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Facets of the polytope of the asymmetric travelling salesman problem with replenishment arcs

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Abstract

The Asymmetric Travelling Salesman Problem with Replenishment Arcs (RATSP) is a new class of problems arising from work related to aircraft routing. Given a digraph with cost on the arcs, a solution of the RATSP, like that of the Asymmetric Travelling Salesman Problem, induces a directed tour in the graph which minimises total cost. However the tour must satisfy additional constraints: the arc set is partitioned into replenishment arcs and ordinary arcs, each node has a non-negative weight associated with it, and the tour cannot accumulate more than some weight limit before a replenishment arc must be used. To enforce this requirement, constraints are needed. We refer to these as replenishment constraints.

In this paper, we review previous polyhedral results for the RATSP and related problems, then prove that two classes of constraints developed in V. Mak and N. Boland [Polyhedral results and exact algorithms for the asymmetric travelling salesman problem with replenishment arcs, Technical Report TR M05/03, School of Information Technology, Deakin University, 2005] are, under appropriate conditions, facet-defining for the RATS polytope.

Keywords: Combinatorial optimisation; Travelling salesman; Polyhedral combinatorics

1. Introduction

The Asymmetric Travelling Salesman Problem with Replenishment Arcs (RATSP) is a generalisation of the well-known Asymmetric Travelling Salesman Problem (ATSP). The problem was introduced by Boland et al. in [9]. Given a digraph $G = (V, A)$ with node set V, arc set A, and costs on the arcs $c \in \mathbb{R}^{|A|}$, not necessarily symmetric, a tour in G is defined to be a sequence that starts from a node, visits each node exactly once, then finishes at the node where the tour started. A solution of the RATSP, like that of the ATSP, induces a tour in G which minimises total cost. However the tour must satisfy additional constraints: the arc set A is partitioned into *replenishment arcs*, denoted by $\mathcal{R}$ and *ordinary arcs*, denoted by $\mathcal{Q} = A \setminus \mathcal{R}$, each node $i \in V$ has a positive weight w_i associated with it, and there is a positive weight limit W; a feasible tour cannot accumulate more than W units of weight before using a replenishment

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arc. We restrict the RATSP to have at least one replenishment arc, i.e. to have $\mathcal{R} \neq \emptyset$, and to have the sum of weights of all nodes greater than the weight limit, i.e. to have $\sum_{i \in V} w_i > W$; otherwise the problem reverts to the usual ATSP.

The RATSP has applications in its own right, and is also of interest for its close connection to vehicle routing problems. It was first applied to model aircraft rotation problems, in which the flights in the schedule are to be linked in the sequence they will be flown by the aircraft. The flight sequence must offer timely opportunities for aircraft maintenance. In the RATSP model, nodes represent flights, arcs represent aircraft connections, weights represent flight times (or some other measure used in maintenance scheduling) and replenishment arcs indicate connections occurring at a maintenance port, with sufficient time available to perform maintenance. For more details of this application, see [8] and [10]. The RATSP can also be viewed as a generalisation of the Asymmetric Capacited Vehicle Routing Problem (ACVRP) (see for example [16,20]) with the fleet size constraint restricting the number of vehicles used relaxed. In this case, the nodes in the RATSP problem represent the customers in the ACVRP, the weights represent customer demands, and from each customer that could end a vehicle route to each customer that could start one there is a replenishment arc, representing vehicle transit via the depot.

Whilst there has been a great deal of work done on algorithms for ATSP and for Capacitated Vehicle Routing Problems (CVRPs) generally, relatively little has been done for RATSP. Exceptions are the column generation approach of [9], the simulated annealing and Lagrangian relaxation heuristics of [21], and the Lagrangian-based branch-and-bound method of [22]. The latter has been found to be the most successful approach computationally, and a key to its success is the Lagrangian relaxation of *strong* valid inequalities to enforce weight feasibility. By "strong" we of course mean that the inequalities are facet-defining for the underlying polytope, under reasonable conditions. Proving this is the central subject of this paper.

There is very little literature on the RATS polytope, although there is, of course, a great deal known about the ATS polytope. See, for example, [3–6,12,14,15,17]. As Mak and Boland [22] show, facet-defining constraints for ATSP can be easily converted to facet-defining constraints for RATSP; we describe this in more detail in Section 3. Naturally facets defined in this way are unlikely to assist in achieving weight feasibility; replenishment arcs and weight feasibility are what distinguishes the RATSP from the ATSP. By contrast, many valid inequalities have been developed for a closely related problem, the CVRPs, which *do* make use of a problem feature closely akin to weight feasibility: vehicle capacity feasibility. Examples can be found in [1,11,19], and Bard et al. [7], who also review such inequalities. However, as Bard et al. [7] comment repeatedly, very little is known about the polyhedral structure of vehicle routing problems, and few of these valid inequalities are known to be facet-defining. As far as we are aware, the only work providing facet-defining results for vehicle routing problems is that of Cornuejols and Harche [11], who study the undirected CVRP and the Graphical Vehicle Routing Problem (GVRP), which is a relaxation of the CVR polytope; and that of Mak and Ernst [24] who show that four classes of inequalities lifted from the D_k^+ and D_k^- inequalities are facet-defining for the VRP-TW under appropriate conditions.

Cornuejols and Harche [11] give a number of results demonstrating that inequalities known as capacity, path, bicycle and wheelbarrow constraints (the latter three come from TSP) can all be facet-defining for the GVRP and CVRP polytopes, under certain conditions. They introduce capacitated forms of the path, bicycle, wheelbarrow and comb inequalities, (the latter also comes from TSP), and prove the validity of these. Capacitated path inequalities are shown to be facet-defining under certain conditions. Their work on capacitated constraints is most relevant to our work. We note that the capacity constraints discussed by Cornuejols and Harche [11] are closely related to the k-path cuts discussed by Kohl et al. [18] for solving Vehicle Routing Problems with Time Windows (VRPTW). The 2-path cuts were applied to solve the VRPTW by Kohl et al. [18] with great success.

The contribution of the work we present here is to prove that two classes of valid inequalities for the RATSP, both of which enforce weight feasibility, are facet-defining under appropriate conditions. We believe our proofs to be of general interest: the techniques we use may help other researchers to studying the polyhedral structure of vehicle routing problems generally. Our proofs use a "double induction" on the number of nodes in the digraph and on the number of nodes in the path defining the inequality: many valid inequalities for combinatorial optimisation problems have this kind of double parameterisation. Furthermore, our proofs apply the idea of general induction for constructing tours for the ATSP, presented in [14], for proving ATSP facet-defining results. As far as we are aware, this is the first application of these ideas outside of [14].

Our paper is structured as follows. In Section 2 we establish our notation, describe the RATSP formally, and present the two classes of valid inequalities which are the focus of this paper. In Section 3, we review previous polyhedral results for the RATSP and in Section 4 we prove that both classes of inequalities are facet-defining under appropriate conditions.

2. Notation and problem description

The RATSP can be represented by a weighted digraph, which we define below.

Definition 2.1. A weighted digraph $\mathcal{G} = (V, A, W, w)$ is a directed graph with node set V, arc set $A = \mathcal{Q} \cup \mathcal{R}$, for $\mathcal{Q}$ the set of ordinary arcs and $\mathcal{R}$ the set of replenishment arcs, weights on nodes $w \in \mathcal{Z}_+^{|V|}$, weight limit $W \in \mathcal{Z}_+$ where $W \geq w_i$ for all $i \in V$, and $W \geq w_i + w_j$ for all $(i, j) \in \mathcal{Q}$.

Note that Definition 2.1 does not restrict $\mathcal{Q}$, since no ordinary arc $(i, j) \in \mathcal{Q}$ with $w_i + w_j > W$ can possibly be used in any feasible solution. Note also that parallel arcs may exists: there may be pairs of nodes that have both ordinary and replenishment arcs. Furthermore, recall our earlier assumptions that $\mathcal{R} \neq \emptyset$ and $\sum_{i \in V} w_i > W$; otherwise the RATSP reverts to ATSP. Our polyhedral analysis will be conducted on the RATS polytope, defined as follows. For the background of polyhedral theory, please refer to Section I.4.3 of [25].

Definition 2.2. Given a weighted digraph $\mathcal{G}$, with arc set A, we define $T_{\mathcal{G}}$ to be the set of indicator vectors $x \in \{0, 1\}^{|A|}$ of weight-feasible tours in $\mathcal{G}$. We define the *RATS polytope* to be the convex hull of $T_{\mathcal{G}}$, written $\mathrm{conv}(T_{\mathcal{G}})$.

Note that in what follows we will often refer to an indicator vector x of a (weight-feasible) tour in $\mathcal{G}$ as simply a (weight-feasible) tour x.

Recall that we allow parallel and replenishment arcs. We adopt the following notation throughout this paper: for a pair of parallel arcs that go from node i to node j, we use $(i, j)^{\mathcal{R}}$ to denote the replenishment arc and use $(i, j)^{\mathcal{Q}}$ to denote the ordinary arc. Later when we present the proofs, we use $\xrightarrow{\mathcal{Q}}$ and $\longrightarrow$ to represent an ordinary arc and a replenishment arc between two nodes respectively. Also, we use $x_{ij}^{\mathcal{Q}}$ and $x_{ij}^{\mathcal{R}}$ to denote the decision variable corresponding to arcs $(i, j)^{\mathcal{Q}}$ and $(i, j)^{\mathcal{R}}$ respectively. In some contexts we may, for simplicity, write $(i, j) \in \mathcal{Q}$ rather than $(i, j)^{\mathcal{Q}} \in \mathcal{Q}$ and $(i, j) \in \mathcal{R}$ rather than $(i, j)^{\mathcal{R}} \in \mathcal{R}$.

Throughout this paper, when we refer to a path, we mean a simple path, unless otherwise stated. We now write a path as a sequence of arcs, of the form $((i_1, i_2)^{\beta_1}, \ldots, (i_k, i_{k+1})^{\beta_k})$ where $\beta_j \in \{\mathcal{Q}, \mathcal{R}\}$ for $j = 1, \ldots, k$. If *all* arcs in the path are replenishment arcs, we may also use $(i_1, \ldots, i_{k+1})^{\mathcal{R}}$ and call the path a *replenishment path*. Likewise for an ordinary path. Furthermore, given $U_1, U_2 \subset V$, $U_1 \cap U_2 = \emptyset$, we use $(U_1, U_2)^{\beta} = \{(i, j) \in \beta \mid i \in U_1, j \in U_2\}$, for $\beta = \{\mathcal{Q}, \mathcal{R}\}$, to denote the ordinary or replenishment cutset between U_1 and U_2; $\delta_{\beta}^{+}(U) = \{(i, j) \in \beta \mid i \in U, j \notin U\}$, for $\beta = \{\mathcal{Q}, \mathcal{R}\}$, to denote the set of ordinary or replenishment arcs that leaves set U; and $\delta_{\beta}^{-}(U) = \{(i, j) \in \beta \mid i \notin U, j \in U\}$, for $\beta = \{\mathcal{Q}, \mathcal{R}\}$, to denote the set of ordinary or replenishment arcs that enters set U. In the case that U consists of a single element, we usually omit the set notation, writing, for example, $\delta_{\mathcal{Q}}^{+}(i)$ rather than $\delta_{\mathcal{Q}}^{+}(\{i\})$. Furthermore, we make the following assumption on the structure of $\mathcal{G}$ throughout this paper.

Assumption 2.1. The set of replenishment arcs induces a complete digraph on V.

Remark 2.1. Computationally, Assumption 2.1 is not restrictive. If some of these replenishment arcs do not exist in the original problem, we can introduce "fake" arcs with high artificial costs. We need this condition for constructing feasible tours in our proofs. From a polyhedral point of view, however, the condition is restrictive, and therefore our study here can be viewed as one for a restricted RATSP. It is our future research direction to study a more general problem.

3. Previous polyhedral results for the RATSP

In contrast to this rich body of work for the ATSP polytope, the only work available on the RATS polytope is that of Boland et al. [9] and that of Mak and Boland [22]. We now provide an overview of this work.

Boland et al. [9] proposed two integer linear programming formulations for the RATSP: a path formulation and an arc formulation. Polyhedral results given for the path formulation were mainly validity results for lifted subtour elimination constraints. A facet-defining result was given, with conditions that are difficult to check in practice. However, the path formulation is not directly relevant to our work here: variables correspond to weight-feasible ordinary paths, and replenishment arcs, so the underlying polytope is rather different to the RATS polytope we have defined here, and furthermore no weight constraints are needed in the formulation. The arc formulation studied in [9] *does* induce the RATS polytope we consider here, however polyhedral results given in [9] are only for the special case with $w_i = 1$ for all $i \in V$ and $W = 2$. We summarise the results of [9] in this case below.

Theorem 3.1. *Given a weighted digraph $\mathcal{G} = (V, A, W, w)$ with $w_i = 1$ for all $i \in V$, $W = 2$, and $(V, \mathcal{R})$ complete, the following statements are true:*

(1) *The RATS polytope* $\operatorname{conv}(T_{\mathcal{G}})$ *has dimension $|V|(|V| - 3) + 1 + |\mathcal{Q}|$.*
(2) *The non-negativity constraints $x_a \geq 0$, for all $a \in A$ are facet-defining for the RATS polytope.*
(3) *The subtour elimination constraints*
$$\sum_{a \in \delta^+_{\mathcal{Q}}(U) \cup \delta^+_{\mathcal{R}}(U)} x_a \geq 1$$
are facet-defining for the RATS polytope for all $U \subset V$, $2 \leq |U| \leq |V| - 2$.
(4) *If $|V| \geq 5$ and $(V, \mathcal{Q})$ is a complete digraph, then the weight violation elimination constraints*
$$\sum_{a \in \delta^+_{\mathcal{Q}}(i) \cup \delta^-_{\mathcal{Q}}(i)} x_a \leq 1$$
are facet-defining for the RATS polytope for all $i \in V$.
(5) *If $\sum_{(i,j) \in \mathcal{R}} \alpha^{\mathcal{R}}_{ij} x^{\mathcal{R}}_{ij} \leq \alpha_0$ is a facet-defining inequality for the ATS polytope on digraph $(V, \mathcal{R})$ that is not equivalent to a non-negativity constraint, then*
$$\sum_{(i,j) \in \mathcal{R}} \alpha^{\mathcal{R}}_{ij} x^{\mathcal{R}}_{ij} + \sum_{(i,j) \in \mathcal{Q}} \alpha^{\mathcal{R}}_{ij} x^{\mathcal{Q}}_{ij} \leq \alpha_0$$
holds for all $x \in T_{\mathcal{G}}$ and is facet-defining for the RATS polytope $\operatorname{conv}(T_{\mathcal{G}})$.

Mak and Boland [22] extend some of these results to the case with general node weights. Two of the constraints which are the principal subject of our work here were also presented in [22], and shown to be valid. We summarise the results of [22] in the remainder of this section.

Theorem 3.2. *Given a weighted digraph $\mathcal{G} = (V, A, W, w)$ with $(V, \mathcal{R})$ complete, statements* (1) *and* (5) *of Theorem* 3.1 *are true.*

In [9] these results were only proved in the special case of unit node weights with $W = 2$, but were proved for general node weights in [22].

We now present the notation needed to define the two classes of replenishment constraints that are the principal subject of this paper. Both are based on simple weight-infeasible ordinary paths. Let $p = (i_1, \ldots, i_k)^{\mathcal{Q}}$ be a weight-infeasible ordinary path. We write $V(p) = \{i_1, \ldots, i_k\}$ and $w(V(p)) = \sum_{j \in V(p)} w_j$.

Definition 3.1. p is a *minimal violation path* if (1) it is weight-infeasible, i.e. $w(V(p)) > W$; and (2) excluding either the first or the last node on the path yields a weight-feasible subpath, i.e. $w(V(p) \backslash \{i_1\}) \leq W$, and $w(V(p) \backslash \{i_k\}) \leq W$.

We use $\mathcal{M}_{\mathcal{G}}$ to denote the set of all minimal violation paths in $\mathcal{G}$.

Definition 3.2. Given any minimal violation path $p = (i_1, \ldots, i_k)^{\mathcal{Q}} \in \mathcal{M}_{\mathcal{G}}$, we define:

(1) $\mathcal{Q}(p) = \{(i_j, i_{j+1}) \in \mathcal{Q} \mid j = 1, \ldots, k - 1\}$.
(2) $\bar{\Delta}(p) = \bigcup_{l=1}^{k-1} \bar{\Delta}_{i_l}(p)$ as the set of weight invalid forward and escaping non-path arcs with respect to p, where $\bar{\Delta}_{i_l}(p) = \left\{(i_l, j) \in \mathcal{Q} \mid j \neq i_1, \ldots, i_{l+1}, \text{ and } \sum_{b=1}^{l} w_{i_b} + w_j > W\right\}$, for $l = 1, \ldots, k - 1$,
(3) $\mathcal{B}(p) = \bigcup_{l=1}^{k-1} \mathcal{B}_{i_l}(p)$ as the set of backward arcs with respect to p, where $\mathcal{B}_{i_l}(p) = \{(i_l, j) \in A \mid j \in \{i_1, \ldots, i_{l-1}\}\}$, for $l = 2, \ldots, k - 1$, and $\mathcal{B}_{i_1}(p) = \emptyset$. Note that $\mathcal{B}(p)$ does not include any backward arcs from the last node in p.

Below are our two classes of replenishment constraints. The propositions below are stated and proved by Mak and Boland [22].

Proposition 3.1. *Constraints*

$$\sum_{a\in\mathcal{Q}(p)} x_a + \sum_{a\in\bar{\Delta}(p)} x_a + \sum_{a\in\mathcal{B}(p)} x_a \le k-2, \tag{1}$$

which we refer to as the $S_1^{\mathcal{G},p}$ *constraints, are valid for the RATS polytope for all* $p = (i_1, \ldots, i_k)^{\mathcal{Q}} \in \mathcal{M}_{\mathcal{G}}$.

Definition 3.3. A set of nodes $\mathcal{L} \subset V$ is defined to be a *minimal violation set* if $w(\mathcal{L}) > W$ and $w(\mathcal{L} \setminus \{l\}) \le W$ for any $l \in \mathcal{L}$.

Proposition 3.2. *Constraints*

$$\sum_{a\in(\mathcal{L},\mathcal{L})\cap\mathcal{Q}} x_a \le |\mathcal{L}| - 2, \tag{2}$$

which we refer to as the $S_2^{\mathcal{G},\mathcal{L}}$ *constraints, are valid for the RATS polytope for all* $\mathcal{L} \subset V$, $\mathcal{L}$ *a minimal violation set.*

Note that if p is an ordinary path with $V(p)$ a minimal violation set, then p is a minimal violation path, but the converse does not necessarily hold. It is not hard to see that (2) is equivalent to the replenishment constraints proposed in [9], and is similar to the 2-path cut of VRPTW, (see, for example [18]) and of ATSP-TW (see, for example, [2]).

The $S_1^{\mathcal{G},p}$ constraints are proven very strong in practice, as numerical results in [22] and [13] show.

4. Polyhedral results for the RATS polytope

In this section, we prove that the $S_1^{\mathcal{G},p}$ and $S_2^{\mathcal{G},\mathcal{L}}$ replenishment constraints are facet-defining under certain conditions. The following defines a condition of our next theorem.

Definition 4.1. A set of nodes $S \subseteq V$ is $\mathcal{Q}$-complete if $\mathcal{Q}$ induces a complete digraph on S.

We now state the theorems.

Theorem 4.1. *Given any weighted-digraph* $\mathcal{G} = (V, A, W, w)$ *with* $|V| \ge 5$ *and any minimal violation set* $\mathcal{L} \subset V$ *defined in* $\mathcal{G}$ *with* $3 \le |\mathcal{L}| < |V|$, *under Assumption 2.1, and provided* $\mathcal{L}$ *is* $\mathcal{Q}$*-complete, replenishment constraint* $S_2^{\mathcal{G},\mathcal{L}}$ *defines a facet of* $\text{conv}(T_{\mathcal{G}})$.

(See Remark 2.1.)

Remark 4.1. Note that the conditions are needed for constructing feasible tours in our proof. For the monotone RATS polytope, however, $\mathcal{L}$ does not need to be $\mathcal{Q}$-complete: it only requires a much less restrictive condition (see [23]).

Theorem 4.2. *For any weighted-digraph* $\mathcal{G} = (V, A, W, w)$ *with* $|V| \ge 5$ *and any minimal violation path* $p = (i_1, \ldots, i_k)^{\mathcal{Q}} \in \mathcal{M}_{\mathcal{G}}$ *with* $3 \le k < |V| - 1$, *under Assumption 2.1, replenishment constraint* $S_1^{\mathcal{G},p}$ *defines a facet of* $\text{conv}(T_{\mathcal{G}})$ *if the following conditions hold: (1) for all* $(i_j, i_l)^{\mathcal{Q}} \in A$ *such that* $j, l \in \{1, \ldots, k\}$, $l \ge j+2$, *and* $\sum_{q=1}^{j} w_{i_q} + w_{i_l} \le W$, *we have that* $\sum_{q=1}^{j} w_{i_q} + \sum_{q=l}^{k} w_{i_q} \le W$; *and (2) if* $(i_k, i_1)^{\mathcal{Q}} \in A$, *then* $(i_2, V \setminus V(p))^{\mathcal{Q}} \cap \bar{\Delta}(p) \ne \emptyset$.

Remark 4.2. Conditions (1) and (2) of Theorem 4.2 are needed for obtaining certain required feasible tours.

Remark 4.3. The theorem does not have an only if characterisation. It is certainly possible that there exists problems with other graph structures such that the $S_1^{\mathcal{G},p}$ constraints are also facet-defining. There are alternatives to Condition (2). For examples, (i) if $(i_k, i_1)^{\mathcal{Q}} \in A$, then there exists $\alpha \in \{2, \ldots, k-1\}$ such that (i_α, i_j), $(i_1, i_{k-1}) \in \bar{\Delta}(p)$ and that $(i_k, i_1, i_{k-1})^{\mathcal{Q}}$ defines a feasible ordinary path; or (ii) if $(i_k, i_1)^{\mathcal{Q}} \in A$, then there exists $\alpha \in \{2, \ldots, k-1\}$ such that $(i_\alpha, i_j), (i_1, i_3) \in \bar{\Delta}(p)$ and that $(i_k, i_1, i_3)^{\mathcal{Q}}$ defines a feasible ordinary path. However, both of these conditions, comparing with Condition (2), are relaxed in one way, but more restrictive in another. Hence we do not consider them in this paper.

Remark 4.4. Both Theorems 4.1 and 4.2 do not hold for $|V| = 4$. For $|V| = 4$, the respective constraints are facet-defining only for the monotone RATS polytope. (See [23].)

We now introduce notation needed for these proofs.

Definition 4.2. For $S = \{i_1, \ldots, i_k\} \subset V$, we write $\{S\}^{\mathcal{R}}$ to denote a replenishment path formed by an arbitrary permutation, π, of the nodes on S, i.e. $\{S\}^{\mathcal{R}} = (i_{\pi(1)}, \ldots, i_{\pi(k)})^{\mathcal{R}}$. Note that under our standing assumption that $(V, \mathcal{R})$ is complete, $\{\mathcal{S}\}^{\mathcal{R}}$ is well defined.

Definition 4.3. Similarly, for $S = \{i_1, \ldots, i_k\} \subset V$, we write $\{S\}^{\mathcal{Q}}$ to denote an ordinary path formed by an arbitrary permutation, π, of the nodes on S, i.e. $\{S\}^{\mathcal{Q}} = (i_{\pi(1)}, \ldots, i_{\pi(k)})^{\mathcal{Q}}$. Note that we must be careful using this notation, as such a path may not exist in the graph for given S.

4.1. Proof of Theorem 4.1

Assumption 4.1. Given a weighted-digraph $\mathcal{G} = (V, A, W, w)$ where $|V| \geq 4$, and any minimal violation set $\mathcal{L} = \{i_1, \ldots, i_k\} \subset V$ with $3 \leq k < |V|$, we assume that there exists an ordinary simple path p that connects all nodes in the set $\mathcal{L}$, i.e. $p = (i_{\pi(1)}, \ldots, i_{\pi(k)})^{\mathcal{Q}}$ for $\{i_{\pi(1)}, \ldots, i_{\pi(k)}\} = \mathcal{L}$.

Remark 4.5. We need this assumption for the purpose of constructing sufficient feasible tours in our polyhedral analysis. In separating the $S_2^{\mathcal{G},\mathcal{L}}$ constraints from integer solutions, it is obvious that if the assumption does not hold, then there will be no replenishment violations within $\mathcal{L}$.

Definition 4.4. Given a weighted-digraph $\mathcal{G} = (V, A, W, w)$, and any minimal violation set $\mathcal{L} = \{i_1, \ldots, i_k\} \subset V$ defined in $\mathcal{G}$, we define the face of $\text{conv}(T_{\mathcal{G}})$ induced by $S_2^{\mathcal{G},\mathcal{L}}$ to be $F_2^{\mathcal{G},\mathcal{L}} = \{x \in \text{conv}(T_{\mathcal{G}}) : \sum_{a \in (\mathcal{L},\mathcal{L})^{\mathcal{Q}}} x_a = |\mathcal{L}| - 2\}$.

Proposition 4.1. *For all weighted-digraphs $\mathcal{G} = (V, A, W, w)$ with $|V| \geq 4$, and any minimal violation set $\mathcal{L}$ defined in $\mathcal{G}$ with $3 \leq |\mathcal{L}| < |V|$, under Assumptions 2.1 and 4.1, $F_2^{\mathcal{G},\mathcal{L}}$ is a proper face of* $\text{conv}(T_{\mathcal{G}})$.

$F_2^{\mathcal{G},\mathcal{L}}$ is a proper face of $\text{conv}(T_{\mathcal{G}})$ if $\emptyset \neq F_2^{\mathcal{G},\mathcal{L}} \neq \text{conv}(T_{\mathcal{G}})$. (Consider $y^1, y^2 \in T_{\mathcal{G}}$, for $y_a^1 = 1$ if $a \in \{(j, j+1)^{\mathcal{Q}} \mid j = 2, \ldots, k-1\} \cup \{(n, 1)^{\mathcal{R}}\} \cup \{(j, j+1)^{\mathcal{R}} \mid j = 1, k, \ldots, n-1\}$ and $y_a^1 = 0$ otherwise; and $y_a^2 = 1$ if $a \in \{(j, j+1)^{\mathcal{R}} \mid j = 1, \ldots, n-1\} \cup \{(n, 1)^{\mathcal{R}}\}$, and $y_a^2 = 0$ otherwise.)

We now prove Theorem 4.1 in two stages. In the first stage, we show that our result is true under the assumption that there exists no ordinary arcs in $\mathcal{G}$ other than those between nodes in $\mathcal{L}$. We do so by a double induction argument. In the second stage of our proof we relax this assumption and show that $S_2^{\mathcal{G},\mathcal{L}}$ is facet-defining in general by constructing a feasible tour for each ordinary arc in $\mathcal{Q} \setminus (\mathcal{L}, \mathcal{L})^{\mathcal{Q}}$ that satisfies the $S_2^{\mathcal{G},\mathcal{L}}$ constraint at equality and uses only this particular ordinary arc and no other ordinary arc in $\mathcal{Q} \setminus (\mathcal{L}, \mathcal{L})^{\mathcal{Q}}$. This completes the proof.

Stage one of proof of Theorem 4.1

Lemma 4.1. *For any weighted-digraph $\mathcal{G} = (V, A, W, w)$ with $|V| \geq 5$ and any minimal violation set $\mathcal{L} \subset V$ with $|\mathcal{L}| = |V| - 1$, under Assumption 2.1, $S_2^{\mathcal{G},\mathcal{L}}$ defines a facet for* $\text{conv}(T_{\mathcal{G}})$ *if the following conditions hold:* (1) *$\mathcal{L}$ is $\mathcal{Q}$-complete, and* (2) *$\mathcal{Q} \setminus (\mathcal{L}, \mathcal{L})^{\mathcal{Q}} = \emptyset$.*

Proof. We make use of the well-known result that a face F defines a facet for a polytope P if $\dim(F) = \dim(P) - 1$ (see Section I.4.3 of [25]), and prove Lemma 4.1 by showing that the dimension of $F_2^{\mathcal{G},\mathcal{L}}$ is one less than the dimension of $\text{conv}(T_{\mathcal{G}})$. From Proposition 4.1, and since Condition 1 of the lemma implies Assumption 4.1, we know that $F_2^{\mathcal{G},\mathcal{L}}$

defines a proper face for $\mathrm{conv}(T_{\mathcal{G}})$, and therefore has dimension at most $\dim(\mathrm{conv}(T_{\mathcal{G}})) - 1$. Hence it remains to show that $F_2^{\mathcal{G},\mathcal{L}}$ has dimension at least $\dim(\mathrm{conv}(T_{\mathcal{G}})) - 1$. We thus show that there are at least $\dim(\mathrm{conv}(T_{\mathcal{G}}))$ affinely independent feasible tours that satisfy constraint $S_2^{\mathcal{G},\mathcal{L}}$ at equality. We show this by induction on $|V|$. By Theorem 3.2, Assumption 2.1 and the conditions of the lemma, $\dim(\mathrm{conv}(T_{\mathcal{G}})) = 2n^2 - 6n + 3$, where $n = |V|$, since we assume $|\mathcal{L}| = |V| - 1$. Our inductive hypothesis, for some $t \geq 5$, is that for any $\mathcal{G}$, $\mathcal{L}$ satisfying the condition of the lemma with $|V| = t$, there exists a subset of tours $\mathcal{X}_{\mathcal{G}} \subset T_{\mathcal{G}}$ defined in $\mathcal{G}$ which contains at least $2t^2 - 6t + 3$ affinely independent feasible tours satisfying constraint $S_2^{\mathcal{G},\mathcal{L}}$ at equality. We first show that our inductive hypothesis is true for the base case where $t = 5$. Then we assume that our inductive hypothesis is true for some $t \geq 5$ and show that it is true for $t + 1$.

Base case $t = 5$. Given any weighted-digraph $\mathcal{G} = (V, A, W, w)$ with $|V| = 5$, and any minimal violation set $\mathcal{L}$ defined in $\mathcal{G}$ with $|\mathcal{L}| = 4$, if $\mathcal{G}$, $\mathcal{L}$ satisfy Assumption 2.1 and the conditions of Lemma 4.1, then we can find 23 affinely independent feasible tours that satisfy $S_2^{\mathcal{G},\mathcal{L}}$ at equality. (See Appendix A.)

Inductive step. Assume that our inductive hypothesis is true for some $t \geq 5$. We now show that our inductive hypothesis is true for $t + 1$.

Let $\mathcal{G}' = (V', A', W', w')$ be any arbitrary weighted-digraph with $|V'| = t + 1$, $A' = \mathcal{Q}' \cup \mathcal{R}'$, and let $\mathcal{L}'$ be any minimal violation set defined in $\mathcal{G}'$ with $|\mathcal{L}'| = t$ such that $\mathcal{G}'$, $\mathcal{L}'$ satisfy the conditions of the lemma. Without loss of generality, say $V' = \{1, \ldots, t, t + 1\}$ and $\mathcal{L}' = \{2, \ldots, t, t + 1\}$. Now let $\mathcal{G} = (V, A, W, w)$ where $V = \{1, \ldots, t\}$, $A = \{(i, j)^{\beta} \in \beta' \mid i, j \in V, \beta \in \{\mathcal{Q}, \mathcal{R}\}\}$, $W = W' - w_{t+1}$, and $w'_i = w_i$ for all $i = 1, \ldots, t$. Also, let $\mathcal{L} = \{2, \ldots, t\}$. Observe that since $\mathcal{L}'$ is a minimal violation set in $\mathcal{G}'$, $\sum_{i\in\mathcal{L}'} w_i = \sum_{i\in\mathcal{L}'} w'_i > W'$, i.e. $\sum_{i\in\mathcal{L}} w_i + w_{t+1} > W' = W + w_{t+1}$. So we get $\sum_{i\in\mathcal{L}} w_i > W$. Furthermore, since $\sum_{i\in\mathcal{L}'\setminus\{j\}} w_i = \sum_{i\in\mathcal{L}'\setminus\{j\}} w'_i \leq W'$ for any $j \in \mathcal{L}$, $\mathcal{L}' \supset \mathcal{L}$ and $\mathcal{L}' = \mathcal{L} \cup \{t + 1\}$, we have $\sum_{i\in\mathcal{L}'\setminus\{j,t+1\}} w_i + w_{t+1} \leq W' = W + w_{t+1}$, and so $\sum_{i\in\mathcal{L}\setminus\{j\}} w_i \leq W$. Hence $\mathcal{L}$ defines a minimal violation set in $\mathcal{G}$. It is not hard to see that $\mathcal{G}$ and $\mathcal{L}$ defined this way satisfy the inductive hypothesis, so by the hypothesis, $\mathcal{X}_{\mathcal{G}}$ exists and there are at least $2t^2 - 6t + 3$ affinely independent feasible solutions defined in $\mathcal{G}$ that satisfy $S_2^{\mathcal{G},\mathcal{L}}$ at equality. Now we attempt to construct $\mathcal{X}_{\mathcal{G}'} \subset T_{\mathcal{G}'}$. This is achieved in two steps. We first modify the tours in $\mathcal{X}_{\mathcal{G}}$, to get tours in $T_{\mathcal{G}'}$ satisfying $S_2^{\mathcal{G}',\mathcal{L}'}$ at equality. We initialise $\mathcal{X}_{\mathcal{G}'}$ with the resulting $2t^2 - 6t + 3$ affinely independent tours in $T_{\mathcal{G}'}$. Then we insert in $\mathcal{X}_{\mathcal{G}'}$ $4t - 4$ new affinely independent feasible solutions defined in $\mathcal{G}'$ that satisfy $S_2^{\mathcal{G}',\mathcal{L}'}$ at equality, each using an arc that has not been used before, to give a total of at least $2(t + 1)^2 - 6(t + 1) + 3$ such tours as required.

Tour modification. We first modify each of the tours in $\mathcal{X}_{\mathcal{G}}$ by inserting the node $t + 1 \in V'$ after node t, so as to obtain $2t^2 - 6t + 3$ affinely independent feasible tours defined in $\mathcal{G}'$ that satisfy $S_2^{\mathcal{G}',\mathcal{L}'}$ at equality. There are two cases. (In what follows, when we describe RATS tours, we use $s_a \xrightarrow{\beta_1} \{U\}^{\beta_2} \xrightarrow{\beta_3} s_t$ for $U \subset V$, $s_a, s_t \notin U$ and distinct, $\beta_1, \beta_2, \beta_3 \in \{\mathcal{Q}, \mathcal{R}\}$, to denote a sequence, and if $U = \emptyset$, the sequence is simply $(s_a, s_t)^{\beta_3}$.)

(1) Note that by Condition (2) of the lemma, no tour in $\mathcal{X}_{\mathcal{G}}$ uses the arc $(t, 1)^{\mathcal{Q}}$. Now, for every tour $\tau \in \mathcal{X}_{\mathcal{G}}$ that uses an ordinary arc $(t, j)^{\mathcal{Q}}$ for some $j \in \{2, \ldots, t - 1\}$, we construct a tour τ' defined in $\mathcal{G}'$ by replacing the arc $(t, j)^{\mathcal{Q}}$ in τ by an ordinary path $(t \xrightarrow{\mathcal{Q}} t + 1 \xrightarrow{\mathcal{Q}} j)$. (See Fig. 1(a).) Clearly, τ' uses $|\mathcal{L}'| - 2$ arcs in $(\mathcal{L}', \mathcal{L}')^{\mathcal{Q}'}$.
(2) For every tour $\tau \in \mathcal{X}_{\mathcal{G}}$ that uses $(t, j)^{\mathcal{R}}$ for some $j \in \{1, \ldots, t - 1\}$, we construct a tour τ' defined in $\mathcal{G}'$ by replacing $(t, j)^{\mathcal{R}}$ with $(t \xrightarrow{\mathcal{Q}} t + 1 \longrightarrow j)$. (See Fig. 1(b).) Hence, τ' uses $|\mathcal{L}'| - 2$ arcs in $(\mathcal{L}', \mathcal{L}')^{\mathcal{Q}'}$.

$\mathcal{X}_{\mathcal{G}'}$ is initialised with each tour in $\mathcal{X}_{\mathcal{G}}$ modified as described in either case.

Tour insertion. As arc $(t, t + 1)^{\mathcal{Q}}$ is used in each of the $2t^2 - 6t + 3$ affinely independent tours defined in $\mathcal{G}'$ we obtained in the tour modification procedure, the following arcs cannot currently exist in the tours in $\mathcal{X}_{\mathcal{G}'}$: $(t, t + 1)^{\mathcal{R}}$, $(t + 1, t)^{\mathcal{Q}}$, $(t + 1, t)^{\mathcal{R}}$. We now insert, in sequence, $4t - 4$ new affinely independent feasible tours defined in $\mathcal{G}'$ that satisfy $S_2^{\mathcal{G}',\mathcal{L}'}$ at equality, each using one of these arcs. We insert the new tours in $\mathcal{X}_{\mathcal{G}'}$ in the following order. In what follows, when we describe a tour, e.g. $\tau'_{(t,j)^{\mathcal{Q}}} = 1 \longrightarrow t + 1 \xrightarrow{\mathcal{Q}} \{\mathcal{L} \setminus \{j, t\}\}^{\mathcal{Q}} \longrightarrow t \xrightarrow{\mathcal{Q}} j \longrightarrow 1$, we mean the arc indicated next to the Greek letter τ', (in this example arc $(t, j)^{\mathcal{Q}}$), is the arc that has never appeared in any of the previously introduced tours.

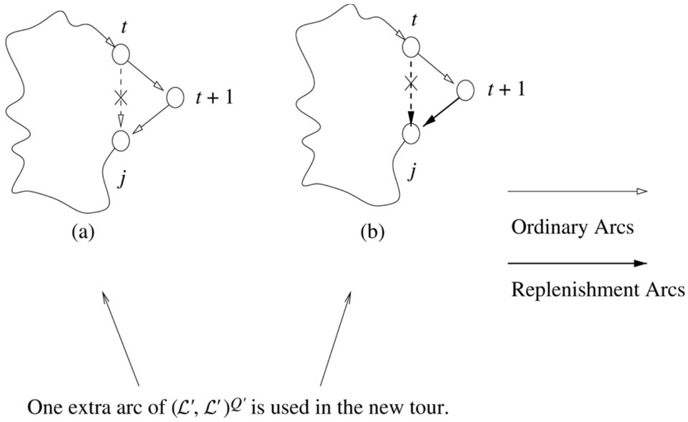

Fig. 1. Case (a) Arc $(t, j)^{\mathcal{Q}}$ in τ is replaced by $(t \xrightarrow{\mathcal{Q}} t+1 \xrightarrow{\mathcal{Q}} j)$ in τ', Case (b) Arc $(t, j)^{\mathcal{R}}$ in τ is replaced by $(t \xrightarrow{\mathcal{Q}} t+1 \longrightarrow j)$ in τ'.

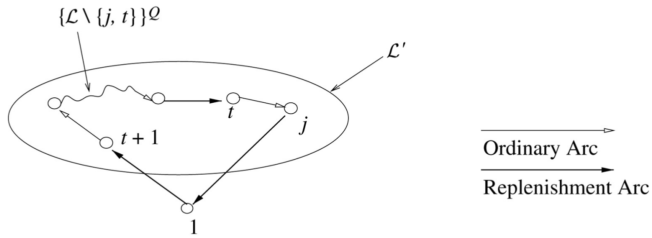

Fig. 2. The tour $\tau'_{(t,j)^{\mathcal{Q}}}$ for $j \in V \setminus \{1, t\}$, Case 1(a).

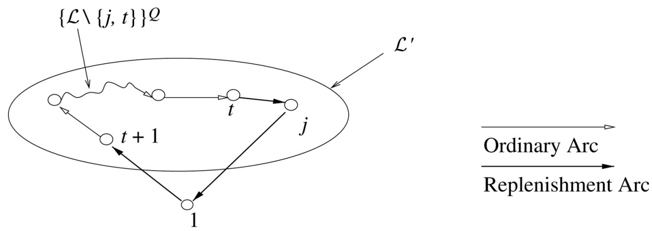

Fig. 3. The tour $\tau'_{(t,j)^{\mathcal{R}}}$ for $j \in V \setminus \{1, t\}$, Case 1(b).

(1) (a) $t-2$ tours each using an ordinary arc that leaves node t but not entering node $t+1$, with arc $(1, t+1)^{\mathcal{R}}$ fixed; $\tau'_{(t,j)^{\mathcal{Q}}} = 1 \longrightarrow t+1 \xrightarrow{\mathcal{Q}} \{\mathcal{L} \setminus \{j, t\}\}^{\mathcal{Q}} \longrightarrow t \xrightarrow{\mathcal{Q}} j \longrightarrow 1$ for $j \in V \setminus \{1, t\}$. (See Fig. 2.)

(b) $t-1$ tours each using a replenishment arc that leaves node t but not entering node $t+1$, again with arc $(1, t+1)^{\mathcal{R}}$ fixed; $\tau'_{(t,j)^{\mathcal{R}}} = 1 \longrightarrow t+1 \xrightarrow{\mathcal{Q}} \{\mathcal{L} \setminus \{j, t\}\}^{\mathcal{Q}} \xrightarrow{\mathcal{Q}} t \longrightarrow j \longrightarrow 1$ for $j \in V \setminus \{1, t\}$. (See Fig. 3.)

(c) $\tau'_{(t,1)^{\mathcal{R}}} = 1 \longrightarrow t+1 \longrightarrow \{\mathcal{L} \setminus \{t\}\}^{\mathcal{Q}} \xrightarrow{\mathcal{Q}} t \longrightarrow 1$.

(2) 1 tour that uses arc $(t, t+1)^{\mathcal{R}}$ (as arc $(t, t+1)^{\mathcal{R}}$ has not appeared in any of the tours previously introduced); $\tau'_{(t,t+1)^{\mathcal{R}}} = t \longrightarrow t+1 \longrightarrow 1 \longrightarrow \{\mathcal{L} \setminus \{t\}\}^{\mathcal{Q}} \xrightarrow{\mathcal{Q}} t$.

(3) 1 tour that uses arc $(t+1, t)^{\mathcal{Q}}$ with arc $(1, t+1)^{\mathcal{R}}$ fixed (as arc $(t+1, t)^{\mathcal{Q}}$ has not appeared in any of the tours previously introduced); $\tau'_{(t+1,t)^{\mathcal{Q}}} = 1 \longrightarrow t+1 \xrightarrow{\mathcal{Q}} t \longrightarrow \{\mathcal{L} \setminus \{t\}\}^{\mathcal{Q}} \longrightarrow 1$. (See Fig. 4.)

(4) 1 tour that uses arc $(t+1, t)^{\mathcal{R}}$ with arc $(1, t+1)^{\mathcal{R}}$ fixed (as arc $(t+1, t)^{\mathcal{R}}$ has not appeared in any of the tours previously introduced); $\tau'_{(t+1,t)^{\mathcal{R}}} = 1 \longrightarrow t+1 \longrightarrow t \xrightarrow{\mathcal{Q}} \{\mathcal{L} \setminus \{t\}\}^{\mathcal{Q}} \longrightarrow 1$. (See Fig. 5.)

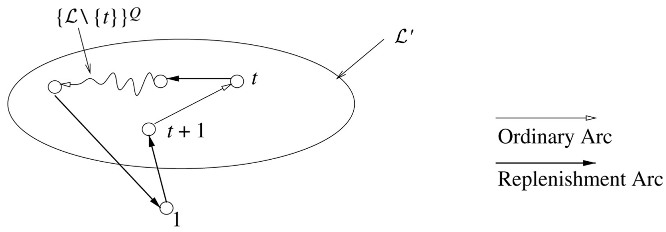

Fig. 4. The tour $\tau'_{(t+1,t)^{\mathcal{Q}}}$, Case 3.

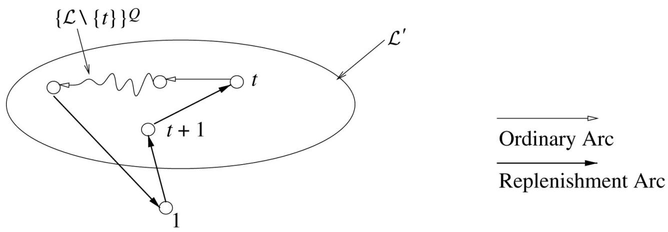

Fig. 5. The tour $\tau'_{(t+1,t)^{\mathcal{R}}}$, Case 4.

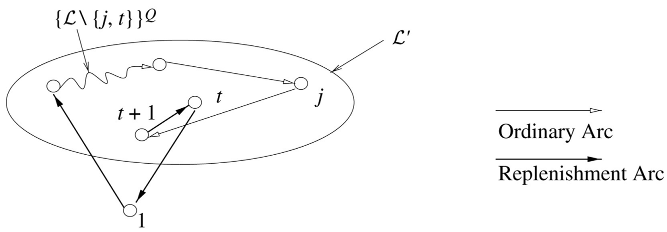

Fig. 6. The tour $\tau'_{(j,t+1)^{\mathcal{Q}}}$ for $j \in V \setminus \{1, t\}$, Case 5(a).

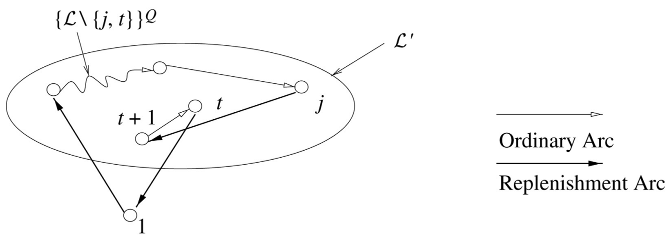

Fig. 7. The tour $\tau'_{(j,t+1)^{\mathcal{R}}}$ for $j \in V \setminus \{1, t\}$, Case 5(b).

(5) (a) $t - 2$ tours each using an ordinary arc that enters node $t + 1$ but comes from neither node t nor node 1 (as none of the previously introduced tours uses an ordinary arc in $\{(j, t)^{\mathcal{Q}} \mid j \in \{2, \ldots, t - 1\}\}$; $\tau'_{(j,t+1)^{\mathcal{Q}}} = j \xrightarrow{\mathcal{Q}} t + 1 \longrightarrow \{t, 1\}^{\mathcal{R}} \longrightarrow \{\mathcal{L} \setminus \{j, t\}\}^{\mathcal{Q}} \xrightarrow{\mathcal{Q}} j$, for $j \in V \setminus \{t, 1\}$. (See Fig. 6.)

(b) $t - 2$ tours each using a replenishment arc that enters node $t + 1$ but comes from neither node t nor node 1 (as none of the previously introduced tours uses a replenishment arc in $\{(j, t)^{\mathcal{R}} \mid j \in \{2, \ldots, t - 1\}\}$. $\tau'_{(j,t+1)^{\mathcal{R}}} = j \longrightarrow t + 1 \xrightarrow{\mathcal{Q}} t \longrightarrow 1 \longrightarrow \{\mathcal{L} \setminus \{j, t\}\}^{\mathcal{Q}} \xrightarrow{\mathcal{Q}} j$, for $j \in V \setminus \{t, 1\}$. (See Fig. 7.) Hence, the lemma is proved. □

Lemma 4.2. *Given any arbitrary $k \in \mathbb{Z}$, $k \geq 3$, for all weighted-digraph $\mathcal{G} = (V, A, W, w)$ with $|V| \geq \max\{5, k+1\}$, and any minimal violation set $\mathcal{L}$ defined on $\mathcal{G}$ with $|\mathcal{L}| = k$, under Assumption 2.1, $S_2^{\mathcal{G},\mathcal{L}}$ defines a facet for* $\mathrm{conv}(T_{\mathcal{G}})$ *if the following conditions hold:* (1) *$\mathcal{L}$ is $\mathcal{Q}$-complete, and* (2) $\mathcal{Q} \setminus (\mathcal{L}, \mathcal{L})^{\mathcal{Q}} = \emptyset$.

Proof. We prove this lemma by showing that $F_2^{\mathcal{G},\mathcal{L}}$ has a dimension of $\dim(\mathrm{conv}(T_{\mathcal{G}}) - 1)$. Again, we are left to show that there are at least $\dim(\mathrm{conv}(T_{\mathcal{G}}))$ affinely independent tours that satisfy constraint $S_2^{\mathcal{G},\mathcal{L}}$ at equality. We show this by induction on $|V|$. Our inductive hypothesis is that there exists a subset $\mathcal{X}_{\mathcal{G}} \subset T_{\mathcal{G}}$ which contains at least $|V|(|V|-3)+1+k(k-1)$ affinely independent feasible solutions defined in $\mathcal{G}$ that satisfy constraint $S_2^{\mathcal{G},\mathcal{L}}$ at equality. We first show that our inductive hypothesis is true for the cases where $|V| = 5$ if $k = 3$, and $|V| = k + 1$ if $k \geq 4$. Then we assume that our inductive hypothesis is true for $|V| = 5, \ldots, t$ if $k = 3$, and $|V| = k + 1, \ldots, t$ if $k \geq 4$, and show that it is true for $|V| = t + 1$.

Base cases. (1) When $k = 3$, we can find exactly 17 affinely independent feasible solutions that satisfy constraint $S_2^{\mathcal{G},\mathcal{L}}$ at equality (see Appendix B). (2) By the result of Lemma 4.1, under Assumption 2.1, and under the conditions of the lemma, $S_2^{\mathcal{G},\mathcal{L}}$ defines a facet for $\mathrm{conv}(T_{\mathcal{G}})$, for any $k \geq 4$ and $|V| = k + 1$, therefore our inductive hypothesis is true for all these cases.

Inductive step. Now we assume that for some $t \geq 5$, our inductive hypothesis is true for $|V| = t$, and $k = \{3, \ldots, t-1\}$, i.e. there exists $\mathcal{X}_{\mathcal{G}} \subset T_{\mathcal{G}}$ containing $t(t-3) + 1 + k(k-1)$ affinely independent vectors defined in $\mathcal{G}$ that satisfy $S_2^{\mathcal{G},\mathcal{L}}$ with equality.

We now show that our inductive hypothesis is true for $|V| = t + 1$.

Let $\mathcal{G}' = (V', A', W', w')$ be any arbitrary weighted-digraph with $|V'| = t + 1$, $A' = \mathcal{Q}' \cup \mathcal{R}'$, and let $\mathcal{L}$ be any minimal violation set defined in $\mathcal{G}'$ with $|\mathcal{L}| = k$ for $k \leq t - 1$ such that Assumption 2.1 and the conditions of the lemma hold. W.l.o.g., say $V' = \{1, \ldots, t, t + 1\}$ and $\mathcal{L} = \{1, \ldots, k\}$. Now let $\mathcal{G} = (V, A, W, w)$ where $V = \{1, \ldots, t\}$, $A = \{(i, j)^{\beta} \in \beta' \mid i, j \in V, \beta \in \{\mathcal{Q}, \mathcal{R}\}\}$ (by Condition 2 of the lemma, $\mathcal{Q} = \mathcal{Q}'$), $W = W'$, and $w_i' = w_i$ for all $i = 1, \ldots, t$. It is not hard to see that $\mathcal{L}$ also defines a minimal violation set in $\mathcal{G}$. Furthermore, $\mathcal{G}$ and $\mathcal{L}$ defined this way satisfy the inductive hypothesis, so by the hypothesis, $\mathcal{X}_{\mathcal{G}}$ exists and there are at least $t(t-3) + 1 + k(k-1)$ affinely independent feasible solutions defined in $\mathcal{G}$ that satisfy $S_2^{\mathcal{G},\mathcal{L}}$ at equality. Now we attempt to construct the claimed $\mathcal{X}_{\mathcal{G}'} \subset T_{\mathcal{G}'}$.

Tour modification. By Condition (2) of the lemma, $(t, j)^{\mathcal{Q}} = \emptyset$ for all $j \in \{1, \ldots, t - 1\}$ do not exist in $\mathcal{G}$. For all $\tau_a \in \mathcal{X}_{\mathcal{G}}$, we construct a tour τ_a' defined in $\mathcal{G}'$ by replacing the arc $(t, j)^{\mathcal{R}}$ in τ_a by a replenishment subpath $t \longrightarrow t+1 \longrightarrow j$ in τ_a'. We now obtained at least $t(t-3)+1+k(k-1)$ affinely independent feasible solutions for $\mathcal{X}_{\mathcal{G}'}'$.

Tour insertion. We need $2t - 2$ new affinely independent feasible tours defined in $\mathcal{G}'$ that satisfy constraint $S_2^{\mathcal{G}',\mathcal{L}}$ at equality. In all the tours we obtained during tour modification, arc $(t, t + 1)^{\mathcal{R}}$ is used, so the following arcs cannot have appeared in these tours: arc $(t + 1, t)^{\mathcal{R}}$, arcs that leave node t but do not enter node $t + 1$, and arcs that enter node $t + 1$ but do not leave t. We insert the following tours sequentially:

(1) (a) $\tau'_{(t,j)^{\mathcal{R}}} = t \longrightarrow j \longrightarrow 1 \longrightarrow t+1 \longrightarrow \{\mathcal{L} \setminus \{1\}\}^{\mathcal{Q}} \longrightarrow \{\bar{\mathcal{L}} \setminus \{t, j\}\}^{\mathcal{R}} \longrightarrow t$, for $j \in \bar{\mathcal{L}} \setminus \{t\}$.

(b) $\tau'_{(t,j)^{\mathcal{R}}} = t \longrightarrow j \xrightarrow{\mathcal{Q}} 1 \longrightarrow t+1 \longrightarrow \{\mathcal{L} \setminus \{1, j\}\}^{\mathcal{Q}} \longrightarrow \{\bar{\mathcal{L}} \setminus \{t\}\}^{\mathcal{R}} \longrightarrow t$, for $j \in \mathcal{L} \setminus \{1\}$.

(2) $\tau'_{(t+1,t)^{\mathcal{R}}} = 1 \longrightarrow t+1 \longrightarrow t \longrightarrow \{\mathcal{L} \setminus \{1\}\}^{\mathcal{Q}} \longrightarrow \{\bar{\mathcal{L}} \setminus \{t\}\}^{\mathcal{R}} \longrightarrow 1$.

(3) (a) $\tau'_{(j,t+1)^{\mathcal{R}}} = j \longrightarrow t+1 \longrightarrow 1 \longrightarrow \{\mathcal{L} \setminus \{1\}\}^{\mathcal{Q}} \longrightarrow \{\bar{\mathcal{L}} \setminus \{j\}\}^{\mathcal{R}} \longrightarrow j$, for $j \in V \setminus \{1, t\}$.

(b) $\tau'_{(j,t+1)^{\mathcal{R}}} = j \longrightarrow t+1 \longrightarrow \{\mathcal{L} \setminus \{j\}\}^{\mathcal{Q}} \longrightarrow \{\bar{\mathcal{L}}\}^{\mathcal{R}} \longrightarrow j$, for $j \in \mathcal{L} \setminus \{1\}$.

At the end of the tour insertion procedure, we have altogether inserted $2t - 2$ new affinely independent feasible tours defined in $\mathcal{G}'$ that satisfy constraint $S_2^{\mathcal{G}',\mathcal{L}}$ at equality. Hence the lemma is proved. □

Stage two of proof of Theorem 4.1

Now, we add in tours each using a distinct arc in $\mathcal{Q} \setminus (\mathcal{L}, \mathcal{L})$. We insert a tour $\tau_{\tilde{a}}$, for each arc $\tilde{a} \in \mathcal{Q} \setminus (\mathcal{L}, \mathcal{L})$, that uses only the arc $\tilde{a}$, but no other arcs in $\{\mathcal{Q} \setminus (\mathcal{L}, \mathcal{L})\} \setminus \{\tilde{a}\}$. We have the following three cases. Let $\bar{\mathcal{L}} = V \setminus \mathcal{L}$.

(1) For each of $(i,j)^{\mathcal{Q}} \in \delta^{+}_{\mathcal{Q}}(\mathcal{L})$, we can construct:

$$\tau_{(i,j)^{\mathcal{Q}}} = i \xrightarrow{\mathcal{Q}} j \longrightarrow \{\bar{\mathcal{L}} \setminus \{j\}\}^{\mathcal{R}} \longrightarrow \{\mathcal{L} \setminus \{i\}\}^{\mathcal{Q}} \longrightarrow i.$$

(2) For each of $(i,j)^{\mathcal{Q}}. \in \delta^{-}_{\mathcal{Q}}(\mathcal{L})$, we can construct:

$$\tau_{(i,j)^{\mathcal{Q}}} = i \xrightarrow{\mathcal{Q}} j \longrightarrow \{\bar{\mathcal{L}} \setminus \{i\}\}^{\mathcal{R}} \longrightarrow \{\mathcal{L} \setminus \{j\}\}^{\mathcal{Q}} \longrightarrow i.$$

(3) Finally, for each of $(i,j)^{\mathcal{Q}} \in (\bar{\mathcal{L}}, \bar{\mathcal{L}})^{\mathcal{Q}}$ (when $|V| \geq |\mathcal{L}| + 2$ only), we can construct:

$$\tau_{(i,j)^{\mathcal{Q}}} = i \xrightarrow{\mathcal{Q}} j \longrightarrow k \longrightarrow \{\mathcal{L} \setminus \{k\}\}^{\mathcal{Q}} \longrightarrow \{\bar{\mathcal{L}} \setminus \{i,j\}\}^{\mathcal{R}} \longrightarrow i,$$

for any arbitrary $k \in \mathcal{L}$.

Since $\mathcal{Q} \setminus (\mathcal{L}, \mathcal{L}) = \delta^{+}_{\mathcal{Q}}(\mathcal{L}) \cup \delta^{-}_{\mathcal{Q}}(\mathcal{L}) \cup (\bar{\mathcal{L}}, \bar{\mathcal{L}})^{\mathcal{Q}}$, we can construct a tour for each of the ordinary arcs in $\mathcal{Q} \setminus (\mathcal{L}, \mathcal{L})$, and get $|\mathcal{Q} \setminus (\mathcal{L}, \mathcal{L})|$ new affinely independent tours. Now, we have in total at least $n(n-3)+1+|\mathcal{Q}|$ affinely independent tours defined in $\mathcal{G}$ that satisfy $S_2^{\mathcal{G},\mathcal{L}}$ at equality. It follows that under Assumption 2.1 and the condition of the theorem, $F_2^{\mathcal{G},\mathcal{L}}$ has dimension at least $n(n-3) + |\mathcal{Q}|$. By Proposition 4.1, $F_2^{\mathcal{G},\mathcal{L}}$ has dimension at most $n(n-3) + |\mathcal{Q}|$, and thus has dimension exactly $n(n-3) + |\mathcal{Q}|$. Hence, the theorem is proved. □

4.2. Proof of Theorem 4.2

In this section, we present polyhedral results for the S_1 replenishment constraints (1). The proof technique is similar to that of S_2 constraints.

Definition 4.5. Given a weighted-digraph $\mathcal{G} = (V, A, W, w)$, and any minimal violation path $p = (i_1, \ldots, i_k)^{\mathcal{Q}} \in \mathcal{M}_{\mathcal{G}}$, we define the face of $\mathrm{conv}(T_{\mathcal{G}})$ induced by $S_1^{\mathcal{G},p}$ to be $F_1^{\mathcal{G},p} = \{x \in \mathrm{conv}(T_{\mathcal{G}}) : \sum_{a \in \mathcal{Q}(p)} x_a + \sum_{a \in \bar{\Delta}(p)} x_a + \sum_{a \in \mathcal{B}(p)} x_a = k - 2\}$.

Proposition 4.2. *For all weighted-digraphs $\mathcal{G}$ with $V = \{1, \ldots, n\}$ where $|V| \geq 4$, and any minimal violation path $p = (i_1, \ldots, i_k)^{\mathcal{Q}} \in \mathcal{M}_{\mathcal{G}}$ with $3 \leq k < |V|$, under Assumption 2.1, $F_1^{\mathcal{G},p}$ is a proper face of* $\mathrm{conv}(T_{\mathcal{G}})$.

Again, we prove the theorem in two stages. In the first stage, we show that our result holds under the assumption that there exists no ordinary arcs in $\mathcal{G}$ other than those in $\mathcal{Q}(p)$ and the proof again comprises a double induction procedure. In the second stage of our proof we show the $S_1^{\mathcal{G},p}$ constraint defines a facet for the RATS polytope when ordinary arcs exist in the graph other than those between nodes in $\mathcal{Q}(p)$.

Stage one of proof of Theorem 4.2

We first present the two lemmas.

Lemma 4.3. *For any weighted-digraph $\mathcal{G} = (V, A, W, w)$ with $V = \{1, \ldots, k+2\}$, and any minimal violation path $p = (i_1, \ldots, i_k)^{\mathcal{Q}} \in \mathcal{M}_{\mathcal{G}}$, under Assumption 2.1, $S_1^{\mathcal{G},p}$ defines a facet for* $\mathrm{conv}(T_{\mathcal{G}})$ *if $\mathcal{Q} = \mathcal{Q}(p)$.*

Proof. Firstly, note that under the condition of the lemma, $|\mathcal{Q}| = k-1$, $\bar{\Delta}(p) = \emptyset$, and $\mathcal{B}(p) \subseteq \mathcal{R}$. We now show that there are at least $\dim(\mathrm{conv}(T_{\mathcal{G}}))$ affinely independent feasible tours that satisfy constraint $S_1^{\mathcal{G},p}$ at equality. We show this by induction on $|V|$. Since $|\mathcal{R}| = |V|(|V| - 1)$ under Assumption 2.1, under the condition of the lemma, and by results of Theorem 3.2, $\dim(\mathrm{conv}(T_{\mathcal{G}})) = n^2 - 2n - 2$, where $n = |V|$. Our inductive hypothesis, for some $t \geq 5$, is that for any $\mathcal{G}$, p satisfying the condition of the lemma with $|V| = t$, there exists a subset of tours $\mathcal{X}_{\mathcal{G}} \subset T_{\mathcal{G}}$ defined in $\mathcal{G}$ which contains at least $t^2 - 2t - 2$ affinely independent feasible tours satisfying constraint $S_1^{\mathcal{G},p}$ at equality. We first show that our inductive hypothesis is true for the base case where $t = 5$. Then we assume that our inductive hypothesis is true for some $t \geq 5$ and show that it is true for $t + 1$.

Base case. Given any weighted-digraph $\mathcal{G} = (V, A, W, w)$ with $|V| = 5$, and any minimal violation path $p = (i_1, i_2, i_3)^{\mathcal{Q}}$ defined in $\mathcal{G}$, if $\mathcal{G}$, p satisfy Assumption 2.1, and the conditions of Lemma 4.3, then we can find 13 affinely independent feasible tours that satisfy the $S_1^{\mathcal{G},p}$ constraint at equality. (See Appendix C.)

Inductive step. Assume that our inductive hypothesis is true for some $t \geq 5$. We now show that our inductive hypothesis is true for $t+1$. Let $\mathcal{G}' = (V', A', W', w')$ be any arbitrary weighted-digraph with $|V'| = t+1$, $A' = \mathcal{Q}' \cup \mathcal{R}'$, and let $p' = (i'_1, \ldots, i'_{t-1})^{\mathcal{Q}}$ be any minimal violation path of length $|V'| - 2$ defined in $\mathcal{G}'$. W.l.o.g., say $V' = \{1, \ldots, t, t+1\}$ and $p' = (3, \ldots, t, t+1)^{\mathcal{Q}}$. Now let $\mathcal{G} = (V, A, W, w)$ where $V = \{1, \ldots, t\}$, $A = \{(i, j)^{\beta} \in \beta' \mid i, j \in V, \beta \in \{\mathcal{Q}, \mathcal{R}\}\}$, $W = \sum_{k=4}^{t-1} w_k + \max\{w_3, w_t\}$, and $w_i = w'_i$ for all $i = 1, \ldots, t$. Also, let $p = (3, \ldots, t)^{\mathcal{Q}}$. Now, since we assumed that $w_i > 0$ for all $i \in V$, we have $\sum_{k=4}^{t} w_k = \sum_{k=4}^{t-1} w_k + w_t \leq \sum_{k=4}^{t-1} w_k + \max\{w_3, w_t\}$, $\sum_{k=3}^{t-1} w_k = \sum_{k=4}^{t-1} w_k + w_3 \leq \sum_{k=4}^{t-1} w_k + \max\{w_3, w_t\}$, and $\sum_{k=3}^{t} w_k = \sum_{k=4}^{t-1} w_k + w_3 + w_t > \sum_{k=4}^{t-1} w_k + \max\{w_3, w_t\}$. Thus, it is not hard to see that $p \in \mathcal{M}_{\mathcal{G}}$, and $\mathcal{G}$, p defined this way satisfy the inductive hypothesis, so by the hypothesis, $\mathcal{X}_{\mathcal{G}}$ exists and there are at least $t^2 - 2t - 2$ affinely independent feasible tours in $\mathcal{G}$ that satisfy $S_1^{\mathcal{G},p}$ at equality. Now we attempt to construct $\mathcal{X}_{\mathcal{G}'} \subset T_{\mathcal{G}'}$.

Tour modification. Observe that by definition of p, and under the condition of the lemma, $\{(t, j)^{\mathcal{Q}} \mid j = 1, \ldots, t-1\} = \emptyset$. Now, for every tour in $\mathcal{X}_{\mathcal{G}}$ that uses the arc $(t, j)^{\mathcal{R}}$ for $j \in \{1, \ldots, t-1\}$, we replace it by $(t \xrightarrow{\mathcal{Q}} t+1 \longrightarrow j)$. Now we have $t^2 - 2t - 2$ affinely independent feasible tours defined in $\mathcal{G}'$ that satisfy $S_1^{\mathcal{G}',p'}$ at equality.

Tour insertion. We now insert, in sequence, $2t - 1$ new affinely independent feasible tours defined in $\mathcal{G}'$ that satisfy $S_1^{\mathcal{G}',p'}$ at equality. We insert the new tours in $\mathcal{X}_{\mathcal{G}'}$ in the following order:

(1) (a) $\tau'_{(t,j)^{\mathcal{R}}} = 1 \longrightarrow t+1 \longrightarrow (j+1, j+2, \ldots, t)^{\mathcal{Q}} \longrightarrow (j, j-1, \ldots, 3)^{\mathcal{R}} \longrightarrow 2 \longrightarrow 1$, for $j = 3, \ldots, t-2$.
  (b) $\tau'_{(t,t-1)^{\mathcal{R}}} = 1 \longrightarrow t+1 \longrightarrow 2 \longrightarrow (t, t-1, \ldots, 3)^{\mathcal{R}} \longrightarrow 1$.
  (c) $\tau'_{(t,1)^{\mathcal{R}}} = 1 \longrightarrow t+1 \longrightarrow 2 \longrightarrow (3, 4, \ldots, t)^{\mathcal{Q}} \longrightarrow 1$, and $\tau'_{(t,2)^{\mathcal{R}}} = 1 \longrightarrow t+1 \longrightarrow (3, 4, \ldots, t)^{\mathcal{Q}} \longrightarrow 2 \longrightarrow 1$.
  (d) $\tau'_{(t,t+1)^{\mathcal{R}}} = 2 \longrightarrow (3, 4, \ldots, t)^{\mathcal{Q}} \longrightarrow t+1 \longrightarrow 1 \longrightarrow 2$.

(2) $\tau'_{(t+1,t)^{\mathcal{R}}} = 1 \longrightarrow t+1 \longrightarrow (t, t-1, \ldots, 3)^{\mathcal{R}} \longrightarrow 2 \longrightarrow 1$.

(3) (a) $\tau'_{(j,t+1)^{\mathcal{R}}} = t+1 \longrightarrow \{1, 2\}^{\mathcal{R}} \longrightarrow (j+1, j+2, \ldots, t)^{\mathcal{Q}} \longrightarrow (3, 4, \ldots, j)^{\mathcal{Q}} \longrightarrow t+1$, for $j = 3, \ldots, t-1$.
  (b) $\tau'_{(2,t+1)^{\mathcal{R}}} = 2 \longrightarrow t+1 \longrightarrow 1 \longrightarrow (3, 4, \ldots, t)^{\mathcal{Q}} \longrightarrow 2$.

Hence the lemma is proved. □

Lemma 4.4. *Given any arbitrary $k \in \mathbb{Z}$, $k \geq 3$, for all weighted-digraphs $\mathcal{G} = (V, A, W, w)$ with $V = \{1, \ldots, t\}$ where $t \geq k+2$ and any minimal violation path $p = (i_1, \ldots, i_k)^{\mathcal{Q}} \in \mathcal{M}_{\mathcal{G}}$, under Assumption 2.1, $S_1^{\mathcal{G},p}$ defines a facet for* $\operatorname{conv}(T_{\mathcal{G}})$ *if $\mathcal{Q} = \mathcal{Q}(p)$.*

Proof. We show that there are at least $\dim(\operatorname{conv}(T_{\mathcal{G}}))$ affinely independent feasible tours that satisfy constraint $S_1^{\mathcal{G},p}$ at equality. We show this by induction on $|V|$. Let $\mathcal{G}' = (V', A', W', w')$ be any arbitrary weight-valid digraph satisfying Assumption 2.1, and the condition of the lemma, with $|V'| = t+1$, $A' = \mathcal{Q}' \cup \mathcal{R}'$, and let $p = (i_1, \ldots, i_k)^{\mathcal{Q}}$ for $k \leq t-2$ be any minimal violation path in $\mathcal{G}'$. W.l.o.g., say $V' = \{1, \ldots, t, t+1\}$ and $p = (1, \ldots, k)^{\mathcal{Q}}$. Now let $\mathcal{G} = (V, A, W, w)$ where $V = \{1, \ldots, t\}$, $A = \{(i, j)^{\beta} \in \beta' \mid i, j \in V, \beta \in \{\mathcal{Q}, \mathcal{R}\}\}$, $W = W'$, and $w_i = w'_i$ for all $i = 1, \ldots, t$. It is not hard to see that p also defines a minimal violation path in $\mathcal{G}$. Furthermore, $\mathcal{G}$ and p defined this way satisfy the inductive hypothesis, so by the hypothesis, $\mathcal{X}_{\mathcal{G}}$ exists and there are at least $t^2 - 3t + k$ affinely independent feasible solutions defined in $\mathcal{G}$ that satisfy $S_1^{\mathcal{G},p}$ at equality. Now we attempt to construct $\mathcal{X}_{\mathcal{G}'} \subset T_{\mathcal{G}'}$.

Tour modification. If $(t, j)^{\mathcal{R}}$, for $j \in \{1, \ldots, t-1\}$, is used in a tour $\tau \in \mathcal{X}_{\mathcal{G}}$, we can construct a tour τ' defined in $\mathcal{G}'$ by replacing the arc $(t, j)^{\mathcal{R}}$ by a replenishment path $(t \longrightarrow t+1 \longrightarrow j)$ in τ'. Note that arcs $(t, t+1)^{\mathcal{R}}$ and $(t+1, j)^{\mathcal{R}}$ do exist in $\mathcal{G}'$ by Assumption 2.1. Hence we get $t^2 - 3t + k$ affinely independent feasible tours defined in $\mathcal{G}'$ that satisfy $S_1^{\mathcal{G}',p}$ at equality.

Tour insertion. We now insert, in sequence, $2t - 2$ new affinely independent feasible tours defined on $\mathcal{G}'$ that satisfy constraint $S_1^{\mathcal{G}',p}$ at equality. In what follows, we use $\mathcal{S} = \{i_1, \ldots, i_k\}$ to represent the set of nodes in p, and define $\overline{\mathcal{S}} = V \setminus \mathcal{S}$.

(1) (a) $\tau'_{(t,j)^{\mathcal{R}}} = 1 \longrightarrow t+1 \longrightarrow \{\overline{\mathcal{S}} \setminus \{t, j\}\}^{\mathcal{R}} \longrightarrow t \longrightarrow j \longrightarrow (k, k-1, \ldots, 1)^{\mathcal{R}}$, for $j \in \{k\} \cup \overline{\mathcal{S}} \setminus \{t\}$.

(b) $\tau'_{(t,j)^{\mathcal{R}}} = 1 \longrightarrow t+1 \longrightarrow \{\overline{\mathcal{S}} \setminus \{t\}\}^{\mathcal{R}} \longrightarrow (j+1, j+2, \ldots, k)^{\mathcal{Q}} \longrightarrow t \longrightarrow (j, j-1, \ldots, 1)^{\mathcal{R}}$, for $j \in \mathcal{S} \setminus \{k\}$.

(2) $\tau'_{(t+1,t)^{\mathcal{R}}} = 1 \longrightarrow t+1 \longrightarrow t \longrightarrow \{\overline{\mathcal{S}} \setminus \{t\}\}^{\mathcal{R}} \longrightarrow (k, k-1, \ldots, 1)^{\mathcal{R}}$.

(3) (a) $\tau'_{(j,t+1)^{\mathcal{R}}} = j \longrightarrow t+1 \longrightarrow t \longrightarrow 1 \longrightarrow (2, \ldots, k)^{\mathcal{Q}} \longrightarrow \{\overline{\mathcal{S}} \setminus \{j, t\}\}^{\mathcal{R}} \longrightarrow j$, for $j \in \overline{\mathcal{S}} \setminus \{t\}$.

(b) $\tau'_{(j,t+1)^{\mathcal{R}}} = j \longrightarrow t+1 \longrightarrow t \longrightarrow \{\overline{\mathcal{S}} \setminus \{t\}\}^{\mathcal{R}} \longrightarrow (k, k-1, \ldots, j+1)^{\mathcal{R}} \longrightarrow (1, 2, \ldots, j)^{\mathcal{Q}}$, for $j \in \mathcal{S} \setminus \{1, k\}$.

(c) $\tau'_{(k,t+1)^{\mathcal{R}}} = k \longrightarrow t+1 \longrightarrow t \longrightarrow (1, 2, \ldots, k-1)^{\mathcal{Q}} \longrightarrow \{\overline{\mathcal{S}} \setminus \{t\}\}^{\mathcal{R}} \longrightarrow k$.

Hence the lemma is proved. □

Stage two of proof of Theorem 4.2

Now, we construct the remaining $|\mathcal{Q} \setminus \mathcal{Q}(p)|$ affinely independent feasible tours defined in $\mathcal{G}$ which satisfy $S_1^{\mathcal{G},p}$ at equality, by inserting a tour for each of the arcs in $\mathcal{Q} \setminus \mathcal{Q}(p)$. We insert a tour $\tau_{\tilde{a}}$, for each arc $\tilde{a} \in \mathcal{Q} \setminus \mathcal{Q}(p)$, that uses the arc $\tilde{a}$ for the first time, so as to guarantee the affine independence of the newly introduced tours. Now we look at these new tours in detail. W.l.o.g., we assume that $V = \{1, \ldots, t\}$ and $p = (1, \ldots, k)^{\mathcal{Q}}$ where $k \le t-2$. Let $\mathcal{S} = V(p) = \{1, \ldots, k\}$, $\overline{\mathcal{S}} = V \setminus \mathcal{S}$, $\mathcal{B}_{\mathcal{Q}}(p) = \mathcal{B}(p) \cap \mathcal{Q}$, and $\mathcal{F}_{\mathcal{Q}}(p) = \{(i, j)^{\mathcal{Q}} \mid i = 1, \ldots, k-2$, and $j = i+2, \ldots, k\}$; we have

$$
\begin{aligned}
&\mathcal{Q} = (\mathcal{S}, \mathcal{S})^{\mathcal{Q}} \cup \delta^+_{\mathcal{Q}}(\mathcal{S}) \cup \delta^-_{\mathcal{Q}}(\mathcal{S}) \cup (\overline{\mathcal{S}}, \overline{\mathcal{S}})^{\mathcal{Q}},\\
&(\mathcal{S}, \mathcal{S})^{\mathcal{Q}} = \mathcal{Q}(p) \cup \mathcal{F}_{\mathcal{Q}}(p) \cup \mathcal{B}_{\mathcal{Q}}(p) \cup (\{k\}, \mathcal{S} \setminus \{k\})^{\mathcal{Q}}\\
&\qquad = \mathcal{Q}(p) \cup \{\mathcal{F}_{\mathcal{Q}}(p) \cap \bar{\Delta}(p)\} \cup \{\mathcal{F}_{\mathcal{Q}}(p) \setminus \bar{\Delta}(p)\} \cup \mathcal{B}_{\mathcal{Q}}(p) \cup (\{k\}, \mathcal{S} \setminus \{k\})^{\mathcal{Q}},\\
&\delta^+_{\mathcal{Q}}(\mathcal{S}) = \{\delta^+_{\mathcal{Q}}(\mathcal{S}) \cap \bar{\Delta}(p)\} \cup \{\delta^+_{\mathcal{Q}}(\mathcal{S}) \setminus \bar{\Delta}(p)\}.
\end{aligned}
$$

Since we assumed that $\mathcal{Q} \setminus \mathcal{Q}(p) = \emptyset$ in Stage One, we have not yet constructed tours using any of the arcs in the set $\mathcal{F}_{\mathcal{Q}}(p) \cup \mathcal{B}_{\mathcal{Q}}(p) \cup (\{k\}, \mathcal{S} \setminus \{k\})^{\mathcal{Q}} \cup \delta^+_{\mathcal{Q}}(\mathcal{S}) \cup \delta^-_{\mathcal{Q}}(\mathcal{S}) \cup (\overline{\mathcal{S}}, \overline{\mathcal{S}})^{\mathcal{Q}}$. So we need to construct a tour for each of these arcs. We have the following cases.

1(a)	$\mathcal{F}_{\mathcal{Q}}(p) \cap \bar{\Delta}(p)$	1(b)	$\mathcal{F}_{\mathcal{Q}}(p) \setminus \bar{\Delta}(p)$		
2(a)	$\mathcal{B}_{\mathcal{Q}}(p)$	2(b)	$(\{k\}, \mathcal{S} \setminus \{1, k\})^{\mathcal{Q}}$	2(c)	$(\mathcal{S} \setminus \{k\}, \overline{\mathcal{S}})^{\mathcal{Q}} \cap \bar{\Delta}(p)$
3(a)	$(k, 1)^{\mathcal{Q}}$	3(b)	$(\mathcal{S} \setminus \{k\}, \overline{\mathcal{S}})^{\mathcal{Q}} \setminus \bar{\Delta}(p)$	3(c)	$(\{k\}, \overline{\mathcal{S}})^{\mathcal{Q}}$
4	$\delta^-_{\mathcal{Q}}(\mathcal{S})$	5	$(\overline{\mathcal{S}}, \overline{\mathcal{S}})^{\mathcal{Q}}$		

Note that: (i) 1(a) is just the set $\{(j, k)^{\mathcal{Q}} \mid \sum_{l=1}^{j} w_l + w_k > W$, for $j = 2, \ldots, k-2$, and $k \ge j+2\}$; (ii) by definition of $\mathcal{G}$, in arc set 3(a), $i_1 \neq \mathcal{S} \setminus \{k\}$; and (iii) $(\{k\}, \overline{\mathcal{S}})^{\mathcal{Q}} \cap \bar{\Delta}(p) = \emptyset$ (for 3(c)).

1(a) $\tau_{(j,k)^{\mathcal{Q}}} = (1, 2, \ldots, j-1)^{\mathcal{Q}} \longrightarrow j \xrightarrow{\mathcal{Q}} k \longrightarrow \{\overline{\mathcal{S}}\}^{\mathcal{R}} \longrightarrow (k-1, k-2, \ldots, j+1)^{\mathcal{R}} \longrightarrow 1$.

1(b) $\tau_{(i,j)^{\mathcal{Q}}} = (1, 2, \ldots, i)^{\mathcal{Q}} \xrightarrow{\mathcal{Q}} (j, j+1, \ldots, k)^{\mathcal{Q}} \longrightarrow \{\overline{\mathcal{S}}\}^{\mathcal{R}} \longrightarrow (i+1, i+2, \ldots, j-1)^{\mathcal{Q}} \longrightarrow 1$.

2(a) $\tau_{(i,j)^{\mathcal{Q}}} = 1 \longrightarrow (i+1, i+2, \ldots, k)^{\mathcal{Q}} \longrightarrow \{\overline{\mathcal{S}}\}^{\mathcal{R}} \longrightarrow (j+1, j+2, \ldots, i)^{\mathcal{Q}} \xrightarrow{\mathcal{Q}} (j, j-1, \ldots, 1)^{\mathcal{R}}$.

2(b) $\tau_{(k,j)^{\mathcal{Q}}} = (1, 2, \ldots, j-1)^{\mathcal{Q}} \longrightarrow \{\overline{\mathcal{S}}\}^{\mathcal{R}} \longrightarrow k \xrightarrow{\mathcal{Q}} (j, j+1, \ldots, k-1)^{\mathcal{Q}} \longrightarrow 1$.

2(c) $\tau_{(k,1)^{\mathcal{Q}}} = k \xrightarrow{\mathcal{Q}} 1 \longrightarrow (k-1, k-2, \ldots, 2)^{\mathcal{R}} \xrightarrow{\mathcal{Q}} j \longrightarrow \{\overline{\mathcal{S}} \setminus \{j\}\}^{\mathcal{R}} \longrightarrow k$. (Note that if $(k, 1)^{\mathcal{Q}} \in A$, by Condition 2 of the theorem, there exists $(2, j)^{\mathcal{Q}} \in \bar{\Delta}(p)$ for $j \in \overline{\mathcal{S}}$.)

3(a) $\tau_{(i,j)^{\mathcal{Q}}} = 1 \longrightarrow i \xrightarrow{\mathcal{Q}} j \longrightarrow (i+1, i+2, \ldots, k)^{\mathcal{Q}} \longrightarrow \{\overline{\mathcal{S}} \setminus \{j\}\}^{\mathcal{R}} \longrightarrow (i-1, i-2, \ldots, 1)^{\mathcal{R}}$.

3(b) $\tau_{(i,j)^{\mathcal{Q}}} = (1, \ldots, i)^{\mathcal{Q}} \xrightarrow{\mathcal{Q}} j \longrightarrow (i+1, i+2, \ldots, k)^{\mathcal{Q}} \longrightarrow \{\overline{\mathcal{S}} \setminus \{j\}\}^{\mathcal{R}} \longrightarrow 1$.

3(c) $\tau_{(k,j)^{\mathcal{Q}}} = (1, \ldots, k-1)^{\mathcal{Q}} \longrightarrow k \xrightarrow{\mathcal{Q}} j \longrightarrow \{\overline{\mathcal{S}} \setminus \{j\}\}^{\mathcal{R}} \longrightarrow 1$.

4 $\tau_{(i,j)^{\mathcal{Q}}} = (1, 2, \ldots, j-1)^{\mathcal{Q}} \longrightarrow (j+1, j+2, \ldots, k)^{\mathcal{Q}} \longrightarrow \{\overline{\mathcal{S}} \setminus \{i\}\}^{\mathcal{R}} \longrightarrow i \xrightarrow{\mathcal{Q}} j \longrightarrow 1.$

5 $\tau_{(i,j)^{\mathcal{Q}}} = 1 \longrightarrow (2, 3 \ldots, k)^{\mathcal{Q}} \longrightarrow i \xrightarrow{\mathcal{Q}} j \longrightarrow \{\overline{\mathcal{S}} \setminus \{i, j\}\}^{\mathcal{R}} \longrightarrow 1.$

Now, we have constructed one tour for each of the arcs in $\mathcal{Q} \setminus \mathcal{Q}(p)$. Therefore at the end of Stage Two, we have in total at least $|V|(|V|-3) + 1 + |\mathcal{Q}|$ affinely independent feasible tours defined in $\mathcal{G}$ that satisfy constraint $S_1^{\mathcal{G},p}$ at equality. By Theorem 3.2 and 4.2, $F_1^{\mathcal{G},p}$ defines a facet for $\text{conv}(T_{\mathcal{G}})$ and the theorem is proved. □

5. Conclusions

Two classes of constraints useful for ensuring replenishment feasibility are considered, and both are proved to be facet-defining for the RATS polytope, under suitable conditions that are easily checked. One of the two classes is a form of the 2-path cuts for vehicle routing problems, described, for example, in [18]: as far as we are aware, these are the first facet-defining results to be proved for constraints of this type. Our proof technique uses a double induction argument, and applies the ideas of Fischetti [14], who provides a general induction technique for proof of ATSP facet-defining results. We believe the proofs we have given to be of general interest as they may help other researchers studying the polyhedral structure of vehicle routing problems generally: this is an area with a shortage of facet-defining results, and the RATSP is a useful problem, providing a kind of "polyhedral bridge" between ATSP and vehicle routing problems.

Our polyhedral results are on RATSPs where replenishment arcs and ordinary arcs within a subset of nodes are assumed to be complete. For future research, it would be of great interest to study the facets of the polytope in its generality.

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Appendix A. Base case 1 of S_2 constraints — $|V| = 5$, $|\mathcal{L}| = 4$

Without loss of generality, we assume that $V = \{1, 2, 3, 4, 5\}$ and $\mathcal{L} = \{1, 2, 3, 4\}$. The $S_2^{\mathcal{G},\mathcal{L}}$ constraint is as follows.

$$x_{12}^{\mathcal{Q}} + x_{13}^{\mathcal{Q}} + x_{14}^{\mathcal{Q}} + x_{21}^{\mathcal{Q}} + x_{23}^{\mathcal{Q}} + x_{24}^{\mathcal{Q}} + x_{31}^{\mathcal{Q}} + x_{32}^{\mathcal{Q}} + x_{34}^{\mathcal{Q}} + x_{41}^{\mathcal{Q}} + x_{42}^{\mathcal{Q}} + x_{43}^{\mathcal{Q}} \leq 2. \tag{A.1}$$

We need to show that there are exactly 23 affinely independent feasible solutions that satisfy constraint (A.1) at equality. We start by constructing a feasible tour that uses exactly two arcs from $\{(1,2)^{\mathcal{Q}}, (1,3)^{\mathcal{Q}}, (1,4)^{\mathcal{Q}}, (2,1)^{\mathcal{Q}}, (2,3)^{\mathcal{Q}}, (2,4)^{\mathcal{Q}}, (3,1)^{\mathcal{Q}}, (3,2)^{\mathcal{Q}}, (3,4)^{\mathcal{Q}}, (4,1)^{\mathcal{Q}}, (4,2)^{\mathcal{Q}}, (4,3)^{\mathcal{Q}}\}$, thus satisfying constraint (A.1) at equality. Then, we sequentially insert 22 other feasible tours that satisfy (A.1) at equality, each using an arc that has not been used before. We use τ_a to denote a tour that uses arc a, which has never been used in any of the previously introduced tours. We start with the following tour:

$$3 \xrightarrow{\mathcal{Q}} 4 \xrightarrow{\mathcal{Q}} 2 \longrightarrow 5 \longrightarrow 1 \longrightarrow 3.$$

Now we insert the following 22 tours sequentially:

$$\tau_{(1,5)^{\mathcal{R}}} = 3 \xrightarrow{\mathcal{Q}} 4 \xrightarrow{\mathcal{Q}} 2 \longrightarrow 1 \longrightarrow 5 \longrightarrow 3,$$

$$\tau_{(4,1)^{\mathcal{Q}}} = 3 \xrightarrow{\mathcal{Q}} 4 \xrightarrow{\mathcal{Q}} 1 \longrightarrow 2 \longrightarrow 5 \longrightarrow 3,$$

$$\tau_{(3,2)^{\mathcal{Q}}} = 3 \xrightarrow{\mathcal{Q}} 2 \longrightarrow 4 \xrightarrow{\mathcal{Q}} 1 \longrightarrow 5 \longrightarrow 3,$$

$$\tau_{(5,4)^{\mathcal{R}}} = 3 \xrightarrow{\mathcal{Q}} 2 \longrightarrow 5 \longrightarrow 4 \xrightarrow{\mathcal{Q}} 1 \longrightarrow 3,$$

$$\tau_{(4,3)^{\mathcal{Q}}} = 3 \xrightarrow{\mathcal{Q}} 2 \longrightarrow 1 \longrightarrow 5 \longrightarrow 4 \xrightarrow{\mathcal{Q}} 3,$$

$$\tau_{(1,4)^{\mathcal{R}}} = 3 \xrightarrow{\mathcal{Q}} 2 \longrightarrow 5 \longrightarrow 1 \longrightarrow 4 \xrightarrow{\mathcal{Q}} 3,$$

$$\tau_{(3,1)^{\mathcal{Q}}} = 4 \xrightarrow{\mathcal{Q}} 3 \xrightarrow{\mathcal{Q}} 1 \longrightarrow 2 \longrightarrow 5 \longrightarrow 4,$$

$$\tau_{(5,2)^{\mathcal{R}}} = 4 \xrightarrow{\mathcal{Q}} 3 \xrightarrow{\mathcal{Q}} 1 \longrightarrow 5 \longrightarrow 2 \longrightarrow 4,$$

$$\tau_{(2,4)^{\mathcal{Q}}} = 4 \xrightarrow{\mathcal{Q}} 3 \longrightarrow 5 \longrightarrow 1 \longrightarrow 2 \xrightarrow{\mathcal{Q}} 4,$$

$$\tau_{(2,3)^{\mathcal{R}}} = 3 \xrightarrow{\mathcal{Q}} 4 \xrightarrow{\mathcal{Q}} 1 \longrightarrow 5 \longrightarrow 2 \longrightarrow 3,$$

$$\tau_{(4,3)^{\mathcal{R}}} = 2 \xrightarrow{\mathcal{Q}} 4 \longrightarrow 3 \xrightarrow{\mathcal{Q}} 1 \longrightarrow 5 \longrightarrow 2,$$

$$\tau_{(4,1)^{\mathcal{R}}} = 2 \xrightarrow{\mathcal{Q}} 4 \longrightarrow 1 \longrightarrow 5 \longrightarrow 3 \xrightarrow{\mathcal{Q}} 2,$$

$$\tau_{(3,1)^{\mathcal{R}}} = 2 \xrightarrow{\mathcal{Q}} 4 \xrightarrow{\mathcal{Q}} 3 \longrightarrow 1 \longrightarrow 5 \longrightarrow 2,$$

$$\tau_{(4,5)^{\mathcal{R}}} = 2 \xrightarrow{\mathcal{Q}} 4 \longrightarrow 5 \longrightarrow 1 \longrightarrow 3 \xrightarrow{\mathcal{Q}} 2,$$

$$\tau_{(2,3)^{\mathcal{Q}}} = 2 \xrightarrow{\mathcal{Q}} 3 \longrightarrow 5 \longrightarrow 1 \longrightarrow 4 \xrightarrow{\mathcal{Q}} 2,$$

$$\tau_{(3,4)^{\mathcal{R}}} = 2 \xrightarrow{\mathcal{Q}} 3 \longrightarrow 4 \xrightarrow{\mathcal{Q}} 1 \longrightarrow 5 \longrightarrow 2,$$

$$\tau_{(4,2)^{\mathcal{R}}} = 2 \xrightarrow{\mathcal{Q}} 3 \xrightarrow{\mathcal{Q}} 1 \longrightarrow 5 \longrightarrow 4 \longrightarrow 2,$$

$$\tau_{(2,1)^{\mathcal{Q}}} = 2 \xrightarrow{\mathcal{Q}} 1 \longrightarrow 4 \xrightarrow{\mathcal{Q}} 3 \longrightarrow 5 \longrightarrow 2,$$

$$\tau_{(1,4)^{\mathcal{Q}}} = 2 \xrightarrow{\mathcal{Q}} 1 \xrightarrow{\mathcal{Q}} 4 \longrightarrow 3 \longrightarrow 5 \longrightarrow 2,$$

$$\tau_{(1,3)^{\mathcal{Q}}} = 2 \xrightarrow{\mathcal{Q}} 1 \xrightarrow{\mathcal{Q}} 3 \longrightarrow 4 \longrightarrow 5 \longrightarrow 2,$$

$$\tau_{(1,2)^{\mathcal{Q}}} = 3 \xrightarrow{\mathcal{Q}} 1 \xrightarrow{\mathcal{Q}} 2 \longrightarrow 4 \longrightarrow 5 \longrightarrow 3, \quad \text{and}$$

$$\tau_{(3,2)^{\mathcal{R}}} = 3 \longrightarrow 2 \longrightarrow 5 \longrightarrow 1 \xrightarrow{\mathcal{Q}} 4 \xrightarrow{\mathcal{Q}} 3.$$

Appendix B. Base case of S_2 constraints — $|V| = 5$, $|\mathcal{L}| = 3$

Given any weighted-digraph $\mathcal{G} = (V, A, W, w)$ with $|V| = 5$, and any minimal violation set $\mathcal{L}$ defined in $\mathcal{G}$ with $|\mathcal{L}| = 3$, if $\mathcal{G}$, $\mathcal{L}$ satisfy Assumption 2.1 and the conditions of Lemma 4.2, then we can find 17 affinely independent feasible tours that satisfy $S_2^{\mathcal{G},\mathcal{L}}$ at equality. Without loss of generality, we assume that $V = \{1, 2, 3, 4, 5\}$ and $\mathcal{L} = \{1, 2, 3\}$. The $S_2^{\mathcal{G},\mathcal{L}}$ constraint is as follows.

$$x_{12}^{\mathcal{Q}} + x_{13}^{\mathcal{Q}} + x_{21}^{\mathcal{Q}} + x_{23}^{\mathcal{Q}} + x_{31}^{\mathcal{Q}} + x_{32}^{\mathcal{Q}} \leq 1. \tag{B.1}$$

We need to show that there are exactly 17 affinely independent feasible solutions that satisfy constraint (B.1) at equality. We first construct feasible tours all using the arc $(1, 2)^{\mathcal{Q}}$, and no other ordinary arcs in $\{(1, 3)^{\mathcal{Q}}, (2, 1)^{\mathcal{Q}}, (2, 3)^{\mathcal{Q}}, (3, 1)^{\mathcal{Q}}, (3, 2)^{\mathcal{Q}}\}$, thus satisfying constraint (B.1) at equality. Now, the problem is reduced to a 4-node ATSP, thus we have the following 6 affinely independent feasible tours (see [14]):

$$1 \xrightarrow{\mathcal{Q}} 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5 \longrightarrow 1, \qquad 1 \xrightarrow{\mathcal{Q}} 2 \longrightarrow 3 \longrightarrow 5 \longrightarrow 4 \longrightarrow 1,$$
$$1 \xrightarrow{\mathcal{Q}} 2 \longrightarrow 4 \longrightarrow 3 \longrightarrow 5 \longrightarrow 1, \qquad 1 \xrightarrow{\mathcal{Q}} 2 \longrightarrow 4 \longrightarrow 5 \longrightarrow 3 \longrightarrow 1,$$
$$1 \xrightarrow{\mathcal{Q}} 2 \longrightarrow 5 \longrightarrow 3 \longrightarrow 4 \longrightarrow 1, \qquad 1 \xrightarrow{\mathcal{Q}} 2 \longrightarrow 5 \longrightarrow 4 \longrightarrow 3 \longrightarrow 1.$$

Now, we insert 11 affinely independent feasible tours that satisfy (B.1) at equality, each using an arc that has not been used before.

$$\tau_{(3,2)^{\mathcal{R}}} = 1 \xrightarrow{\mathcal{Q}} 3 \longrightarrow 2 \longrightarrow 5 \longrightarrow 4 \longrightarrow 1,$$

$$\tau_{(5,2)^{\mathcal{R}}} = 1 \xrightarrow{\mathcal{Q}} 3 \longrightarrow 5 \longrightarrow 2 \longrightarrow 4 \longrightarrow 1,$$

$$\tau_{(2,1)^{\mathcal{R}}} = 1 \xrightarrow{\mathcal{Q}} 3 \longrightarrow 4 \longrightarrow 5 \longrightarrow 2 \longrightarrow 1,$$
$$\tau_{(4,2)^{\mathcal{R}}} = 1 \xrightarrow{\mathcal{Q}} 3 \longrightarrow 4 \longrightarrow 2 \longrightarrow 5 \longrightarrow 1,$$
$$\tau_{(1,5)^{\mathcal{R}}} = 2 \xrightarrow{\mathcal{Q}} 1 \longrightarrow 5 \longrightarrow 3 \longrightarrow 4 \longrightarrow 2,$$
$$\tau_{(1,4)^{\mathcal{R}}} = 2 \xrightarrow{\mathcal{Q}} 1 \longrightarrow 4 \longrightarrow 3 \longrightarrow 5 \longrightarrow 2,$$
$$\tau_{(1,3)^{\mathcal{R}}} = 2 \xrightarrow{\mathcal{Q}} 1 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5 \longrightarrow 2,$$
$$\tau_{(2,3)^{\mathcal{Q}}} = 2 \xrightarrow{\mathcal{Q}} 3 \longrightarrow 1 \longrightarrow 5 \longrightarrow 4 \longrightarrow 2,$$
$$\tau_{(1,2)^{\mathcal{R}}} = 2 \xrightarrow{\mathcal{Q}} 3 \longrightarrow 5 \longrightarrow 4 \longrightarrow 1 \longrightarrow 2,$$
$$\tau_{(3,1)^{\mathcal{Q}}} = 3 \xrightarrow{\mathcal{Q}} 1 \longrightarrow 5 \longrightarrow 2 \longrightarrow 4 \longrightarrow 3, \quad \text{and}$$
$$\tau_{(3,2)^{\mathcal{Q}}} = 3 \xrightarrow{\mathcal{Q}} 2 \longrightarrow 5 \longrightarrow 1 \longrightarrow 4 \longrightarrow 3.$$

Appendix C. Base case of S_1 constraints — $|V| = 5$, $k = 3$

Without loss of generality, we assume that $V = \{1, 2, 3, 4, 5\}$ and $p = (1, 2, 3)^{\mathcal{Q}}$. The $S_1^{\mathcal{G},p}$ constraint is given as:

$$x_{12}^{\mathcal{Q}} + x_{23}^{\mathcal{Q}} + x_{21}^{\mathcal{R}} \leq 1. \tag{C.1}$$

We first construct six affinely independent feasible tours each using the arc $(2, 1)^{\mathcal{R}}$, and using no other arcs in $\{(1, 2)^{\mathcal{Q}}, (2, 3)^{\mathcal{Q}}\}$. Note that in these six tours, constraint (C.1) is satisfied at equality. Now, the problem is reduced to a 4-node ATSP, thus we have the following 6 affinely independent feasible tours (see [14]):

$$\begin{array}{ll}
2 \longrightarrow 1 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5 \longrightarrow 2, & 2 \longrightarrow 1 \longrightarrow 3 \longrightarrow 5 \longrightarrow 4 \longrightarrow 2, \\
2 \longrightarrow 1 \longrightarrow 4 \longrightarrow 3 \longrightarrow 5 \longrightarrow 2, & 2 \longrightarrow 1 \longrightarrow 4 \longrightarrow 5 \longrightarrow 3 \longrightarrow 2, \\
2 \longrightarrow 1 \longrightarrow 5 \longrightarrow 3 \longrightarrow 4 \longrightarrow 2, & 2 \longrightarrow 1 \longrightarrow 5 \longrightarrow 4 \longrightarrow 3 \longrightarrow 2.
\end{array}$$

Then, we sequentially insert 7 affinely independent feasible tours that satisfy (C.1) at equality, each using an arc that has not been used before.

$$\tau_{(5,1)^{\mathcal{R}}} = 2 \xrightarrow{\mathcal{Q}} 3 \longrightarrow 5 \longrightarrow 1 \longrightarrow 4 \longrightarrow 2,$$
$$\tau_{(1,2)^{\mathcal{R}}} = 2 \xrightarrow{\mathcal{Q}} 3 \longrightarrow 4 \longrightarrow 5 \longrightarrow 1 \longrightarrow 2,$$
$$\tau_{(4,1)^{\mathcal{R}}} = 2 \xrightarrow{\mathcal{Q}} 3 \longrightarrow 5 \longrightarrow 4 \longrightarrow 1 \longrightarrow 2,$$
$$\tau_{(3,1)^{\mathcal{R}}} = 2 \xrightarrow{\mathcal{Q}} 3 \longrightarrow 1 \longrightarrow 5 \longrightarrow 4 \longrightarrow 2,$$
$$\tau_{(2,5)^{\mathcal{R}}} = 1 \xrightarrow{\mathcal{Q}} 2 \longrightarrow 5 \longrightarrow 4 \longrightarrow 3 \longrightarrow 1,$$
$$\tau_{(2,4)^{\mathcal{R}}} = 1 \xrightarrow{\mathcal{Q}} 2 \longrightarrow 4 \longrightarrow 5 \longrightarrow 3 \longrightarrow 1,$$
$$\tau_{(2,3)^{\mathcal{R}}} = 1 \xrightarrow{\mathcal{Q}} 2 \longrightarrow 3 \longrightarrow 5 \longrightarrow 4 \longrightarrow 1.$$

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