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On the existence of a common solution X to the matrix equations $A_i X B_j = C_{ij}$, $(i, j) \in \Gamma$

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Abstract

In this paper conditions are derived for the existence of a common solution X to the matrix equations $A_i X B_j = C_{ij}$, $(i, j) \in \Gamma$, where the matrices A_i, B_j, C_{ij} and X have suitable dimensions and the (i, j)'s are index pairs in some set Γ. The purpose of this paper is to present, for certain specific sets of index pairs Γ, verifiable necessary and sufficient solvability conditions that are stated directly in terms of the matrices and that do not use Kronecker products.

Keywords: Common solution; Radical; Matrix equations

1. Introduction

In this paper we study the set of linear matrix equations

$$A_i X B_j = C_{ij}, \quad (i, j) \in \Gamma,$$

where A_i, B_j and C_{ij} are given matrices, with X the unknown matrix and where Γ denotes a set of index pairs. We assume that all the matrices have suitable dimensions.

In this paper we are concerned with conditions for the existence of a matrix X being a common solution to all the matrix equations simultaneously. We do not want to use Kronecker products, but we want to derive conditions in which the matrices occur directly. This may give more insight in why a common solution exists. Also if

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the existence of a common solution is part of a larger problem, then conditions stated directly in terms of the matrices might be more preferable. See for instance [7,8].

Our approach is in contrast with the approach in [9], in which the same set of linear equations is studied as in this paper. Opposite to our paper, the results in [9] are not just simply stated in terms of the original matrices, but require the use of generalized inverses. Further, the conditions for existence of a common solution in [9] are obtained, and have to be verified, in a finite iterative process. It can be shown that the process can be described in a much simpler way than given in [9] and that the outcome of the process crucially depends on the choice for a solution of the equation in the first induction step. These aspects are not mentioned and as such the results in [9] seem to be incomplete.

For the case that $\Gamma = \{(1, 2), (2, 1)\}$ or, which after a renumbering is the same, for $\Gamma = \{(1, 1), (2, 2)\}$, we have presented verifiable necessary and sufficient conditions for the existence of a common solution in [7]. In this paper we recall these conditions and extend the conditions in two directions. First, we recall from [8] necessary and sufficient conditions for the existence of a common solution for the case that $\Gamma = \{(i, j) \mid i, j \in \underline{k},\ i \neq j\}$ for an arbitrary integer $k \geqslant 2$, where $\underline{k} = \{1, 2, \ldots, k\}$. We have included these results for completeness and general interest. Secondly, for the case that $\Gamma = \{(i, i) \mid i \in \underline{k}\}$, i.e. $\Gamma = \{(i, j) \mid i, j \in \underline{k},\ i = j\}$, for an arbitrary integer $k \geqslant 2$, we present necessary and sufficient conditions provided some additional assumption is satisfied.

The outline of this paper is as follows. In Section 2 we review some known results, introduce the radical of a family of linear subspaces and we present some useful observations. In Section 3 we recall the results for the case that $\Gamma = \{(i, j) \mid i, j \in \underline{k},\ i \neq j\}$. We use the notion of radical to formulate the assumption under which we can prove our result for the case that $\Gamma = \{(i, i) \mid i \in \underline{k}\}$. We derive this result in Section 4. It consists of verifiable necessary and sufficient conditions for the existence of a common solution to the above linear matrix equations. We conclude the paper with some remarks concerning possible further research in Section 5.

2. Known results, radicals, useful observations

2.1. Known results

To derive the solvability conditions in this paper we use the next three lemmas. In the lemmas A, B, C and D are given matrices of suitable dimensions.

Lemma 2.1. *The following statements are equivalent*:

1. *There is a matrix X such that $AXB = C$.*
2. $\operatorname{im} C \subseteq \operatorname{im} A$ *and* $\ker B \subseteq \ker C$.
3. $UC = 0$ *and* $CV = 0$ *for all matrices U and V such that $UA = 0$ and $BV = 0$.*

Proof. See any textbook on matrix theory. For instance, [1] or [4]. See also [5]. □

Lemma 2.2. *There is a matrix X such that $AX = C$ and $XB = D$ if and only if* $\operatorname{im} C \subseteq \operatorname{im} A$, $\ker B \subseteq \ker D$ *and* $AD = CB$.

Proof. See [4, p. 25]. □

Lemma 2.3. $\operatorname{rank}\begin{bmatrix} C & A \\ B & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$ *if and only if* $C \ker B \subseteq \operatorname{im} A$.

Proof. See [7]. □

2.2. *Radical of a family of subspaces*

Most of the material in this subsection can be found in [6]. Let $\mathscr{L} = \{\mathscr{L}_i \mid i \in \underline{k}\}$ be a family of linear subspaces in some vector space $\mathscr{X}$. We say that the family $\mathscr{L}$ is independent in $\mathscr{X}$ if $\mathscr{L}_i \cap (\sum_{j \in \underline{k}, j \neq i} \mathscr{L}_j) = 0$ for all $i \in \underline{k}$. It can be shown that $\mathscr{L}$ is an independent family if and only if any vector $a \in \sum_{j \in \underline{k}} \mathscr{L}_j$ can uniquely be decomposed as $a = \sum_{j \in \underline{k}} a_j$ with $a_i \in \mathscr{L}_i$ for all $i \in \underline{k}$. Following [6] we define the radical of the family $\mathscr{L}$ to be the subspace

$$\mathscr{L}^{\vee} = \sum_{i \in \underline{k}} \left(\mathscr{L}_i \cap \left(\sum_{j \in \underline{k}, j \neq i} \mathscr{L}_j \right) \right).$$

Clearly, $\mathscr{L}$ is an independent family if and only if $\mathscr{L}^{\vee} = 0$. In [6] it is shown that the radical $\mathscr{L}^{\vee}$ is the smallest subspace $\mathscr{L}_0$ in $\mathscr{X}$ with the property that the family of quotient spaces $\{(\mathscr{L}_i + \mathscr{L}_0)/\mathscr{L}_0 \mid i \in \underline{k}\}$ in $\mathscr{X}/\mathscr{L}_0$ is independent. If $\mathscr{L}^{\vee} \subseteq \mathscr{L}_i$ for all $i \in \underline{k}$, the above implies that there exists a full column rank matrix L partitioned as $L = [L_0, L_1, \ldots, L_k]$ with $\mathscr{L}^{\vee} = \operatorname{im} L_0$ and $\mathscr{L}_i = \operatorname{im}[L_0, L_i]$ for all $i \in \underline{k}$. Finally, we note that $\mathscr{L}^{\vee} \subseteq \mathscr{L}_i$ for all $i \in \underline{k}$ if and only if $\mathscr{L}^{\vee} = \bigcap_{i \in \underline{k}} \mathscr{L}_i$.

2.3. *Some useful observations*

In the proof of our result in Section 4 we make use of the following observations. In the observations all matrices have suitable dimensions.

1. If Q is a matrix of suitable dimensions then
   (i) $C \ker B \supseteq CQ \ker BQ$,
   (ii) if $\ker B \subseteq \ker C$ then $\ker BQ \subseteq \ker CQ$.
2. $[C_1, C_2] \ker [B_1, B_2] = [C_1, C_1 + C_2] \ker [B_1, B_1 + B_2]$

(The inclusion $\supseteq$ follows from (i) above with $C = [C_1, C_2]$, $B = [B_1, B_2]$ and $Q = \begin{bmatrix} I & I \\ 0 & I \end{bmatrix}$ where I is an identity matrix of suitable dimensions. The other inclusion $\subseteq$ follows with $C = [C_1, C_1 + C_2]$, $B = [B_1, B_1 + B_2]$ and $Q = \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix}$).

3. If B_1 has full column rank then $[C_1, C_2] \ker [B_1, 0] = \operatorname{im} C_2$.

3. $\Gamma = \{(i, j) \mid i, j \in \underline{k} \text{ with } i \neq j\}$

In this section we consider the $k^2 - k$ linear matrix equations $A_i X B_j = C_{ij}$ for $i, j \in \underline{k}$ with $i \neq j$, and we present verifiable necessary and sufficient conditions for the existence of a common solution. To state and derive the conditions we denote

$$B = [B_1, B_2, \ldots, B_k] \text{and} \quad A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{bmatrix}.$$

Furthermore, for $i \in \underline{k}$ we denote

$$\check{B}_i = [B_1, \ldots, B_{i-1}, B_{i+1}, \ldots, B_k], \quad \Lambda_i = [C_{i1}, \ldots, C_{ii-1}, C_{ii+1}, \ldots, C_{ik}],$$

$$\check{A}_i = \begin{bmatrix} A_1 \\ \vdots \\ A_{i-1} \\ A_{i+1} \\ \vdots \\ A_k \end{bmatrix}, \quad \Delta_i = \begin{bmatrix} C_{1i} \\ \vdots \\ C_{i-1i} \\ C_{i+1i} \\ \vdots \\ C_{ki} \end{bmatrix}$$

and

$$\Gamma_i = \begin{bmatrix} 0 & \cdots & 0 & -C_{1i} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & -C_{i-1i} & 0 & \cdots & 0 \\ C_{i1} & \cdots & C_{ii-1} & 0 & C_{ii+1} & \cdots & C_{ik} \\ 0 & \cdots & 0 & -C_{i+1i} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & -C_{ki} & 0 & \cdots & 0 \end{bmatrix}.$$

Now we can recall the next result from [8]. We have included the result and its proof for reasons of completeness.

Theorem 3.1. *There exists a matrix X such that $A_i X B_j = C_{ij}$ for all $i, j \in \underline{k}$ with $i \neq j$ if and only if* $\operatorname{im} \Delta_i \subseteq \operatorname{im} \check{A}_i$, $\ker \check{B}_i \subseteq \ker \Lambda_i$, *and* $\Gamma_i \ker B \subseteq \operatorname{im} A$ *for all* $i \in \underline{k}$.

To prove the above theorem we follow the line of Mitra [2] and we need some additional notation. In the remainder of this section we assume that U is a matrix such that $\operatorname{im} A \subseteq \ker U$, and that U is partitioned as $U = [U_1, U_2, \ldots, U_k]$ in such a way that the products $U_i A_i$ are defined for $i \in \underline{k}$. Similarly, we assume that V is a matrix such that $\ker B \subseteq \operatorname{im} V$ and that V is partitioned as $V^{\mathrm{T}} = [V_1^{\mathrm{T}}, V_2^{\mathrm{T}}, \ldots, V_k^{\mathrm{T}}]$ (T means transpose) such that the products $B_i V_i$ are defined for all $i \in \underline{k}$. Furthermore, given matrices $Y_1, Y_2, \ldots, Y_k$, we denote

$$C(Y_1, Y_2, \ldots, Y_k) = \begin{bmatrix} Y_1 & C_{12} & \cdots & C_{1k} \\ C_{21} & Y_2 & \cdots & C_{2k} \\ \vdots & \vdots & & \vdots \\ C_{k1} & C_{k2} & \cdots & Y_k \end{bmatrix}.$$

Proof of Theorem 3.1

(only if)-part.

Assume that X satisfies $A_i X B_j = C_{ij}$ for all $i, j \in \underline{k}$ with $i \neq j$. Given any $i \in \underline{k}$, it is immediate from the definitions that $\check{A}_i X B_i = \Delta_i$ and $A_i X \check{B}_i = \Lambda_i$. So, from Lemma 2.1 it follows that $\operatorname{im} \Delta_i \subseteq \operatorname{im} \check{A}_i$ and $\ker \check{B}_i \subseteq \ker \Lambda_i$ for all $i \in \underline{k}$. Furthermore, it is clear that there are matrices $Y_1, Y_2, \ldots, Y_k$ such that $AXB = C(Y_1, Y_2, \ldots, Y_k)$. For instance, take $Y_i = A_i X B_i$ for all $i \in \underline{k}$. Using Lemma 2.1 it therefore follows that there are matrices $Y_1, Y_2, \ldots, Y_k$ such that $UC(Y_1, Y_2, \ldots, Y_k) = 0$ and $C(Y_1, Y_2, \ldots, Y_k)V = 0$. By the definition of $C(Y_1, Y_2, \ldots, Y_k)$ the latter two equations mean that $-U_i Y_i = \sum_{j \in \underline{k}, j \neq i} U_j C_{ji}$ and $-Y_i V_i = \sum_{j \in \underline{k}, j \neq i} C_{ij} V_j$ for all $i \in \underline{k}$.

Now assume that $i = 1$. (Other values of i can be treated in a similar way.) Then it follows that there exists a matrix Y_1 that is a common solution to a pair of linear matrix equations of the same type as in Lemma 2.2. From Lemma 2.2 it therefore follows that $U_1(C_{12}V_2 + C_{13}V_3 + \cdots + C_{1k}V_k) = (U_2 C_{21} + U_3 C_{31} + \cdots + U_k C_{k1})V_1$. This expression can also be written as $U\Gamma_1 V = 0$, or, equivalently, as $\Gamma_1 \ker B \subseteq \operatorname{im} A$. Since the above procedure may be repeated for any value of $i \in \underline{k}$ the proof of the (only if)-part is now completed.

(if)-part.

Because $\operatorname{im} \Delta_1 \subseteq \operatorname{im} \check{A}_1$, there exists a matrix Z_1 such that $C_{j1} = A_j Z_1$ for all $j = 2, 3, \ldots, k$. Hence, there exists a matrix Z_1 such that $U_2 C_{21} + U_3 C_{31} + \cdots + U_k C_{k1} = (U_2 A_2 + U_3 A_3 + \cdots + U_k A_k) Z_1 = -U_1 A_1 Z_1$. Here the last equality is due to the fact that $U_1 A_1 + U_2 A_2 + \cdots + U_k A_k = UA = 0$. So, it follows that

$\operatorname{im}(U_2C_{21} + U_3C_{31} + \cdots + U_kC_{k1}) \subseteq \operatorname{im} U_1$. Similarly, $\ker \check{B}_1 \subseteq \ker \Lambda_1$ implies that $\ker V_1 \subseteq \ker(C_{12}V_2 + C_{13}V_3 + \cdots + C_{1k}V_k)$.

Finally, as already indicated in the (only if)-part, it follows from $\Gamma_1 \ker B \subseteq \operatorname{im} A$ that $U_1(C_{12}V_2 + C_{13}V_3 + \cdots + C_{1k}V_k) = (U_2C_{21} + U_3C_{31} + \cdots + U_kC_{k1})V_1$. By Lemma 2.2 it is now clear that there exists a matrix Y_1 such that $-U_1Y_1 = U_2C_{21} + U_3C_{31} + \cdots + U_kC_{k1}$ and $-Y_1V_1 = C_{12}V_2 + C_{13}V_3 + \cdots + C_{1k}V_k$.

Clearly, the above can be repeated for any value of $i \in \underline{k}$ showing that there exist matrices $Y_1, Y_2, \ldots, Y_k$ such that $-U_iY_i = \sum_{j\in\underline{k}, j\neq i} U_jC_{ji}$ and $-Y_iV_i = \sum_{j\in\underline{k}, j\neq i} C_{ij}V_j$ for all $i \in \underline{k}$. Hence, there are matrices $Y_1, Y_2, \ldots, Y_k$ such that $UC(Y_1, Y_2, \ldots, Y_k) = 0$ and $C(Y_1, Y_2, \ldots, Y_k)V = 0$. By Lemma 2.1 it follows that the matrices $Y_1, Y_2, \ldots, Y_k$ are such that $AXB = C(Y_1, Y_2, \ldots, Y_k)$ for some matrix X. From the definition of $C(Y_1, Y_2, \ldots, Y_k)$ it is now immediate that this matrix X satisfies $A_iXB_j = C_{ij}$ for all $i, j \in \underline{k}$ with $i \neq j$. This concludes the proof of the (if)-part. □

Using Lemma 2.3 the next corollary follows immediately from Theorem 3.1.

Corollary 3.2. *There exists a matrix X such that $A_iXB_j = C_{ij}$ for all $i, j \in \underline{k}$ with $i \neq j$ if and only if for all $i \in \underline{k}$,*

$$\operatorname{rank} \check{A}_i = \operatorname{rank}[\check{A}_i \;\; \Delta_i], \quad \operatorname{rank} \check{B}_i = \operatorname{rank} \begin{bmatrix} \check{B}_i \\ \Lambda_i \end{bmatrix},$$
$$\operatorname{rank} \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} \Gamma_i & A \\ B & 0 \end{bmatrix}.$$

For $k = 2$ the result in Theorem 3.1 can be rewritten into the next result presented in [7].

Corollary 3.3. *There exists a matrix X such that $A_1XB_2 = C_{12}$ and $A_2XB_1 = C_{21}$ if and only if* $\operatorname{im} C_{12} \subseteq \operatorname{im} A_1$, $\operatorname{im} C_{21} \subseteq \operatorname{im} A_2$, $\ker B_1 \subseteq \ker C_{21}$, $\ker B_2 \subseteq \ker C_{12}$ *and*

$$\begin{bmatrix} C_{12} & 0 \\ 0 & -C_{21} \end{bmatrix} \ker[B_2 \quad B_1] \subseteq \operatorname{im} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}.$$

4. $\Gamma = \{(i, i) \mid i \in \underline{k}\}$

In this section we consider the k linear matrix equations $A_iXB_i = C_{ii}$ for $i \in \underline{k}$. We denote $\mathscr{L}_i = \operatorname{im} B_i$, $i \in \underline{k}$, and we write $\mathscr{L}^\vee$ for the radical of the family $\{\mathscr{L}_i \mid i \in \underline{k}\}$. Our standing assumption throughout this section is that $\mathscr{L}^\vee = \bigcap_{i\in\underline{k}} \mathscr{L}_i$. Under this assumption we are able to derive verifiable necessary and sufficient conditions for the existence of a common solution X to the above equations.

Theorem 4.1. *Under the assumption that* $\mathscr{L}^{\vee} = \bigcap_{i \in \underline{k}} \mathscr{L}_i$ *the following holds. There is a matrix* X *such that* $A_i X B_i = C_{ii}$ *for all* $i \in \underline{k}$ *if and only if* $\operatorname{im} C_{ii} \subseteq \operatorname{im} A_i$, $\ker B_i \subseteq \ker C_{ii}$ *for all* $i \in \underline{k}$ *and*

$$\begin{bmatrix} C_{11} & 0 & \cdots & 0 \\ 0 & C_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_{kk} \end{bmatrix} \ker \begin{bmatrix} B_1 & -B_2 & 0 & \cdots & 0 \\ 0 & B_2 & -B_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & B_{k-1} & -B_k \end{bmatrix} \subseteq \operatorname{im} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{bmatrix}.$$

Proof

(only if)-part.

Assume that X satisfies $A_i X B_i = C_{ii}$ for all $i \in \underline{k}$. By Lemma 2.1 this means that $\operatorname{im} C_{ii} \subseteq \operatorname{im} A_i$, $\ker B_i \subseteq \ker C_{ii}$ for all $i \in \underline{k}$. This proves the first $2k$ subspace inclusions mentioned in the conditions of the theorem. To prove the last subspace inclusion consider the vector

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \end{bmatrix} \in \ker \begin{bmatrix} B_1 & -B_2 & 0 & \cdots & 0 \\ 0 & B_2 & -B_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & B_{k-1} & -B_k \end{bmatrix}.$$

Then $B_1 u_1 = B_2 u_2 = \cdots = B_k u_k =: b$. Because X is a common solution it follows that

$$\begin{bmatrix} C_{11} & 0 & \cdots & 0 \\ 0 & C_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_{kk} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \end{bmatrix} = \begin{bmatrix} A_1 X B_1 u_1 \\ A_2 X B_2 u_2 \\ \vdots \\ A_k X B_k u_k \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{bmatrix} X b \in \operatorname{im} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{bmatrix}.$$

Hence, also the last subspace inclusion follows and the proof of the (only if)-part is completed.

(if)-part.

Because of the assumption that $\mathscr{L}^{\vee} = \bigcap_{i \in \underline{k}} \mathscr{L}_i$ it follows that $\mathscr{L}^{\vee} \subseteq \mathscr{L}_i$ for all $i \in \underline{k}$. In Section 2 we have seen that then there exists a full column rank matrix $L = [L_0, L_1, \ldots, L_k]$ such that $\mathscr{L}^{\vee} = \operatorname{im} L_0$ and $\mathscr{L}_i = \operatorname{im} [L_0, L_i]$ for all $i \in \underline{k}$. Recall that $[L_0, L_i]$ has full column rank and that $\mathscr{L}_i = \operatorname{im} B_i$ for all $i \in \underline{k}$. It follows that there exist full column rank matrices Q_i such that $B_i Q_i = [L_0, L_i]$ for all $i \in \underline{k}$. Correspondingly, partition $C_{ii} Q_i = [\bar{C}_{ii}, \hat{C}_{ii}]$ for all $i \in \underline{k}$. We claim that

$$\begin{bmatrix} C_{11} & 0 & \cdots & 0 \\ 0 & C_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_{kk} \end{bmatrix} \ker \begin{bmatrix} B_1 & -B_2 & 0 & \cdots & 0 \\ 0 & B_2 & -B_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & B_{k-1} & -B_k \end{bmatrix} \supseteq \operatorname{im} \begin{bmatrix} \bar{C}_{11} \\ \bar{C}_{22} \\ \vdots \\ \bar{C}_{kk} \end{bmatrix}.$$

For reasons of simplicity and space limitations we prove this claim for $k = 3$ only. For other values of k similar proofs can be given. In the proof below the observations of Section 2.3 will be used. First note that

$$\begin{bmatrix} C_{11} & 0 & 0 \\ 0 & C_{22} & 0 \\ 0 & 0 & C_{33} \end{bmatrix} \ker \begin{bmatrix} B_1 & -B_2 & 0 \\ 0 & B_2 & -B_3 \end{bmatrix}$$

$$\supseteq \begin{bmatrix} C_{11}Q_1 & 0 & 0 \\ 0 & C_{22}Q_2 & 0 \\ 0 & 0 & C_{33}Q_3 \end{bmatrix} \ker \begin{bmatrix} B_1Q_1 & -B_2Q_2 & 0 \\ 0 & B_2Q_2 & -B_3Q_3 \end{bmatrix}$$

$$= \begin{bmatrix} \bar{C}_{11} & \hat{C}_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{C}_{22} & \hat{C}_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{C}_{33} & \hat{C}_{33} \end{bmatrix}$$

$$\times \ker \begin{bmatrix} L_0 & L_1 & -L_0 & -L_2 & 0 & 0 \\ 0 & 0 & L_0 & L_2 & -L_0 & -L_3 \end{bmatrix}.$$

Now adding in both matrices in the above expression the first column to the third, and next the third to the fifth, we obtain that the above subspace is equal to

$$\begin{bmatrix} \bar{C}_{11} & \hat{C}_{11} & \bar{C}_{11} & 0 & \bar{C}_{11} & 0 \\ 0 & 0 & \bar{C}_{22} & \hat{C}_{22} & \bar{C}_{22} & 0 \\ 0 & 0 & 0 & 0 & \bar{C}_{33} & \hat{C}_{33} \end{bmatrix} \ker \begin{bmatrix} L_0 & L_1 & 0 & -L_2 & 0 & 0 \\ 0 & 0 & L_0 & L_2 & 0 & -L_3 \end{bmatrix}$$

Because the matrix $[L_0, L_1, L_2, L_3]$ has full column rank the matrix

$$\begin{bmatrix} L_0 & L_1 & 0 & -L_2 & 0 \\ 0 & 0 & L_0 & L_2 & -L_3 \end{bmatrix}$$

also has full column rank. Therefore, it follows that the last subspace above is equal to

$$\operatorname{im} \begin{bmatrix} \bar{C}_{11} \\ \bar{C}_{22} \\ \bar{C}_{33} \end{bmatrix},$$

which proves our claim for the case $k = 3$. With our claim for general k, it now follows that

$$\operatorname{im} \begin{bmatrix} \bar{C}_{11} \\ \bar{C}_{22} \\ \vdots \\ \bar{C}_{kk} \end{bmatrix} \subseteq \operatorname{im} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{bmatrix}.$$

Because L_0 has full column rank, also automatically

$$\ker L_0 \subseteq \ker \begin{bmatrix} \bar{C}_{11} \\ \bar{C}_{22} \\ \vdots \\ \bar{C}_{kk} \end{bmatrix}.$$

Furthermore, since $\operatorname{im} C_{ii} \subseteq \operatorname{im} A_i$, $\ker B_i \subseteq \ker C_{ii}$ for all $i \in \underline{k}$, it follows that $\operatorname{im} C_{ii} Q_i \subseteq \operatorname{im} A_i$, $\ker B_i Q_i \subseteq \ker C_{ii} Q_i$ for all $i \in \underline{k}$. Hence, we have that $\operatorname{im} \hat{C}_{ii} \subseteq \operatorname{im} A_i$ for all $i \in \underline{k}$, and since L_i has full column rank we also automatically have $\ker L_i \subseteq \ker \hat{C}_{ii}$ for all $i \in \underline{k}$. By lemma 2.1 it follows that there are matrices X_0, $X_1, \ldots, X_k$ such that

$$\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{bmatrix} X_0 L_0 = \begin{bmatrix} \bar{C}_{11} \\ \bar{C}_{22} \\ \vdots \\ \bar{C}_{kk} \end{bmatrix}$$

and $A_i X_i L_i = \hat{C}_{ii}$ for all $i \in \underline{k}$. Because $[L_0, L_1, \ldots, L_k]$ has full column rank it is now possible to define the matrix X as follows:

$$X[L_0, L_1, \ldots, L_k] = [X_0 L_0, X_1 L_1, \ldots, X_k L_k].$$

Then we have $A_i X[L_0, L_i] = A_i[X_0 L_0, X_i L_i] = [A_i X_0 L_0, A_i X_i L_i] = [\bar{C}_{ii}, \hat{C}_{ii}]$ for all $i \in \underline{k}$. Recall that $\operatorname{rank} B_i = \operatorname{rank}[L_0, L_i]$ and that $[L_0, L_i]$ has full column rank for all $i \in \underline{k}$. This means that there are full row rank matrices $R_1, R_2, \ldots, R_k$ such that $B_i = [L_0, L_i] R_i$ for all $i \in \underline{k}$. Furthermore, the matrices $R_1, R_2, \ldots, R_k$ are such that $R_i Q_i = I$ and $C_{ii} = [\bar{C}_{ii}, \hat{C}_{ii}] R_i$ for all $i \in \underline{k}$. To see the latter, note that $B_i = [L_0, L_i] R_i = B_i Q_i R_i$ for all $i \in \underline{k}$. Hence, for all $i \in \underline{k}$ we have $B_i(I - Q_i R_i) = 0$, implying that $\operatorname{im}(I - Q_i R_i) \subseteq \ker B_i$ for all $i \in \underline{k}$. Since $\ker B_i \subseteq \ker C_{ii}$ for all $i \in \underline{k}$, it follows that also $C_{ii}(I - Q_i R_i) = 0$ for all $i \in \underline{k}$. This clearly implies that $C_{ii} = C_{ii} Q_i R_i = [\bar{C}_{ii}, \hat{C}_{ii}] R_i$ for all $i \in \underline{k}$. So, it follows that $A_i X B_i = A_i X[L_0, L_i] R_i = [\bar{C}_{ii}, \hat{C}_{ii}] R_i = C_{ii}$ for all $i \in \underline{k}$. This concludes the proof of the (if)-part. □

In this section we developed necessary and sufficient conditions for the existence of a common solution to the k matrix equations $A_i X B_i = C_{ii}$ for $i \in \underline{k}$. We were able to derive the conditions by assuming that

$$\sum_{i \in \underline{k}} \left(\operatorname{im} B_i \cap \left(\sum_{j \in \underline{k}, j \neq i} \operatorname{im} B_j \right) \right) = \bigcap_{i \in \underline{k}} \operatorname{im} B_i. \tag{1}$$

For $k = 2$ the assumption is always satisfied. Indeed, then the left hand side and the right hand side are both equal to $\operatorname{im} B_1 \cap \operatorname{im} B_2$. From Theorem 4.1 it therefore easily follows that there is a matrix X such that $A_1 X B_1 = C_{11}$ and $A_2 X B_2 = C_{22}$ if and only if $\operatorname{im} C_{ii} \subseteq \operatorname{im} A_i$, $\ker B_i \subseteq \ker C_{ii}$ for $i = 1, 2$, and

$$\begin{bmatrix} C_{11} & 0 \\ 0 & C_{22} \end{bmatrix} \ker \begin{bmatrix} B_1 & -B_2 \end{bmatrix} \subseteq \operatorname{im} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}.$$

This result is comparable with Corollary 3.3 and the result presented in [7].

Of course, we also can derive the "dual" solvability conditions when we assume that

$$\bigcap_{i \in \underline{k}} \left(\ker A_i + \left(\bigcap_{j \in \underline{k}, j \neq i} \ker A_j \right) \right) = \sum_{i \in \underline{k}} \ker A_i. \tag{2}$$

5. Conclusions and remarks

In this paper we presented conditions for the existence of a common solution X to the matrix equations $A_i X B_j = C_{ij}$, $(i, j) \in \Gamma$, where the matrices A_i, B_j, C_{ij} and X have suitable dimensions and either $\Gamma = \{(i, j) \mid i, j \in \underline{k}, i \neq j\}$ or $\Gamma = \{(i, i) \mid i \in \underline{k}\}$, i.e. $\Gamma = \{(i, j) \mid i, j \in \underline{k}, i = j\}$, in which $\underline{k} = \{1, 2, \ldots, k\}$. The purpose of this paper has been to derive for these two sets Γ necessary and sufficient conditions for the existence of a common solution to the equations $A_i X B_j = C_{ij}$, $(i, j) \in \Gamma$, that can be verified and that are stated directly in terms of the known matrices and that do not use Kronecker products.

For the case that $\Gamma = \{(i, j) \mid i, j \in \underline{k}, i \neq j\}$ we have been completely successful and have been able to derive verifiable necessary and sufficient conditions for the existence of a common solution. For the case that $\Gamma = \{(i, i) \mid i \in \underline{k}\}$ we have derive verifiable necessary and sufficient conditions for the existence of a common solution provided the additional assumption (1) is satisfied. Dual conditions can be obtained when assumption (2) is satisfied.

It is still an open problem which are the necessary and sufficient conditions for the existence of a common solution in the case that neither assumption (1) nor assumption (2) is satisfied.

A further point of research might be the characterization of *all* solutions X, in particular the maximal rank ones, common to the equations $A_i X B_j = C_{ij}$, $(i, j) \in \Gamma$, given that at least one common solution exists. See for instance [2] for the case $k = 2$.

In this paper all equations were considered to be equations over a field. It might be interesting to see how the results of this paper can be extended to equations over a ring. See for instance [3] for the case $k = 2$.

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