0}, d_K)$, each point $\omega \in \Omega^{\mathbb{N}_0}$ having exactly N preimages under Σ. Regarding the forward dynamic, Σ has N fixed points: $\omega = (\bar{n})$, $0 \leqslant n \leqslant N-1$, where *the overbar denotes indefinite repetition throughout*.

The corresponding (invertible) two-sided shift transformation on the two-sided sequence (or *bisequence*) space

$$\Omega^{\mathbb{Z}} = \{\omega = (\omega_n)_{n \in \mathbb{Z}}\colon \omega_n \in \Omega\},$$

is defined as $\Sigma : \omega \mapsto \omega'$ with $\omega'_n = \omega_{n+1}$, $n \in \mathbb{Z}$. The cylinder sets are given now as $\{\omega \in \Omega^{\mathbb{Z}}\colon \omega_k = a_k,\ |k| \leqslant n\}$ and

$$d_K(\omega, \omega') = \sum_{n \in \mathbb{Z}} \frac{\delta(\omega_n, \omega'_n)}{K^{|n|}},$$

with $K > 3$.

Let T be a measure preserving map on a probability space $(X, \mathcal{F}, \mu)$ and $\alpha = \{A_0, \ldots, A_{N-1}\}$ be a generating partition of the sigma-algebra $\mathcal{F}$ with respect to T, i.e., the subsets of the form $A_{a_0} \cap T^{-1} A_{a_1} \cap \cdots \cap T^{-n} A_{a_n}$ generate $\mathcal{F}$. Assume moreover that for every sequence $(A_{a_n})_{n \in \mathbb{N}_0}$, the set $\bigcap_{n=0}^{\infty} T^{-n} A_{a_n}$ contains at most one point of X; this assumption is fulfilled by any positively expansive continuous map or expansive homeomorphism on compact metric spaces (in particular, by the one-sided and two-sided transformations we considered above) and implies that the coding map Φ to be defined in (3)–(4) is one-to-one. Define now on the cylinder sets of $\Omega^{\mathbb{N}_0}$ the measure

$$m_T(C_{a_0,\ldots,a_n}) = \mu\big(A_{a_0} \cap T^{-1} A_{a_1} \cap \cdots \cap T^{-n} A_{a_n}\big).$$

For $\omega \in \Omega^{\mathbb{N}_0}$ define the *coding map* $\Phi : X \to \Omega^{\mathbb{N}_0}$ by

$$\Phi(x) = (\omega_0, \ldots, \omega_n, \ldots), \tag{3}$$

where

$$\omega_n = a_n \in \Omega \quad \text{if } T^n(x) \in A_{a_n},\ n \geqslant 0. \tag{4}$$

Then $\Phi : (X, \mathcal{F}, \mu) \to (\Omega^{\mathbb{N}_0}, \mathcal{B}, m_T)$ is measure-preserving (since, by definition, $\Phi^{-1}(C_{a_0,\dots,a_n}) = A_{a_0} \cap T^{-1}A_{a_1} \cap \dots \cap T^{-n}A_{a_n}$) and, moreover,

$$\Phi \circ T = \Sigma \circ \Phi, \tag{5}$$

i.e., T and Σ are isomorphic and, hence, $(X, \mathcal{F}, \mu, T)$ and $(\Omega^{\mathbb{N}_0}, \mathcal{B}, m_T, \Sigma)$ are dynamically equivalent.

One interesting consequence of this construction is that the coded orbits of T contain any arbitrary pattern. Indeed, given any N-symbol *pattern* of length $L \geqslant 1$, $a_0^{L-1} := a_0 a_1 \dots a_{L-1}$ with symbols $a_n \in \{0, 1, \dots, N-1\}$, choose

$$x_0 \in \bigcap_{n=0}^{L-1} T^{-n} A_{a_n}.$$

Then $\Phi(x_0) \in C_{a_0,\dots,a_{L-1}}$ and this for any $L \geqslant 1$. Letting $L \to \infty$, we conclude that the coding map Φ associates to each orbit $orb(x) = \{T^n(x)\colon\ n \geqslant 0\}$ a unique, infinitely long pattern of symbols from $\{0, 1, \dots, N-1\}$, namely, $\Phi(x)$, for almost all $x \in X$.

In the special case of invertible maps $T : X \to X$, both T and T^{-1} are measurable and all the above generalizes to two-sided sequences.

Example 2. As a standard example (that it is going to be our workhorse), take $X = [0, 1]$, $\mathcal{F}$ the Borel sigma-algebra restricted to $[0, 1]$, $d\mu = \frac{1}{\pi\sqrt{x(1-x)}}\,dx$, $f(x) = 4x(1-x)$, the *logistic map*, and $\alpha = \{A_0 = [0, \frac{1}{2}), A_1 = [\frac{1}{2}, 1]\}$ (it is irrelevant whether the midpoint $\frac{1}{2}$ belongs to the left or to the right partition element). Then $\Phi(\frac{1}{4}) = (0, \bar{1})$, $\Phi(\frac{1}{2}) = (1, 1, \bar{0})$ and $\Phi(\frac{3}{4}) = (\bar{1})$. Observe for further reference that $\Phi(\frac{1}{4}) < \Phi(\frac{1}{2}) < \Phi(\frac{3}{4})$, where $<$ stands for the lexicographical order of $\{0, 1\}^{\mathbb{N}_0}$, but, e.g., $\Phi(\frac{1}{2}) > \Phi(1) = (1, \bar{0})$, hence the coding map $\Phi : [0, 1] \to \{0, 1\}^{\mathbb{N}_0}$ does not preserve the order structure. The fixed points of f are $0 = \Phi^{-1}((\bar{0}))$ and $\frac{3}{4} = \Phi^{-1}((\bar{1}))$.

3. Forbidden order patterns

In the previous section, we saw that the symbolic dynamics of maps defines any symbol pattern of any length, under rather general assumptions. In this section we will see that the situation is not quite the same when considering order patterns.

Let $(X, <)$ be a totally ordered set and $T : X \to X$ a map. Given $x \in X$, the orbit of x is the set $\{T^n(x)\colon\ n \in \mathbb{N}_0\}$, where $T^0(x) \equiv x$ and $T^n(x) \equiv T(T^{n-1}(x))$. If x is not a periodic point of period less than $L \geqslant 2$, we can then associate with x an order pattern of length L, as follows. We say that x *defines the order pattern* $\pi = \pi(x) = [\pi_0, \dots, \pi_{L-1}]$, where $\{\pi_0, \dots, \pi_{L-1}\}$ is a permutation of $\{0, 1, \dots, L-1\}$, if

$$T^{\pi_0}(x) < T^{\pi_1}(x) < \dots < T^{\pi_{L-1}}(x).$$

Alternatively, we say that x *is of type* π or that π *is realized by* x. Thus, π is just a permutation on $\{0, 1, \dots, L-1\}$, given by $0 \mapsto \pi_0, \dots, L-1 \mapsto \pi_{L-1}$, that encapsulates the order of the points $x_n = T^n(x)$, $0 \leqslant n \leqslant L-1$. The set of order patterns of length L or, equivalently, the set of permutations on $\{0, 1, \dots, L-1\}$ will be denoted by $\mathcal{S}_L$. Furthermore set

$$P_\pi = \{x \in X\colon\ x \text{ defines } \pi \in \mathcal{S}_L\}.$$

A plain difference between symbol patterns and order patterns of length L is their cardinality: the former grow exponentially with L (exactly as N^L, where N is the number of symbols) while the latter do super-exponentially. Specifically,

$$|\mathcal{S}_L| = L! \propto e^{L(\ln L-1)+(1/2)\ln 2\pi L} \tag{6}$$

(Stirling's formula), where, as usual, $|\cdot|$ denotes cardinality and $\propto$ means "asymptotically." Although one can construct functions whose orbits realize any possible order pattern (see below), numerical simulations support the conjecture that order patterns, like symbol patterns, grow only exponentially for 'well-behaved' functions [6]. In fact, if I is a closed interval of $\mathbb{R}$ and $f: I \to I$ is *piecewise monotone* (i.e., there is a finite partition of I into intervals such that f is continuous and strictly monotone on each of those intervals), then one can prove [6] that

$$\bigl|\{\pi \in \mathcal{S}_L:\ P_\pi \neq \varnothing\}\bigr| \propto e^{L h_{\mathrm{top}}(f)}, \tag{7}$$

where $h_{\mathrm{top}}(f)$ is the topological entropy of f. From (6) and (7) we conclude:

Proposition 1. *If f is a piecewise monotone map on a closed interval $I \subset \mathbb{R}$, then there are $\pi \in \mathcal{S}_L$, $L \geqslant 2$, such that $P_\pi = \varnothing$.*

Order patterns that do not appear in any orbit of f are called *forbidden patterns*, at variance with the *allowed patterns*, for which there are intervals of points that realize them.

Example 3. As a simple illustration borrowed from [2], consider again the logistic map. For $L = 2$ we have

$$P_{[0,1]} = \left(0, \frac{3}{4}\right), \qquad P_{[1,0]} = \left(\frac{3}{4}, 1\right).$$

Observe that the endpoints of P_π are period-1 (i.e., fixed) points (0 and $\frac{3}{4}$) or preimages of them ($f(1) = 0$). But already for $L = 3$ ($f^2(x) = -64x^4 + 128x^3 - 80x^2 + 16x$) there are permutations that are not realized (see Fig. 1):

$$P_{[0,1,2]} = \left(0, \frac{1}{4}\right), \qquad P_{[0,2,1]} = \left(\frac{1}{4}, \frac{5-\sqrt{5}}{8}\right), \qquad P_{[2,0,1]} = \left(\frac{5-\sqrt{5}}{8}, \frac{3}{4}\right),$$
$$P_{[1,0,2]} = \left(\frac{3}{4}, \frac{5+\sqrt{5}}{8}\right), \qquad P_{[1,2,0]} = \left(\frac{5+\sqrt{5}}{8}, 1\right), \qquad P_{[2,1,0]} = \varnothing.$$

In going from $\pi \in \mathcal{S}_2$ to $\pi \in \mathcal{S}_3$, we see that $P_{[0,1]}$ splits into the subintervals $P_{[0,1,2]}$, $P_{[0,2,1]}$ and $P_{[2,0,1]}$ at the eventually periodic point $\frac{1}{4}$ (preimage of $\frac{3}{4}$) and at the period-2 point $\frac{5-\sqrt{5}}{8}$. Likewise, $P_{[1,0]}$ splits into $P_{[1,0,2]}$ and $P_{[1,2,0]}$ at the period-2 point $\frac{5+\sqrt{5}}{8}$.

From a different perspective, as we move rightward in Fig. 1 from the neighborhood of 0, where $x < f(x) < f^2(x)$, the curves $y = f(x)$ and $y = f^2(x)$ cross at $x = \frac{1}{4}$, what causes the first swap: $[0,1,2]$ transforms to $[0,2,1]$. In general, the crossings at $x = \frac{1}{4}$, $\frac{5-\sqrt{5}}{8}$ and $\frac{5+\sqrt{5}}{8}$ between $f^{\pi(i)}$ and $f^{\pi(i+1)}$ causes the exchange of $\pi(i)$ and $\pi(i+1)$ in the pre-crossing pattern. At $x = \frac{3}{4}$ all three curves cross and $[2,0,1]$ goes over to $[1,0,2]$.

The absence of $\pi = [2,1,0]$ triggers, in turn, an avalanche of longer missing patterns. To begin with, the pattern $[*,2,*,1,*,0,*]$ (where the wildcard $*$ stands eventually for any other entries of the pattern) cannot be realized by any $x \in [0,1]$ since the inequality

$$f^2(x) < f(x) < x \tag{8}$$

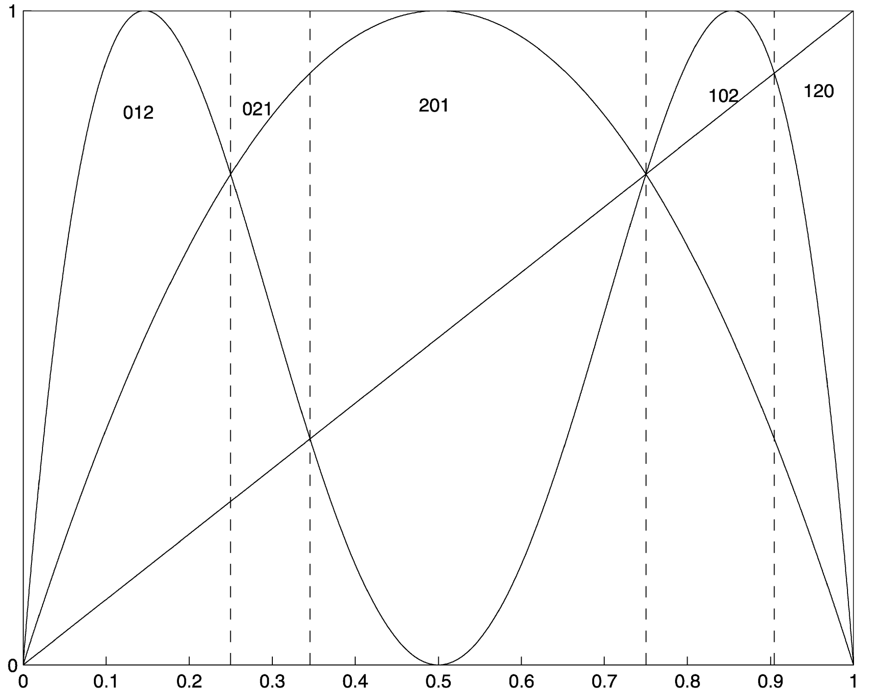

Fig. 1. The sets P_π, $\pi \in \mathcal{S}_3$, are graphically obtained by raising vertical lines at the crossing points of the curves $y = x$, $y = f(x)$, and $y = f^2(x)$. The three digits on the top are shorthand for order patterns (e.g., 012 stands for [0, 1, 2]). We see that $P_{[2,1,0]} = \varnothing$.

cannot occur. By the same token, the patterns $[*, 3, *, 2, *, 1, *]$, $[*, 4, *, 3, *, 2, *]$, and, more generally, $[*, n+2, *, n+1, *, n, *] \in \mathcal{S}_L$, $0 \leqslant n \leqslant L-3$, cannot be realized either for the same reason (substitute x by $f^n(x)$ in (8)).

The same follows for the *tent map* $\Lambda : [0,1] \to [0,1]$,

$$\Lambda(x) = \begin{cases} 2x, & 0 \leqslant x \leqslant \frac{1}{2}, \\ 2 - 2x, & \frac{1}{2} \leqslant x \leqslant 1. \end{cases} \tag{9}$$

In fact, if λ is the Lebesgue measure, $d\mu = \frac{1}{\pi\sqrt{x(1-x)}}\,dx$ is (as in Example 2) the invariant measure of the logistic map $f(x) = 4x(1-x)$, and $\phi : ([0,1], \lambda) \to ([0,1], \mu)$ is the measure preserving isomorphism given by

$$\phi(x) = \sin^2\left(\frac{\pi}{2}x\right), \tag{10}$$

then the dynamical systems $([0,1], \mathcal{B}, \lambda, \Lambda)$ and $([0,1], \mathcal{B}, \mu, f)$, where $\mathcal{B}$ is the Borel sigma-algebra restricted to the interval [0, 1], are *isomorphic* (or conjugate) by means of ϕ, i.e., $f \circ \phi = \phi \circ \Lambda$. Since, moreover, ϕ is strictly increasing, forbidden patterns for f correspond to forbidden patterns for Λ in a one-to-one way.

From the last paragraph it should be clear that isomorphic dynamical systems need not have the same forbidden patterns: the isomorphism (ϕ above) must also preserve the linear order of both spaces (supposing both spaces are linearly ordered), and this will be in general not the case. For example, the λ-preserving *shift map* $S_2 : x \mapsto 2x \pmod 1$, $0 \leqslant x \leqslant 1$, has no forbidden patterns of length 3, although it is isomorphic to the logistic and tent maps (the isomorphism with f is proved via the semi-conjugacy $\varphi : ([0,1],\lambda) \to ([0,1],\mu)$, $\varphi(x) = \sin^2 \pi x$, which does not preserve order on account of being increasing on $(0,\frac{1}{2})$ and decreasing on $(\frac{1}{2},1)$). The same happens with the logistic map and the $(\frac{1}{2},\frac{1}{2})$-Bernoulli shift, a model for tossing of a fair coin, because, as we saw in Example 2, the corresponding isomorphy (actually, the coding map) $\Phi : [0,1] \to \{0,1\}^{\mathbb{N}_0}$ is not order-preserving.

Two isomorphic dynamical systems, whose phase spaces are linearly ordered, are called *order-isomorphic* if the isomorphism between them is also an order-isomorphism (i.e., it also preserves the order structure). It is obvious that two order-isomorphic systems (like those defined by the logistic and the tent map) have the same order patterns.

Proposition 2. *Given $X_1, X_2 \subset \mathbb{R}$ endowed with the standard Borel sigma-algebra $\mathcal{B}$, suppose that the dynamical systems $(X_1,\mathcal{B},\mu_1,f_1)$ and $(X_2,\mathcal{B},\mu_2,f_2)$ are isomorphic via a continuous map $\phi : X_1 \to X_2$. If f_1 is topologically transitive and, for all $x \in X_1$, both x and $\phi(x)$ define the same order patterns, then ϕ is order-preserving.*

Proof. We want to prove that if $x, x' \in X_1$ and $x < x'$, then $\phi(x) < \phi(x')$. Because of continuity, for all $\varepsilon > 0$ there exists $0 < \delta < \frac{x'-x}{2}$ such that $|y-x| < \delta \Rightarrow |\phi(y)-\phi(x)| < \frac{\varepsilon}{2}$ and $|y'-x'| < \delta \Rightarrow |\phi(y')-\phi(x')| < \frac{\varepsilon}{2}$. On the other hand, transitiveness implies that, given x, x' and δ as above, there exists $x_0 \in X_1$, $N = N(x,\delta)$ and $N' = N'(x',\delta)$ such that $|f_1^N(x_0) - x| < \delta$ and $|f_1^{N'}(x_0) - x'| < \delta$. Thus $f_1^N(x_0) < f_1^{N'}(x_0)$ and, by assumption, $\phi \circ f_1^N(x_0) = f_1^N(\phi(x_0)) < f_2^{N'}(\phi(x_0)) = \phi \circ f_1^{N'}(x_0)$ holds. Choose now $y = f_1^N(x_0)$, $y' = f_1^{N'}(x_0)$ to deduce

$$\phi(y) < \phi(y') \leqslant \phi(x') + \left|\phi(y') - \phi(x')\right| \leqslant \phi(x') + \frac{\varepsilon}{2},$$

where ε is arbitrary. If we choose now $\varepsilon < \frac{|\phi(x)-\phi(x')|}{2}$, then it follows $\phi(x) < \phi(x')$, since $|\phi(y) - \phi(x)| < \frac{\varepsilon}{2}$. $\square$

Finally, observe that the setting we are considering is more general than the setting of Kneading Theory since our functions need not be continuous (but only piecewise-continuous). Under some assumptions [10], the kneading invariants completely characterize the order-isomorphy of continuous maps.

4. Outgrowth forbidden patterns

According to Proposition 1, for every piecewise monotone interval map on $\mathbb{R}$, $f : I \to I$, there exist $\pi \in \mathcal{S}_L$, $L \geqslant 2$, which cannot occur in any orbit. We call them *forbidden patterns* for f and recall how their absence pervades all longer patterns in form of *outgrowth forbidden patterns* (see Example 3). Since $\pi = [\pi_0,\ldots,\pi_{L-1}]$ is forbidden for f, then the $2(L+1)$ patterns of length $L+1$,

$$\begin{array}{llll} [L,\pi_0,\ldots,\pi_{L-1}], & [\pi_0,L,\pi_1,\ldots,\pi_{L-1}], & \ldots, & [\pi_0,\ldots,\pi_{L-1},L], \\ [0,\pi_0+1,\ldots,\pi_{L-1}+1], & [\pi_0+1,0,\pi_1+1,\ldots,\pi_{L-1}+1], & \ldots, & \\ [\pi_0+1,\ldots,\pi_{L-1}+1,0], & & & \end{array}$$

are also forbidden for f. Assume for the time being that all these forbidden patterns belonging to the "first generation" are all different. Then, proceeding similarly as before, we would find

$$2(L+1)\times 2(L+2)=2^2(L+1)(L+2)$$

forbidden patterns of length $L+2$ in the second generation and, in general,

$$2^m(L+1)\dots(L+m)=2^n\frac{(L+m)!}{L!}$$

forbidden patterns of length $L+m$ in the mth generation, provided that all forbidden patterns up to (and including) the mth generation are different. Observe that all these forbidden patterns generated by π have the form

$$[*,\pi_0+n,*,\pi_1+n,*,\dots,*,\pi_{L-1}+n,*]\in\mathcal{S}_N \tag{11}$$

with $n=0,1,\dots,N-L$, where $N-L\geqslant 1$ is the number of wildcards $*\in\{0,1,\dots,n-1, L+n,\dots,N-1\}$ (with $*\in\{L,\dots,N-1\}$ if $n=0$ and $*\in\{0,\dots,N-L-1\}$ if $n=N-L$).

A better upper bound on the number of outgrowth forbidden patterns of length N of π is obtained using the following reasoning. For fixed n, the number of outgrowth patterns of π of the form (11) is $N!/(N-L)!$. This is because out of all possible permutations of the numbers $\{0,1,\dots,N-1\}$, we only count those that have the entries $\{\pi_0+n,\pi_1+n,\dots,\pi_{L-1}+n\}$ in the required order. Next, note that we have $N-L+1$ choices for the value of n. Each choice generates a set of $N!/(N-L)!$ outgrowth patterns. These sets are not necessarily disjoint, but an upper bound on the size of their union, i.e., the set of all outgrowth forbidden patterns of length N of π, is given by

$$(N-L+1)\frac{N!}{(N-L)!}.$$

A weak form of the converse holds also true: if $[L,\pi_0,\dots,\pi_{L-1}],[\pi_0,L,\dots,\pi_{L-1}],\dots,[\pi_0,\dots,\pi_{L_0-1},L]\in\mathcal{S}_{L+1}$ are forbidden, then $[\pi_0,\dots,\pi_{L-1}]\in\mathcal{S}_L$ is also forbidden.

Forbidden patterns that are not outgrowth patterns of other forbidden patterns of shorter length are called *forbidden root patterns* since they can be viewed as the root of the tree of forbidden patterns spanned by the outgrowth patterns they generate, branching taking place when going from one length (or generation) to the next.

Example 4. If f is the logistic map, then

$$f^3(x)=-16\,384x^8+65\,536x^7-106\,496x^6+90\,112x^5-42\,240x^4+10\,752x^3 \\ -1344x^2+64x.$$

In Fig. 2, which is Fig. 1 with the curve $y=f^3(x)$ super-imposed, we can see the 12 allowed patterns of length 4 of the logistic map. Since there are 24 possible patterns of length 4, we conclude that 12 of them are forbidden. The outgrowth patterns of [2, 1, 0], the only forbidden pattern of length 3, are (see (11)):

$$\begin{array}{lllll}(n=0) & [3,2,1,0], & [2,3,1,0], & [2,1,3,0], & [2,1,0,3],\\ (n=1) & [0,3,2,1], & [3,0,2,1], & [3,2,0,1], & [3,2,1,0].\end{array}$$

Observe that the pattern [3, 2, 1, 0] is repeated. Therefore, the remaining five forbidden patterns of length 4 are root patterns.

In Fig. 2 one can also follow the first two splittings of the intervals P_π:

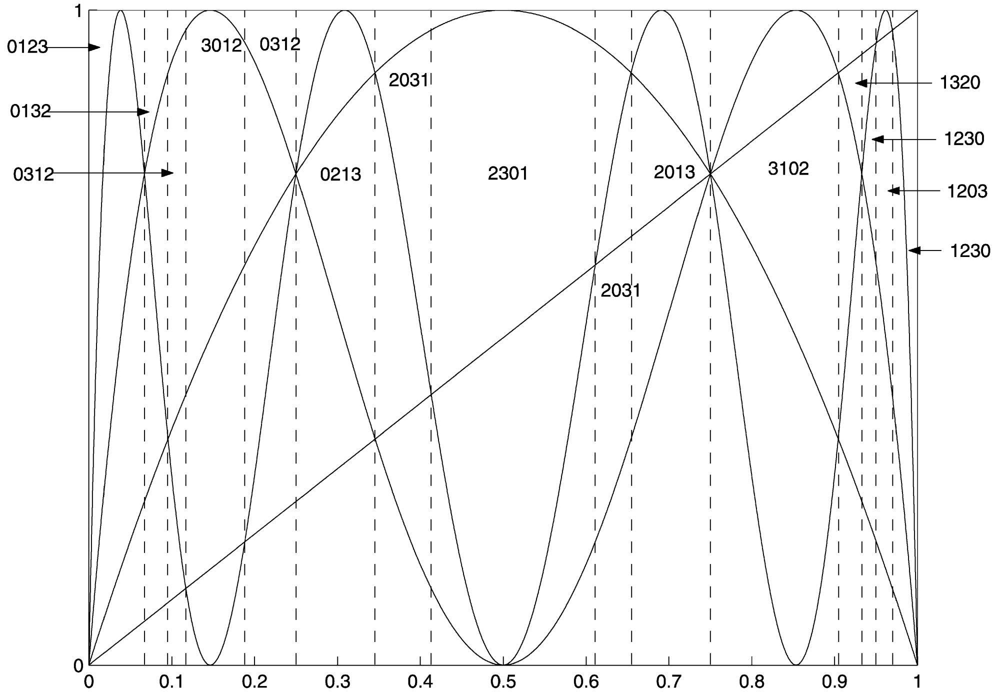

Fig. 2. The twelve allowed order patterns of length 4 for the logistic map. Note the two components of $P_{[0,3,1,2]}$, $P_{[2,0,3,1]}$ and $P_{[1,2,3,0]}$.

$$P_{[0,1]} \to \begin{cases} P_{[0,1,2]} \to P_{[0,1,2,3]}, P_{[0,1,3,2]}, P_{[0,3,1,2]}, P_{[3,0,1,2]}, \\ P_{[0,2,1]} \to P_{[0,2,1,3]}, \\ P_{[2,0,1]} \to P_{[2,0,1,3]}, P_{[2,0,3,1]}, P_{[2,3,0,1]}, \end{cases}$$

$$P_{[1,0]} \to \begin{cases} P_{[1,0,2]} \to P_{[3,1,0,2]}, \\ P_{[1,2,0]} \to P_{[1,2,0,3]}, P_{[1,2,3,0]}, P_{[1,3,2,0]}. \end{cases}$$

The splitting of the intervals P_π can be understood in terms of periodic points and their preimages. Thus, the splitting of $P_{[0,1]}$ is due to the points $\frac{1}{4}$ (first preimage of the period-1 point $\frac{3}{4}$) and $\frac{5-\sqrt{5}}{8}$ (a period-2 point); the second period-2 point, $\frac{5-\sqrt{5}}{8}$, is responsible for the splitting of $P_{[1,0]}$. On the contrary, $P_{[0,2,1]}$ and $P_{[1,0,2]}$ do not split because they contain neither period-3 point nor first preimages of period-2 points nor second preimages of fixed points.

Given the permutation $\sigma \in \mathcal{S}_N$, we say that σ contains the *consecutive pattern* $\tau = [\tau_0, \tau_1, \dots, \tau_{L-1}] \in \mathcal{S}_L$, $L < N$, if it contains a consecutive subsequence order-isomorphic to τ. Alternatively, we say that σ avoids the *consecutive pattern* τ if it contains no consecutive subsequence order-isomorphic to τ [8].

Suppose now $\sigma \in \mathcal{S}_N$, $\pi \in \mathcal{S}_L$, $L < N$, and

$$\begin{aligned} &\pi(p_0) = 0, \quad &&\pi(p_1) = 1, \quad &&\dots, \quad &&\pi(p_{L-1}) = L-1, \\ &\sigma(s_0) = n, \quad &&\sigma(s_1) = 1+n, \quad &&\dots, \quad &&\sigma(s_{L-1}) = L-1+n, \end{aligned}$$

with $n \in \{0, 1, \dots, N-L\}$. Then, the sequences $p_0, p_1, \dots, p_{L-1}$ and $s_0, s_1, \dots, s_{L-1}$ are consecutive subsequences of π^{-1} and σ^{-1} (starting at positions 0 and n), respectively. If, moreover, σ is an outgrowth pattern of π (see (11)), then $s_0, s_1, \dots, s_{L-1}$ is order-isomorphic to $p_0, p_1, \dots, p_{L-1}$. It follows that $\sigma \in \mathcal{S}_N$ is an outgrowth pattern of $\pi = [\pi_0, \dots, \pi_{L-1}]$ if σ^{-1} contains π^{-1} as a consecutive subsequence. Hence, the allowed patterns for f are the permutations that avoid all such consecutive subsequences for every forbidden root pattern of f.

Example 5. Take $\pi = [2, 0, 1]$ to be a forbidden pattern for a certain function f. Then $\sigma = [4, 2, 1, 5, 3, 0]$ is an outgrowth pattern of π because it contains the subsequence $4, 2, 3$ ($n = 2$). Equivalently, $\sigma^{-1} = [5, 2, 1, 4, 0, 3]$ contains the consecutive pattern $1, 4, 0$ (starting at location σ_2^{-1}), which is order-isomorphic to $\pi^{-1} = [1, 2, 0]$.

Let $\mathrm{out}(\pi)$ denote the family of outgrowth patterns of the forbidden pattern π,

$$\mathrm{out}_N(\pi) = \mathrm{out}(\pi) \cap \mathcal{S}_N = \{\sigma \in \mathcal{S}_N\colon\ \sigma^{-1} \text{ contains } \pi^{-1} \text{ as a consecutive pattern}\},$$

and

$$\mathrm{avoid}_N(\pi) = \mathcal{S}_N \setminus \mathrm{out}_N(\pi) = \{\sigma \in \mathcal{S}_N\colon\ \sigma^{-1} \text{ avoids } \pi^{-1} \text{ as a consecutive pattern}\},$$

where $\setminus$ stands for set difference. The fact that some of the outgrowth patterns of a given length will be the same and that this depends on π, makes the analytical calculation of $|\mathrm{out}_N(\pi)|$ extremely complicated. Yet, from [8] we know that there are constants $0 < c, d < 1$ such that

$$c^N N! < |\mathrm{avoid}_N(\pi)| < d^N N!$$

(for the first inequality, $L \geqslant 3$ is needed). This implies that

$$(1 - d^N)N! < |\mathrm{out}_N(\pi)| < (1 - c^N)N!. \tag{12}$$

This factorial growth with N can be exploited in practical applications to tell random from deterministic time series with, in principle, arbitrarily high probability. As said in the Introduction, these practical aspects are beyond the scope of this paper, but let us bring up here the following, related point. In the case of real (hence, *finite*) randomly generated sequences, a given order pattern $\pi \in \mathcal{S}_L$ can be missing with non-vanishing probability. We call *false forbidden patterns* such missing order patterns in finite random sequences without constraints, to distinguish them from the 'true' forbidden patterns of deterministic (finite or infinite) sequences. True and false forbidden patterns of self maps on one-dimensional intervals have been studied in [3].

5. Order patterns and one-sided shifts

The general study of order patterns and forbidden patterns is quite difficult. Analytical results seem to be only feasible for particular maps. In this and next sections we will consider the one- and two-sided shifts since, owing to their simple structure, they can be analyzed with greater detail. As we saw in Section 2, shifts are continuous maps (automorphisms if two-sided) on compact metric spaces $(\{0, 1, \dots, N-1\}^{\mathbb{N}_0}, d_K)$ (correspondingly, $(\{0, 1, \dots, N-1\}^{\mathbb{Z}}, d_K)$) that can be lexicographically ordered:

$$\omega < \omega' \quad \Leftrightarrow \quad \begin{cases} \omega_0 < \omega'_0 \\ \text{or} \\ \omega_0 = \omega'_0, \dots,\ \omega_{n-1} = \omega'_{n-1} \text{ and } \omega_n < \omega'_n\ (n \geqslant 1), \end{cases}$$

If $\overline{\mathcal{N}}$ denotes the countable, dense and Σ-invariant set of ω eventually terminating in an infinite string of $(N-1)$s except the sequence $(\overline{N-1})$, then the map $\psi : \{0, 1, \ldots, N-1\}^{\mathbb{N}_0} \backslash \overline{\mathcal{N}} \to [0, 1]$ defined by

$$\psi : (\omega_n)_{n \in \mathbb{N}_0} \mapsto \sum_{n=0}^{\infty} \omega_n N^{-(n+1)} \tag{13}$$

is one-to-one and *order-preserving*; moreover, ψ^{-1} is also order-preserving. As a matter of fact, the lexicographical order in $\{0, 1, \ldots, N-1\}^{\mathbb{N}_0} \backslash \overline{\mathcal{N}}$ corresponds via ψ to the standard order (induced by the positive numbers) in the interval $[0, 1]$. Although not important for our purposes, let us point out that ψ is continuous, but ψ^{-1} is not. Since the map

$$S_N = \psi \circ \Sigma \circ \psi^{-1} : [0, 1] \to [0, 1], \tag{14}$$

where Σ is the shift on N symbols, is piecewise linear and $\overline{\mathcal{N}}$ is dense, it follows (Proposition 1) that Σ will have forbidden order patterns (although Σ has no forbidden *symbol* pattern, see Section 2). In particular, if Σ is the $(\frac{1}{N}, \ldots, \frac{1}{N})$-Bernoulli shift, then S_N is the Lebesgue-measure preserving *sawtooth map* $S_N : x \mapsto Nx \pmod 1$. Observe that only sequences that are not eventually periodic define order patterns of any length.

What is the structure of the allowed order patterns? It is easy to convince oneself (see Example 6 below) that, given $\omega = (\omega_0, \ldots, \omega_{L-1}, \ldots) \in \{0, 1, \ldots, N-1\}^{\mathbb{N}_0}$ of type $\pi \in \mathcal{S}_L$, π can be decomposed into, in general, N blocks,

$$[\pi_0, \ldots, \pi_{k_0-1}; \pi_{k_0}, \ldots, \pi_{k_0+k_1-1}; \ldots; \pi_{k_0+\cdots+k_{N-2}}, \ldots, \pi_{k_0+\cdots+k_{N-2}+k_{N-1}-1}], \tag{15}$$

the at most $N-1$ semicolons separating the different blocks, where $k_n \geqslant 0$, $0 \leqslant n \leqslant N-1$, is the number of symbols $n \in \{0, 1, \ldots, N-1\}$ in ω_0^{L-1} ($k_n = 0$ if none, with the corresponding block missing) and $k_0 + \cdots + k_{N-1} = L$. Moreover, these blocks obey the following basic restrictions.

(R1) The first (leftmost) block, $\pi_0, \ldots, \pi_{k_0-1}$, contains the locations of the 0s in ω_0^{L-1}. Each 0-run (i.e., a segment of two or more consecutive 0s contained in or intersected by ω_0^{L-1}), if any, contributes an *increasing* subsequence $\pi_i, \pi_i + 1, \pi_i + 2, \ldots$ (as long as the 0-run), which is possibly intertwined with other entries of this block.

(R2) The last (rightmost) block, $\pi_{k_0+\cdots+k_{N-2}}, \ldots, \pi_{k_0+\cdots+k_{N-2}+k_{N-1}-1}$, contains the locations of the $(N-1)$s in ω_0^{L-1}. Each $(N-1)$-run contained in or intersected by ω_0^{L-1}, if any, contributes a *decreasing* subsequence $\pi_{k_0+\cdots+k_{N-2}+i}, \pi_{k_0+\cdots+k_{N-2}+i} - 1, \ldots$ (as long as the $(N-1)$-run), which is possibly intertwined with other entries of this block.

(R3) Every intermediate block, $\pi_{k_0+\cdots+k_{j-1}}, \ldots, \pi_{k_0+\cdots+k_{j-1}+k_j-1}$, $1 \leqslant j \leqslant N-2$, contains the locations of the js in ω_0^{L-1}. Each j-run contained in or intersected by ω_0^{L-1}, if any, contributes a subsequence of the same length as the run, that is increasing $(\pi_{k_0+\cdots+k_{j-1}+i}, \pi_{k_0+\cdots+k_{j-1}+i} + 1, \ldots)$ if the run is followed by a symbol $> j$, or *decreasing* $(\pi_{k_0+\cdots+k_{j-1}+i}, \pi_{k_0+\cdots+k_{j-1}+i} - 1, \ldots)$ if the run is followed by a symbol $< j$. These subsequences may be intertwined with other entries of the same block.

(R4) If the entries $\pi_m \leqslant L-2$ and $\pi_n \leqslant L-2$ belong to the *same block of* $\pi \in \mathcal{S}_L$, and π_m appears on the left of π_n (i.e., $0 \leqslant m < n \leqslant L-1$), then $\pi_m + 1$ appears also on the left of $\pi_n + 1$ (i.e., $\pi_m + 1 = \pi_{m'}$, $\pi_n + 1 = \pi_{n'}$ and $0 \leqslant m' < n' \leqslant L-1$).

In (R4), $\pi_m + 1$ and $\pi_n + 1$ may appear in the same block or in different blocks. Let us mention at this point that (R4) implies some simple consequences for the relative locations of increasing and decreasing subsequences within the same block and their continuations (if any) outside the block. In particular:

(A) If $\pi_i, \pi_i + 1, \ldots, \pi_i + l - 1$, $1 \leqslant l \leqslant L - 1$, is an increasing subsequence within the same block of $\pi \in \mathcal{S}_L$ with $\pi_i + l < L$, then $\pi_i + l$ is on the right of $\pi_i + l - 1$ (i.e., $\pi_i + l - 1 = \pi_m$, $\pi_i + l = \pi_n$, and $m < n$).
(B) If $\pi_i, \pi_i - 1, \ldots, \pi_i - l + 1$, $1 \leqslant l \leqslant L - 1$, is a decreasing subsequence within the same block of $\pi \in \mathcal{S}_L$ with $\pi_i < L - 1$, then $\pi_i + 1$ is on the left of π_i (i.e., $\pi_i + 1 = \pi_j$ with $j < i$).
(C) If $\pi_i, \pi_i \pm 1, \ldots, \pi_i \pm l \mp 1$ and $\pi_j, \pi_j \pm 1, \ldots, \pi_j \pm h \mp 1$, $1 \leqslant l, h \leqslant L - 1$, are two subsequences with the same monotony (upper signs for increasing, lower signs for decreasing subsequences) within the same block of $\pi \in \mathcal{S}_L$, then they are fully separated or, if intertwined, then it may not happen that two or more entries of one of them are between two entries of the other.

Example 6. Take in $\{0, 1, 2\}^{\mathbb{N}_0}$ the sequence

$$\omega = (|_0 2|_1 1|_2 1|_3 1|_4 2|_5 2|_6 0|_7 0|_8 1|_9 1|_{10} 0|_{11} 0|_{12} 2|_{13} 2|2|1 \ldots), \tag{16}$$

where $|_k b$ indicates that the entry $b \in \{0, 1, 2\}$ is at place k. Then ω defines the order pattern

$$\pi = [6, 10, 7, 11; 9, 8, 1, 2, 3; 5, 0, 4, 13, 12] \in \mathcal{S}_{14},$$

where the first block, $\pi_0^3 = 6, 10, 7, 11$, is set by the $k_0 = 4$ symbols 0 in ω_0^{13}, which appear grouped in two runs, ω_6^7 and ω_{10}^{11} (note the two increasing subsequences 6, 7 and 10, 11 in this block); the intermediate block, $\pi_4^8 = 9, 8, 1, 2, 3$, comes from the $k_1 = 5$ symbols 1 in ω_0^{13}, grouped also in two runs, ω_1^3, followed by the symbol $2 > 1$, and ω_8^9, followed by the symbol $0 < 1$ (note the corresponding increasing subsequence 1, 2, 3, and decreasing subsequence 9, 8, in this block); finally, the last block $\pi_9^{13} = 5, 0, 4, 13, 12$ accounts for the $k_2 = 5$ appearances of the symbol 2 in ω_0^{13} (the decreasing subsequences 5, 4 and 13, 12 come from the runs ω_4^5 and ω_{12}^{13}, respectively, where ω_{12}^{13} is the intersection within ω_0^{13} of a longer 2-run). (R4) is easily checked to be fulfilled.

Observe that two sequences ω, ω' with $\omega_0^{L-1} \neq \omega_0'^{L-1}$ may define the same order pattern of length L, while two sequences ω, ω' with $\omega_0^{L-1} = \omega_0'^{L-1}$ may define different order patterns of length L (depending on $\omega_{L-1}, \ldots,$ and $\omega'_{L-1}, \ldots$).

Proposition 3. *The one-sided shift on $N \geqslant 2$ symbols has no forbidden patterns of length $L \leqslant N + 1$.*

Proof. First of all, note that if $\omega = (\omega_0, \omega_1, \omega_2, \ldots)$ is of type $\pi = [\pi_0, \pi_1, \ldots, \pi_N]$, then the point $\bar{\omega} = (N - 1 - \omega_0, N - 1 - \omega_1, N - 1 - \omega_2, \ldots)$ is of type $\pi_{\text{mirrored}} = [\pi_N, \pi_{N-1}, \ldots, \pi_1, \pi_0]$.

Given $\pi = [\pi_0, \pi_1, \ldots, \pi_N]$, we can therefore assume, without loss of generality, that $\pi_0 < \pi_N$. Consider two cases.

- If $\pi_N \neq N$, then there is some $l \in \{1, 2, \ldots, N - 1\}$ such that $\pi_l = N$. In this case, the point $\omega = (\omega_0, \omega_1, \ldots) \in \{0, 1, \ldots, N - 1\}^{\mathbb{N}_0}$, where

$$\omega_{\pi_0} = 0, \qquad \omega_{\pi_1} = 1, \qquad \ldots, \qquad \omega_{\pi_{l-1}} = l-1,$$
$$\omega_{\pi_l} = l-1, \qquad \omega_{\pi_{l+1}} = l, \qquad \ldots, \qquad \omega_{\pi_{N-1}} = N-2, \qquad \omega_{\pi_N} = N-1,$$
$$\omega_{N+1} = \omega_{N+2} = N-1,$$

is of type π. Indeed, it is enough to note that

$$\Sigma^{\pi_{l-1}}(\omega) = (l-1, \omega_{\pi_{l-1}+1}, \ldots) < (l-1, N-1, N-1, \ldots) = \Sigma^N(\omega) = \Sigma^{\pi_l}(\omega).$$

- If $\pi_N = N$, let us first assume that $\pi_0 \neq 0$. Then there is $k \in \{1, 2, \ldots, N-1\}$ such that $\pi_k + 1 = \pi_0$. In this case, the point $\omega = (\omega_0, \omega_1, \ldots) \in \{0, 1, \ldots, N-1\}^{\mathbb{N}_0}$, where

$$\omega_{\pi_0} = 0, \qquad \omega_{\pi_1} = 1, \qquad \ldots, \qquad \omega_{\pi_{k-1}} = k-1, \qquad \omega_{\pi_k} = k,$$
$$\omega_{\pi_{k+1}} = k, \qquad \omega_{\pi_{k+2}} = k+1, \qquad \ldots, \qquad \omega_{\pi_{N-1}} = N-2, \qquad \omega_{\pi_N} = N-1,$$
$$\omega_{N+1} = N-1,$$

is of type π. This is clear because

$$\Sigma^{\pi_k}(\omega) = (k, 0, \ldots) < (k, \omega_{\pi_{k+1}+1}, \ldots) = \Sigma^{\pi_{k+1}}(\omega).$$

In the case that $\pi_0 = 0$, then there is $l \in \{1, 2, \ldots, N-1\}$ such that $\pi_l = N-1$. Now the point $\omega = (\omega_0, \omega_1, \ldots) \in \{0, 1, \ldots, N-1\}^{\mathbb{N}_0}$, where

$$\omega_{\pi_0} = 0, \qquad \omega_{\pi_1} = 1, \qquad \ldots, \qquad \omega_{\pi_{l-1}} = l-1, \qquad \omega_{\pi_l} = l-1,$$
$$\omega_{\pi_{l+1}} = l, \qquad \ldots, \qquad \omega_{\pi_{N-1}} = N-2, \qquad \omega_{\pi_N} = N-1,$$

is of type π, since

$$\Sigma^{\pi_{l-1}}(\omega) = (l-1, \omega_{\pi_{l-1}+1}, \ldots) < (l-1, N-1, \ldots) = \Sigma^{N-1}(\omega) = \Sigma^{\pi_l}(\omega). \qquad \square$$

Next we are going to show that the one-sided shift on N symbols has forbidden patterns (more specifically, forbidden *root* patterns) of any length $L \geqslant N+2$. In order to construct explicit instances, we need first to introduce some notation and definitions.

Consider a partition of the sequence $0, 1, \ldots, L-1$ of the form

$$p_1 < p_2 < \cdots < p_d < \cdots < p_D, \tag{17}$$

where

$$p_d = e_d, e_d + 1, \ldots, e_d + h_d - 1, \tag{18}$$

$1 \leqslant d \leqslant D$, $D \geqslant 2$, with (i) $h_d \geqslant 1$, $h_1 + \cdots + h_D = L$, (ii) $e_1 = 0$, $e_D + h_D - 1 = L-1$, and (iii) $e_d + h_d = e_{d+1}$ for $1 \leqslant d \leqslant D-1$, i.e., the *follower* of p_d, $e_d + h_d$, $d \leqslant D-1$, is the first element of p_{d+1}, e_{d+1}. We call (17) a partition of $0, 1, \ldots, L-1$ in D segments, (18) being an *increasing segment*, and denote by $\overleftarrow{p_d}$ the *decreasing* or *reversed segment*

$$\overleftarrow{p_d} = e_d + h_d - 1, \ldots, e_d + 1, e_d.$$

We also call e_d the first element of $\overleftarrow{p_d}$ and e_{d+1} the follower of $\overleftarrow{p_d}$.

Since a segment p_d is nothing else but a special case of a subsequence $\pi_i, \pi_i + 1, \ldots, \pi_i + l - 1$, where $0 \leqslant \pi_i = e_d \leqslant L - l$ if p_d is increasing and $l + 1 \leqslant \pi_i = e_d \leqslant L - 1$ if p_d is decreasing, and $\pi_i \pm 1 = \pi_{i+1}, \ldots, \pi_i \pm l \mp 1 = \pi_{i+l-1}$, respectively, the consequences (A)–(C) of the restriction (R4) apply as well. In the proof of the existence of forbidden root patterns below (Lemmas 1 and 2, and Proposition 4) we are going to use (A) and (B) in the following, particularized version (that will be also referred to as (R4)): *The follower* (*if any*) *of an increasing segment* p_n (*correspondingly, decreasing segment* $\overleftarrow{p_n}$) *in an allowed pattern* π *appears always to the right of* p_n (*correspondingly, to the left of* $\overleftarrow{p_n}$).

Definition. Consider the partition (17) of $0, 1, \ldots, L-1$ in segments.

1. We call
$$\pi = [p_1, p_3, \ldots, \overleftarrow{p_4}, \overleftarrow{p_2}] \quad \text{and} \quad \pi_{\text{mirrored}} = [p_2, p_4, \ldots, \overleftarrow{p_3}, \overleftarrow{p_1}] \tag{19}$$
a *tent pattern* of length L.
2. We call
$$\pi = [\ldots, \overleftarrow{p_3}, \overleftarrow{p_1}, p_2, p_4, \ldots] \quad \text{and} \quad \pi_{\text{mirrored}} = [\ldots, \overleftarrow{p_4}, \overleftarrow{p_2}, p_1, p_3, \ldots] \tag{20}$$
a *spiralling pattern* of length L.

Observe that the relation between partitions of $0, 1, \ldots, L-1$ in segments and spiralling patterns of length L is one-to-one except when $p_1 = 0$ ($h_1 = 1$). In this case, $\overleftarrow{p_1}, p_2 = 0, 1, \ldots, e_2 + h_2 - 1$ can be taken for $p_1' \equiv 0, 1, \ldots, e_2 + h_2 - 1$ ($h_1' = h_2 + 1$).

Lemma 1. *If* $N \geqslant 2$ *is the number of symbols and* π *is a tent pattern with* D *segments, then* π *is forbidden if and only if* $D \geqslant N + 2$.

Proof. Consider the tent pattern $\pi = [p_1, p_3, \ldots, \overleftarrow{p_4}, \overleftarrow{p_2}]$. To begin with, the last entry $h_1 - 1$ of p_1 and the first entry e_3 of p_3 may not be in the same block, otherwise the restriction (R4) would be violated (e_2 should be on the left of $e_3 + 1$ if $h_3 \geqslant 2$ or on the left of e_4 if $h_3 = 1$). Thus we separate them with a first semicolon:

$$\pi = [p_1; p_3, \ldots, \overleftarrow{p_4}, \overleftarrow{p_2}].$$

Observe that the resulting leftmost block, p_1, complies with the restriction (R1). Consider now the followers of $\overleftarrow{p_2}$ and $\overleftarrow{p_4}$ to conclude similarly that we need to separate these segments by a second semicolon:

$$\pi = [p_1; p_3, \ldots, \overleftarrow{p_4}; \overleftarrow{p_2}].$$

The resulting rightmost block satisfies (R2).

The procedure continues along the same lines: in the kth step, (R4) requires a kth semicolon between the segments p_k and p_{k+2}, so that, if $D \geqslant N + 1$, the $(N-1)$th semicolon will separate p_{N-1} and p_{N+1}. All these intermediary blocks trivially fulfill the restriction (R3).

Finally, if $D = N + 1$, the 'central' block $p_N \overleftarrow{p_{N+1}}$ (N odd) or $p_{N+1} \overleftarrow{p_N}$ (N even) complies with (R3) and (R4) and hence π is allowed. A further segment p_{N+2} would require an Nth semicolon to separate p_N and p_{N+1} in order not to violate (R4).

The proof for π_{mirrored} is completely analogue. □

Lemma 2. *If* $N \geqslant 2$ *is the number of symbols and* π *is a spiralling pattern with* D *segments and* $h_1 \geqslant 2$ (*i.e.,* $p_1 = 0, 1, \ldots$), *then*

1. π *is forbidden if and only if* (a) $D = N$ *and* $h_D \geqslant 2$, *or* (b) $D \geqslant N + 1$;
2. π *is allowed if and only if* (a′) $D < N$, *or* (b′) $D = N$ *and* $h_D = 1$.

Part 2 of Lemma 2, which is the logical negation of part 1, has been explicitly formulated for further references.

Proof. Consider the spiralling pattern (20). To begin with, the entries $h_1 - 1$ and $h_1 - 2$ of $\overleftarrow{p_1} = h_1 - 1, \dots, 1, 0$ may not be in the same block, otherwise the restriction (R4) would be violated (e_2 should be on the left of $h_1 - 1$). Thus we separate them with a first semicolon:

$$\pi = [\dots, \overleftarrow{p_3}, h_1 - 1; h_1 - 2, \dots, 1, 0, p_2, p_4, \dots].$$

From here on, three possibilities can occur that we illustrate in a general step of even order. (i) If $p_{2\nu}$ consists of more than one element (i.e., $h_{2\nu} \geqslant 2$), then we apply (R4) to $p_{2\nu}$ to conclude that we need a semicolon between $e_{2\nu} + h_{2\nu} - 2$ and $e_{2\nu} + h_2 - 1$ (since the follower of $p_{2\nu}$, i.e., the first entry of $\overleftarrow{p_{2\nu+1}}$, is on the wrong side). (ii) If $p_{2\nu}$ consists of one element ($h_{2\nu} = 1$) and $p_{2\nu-2}$ consists of more than one element ($h_{2\nu-2} \geqslant 2$), then we apply (R4) to the pair $p_{2\nu} = e_{2\nu}$ and $e_{2\nu-2} + h_{2\nu-2} - 1$, the last element of $p_{2\nu-2}$, which has been separated with a semicolon from the rest of elements in $p_{2\nu-2}$ two steps earlier. (iii) If both $p_{2\nu}$ and $p_{2\nu-2}$ consist of a single element ($h_{2\nu} = h_{2\nu-2} = 1$), apply (R4) to the pair $p_{2\nu-2} = e_{2\nu-2} < p_{2\nu} = e_{2\nu}$ to infer the need for a semicolon separating them (since $e_{2\nu-2} + 1 = e_{2\nu-1}$, the first element of $\overleftarrow{p_{2\nu-1}}$, is on the right of $e_{2\nu} + 1 = e_{2\nu+1}$, the first element of $\overleftarrow{p_{2\nu+1}}$). As a general rule, we need one semicolon per segment $p_{2\nu}$ or $\overleftarrow{p_{2\nu+1}}$ as long as there are still a posterior segment $\overleftarrow{p_{2\nu+1}}$ or $p_{2\nu+2}$, respectively, on the ‘wrong’ side. Note that all (intermediary) blocks ensued so far comply with (R3).

Following this way, we run out of the $N - 1$ semicolons we may use (corresponding to the N symbols), after having considered the segment p_{N-1}. Yet if $D = N$ and $h_N \geqslant 2$, then p_N will violate (R1) if N is odd, or (R2) if N is even. If $D \geqslant N + 1$, then the segment p_{N+1} will be on the wrong side of p_N and the pattern will not comply with (R4).

The proof for π_{mirrored} is completely analogue. □

The constructive, stepwise procedure used in the proofs of Lemmas 1 and 2 can be used *mutatis mutandis* in general to decompose any ordinal pattern into well-formed (i.e., complying with (R1)–(R4)) blocks. For instance, one could start from the leftmost entry and move on rightward one entry at a time, inserting a semicolon between the current and the previous entry whenever necessary to enforce the restrictions (R1)–(R4).

Proposition 4. *The following patterns of length $L \geqslant N + 2$, together with their corresponding mirrored patterns, are forbidden root patterns.*

1. *The tent patterns with $N + 2$ segments*

$$[0, p_3, \dots, p_N, L - 1, \overleftarrow{p_{N+1}}, \dots, \overleftarrow{p_2}] \tag{21}$$

if N is odd, or

$$[0, p_3, \dots, p_{N+1}, L - 1, \overleftarrow{p_N}, \dots, \overleftarrow{p_2}] \tag{22}$$

if N is even. Here $p_1 = 0$, and $p_{N+2} = L - 1$.

2. *The spiralling pattern with $N + 1$ segments*

$$[L - 2, \overleftarrow{p_{N-2}}, \dots, \overleftarrow{p_3}, 1, 0, p_2, \dots, p_{N-1}, L - 1] \tag{23}$$

if N is odd, or

$$[L - 1, \overleftarrow{p_{N-1}}, \dots, \overleftarrow{p_3}, 1, 0, p_2, \dots, p_{N-2}, L - 2], \tag{24}$$

if N is even. Here $p_1 = 0, 1$, $p_N = L - 2$, and $p_{N+1} = L - 1$.

3. *The spiralling pattern with N segments*

$$[L-1, L-2, \overleftarrow{p_{N-2}}, \ldots, \overleftarrow{p_3}, 1, 0, p_2, \ldots, p_{N-1}] \tag{25}$$

if N is odd, or

$$[\overleftarrow{p_{N-1}}, \ldots, \overleftarrow{p_3}, 1, 0, p_2, \ldots, p_{N-2}, L-2, L-1], \tag{26}$$

if N is even. Here $p_1 = 0, 1$, and $p_N = L-2, L-1$.

Of course, the cases 2 and 3 are related to the two possibilities in Lemma 2.

Proof. First of all, remember that given a forbidden pattern

$$[\pi_0, \ldots, \pi_{L-2}] \in \mathcal{S}_{L-1},$$

its outgrowth patterns of length L have the form (*Group A*)

$$[L-1, \pi_0, \ldots, \pi_{L-2}], \qquad [\pi_0, L-1, \ldots, \pi_{L-2}], \qquad \ldots, \qquad [\pi_0, \ldots, \pi_{L-2}, L-1],$$

or the form (*Group B*)

$$[0, \pi_0+1, \ldots, \pi_{L-2}+1], \qquad [\pi_0+1, 0, \ldots, \pi_{L-2}+1], \qquad \ldots,$$
$$[\pi_0+1, \ldots, \pi_{L-2}+1, 0].$$

1. This case is trivial. Any tent pattern made out of $N+2$ segments is forbidden according to Lemma 1. Moreover, since the entries $L-1$ and 0 in patterns (21) and (22) are segments on their own, the number of segments D of the these tent patterns will fall below the threshold value $D = N+2$ once $L-1$ (Group A) or 0 (Group B) are deleted.

2. Only (23) will be considered here, the proof for (24) and their mirrored patterns being completely analogue. That (23) is forbidden follows readily from Lemma 2(b). To prove that π is also a root pattern, we need to show that it is not the outgrowth of any forbidden pattern of shorter length.

There are two possibilities. Suppose first that π is an outgrowth forbidden pattern of Group A. Deletion of the entry $L-1$ yields then the spiralling pattern

$$[L-2, \overleftarrow{p_{N-2}}, \ldots, \overleftarrow{p_3}, 1, 0, p_2, \ldots, p_{N-1}],$$

which is allowed on account of having N segments, $h_1 = 2$, and a last segment $p_N = L-2$ of length 1 (Lemma 2(b′)).

Thus, suppose that π is an outgrowth forbidden pattern of Group B. In this case, after removing the entry 0 and subtracting 1 from the remaining entries we are left with the pattern

$$\left[L-3, \overleftarrow{p'_{N-2}}, \ldots, \overleftarrow{p'_3}, 0, p'_2, \ldots, p'_{N-1}, L-2\right], \tag{27}$$

where $p'_d = e_d - 1, \ldots, e_d + h_d - 2$, $2 \leqslant d \leqslant N+1$. Since $p'_1 = 0$ ($h'_1 = h_1 - 1 = 1$) and $p'_2 = 1, \ldots$ ($h'_2 = h_2 \geqslant 1$), we can merge p'_1 and p'_2 into the new segment $p''_1 \equiv 0, 1, \ldots$, so that (27) is a spiralling pattern with $h''_1 \geqslant 2$ and the following N segments: $p''_1, p'_3, \ldots, p'_{N-1}, p'_N = L-3$, $p_{N+1} = L-2$. According to Lemma 2(b′), the order pattern (27) is allowed.

3. This case uses Lemma 2(a)–(a′) instead. The proof proceeds similarly to case 2. $\square$

Example 7. For $N = 2n+1$, Proposition 4 provides the following six forbidden patterns of minimal length $L = N+2$:

$$[0, 2, \dots, 2n, 2n+2, 2n+1, \dots, 3, 1],$$
$$[2n+1, 2n-1, \dots, 1, 0, 2, \dots, 2n, 2n+2],$$
$$[2n+2, 2n+1, \dots, 1, 0, 2, \dots, 2n-2, 2n],$$

and their mirrored patterns. For $N = 2n$, the six forbidden patterns of minimal length $L = N + 2$ provided by Proposition 4 are:

$$[0, 2, \dots, 2n, 2n+1, \dots, 3, 1],$$
$$[2n+1, 2n-1, \dots, 1, 0, 2, \dots, 2n-2, 2n],$$
$$[2n-1, 2n-3, \dots, 1, 0, 2, \dots, 2n, 2n+1],$$

and their mirrored patterns. In particular, for $N = 2$ we obtain the following minimal-length forbidden patterns:

$$\begin{array}{ll} [0, 2, 3, 1], & [1, 3, 2, 0], \\ [3, 1, 0, 2], & [2, 0, 1, 3], \\ [1, 0, 2, 3], & [3, 2, 0, 1]. \end{array}$$

We conjecture that the only forbidden patterns of minimal length $L = N + 2$ are the six patterns delivered by Proposition 4 after setting $p_k = k - 1$ (respectively $p_k = k$) in those segments not explicitly given in the tent patterns (21)–(22) (respectively in the spiralling patterns (23)–(26)).

Corollary 1. *For every $K \geqslant 2$ there are self maps on the interval $[0, 1]$ without forbidden patterns of length $L \leqslant K$.*

Proof. Let $S_N = \psi \circ \Sigma \circ \psi^{-1} : [0, 1] \to [0, 1]$ be the map (14). Since ψ is an order-isomorphy, S_N and Σ, the shift on N symbols, have the same forbidden patterns. Therefore, if $N + 1 \leqslant K$, then S_N has no forbidden patterns of length $L \leqslant K$ because of Proposition 3. $\square$

It follows that *there are interval maps on $\mathbb{R}^n$ without forbidden patterns*. For example, one can decompose $[0, 1]$ in infinite many half-open intervals (of vanishing length), $[0, 1] = \bigcup_{N=2}^{\infty} I_N$ and define on each I_N a properly scaled version of S_N, $\tilde{S}_N : I_N \to I_N$. In $\mathbb{R}^2$ one can perform the said decomposition along the 1-axis and define on $I_N \times [0, 1]$ the function $(\tilde{S}_N, \mathrm{Id})$. Now, Eq. (7) shows that adding some natural assumption, like piecewise monotony, can make all the difference.

6. Order patterns and two-sided shifts

Consider now the bisequence space, $\{0, 1, \dots, N-1\}^{\mathbb{Z}}$, endowed with the lexicographical (or product) order. With the notation ω_- for the *left sequence* $(\omega_{-n})_{n \in \mathbb{N}}$ of $\omega \in \{0, 1, \dots, N-1\}^{\mathbb{Z}}$ and ω_+ for the *right sequence* $(\omega_n)_{n \in \mathbb{N}_0}$, we have

$$\omega < \omega' \quad \Leftrightarrow \quad \begin{cases} \omega_+ < \omega'_+ \\ \text{or} \\ \omega_- < \omega'_- \text{ if } \omega_+ = \omega'_+, \end{cases}$$

where $<$ between right (respectively left) sequences denotes lexicographical order in $\{0, 1, \dots, N-1\}^{\mathbb{N}_0}$ (respectively $\{0, 1, \dots, N-1\}^{\mathbb{N}}$). Thus, the lexicographical order for bisequences is defined most of the time by the right sequences of the points being compared, except when they coincide, in which case the order is defined by their left sequences. If we map $\{0, 1, \ldots, N-1\}^{\mathbb{Z}}$ onto $[0, 1] \times [0, 1] \equiv [0, 1]^2$ via

$$(\omega_-, \omega_+) \mapsto \left(\sum_{n=1}^{\infty} \omega_{-n} N^{-n}, \sum_{n=0}^{\infty} \omega_n N^{-(n+1)} \right),$$

we find that lexicographical order in $\{0, 1, \ldots, N-1\}^{\mathbb{Z}}$ corresponds to lexicographical order in $[0, 1]^2$, which results thereby foliated into a continuum of copies of $([0, 1], <)$. In order for this map to be one-to-one, we have to exclude the countable set $\overline{\overline{\mathcal{N}}}$ of all bisequences terminating in an infinite string of $(N-1)$s in either direction.

In relation with the order patterns defined by the orbits of two-sided sequences,

$$\begin{aligned} &\Sigma^i(\omega) < \Sigma^j(\omega) \\ &\quad\Leftrightarrow \quad \begin{cases} (\omega_i, \omega_{i+1}, \ldots) < (\omega_j, \omega_{j+1}, \ldots) \\ \text{or} \\ (\omega_{i-1}, \omega_{i-2}, \ldots) < (\omega_{j-1}, \omega_{j-2}, \ldots) \text{ if } (\omega_i, \omega_{i+1}, \ldots) = (\omega_j, \omega_{j+1}, \ldots), \end{cases} \end{aligned}$$

where $i, j \geqslant 0$, $i \neq j$. It follows that the 'exceptional' condition $(\omega_i, \omega_{i+1}, \ldots) = (\omega_j, \omega_{j+1}, \ldots)$ occurs if and only if $\Sigma^{|i-j|}(\omega_+) = \omega_+$, i.e., when the right sequence ω_+ of $\omega \in \{0, 1, \ldots, N-1\}^{\mathbb{Z}}$ is periodic from the entry $\min\{i, j\}$ on with period $p = |i-j|$.

Proposition 5. *The two-sided shift on $N \geqslant 2$ symbols has no forbidden patterns of length $L \leqslant N+1$ and has forbidden root patterns for $L \geqslant N+2$.*

Proof. The one-sided sequence $\omega_+ \in \{0, 1, \ldots, N-1\}^{\mathbb{N}_0}$ defines an order pattern π of length L,

$$\Sigma^{\pi_0}(\omega_+) < \Sigma^{\pi_1}(\omega_+) < \cdots < \Sigma^{\pi_{L-1}}(\omega_+),$$

if and only if the two-sided sequences $\omega = (\omega_-, \omega_+)$, with $\omega_- \in \{0, 1, \ldots, N-1\}^{\mathbb{N}}$ arbitrary, define the same order pattern. □

Example 8. Let $I^2 = [0, 1] \times [0, 1]$ endowed with the induced Lebesgue measure λ and $B : I^2 \to I^2$ the λ-invariant *baker's map*,

$$B(x, y) = \begin{cases} (2x, \frac{1}{2}y), & 0 \leqslant x < \frac{1}{2}, \\ (2x-1, \frac{1}{2}y + \frac{1}{2}), & \frac{1}{2} \leqslant x \leqslant 1. \end{cases}$$

A generating partition of (I^2, λ, B) is: $A_0 = [0, \frac{1}{2}) \times [0, 1]$ and $A_1 = [\frac{1}{2}, 1] \times [0, 1]$. For Σ take the two-sided $(\frac{1}{2}, \frac{1}{2})$-Bernoulli shift. Then B and Σ are isomorphic via the λ-invariant coding map $\Phi : I^2 \to \{0, 2\}^{\mathbb{Z}} \setminus \overline{\overline{\mathcal{N}}}$, given by

$$\Phi(x) = (\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots),$$

where $\omega_n = a_n$ if $B^n(x) \in A_{a_n}$, $n \in \mathbb{Z}$. Since Φ preserves order (in fact, Φ is the inverse of the order-preserving map $(\omega_-, \omega_+) \mapsto (\sum_{n=0}^{\infty} \omega_{-n} 2^{-(n+1)}, \sum_{n=1}^{\infty} \omega_n 2^{-n})$, sequences ending with $\overline{1}$ excluded), we conclude that the baker's transformation has no forbidden patterns of length $\leqslant 3$.

Acknowledgments

We are very thankful to the referees for their valuable comments. J.M.A. was financially supported by the Spanish Ministry of Education and Science (Grant MTM2005-04948).

References

[1] J.M. Amigó, M.B. Kennel, L. Kocarev, The permutation entropy rate equals the metric entropy rate for ergodic information sources and ergodic dynamical systems, Phys. D 210 (2005) 77–95.

[2] J.M. Amigó, L. Kocarev, J. Szczepanski, Order patterns and chaos, Phys. Lett. A 355 (2006) 27–31.

[3] J.M. Amigó, S. Zambrano, M.A.F. Sanjuán, True and false forbidden patterns in deterministic and random dynamics, Europhys. Lett. 79 (2007), 5001-p1–p5.

[4] E. Babson, E. Steingrímsson, Generalized permutation patterns and a classification of the Mahonian statistics, Sém. Lothar. Combin. 44 (2000) 18, Article B44b.

[5] G.L. Baker, J.P. Gollub, Chaotic Dynamics: An Introduction, Cambridge University Press, Cambridge, 1996.

[6] C. Bandt, G. Keller, B. Pompe, Entropy of interval maps via permutations, Nonlinearity 15 (2002) 1595–1602.

[7] M.F. Doherty, J.M. Ottino, Chaos in deterministic systems: Strange attractors, turbulence and applications in chemical engineering, Chem. Engin. Sci. 43 (1988) 139–183.

[8] S. Elizalde, M. Noy, Consecutive patterns in permutations, Adv. in Appl. Math. 30 (2003) 110–125.

[9] S. Elizalde, Asymptotic enumeration of permutations avoiding generalized patterns, Adv. in Appl. Math. 36 (2006) 138–155.

[10] W. de Melo, S. van Strien, One-Dimensional Dynamics, Springer-Verlag, Berlin, 1993.

[11] A.N. Sarkovskii, Coexistence of cycles of a continuous map of a line into itself, Ukrainian Math. J. 16 (1964) 61–71.

[12] P. Walters, An Introduction to Ergodic Theory, Springer-Verlag, New York, 1982.