On the Finite Imprimitive Unitary Reflection Groups

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Let \( V = \mathbb{C}^n \), the complex vector space of dimension \( n \) with standard unitary inner product. A reflection in \( V \) is a linear transformation of \( V \) of finite order with exactly \( n - 1 \) eigenvalues equal to 1, and an unitary reflection group \( R \) on \( V \) is a group generated by reflections in \( V \). We say that \( R \) is imprimitive if \( V \) is the direct sum of nontrivial linear subspaces \( V = V_1 \oplus V_2 \oplus \cdots \oplus V_t \), where each \( V_i \) is invariant under \( R \), and \( \{ V_i \mid i = 1, \ldots, t \} \) is called a system of imprimitivity for \( R \).

Let \( \Pi_n \) be the group of all \( n \times n \) complex permutation matrices, and let \( A(m, p, n) \) (\( m, p \) integers, \( p \) dividing \( m \)) be the group of diagonal \( n \times n \) complex matrices whose diagonal elements are powers of some (fixed) primitive \( m \)th root of unity \( \xi \), and whose determinants are \( m/p \)th roots of unity. Then \( \Pi_n \) normalizes \( A(m, p, n) \), and \( G(m, p, n) = A(m, p, n) \Pi_n \) is a semidirect product. In [2, pp. 10–11], it is proved that if \( R \) is any finite imprimitive unitary reflection group with system of imprimitivity \( \{ V_i \mid i = 1, \ldots, t \} \), then \( t = n \geq 2 \), and \( R \) is isomorphic to \( G(m, p, n) \) for some values of \( m, p \).

\( G(m, 1, n) \) is the wreath product \( C_m \wr S_n \) of the cyclic group \( C_m \) of order \( m \) and the symmetric group \( S_n \) on \( n \) elements, and has a presentation

\[
G(m, 1, n) = \langle r_1, \ldots, r_{n-1}, w_1, \ldots, w_n \mid r_i^2 = (r_ir_{i+1})^3 = (w_j)^2 = 1, |i - j| \geq 2, w_i^m = 1, w_iw_j = w_jw_i, r_iw_i = w_{i+1}r_i, r_iw_j = w_jr_i, j \neq i, i + 1 \rangle.
\]

(See [5, p. 616].)

It is sometimes called the generalized symmetric group because we may identify \( r_i \) with the transposition \( (i i + 1) \) and \( w_i \) with the mapping \( (i) \), and thus \( G(m, 1, n) \) is the group permuting the letters \( \{1, \ldots, n\} \) as well as multiplying any number of them by some power of \( \xi \). Thus, the above relations imply that if \( g \in S_n = \langle r_i \mid i = 1, \ldots, n - 1 \rangle \cong \Pi_n \), then \( gw_i g^{-1} = w_{g(i)}, i = 1, \ldots, n \).

Let \( C_m^n \) be the subgroup of \( G(m, 1, n) \) generated by \( \{ w_1, \ldots, w_n \} \). Then \( C_m^n \cong \mathbb{Z}^n \times C_m \triangleleft G(m, 1, n) \). If \( p \) is any integer dividing \( m \), define \( C_{m,p}^n = \mathbb{Z}^n \times C_m \triangleleft G(m, 1, n) \).
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\{ w_1^a w_2^b \cdots w_n^c \mid \sum_{i=1}^n a_i \equiv 0 \pmod{p} \}. It is easy to see that \( C_{m,p}^n \cong A(m, p, n) \), and that \( G(m, p, n) \cong \{ wr | w \in C_{m,p}^n, r \in S_n \} \triangleleft G(m, 1, n) \).

The main aim of this paper is to determine the conjugacy classes and irreducible representations of \( G(m, p, n) \) in terms of those of \( G(m, 1, n) \). ("Representations" will always be taken to mean ordinary representations, that is representations with identity factor set, unless it is stated to the contrary.)

The conjugacy classes are dealt with in Section 2. Our main tool for the problem of the irreducible representations will be Clifford’s theory of inducing from normal subgroups, and the most important results of his work are given in Section 3. We will however, quote without proof other results from Clifford’s work, and the reader is referred to the original paper [4] for the details.

In Sections 4 and 5, we apply this theory to the irreducible representations of \( C_{m,p}^n \), and in Section 6, we look at the case \( p = 1 \); in particular at the irreducible components of the restriction to \( G(m, p, n) \) of the irreducible representations of \( G(m, 1, n) \). Finally, we apply our work in Section 7 to derive some well-known results on the group \( G(2, 2, n) \), which is isomorphic to the Weyl group of type \( D_n \).

The symbol \( \Box \) will be used throughout to denote the end of a proof.

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Let \( \sigma \in G(m, 1, n) \). Then \( \sigma \) may be expressed (uniquely) as the product of disjoint cycles \( \sigma = \theta_1 \cdots \theta_t \), where

\[
\theta_i = \begin{pmatrix}
\xi^{k_{i1}} b_{i1} & b_{i2} & \cdots & b_{it_i} \\
\xi^{k_{i2}} b_{i2} & \xi^{k_{i3}} b_{i3} & \cdots & \xi^{k_{it_i}} b_{it_i}
\end{pmatrix}, \quad b_{ij} \in \{1, \ldots, n\}, \ k_{ij} \in \{1, \ldots, m\}.
\]

When \( \theta_i \) is a cycle of this form, we define \( \text{diag} \theta_i = \begin{pmatrix} b_{i1} & b_{i2} & \cdots & b_{it_i} \end{pmatrix} \).

Let \( f(\theta_i) = \sum_{j=1}^t k_{ij} \), and put \( f(\sigma) = \sum_{i=1}^t f(\theta_i) \). Define \( a_{rs}(\sigma) \) to be the number of cycles \( \theta_r \) of \( \sigma \) of length \( s \) such that \( f(\theta_i) = r \pmod{m} \) for \( 1 \leq r \leq m, 1 \leq s \leq n \). The \( m \times n \) matrix \( (a_{rs}(\sigma)) \) is called the type \( Ty(\sigma) \) of \( \sigma \).

**Lemma 1.** \( \sigma, \sigma_1 \in G(m, 1, n) \) are conjugate in \( G(m, 1, n) \) if and only if \( Ty(\sigma) = Ty(\sigma_1) \).

**Proof.** See [7, p. 44]. \( \Box \)

**Lemma 2.** Let \( \sigma = \theta_1 \cdots \theta_t \in G(m, 1, n) \) be as above. There exists \( \tau \in G(m, 1, n) \) such that

\[
\tau C_{G(m,1,n)}(\sigma)^{-1} = C_{G(m,1,n)}(\tau \sigma \tau^{-1}) = \left\{ \prod_{i=1}^t (\tau \theta_i \tau^{-1})^{a_i} (\text{diag} \tau \theta_i \tau^{-1})^{b_i} \phi | a_i, b_i \in \mathbb{Z} \right\},
\]

where \( \phi \) permutes the sets of symbols in cycles of \( \tau \sigma \tau^{-1} \) of similar type as they stand.
Proof. See [7, p. 55].

**Lemma 3.** Let \( \sigma = \theta_1 \cdots \theta_i \in G(m, 1, n) \) be as above, and let \( \nu \in C_{G(m, 1, n)}(\sigma) \). Then \( f(\nu) \equiv \sum_{i=1}^{t} (a_i f(\theta_i) + b_i l_i) \pmod{m} \), for some \( a_i, b_i \in \mathbb{Z} \).

Proof. \( \nu \in C_{G(m, 1, n)}(\sigma) \) if and only if \( \tau \nu^{-1} \in C_{G(m, 1, n)}(\tau \sigma \tau^{-1}) \), where \( \tau \in G(m, 1, n) \) is the element defined in Lemma 2. Thus \( f(\nu) \equiv f(\tau \nu^{-1})(\mod m) \equiv \sum_{i=1}^{t} (a_i f(\tau \theta_i^{-1}) + b_i l_i) \pmod{m} \) for some \( a_i, b_i \in \mathbb{Z} \) (since \( f(\phi) = 0 \equiv \sum_{i=1}^{t} (a_i f(\theta_i) + b_i l_i) \pmod{m} \)).

**Theorem 1.** Let \( p \) be any integer dividing \( m \), and let \( \sigma = \theta_1 \cdots \theta_i \in G(m, 1, n) \) be as above. Put \( d(s) = (f(\theta_1), \ldots, f(\theta_i), l_1, \ldots, l_i, p) \), where \((\ldots)\) denotes H.C.F. Then \( f(\nu) \equiv s \pmod{p} \) has a solution with \( \nu \in C_{G(m, 1, n)}(\sigma) \) if and only if \( d(s) \) divides \( s \).

Proof. The proof depends on a well-known result from number theory which states that \( a_1 f(\theta_i) + \cdots + a_t f(\theta_i) + b_1 l_1 + \cdots + b_t l_t \equiv s \pmod{p} \) has a solution with \( a_i, b_i \in \mathbb{Z} \) if and only if \( d(s) \) divides \( s \) (see [9, p. 53]).

Let \( \nu \in C_{G(m, 1, n)}(\sigma) \). Then by Lemma 3, \( f(\nu) \equiv \sum_{i=1}^{t} (a_i f(\theta_i) + b_i l_i) \pmod{p} \), for some \( a_i, b_i \in \mathbb{Z} \), and thus \( d(s) \) divides \( f(\nu) \).

Conversely, if \( d(s) \) divides \( s \), choose \( a_i, b_i \in \mathbb{Z} \) such that \( \sum_{i=1}^{t} (a_i f(\theta_i) + b_i l_i) \equiv s \pmod{p} \), and let \( \nu = \prod_{i=1}^{t} \theta_i^{a_i}(\text{diag } \theta_i)^{b_i} \in C_{G(m, 1, n)}(\sigma) \).

**Theorem 2.** Let \( \sigma \in G(m, p, n) \), \( \rho, \tau \in G(m, 1, n) \). Then

(i) \( \rho \sigma^{-1} \sim_{G(m, p, n)} \sigma \) if and only if \( f(\rho) \equiv 0 \pmod{d(\sigma)} \),

(ii) \( \rho \sigma^{-1} \sim_{G(m, p, n)} \tau \sigma^{-1} \) if and only if \( f(\rho) \equiv f(\tau)(\mod d(\sigma)) \).

Proof. (i) We show that \( \rho C_{G(m, 1, n)}(\sigma) \cap G(m, p, n) \neq \emptyset \) if and only if \( f(\rho) \equiv 0 \pmod{d(\sigma)} \), and then apply Lemma 4.

Assume \( f(\rho) = rd(\tau) \) for some \( r \in \mathbb{Z} \). By Theorem 1, there exists \( \nu \in C_{G(m, 1, n)}(\sigma) \) such that \( f(\nu) \equiv -rd(\tau) \pmod{p} \), and thus \( f(\rho \nu) \equiv 0 \pmod{p} \), which implies that \( \rho \nu \in \rho C_{G(m, 1, n)}(\sigma) \cap G(m, p, n) \).

Conversely, let \( \nu \in C_{G(m, 1, n)}(\sigma) \) be such that \( \rho \nu \in G(m, p, n) \). Then \( f(\rho \nu) \equiv 0 \pmod{p} \) and \( f(\nu) \equiv rd(\tau) \pmod{p} \) for some \( r \in \mathbb{Z} \) by Theorem 1. Thus \( f(\rho) \equiv -rd(\tau) \pmod{p} \), and hence \( d(\sigma) \) divides \( f(\rho) \).

(ii) now follows immediately from (i).

Our main result comes as a simple corollary to this theorem.
THEOREM 3. Denote the conjugacy class of $G(m, 1, n)$ containing $\sigma$ by $\operatorname{Ccl}_{G(m, 1, n)}(\sigma)$. Then if $\sigma \in G(m, p, n)$, $\operatorname{Ccl}_{G(m, 1, n)}(\sigma)$ splits into the union of $d(\sigma)$ conjugacy classes in $G(m, p, n)$ with representative elements $\{\rho_i \rho_i^{-1} \mid i = 0, \ldots, d(\sigma) - 1\}$, where $\rho_i = (\lambda e_i, \lambda)$ for $i = 0, \ldots, d(\sigma) - 1$.

At this stage, we give a brief resume of the main results of Clifford's paper on inducing representations from normal subgroups. Full details may be found in [4].

Let $H < G$, and let $T$ be an irreducible representation of $H$. For $x \in H$, $g \in G$, we define $T^g(x) = T(\sigma x) = T(g x g^{-1})$. Then $T^g$ is an irreducible representation of $H$.

DEFINITION 1. Two irreducible representations $T, S$ of $H$ are $G$-conjugate if and only if there exists some $g \in G$ such that $T^g \simeq S$. ($\simeq$ denotes equivalence of representations.) Let $I(T) = \{g \in G \mid T^g \simeq T\}$. Then $H < I(T)$, and $I(T)/H = F(T)$ say. For each $g \in I(T)$, there exists $\hat{T}(g)$ such that

1. $\hat{T}(x) = T(x)$ for all $x \in H$,
2. $T^g(x) = \hat{T}(g) T(x) \hat{T}(g)^{-1}$ for all $g \in I(T), x \in H$,
3. $\hat{T}(g) \hat{T}(g') = \alpha(g, g') \hat{T}(gg')$ for all $g, g' \in I(T)$, where $\alpha: I(T) \times I(T) \to \mathbb{C}$ satisfies $\alpha(xg, x'g') = \alpha(g, g')$ for all $x, x' \in H, g, g' \in I(T)$. $\hat{T}$ is a projective representation of $I(T)$ with factor set $\alpha$, and is called the extension of $T$ to $I(T)$.

THEOREM 4 (Clifford). (i) Let $T$ be an irreducible representation of $H$, and $\hat{T}$ the extension of $T$ to $I(T)$. If $T$ has factor set $\alpha$, and $P$ is any irreducible projective representation of $F(T)$ with factor set $\alpha^{-1}$, then $\hat{T} \otimes P$ (representing $\otimes$ as tensor product) is an irreducible representation of $I(T)$, where $P(g) = P(Hg)$ for all $g \in I(T)$. Further, if $(\ ) \uparrow G$ denotes the representation induced in $G$, then $(\hat{T} \otimes P) \uparrow G$ is an irreducible representation of $G$.

(ii) For fixed $T$, $\{(\hat{T} \otimes P) \uparrow G \mid P \text{ is an irreducible projective representation of } F(T) \text{ with factor set } \alpha^{-1}\}$ is called a set of $H$-associate representations of $G$, and if $T$ runs through a set of representative elements from each class of $G$-conjugate representations of $H$, the union of the corresponding sets of $H$-associate representations of $G$ is a full set of inequivalent irreducible representations of $G$.

(iii) If $S$ is another irreducible representation of $H$, and $Q$ an irreducible projective representation of $F(S)$ with appropriate factor set, then $(\hat{T} \otimes P \uparrow G) \simeq (S \otimes Q) \uparrow G$ if and only if there exists $g \in G$ such that (a) $S \simeq T^g$, and (b) $Q \simeq P^g$ as representation $S$ of $F(S)$. 
We first apply these results to the case where \( H = C_{m,p}^n \), and \( G = G(m, p, n) = C_{m,p}^n S_n \) (semidirect product). The irreducible representations of \( C_{m,p}^n \) are all one-dimensional, and to emphasize this, we will denote them by \( \{ \chi \} \). If \( \chi \) is such a representation, and \( F_\chi(\chi) = \{ g \in S_n \mid \chi^g = \chi \} \); then \( I(\chi) = C_{m,p}^n F_\chi(\chi) \) (semidirect product), and thus \( F(\chi) = I(\chi)/C_{m,p}^n \cong F_\chi(\chi) \).

In this case, \( x \) is easily constructed by defining \( \chi(wg) = \chi(w) \) for all \( w \in C_{m,p}^n \), \( g \in F_\chi(\chi) \). Then \( \chi(wgw'g') = \chi(wgwg') = \chi(gw\chi) = \chi(g) \chi(w) \chi(w') \) (since \( g \in F_\chi(\chi) \), \( = \chi(gw) \chi(wg) \chi(g') \), and thus \( \chi \) is an ordinary representation of \( I(\chi) \) with identity factor set.

In the next two paragraphs, we shall consider the structure of \( F_\chi(\chi) \), and determine its irreducible representations.

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Let \( \chi = \chi_1 \cdots \chi_n \), \( \psi = \psi_1 \cdots \psi_n \) be irreducible representations of \( C_{m,n}^n \). We say \( \chi \sim \psi \) if and only if

1. \( \chi(\psi_i^{-1}) = \psi(\chi_i^{-1}) \) for all \( i, j = 1, \ldots, n \), and
2. \( \chi_i \psi_i L_i^{-1} = 1 \) for \( i = 1, \ldots, n \), where \( \chi_i = \chi(w_i) = \chi_i(w_i) \) for \( i = 1, \ldots, n \).

**Lemma 5.** \( \sim \) is an equivalence relation on the set of irreducible representations of \( C_{m,p}^n \), and the equivalence classes \( E_\sim(\chi) \) correspond to representations of \( C_{m,n}^n \) which give the same representation of \( C_{m,p}^n \).

**Proof.** It is trivial to prove that \( \sim \) is an equivalence relation. \( C_{m,p}^n \) is generated by \( \{ w_i w_i^{-1}, w_i \mid i, j = 1, \ldots, n \} \), and thus \( \chi \) and \( \psi \) are identical on \( C_{m,p}^n \) if and only if

1. \( \chi(w_i w_i^{-1}) = \psi(w_i w_i^{-1}) \) for all \( i, j = 1, \ldots, n \), and
2. \( \chi(w_i) = \psi(w_i) \) for all \( i = 1, \ldots, n \), which is precisely the condition for \( \chi \) and \( \psi \) to be equivalent under \( \sim \).

Thus the irreducible representations of \( C_{m,p}^n \) may be derived by restricting to \( C_{m,p}^n \) one irreducible representation of \( C_{m,n}^n \) from each \( E_\sim(\chi) \).

Let \( \chi = \chi_1 \cdots \chi_n \) be an irreducible representation of \( C_{m,n}^n \), and let \( g \in S_n \). We want to know when \( \chi^g = \chi \) as representations of \( C_{m,p}^n \). This will be so if and only if

1. \( \chi^g(w_i w_i^{-1}) = \chi(w_i w_i^{-1}) \) for all \( i, j = 1, \ldots, n \), and
2. \( \chi^g(w_i) = \chi(w_i) \) for all \( i = 1, \ldots, n \), that is if and only if
   1. \( \chi_{\sigma(i)} \chi_i L_i^{-1} = \chi_{\sigma(i)} L_i^{-1} \), and
   2. \( (\chi_{\sigma(i)} L_i^{-1})^i = 1 \) for all \( i, j = 1, \ldots, n \).
Let $g, g' \in F_p(\chi)$. Then there exist $\theta, \theta' \in C_p$ (the group of $p$th roots of 1) such that $\bar{\chi}_0(i) = \theta, \bar{\chi}'_0(i) = \theta'$ for all $i = 1, \ldots, n$. Thus $\bar{\chi}_0'(i) = \theta_0 \bar{\chi}'_0(i) = \theta_0 \theta'$ for all $g, g' \in F_p(\chi)$.

Let $F_p(\chi) = \{g \in F_p(\chi) | \theta_0 = 1\}$. Then $F_p(\chi) \triangleleft F_p(\chi)$, and $F_p(\chi)/\Theta(\chi) \cong \Theta(\chi) = \{g \in F_p(\chi) | \theta_0 = 1\}$. At this point, it is useful to introduce a characterization of the group $F_p(\chi)$.

**Definition 2.** Let $\chi = \chi_1 \cdots \chi_n$ be an irreducible representation of $C_m^n$, and let $\mathcal{P}_i = \{j \mid \chi_j = \xi^i\}$, for $i = 1, \ldots, m$. (It is of course possible that $\mathcal{P}_i = \emptyset$ for some $i$.) We say that $\chi$ is of type $[t_1, \ldots, t_n]$ if $t_i = |\mathcal{P}_i|$, $i = 1, \ldots, m$, and we will write $\mathcal{P}_i = \{n_{i1}, \ldots, n_{in}\}$.

From this definition, we see that if $\chi$ is of type $[t_1, \ldots, t_n]$, then $F_p(\chi) = S(1) \times S(2) \times \cdots \times S(m)$, where $S(i)$ is the (symmetric) group permuting the $t_i$ indices in $\mathcal{P}_i$. Clearly $F_p(\chi)$ is independent of $p$, and we will merely write $F(\chi)$.

**Lemma 6.** Let $k \in \{1, \ldots, m\}$. There exists $g \in F_p(\chi)$ such that $\theta_0 = \xi^k$ if and only if

(i) $q$ divides $k$, where $q = m/p$, and

(ii) $t_i = t_{i+k}$, $i = 1, \ldots, m$, where $j$ denotes the residue of $j$ (mod $m$).

**Proof.** Assume there exists $g \in F_p(\chi)$ such that $\theta_0 = \xi^k$. Then $1 = \theta_0^p = \xi^{pk}$, and thus $q$ divides $k$. Let $j \in \mathcal{P}_i$. Then $\bar{\chi}_j = \xi^i$, and $\bar{\chi}_0(i) = \xi^k \bar{\chi}_j = \xi^{k+i}$, and thus $g(j) \in \mathcal{P}_{k+i}$. Hence $g(\mathcal{P}_i) = \mathcal{P}_{k+i}$, and $t_i = t_{i+k}$, $i = 1, \ldots, m$.

Conversely, assume $q$ divides $k$ and $t_i = t_{i+k}$ for $i = 1, \ldots, m$. Define $h \in S_n$ by $h(n_{ij}) = n_{i+k, j}$ for $j = 1, \ldots, t_i$, $i = 1, \ldots, m$. Then $\bar{\chi}_h(n_{ij}) = \bar{\chi}_{n_{i+k, j}} = \xi^{i+k} = \xi^k \bar{\chi}_j$ for all $i = 1, \ldots, n$. Put $\theta_h = \xi^k$. Then $\bar{\chi}_h = \theta_h \bar{\chi}_h$, $i = 1, \ldots, n$, and as $g$ divides $k$, $\theta_0^p = 1$. Thus $h \in F_p(\chi)$. □

**Theorem 5.** Let $k$ be the least positive integer such that

(i) $q$ divides $k$, and

(ii) $t_i = t_{i+k}$ for all $i = 1, \ldots, m$, and let $h$ be defined as in Lemma 6 for this value of $k$. Then $\theta_h$ generates $\Theta(\chi)$. Further, $k$ divides $m$, and thus, $F_p(\chi) = \bigcup_{i=0}^{l-1} F(\chi) h^i$, where $l = m/k$.

**Proof.** If $g \in F_p(\chi)$, write $\theta_0 = \xi^{a_i}$, $1 \leq a_i \leq m$. Since $\Theta(\chi)$ is cyclic, it is generated by $\theta_0$ when $a_0$ takes least positive value, and hence by Lemma 6, when $a_0$ is the least positive integer such that $q$ divides $a_0$, and $t_i = t_{i+k}$ for all $i = 1, \ldots, m$. Thus $\Theta(\chi)$ is generated by $\theta_h$.

Let $m = ak + b$, where $a$ is a positive integer, and $0 \leq b < k$. Then $t_{i+b} = t_{i+m-ak} = t_i$ for all $i = 1, \ldots, m$, and by the minimality of $k$, $b = 0$. Thus $k$ divides $m$ and $|\Theta(\chi)| = m/k = l$, which gives the result.
Henceforth, $h$ will always denote the element of $F_p(\chi)$ defined in this way.

We now turn our attention to deriving the conjugates of a given representation $\chi$.

**Theorem 6.** Let $\chi, \psi$ be irreducible representations of $C_n^m$. $\chi \downarrow C_{n,p}$ and $\psi \downarrow C_{n,p}$ are $G(m, p, n)$-conjugate representations if and only if there exists $\mu \in E_\sim(\psi)$ such that type $\mu = \text{type} \chi$. ($\downarrow$ denotes restriction.)

**Proof.** Assume that $\chi \downarrow C_{n,p}$ and $\psi \downarrow C_{n,p}$ are $G(m, p, n)$-conjugate representations. Then there exists $g \in S_n$ such that $\psi = \chi \circ$ on $C_{n,p}$, and thus $\chi \circ \in E_\sim(\psi)$ is such that type $\chi \circ = \text{type} \chi$. On the other hand, if there exists some $\mu \in E_\sim(\psi)$ such that type $\mu = \text{type} \chi$, we may choose $g \in S_n$ such that $\chi \circ \mu = \mu$ as representations of $C_n^m$, and thus $\chi \downarrow C_{n,p}$ and $\psi \downarrow C_{n,p}$ are $G(m, p, n)$-conjugate representations.

We are now faced with the problem of determining the irreducible representations of $F_p(\chi)$. Our method will be another application of Clifford's theory, and we will use the notation already developed without further reference. Thus $F_p(\chi) = \bigcup_{i=0}^{l-1} \overline{F(\chi)} h^i$, where $\overline{F(\chi)} = S(1) \times \cdots \times S(m)$, and $h \in S_n$ is defined by $h(n_{ij}) = n(i+k)j$ for $j = 1, \ldots, t_i$, $i = 1, \ldots, m$.

If $x \in S(i)$, $h^i x h^{-i} \in S(i+rk)$, and the map $(h^r)^* : S(i) \rightarrow S(i+rk)$ given by $(h^r)^* (x) = h^r x h^{-r}$ is an isomorphism. Thus any $x \in S(i)$ may be written (uniquely) as the product $x = x_1 x_2 \cdots x_k h_{x_1} h_{x_2} \cdots h_{x_k} h_{x_{i-1}} \cdots h_{x_{i-1}} x_{i+rk}$, where $x_i \in S(i)$, and hence $h_{x_i} x_i \in S(i+rk)$ for $i = 1, \ldots, k$, $r = 0, \ldots, l - 1$. Similarly, if $\{T_{i,r} | j = 1, \ldots, a_i\}$ is a full set of inequivalent irreducible representations of $S(i)$, $i = 1, \ldots, k$, then a full set of inequivalent irreducible representations of $S(i+rk)$, $r = 0, \ldots, l - 1$ is given by $\{T_{i,r} h^{i-r} h_{x_i} x_i | j = 1, \ldots, a_i\}$, where $T_{i,r} h^{i-r} h_{x_i} x_i \rightarrow T_{i,r} h_{x_i} x_i$. Thus we have proved:

**Theorem 7.** Let $x \in \overline{F(\chi)}$ be as above. A full set of inequivalent irreducible representations of $\overline{F(\chi)}$ is given by

$$\{P(x) = P_{i_1}(x_{i_1}) \otimes \cdots \otimes P_{k_1}(x_{i_1}) \otimes \cdots \otimes P_{i_{l-1}}(x_{i_{l-1}}) \otimes \cdots \otimes P_{k_{l-1}}(x_{k_{l-1}}), \}
$$

where $P_{i_1} \in \{T_{i_1} | j = 1, \ldots, a_i\}$, $i = 1, \ldots, k$, $r = 0, \ldots, l - 1$. When $P$ is of this form, we write $P = (P_{i_1} | i = 1, \ldots, k, r = 0, \ldots, l - 1)$, or simply $P = (P_{i_1})$.

**Lemma 7.** Let $P = (P_{i_1})$ be an irreducible representation of $\overline{F(\chi)}$. Then $P h^j \simeq P$ (as representations of $\overline{F(\chi)}$) if and only if $P_{i_1} = P_{i_1 + j}$ for all $i = 1, \ldots, k$, $r = 0, \ldots, l - 1$, where $j$ denotes the residue of $j$ (mod $l$).
Proof. Let $\chi$ be the character of $P$, and $\chi_i$ the character of $P_i$. Then $\chi^{t^b} = \chi$ implies that $\chi^{t^b}(x_i) = \chi(x_i)$ for all $x_i \in S(i)$, $i = 1, \ldots, k$, $r = 0, \ldots, l - 1$. In other words, $\chi_i(x_i^{t^b}) = \chi_i(x_i)$ for all $x_i \in S(i)$, which means that $\chi_i^{t^b} = \chi_i$ for all $i = 1, \ldots, k$, $r = 0, \ldots, l - 1$. However, since $\{x_i^{t^b} | x_i \in S(i), i = 1, \ldots, k, r = 0, \ldots, l - 1\}$ generates $F(\chi)$, we see that this condition is sufficient to imply that $\chi^{t^b}(x) = \chi(x)$ for all $x \in F(\chi)$, and hence that $P_i^{t^b} \simeq P$. 

The following two results now come as simple corollaries to this lemma.

**Theorem 8.** Two irreducible representations $P = (P_i)$ and $Q = (Q_i)$ of $\bar{F}(\chi)$ are $F(\chi)$-conjugate if and only if there exists some integer $s$ such that $P_i = Q_i^{t^s}$ for all $i = 1, \ldots, k$, $r = 0, \ldots, l - 1$.

**Proof.** This is immediate from Lemma 7.

**Theorem 9.** Let $P = (P_i)$ be an irreducible representation of $\bar{F}(\chi)$, and let $b$ be the least positive integer such that $P_i = P_i^{t^b}$ for all $i = 1, \ldots, k$, $r = 0, \ldots, l - 1$. Then $b$ divides $l$ and $I(P) = \{x \in F_p(\chi) | P^x \simeq P\} = \bigcup_{c=0}^{l-1} F(\chi) h^{t^c}$, where $c = l/b$.

**Proof.** The proof follows from Lemma 7, arguing as in the proof of Theorem 5.

We now determine the extension $\bar{P}$ to $I(P)$ of the representation $P = (P_i)$ of $\bar{F}(\chi)$. (We use the notation $\bar{P}$ to prevent any confusion with the work of Section 4.) Let $x = x_1 x_2 \cdots x_k \cdots h^{b-1} x_k^{-1} \cdots h^{b-1} x_k^{-1}$, and let $b$ be as in Theorem 9. Then

$$P(x) = (P_1(x_k^{-1}) \otimes \cdots \otimes P_k(x_k^{-1}) \otimes \cdots \otimes P_{k-1}(x_k^{-1}) \otimes \cdots \otimes P_{k-2}(x_k^{-1}) \otimes \cdots \otimes P_{k-1}(x_k^{-1})) \otimes \cdots \otimes (P_{1-1}(x_k^{-1}) \otimes \cdots \otimes P_k(x_k^{-1}) \otimes \cdots \otimes P_1(x_k^{-1}) \otimes \cdots \otimes P_{k-1}(x_k^{-1})),$$

and

$$P_i^{t^b}(x) = P_i^{(t^b)x} = P_i(x_k^{-1}) \otimes \cdots \otimes P_k(x_k^{-1}) \otimes \cdots \otimes P_{k-1}(x_k^{-1}) \otimes \cdots \otimes P_{k-2}(x_k^{-1}) \otimes \cdots \otimes P_{k-1}(x_k^{-1}) \otimes \cdots \otimes P_{1-1}(x_k^{-1}) \otimes \cdots \otimes P_k(x_k^{-1}) \otimes \cdots \otimes P_1(x_k^{-1}) \otimes \cdots \otimes P_{k-1}(x_k^{-1}),$$

since $P_i = P_i^{t^b}$ for all $i = 1, \ldots, k$, $r = 0, \ldots, l - 1$.

Let $t = \prod_{i=1}^k \prod_{r=0}^{l-1} (\deg P_i)$, and let $E_{rs}$ be the $t \times t$ matrix with $1$ in the $rs$th position, and zero elsewhere. Put $M = \sum_{a=1}^d \sum_{r=1}^t E_{a_1 a_2} \otimes E_{a_3 a_4} \otimes \cdots \otimes E_{a_d a_1}$. Then using the fact that $E_{rs} E_{nt} = \delta_{su} E_{rn}$, we may easily prove:
**Lemma 8.** (i) \( M_s = \sum_{a_1, \ldots, a_s=1}^t E_{a_1} \otimes \cdots \otimes E_{a_s}, \) for \( s = 0, 1, 2, \ldots \).

(ii) \( M^0 \) is the identity matrix.

(iii) \( M(A \otimes I \otimes \cdots \otimes I)M^{-1} = I \otimes A \otimes I \otimes \cdots \otimes I \)

\( M(I \otimes A \otimes I \otimes \cdots \otimes I)M^{-1} = I \otimes I \otimes A \otimes I \otimes \cdots \otimes I \)

for any \( t \times t \) matrix \( A \),

\( M(I \otimes \cdots \otimes I \otimes A)M^{-1} = A \otimes I \otimes \cdots \otimes I \)

where \( I \) is the identity \( t \times t \) matrix.

Put \( \bar{P}(h^s) = M_s, s = 0, \ldots, c - 1, \) and \( \bar{P}(xh^s) = P(x) M_s, x \in \bar{F}(\chi) \). Then Lemma 8 implies that \( \bar{P}(g) P(x) \bar{P}(g)^{-1} = P^g(x) \) for all \( x \in \bar{F}(\chi), g \in I(P) \), and that \( \bar{P} \) is an ordinary representation of \( I(P) \) with identity factor set. Since \( F_p(\chi) / \bar{F}(\chi) \) is a cyclic group, the application of Clifford's theory becomes especially simple, and we have the following result.

**Lemma 9.** Let \( \zeta \) be a primitive \( l \)th root of 1. We define

\( P_{(\zeta)}(xh^i) = \zeta^i (\bar{P} \uparrow F_p(\chi))(x) \) for all \( x \in \bar{F}(\chi), i = 1, \ldots, \ell. \)

Then \( \{ P_{(\zeta)} | j = 1, \ldots, c \} \) is a full set of inequivalent irreducible representations of \( F_p(\chi) \) which are \( \bar{F}(\chi) \)-associate to \( \bar{P} \uparrow F_p(\chi) \).

**Proof.** See [4, pp. 547–548].

Thus, by letting \( \bar{P} \) run through a set of representative elements of the \( F_p(\chi) \)-conjugate representations of \( \bar{F}(\chi) \) (which we know by Theorem 8), we get a full set of inequivalent irreducible representations of \( F_p(\chi) \) (see Theorem 4 (ii)).

Applying the results of Section 5 to the case \( p = 1 \), we get the following result, which is proved in [7, pp. 89–95].

**Lemma 10.** Let \( \chi \) and \( \psi \) be irreducible representations of \( C_m^* \). Then

(i) \( F_1(\chi) = \bar{F}(\chi) \),

(ii) \( \chi \) and \( \psi \) are \( G(m, 1, n) \)-conjugate if and only if type \( \chi = \text{type } \psi \).

**Proof.** Since \( p = 1, q = m \), and thus \( k = m \) by Theorem 5, which proves (i). The fact that two irreducible representations of \( C_m^* \) are \( G(m, 1, n) \)-conjugate if and only if they have the same type follows from Theorem 6 and the fact that in this case, \( E_\infty(\chi) = \{ \chi \} \) for all irreducible representations \( \chi \) of \( C_m^* \).
This result now leads us naturally to our main consideration: What are the irreducible components of the restriction to $G(m, p, n)$ of a given irreducible representation of $G(m, 1, n)$, where $p$ is any integer dividing $m$? Before giving a solution to this problem, we prove some useful lemmas.

**Lemma 11.** Let $p$ divide $m$, and let $\chi$ be an irreducible representation of $C_m^n$. Denote the extension of $\chi$ to $C_m^n, \bar{F}(\chi)$ by $\tilde{\chi}$, and the extension of $\chi$ to $C_m^n, \bar{F}(\chi)$ by $\chi'$. If $P$ is an irreducible representation of $\bar{F}(\chi)$ with character $\lambda$, let $\lambda$ be the character of the representation $P$ of $C_m^n, \bar{F}(\chi)$. Then

1. $\chi' = \tilde{\chi}$ on $C_m^n, \bar{F}(\chi) \cap C_m^n, \bar{F}(\chi) = C_m^n, \bar{F}(\chi)$,
2. $(\chi', \lambda) \uparrow G(m, 1, n) \downarrow G(m, p, n) = (\tilde{\chi}(\lambda) \downarrow C_m^n, \bar{F}(\chi)) \uparrow G(m, p, n)$

**Proof.** (i) follows from the definitions of $\chi'$ and $\tilde{\chi}$.

(ii) $((\chi', \lambda) \downarrow G(m, 1, n)) \downarrow G(m, p, n) = ((\chi, \lambda) \downarrow C_m^n, \bar{F}(\chi)) \uparrow G(m, p, n)$ by Mackey's subgroup theorem, see [3, p. 221,]

$$= \sum_{\chi \in \bar{F}(\chi)} (\chi, \lambda) \downarrow \chi \uparrow G(m, p, n)$$

**Lemma 12.** Let $\{P_{(i)} \mid i = 1, \ldots, c\}$ be the irreducible representations of $F_p(\chi)$ described in Lemma 9, and let $\lambda_{(i)}(w)$ be the character of the representation $P_{(i)}(w)$ defined by $\lambda_{(i)}(w) = P_{(i)}(w)$ for all $w \in C_m, \bar{F}(\chi)$, $i = 1, \ldots, c$. Then $(\lambda \downarrow C_m^n, \bar{F}(\chi)) \uparrow C_m^n, \bar{F}(\chi) = \sum_{i=1}^c \lambda_{(i)}(w)$.

**Proof.** Let $w \in C_m, \bar{F}(\chi)$. Then

$$(\lambda \downarrow C_m^n, \bar{F}(\chi)) \uparrow C_m^n, \bar{F}(\chi) = \sum_{\chi \in \bar{F}(\chi)} \lambda \downarrow \chi \uparrow C_m^n, \bar{F}(\chi)$$

Now $\lambda$ is an irreducible component of the restriction to $\bar{F}(\chi)$ of some irreducible character $\nu$ of $F_p(\chi)$ if and only if $\nu$ is an $\bar{F}(\chi)$-associate of $(\lambda \uparrow F_p(\chi))$, where $\bar{\lambda}$ is the character of the representation $\bar{F}$ of $I(P)$ defined in Section 5, and in this case, $\lambda$ always appears with multiplicity 1 (see [4, p. 547]). Thus by the Frobenius
reciprocity theorem, the irreducible components of \((\lambda \uparrow F_{\varphi}(\chi))\) are precisely the \(\overline{F}(\chi)\)-associates of \((\lambda \uparrow F_{\varphi}(\chi))\), each appearing once and once only. The result now follows by Lemma 9. \(\square\)

The above two lemmas lead to our main theorem, which is enunciated in terms of representations.

**Theorem 10.** Let the notation be as in Lemmas 11 and 12 above. Then

(i) The irreducible components of \(\left( (\chi' \otimes P) \uparrow G(m, 1, n) \right) \downarrow G(m, p, n)\) are \(\left\{ (\chi \otimes P_{\varphi}(\psi)) \uparrow G(m, p, n) \right\}_{i = 1, \ldots, c} \), where each component appears with multiplicity 1. Thus in particular, \(\left( (\chi' \otimes P) \uparrow G(m, 1, n) \right) \downarrow G(m, p, n)\) is irreducible if and only if \(I(P) = \overline{F}(\chi)\).

(ii) If \(\psi\) is another irreducible representation of \(C_m^n\), and \(Q\) an irreducible representation of \(\overline{F}(\psi)\), then

\[
\left( (\chi' \otimes P) \uparrow G(m, 1, n) \right) \downarrow G(m, p, n) \cong \left( (\psi' \otimes Q) \uparrow G(m, 1, n) \right) \downarrow G(m, p, n)
\]

if and only if there exists \(g \in S_n\) such that

(a) \(\psi = \chi^g\) as representations of \(C_m^n\), that is \(\psi \downarrow C_m^n\) and \(\chi \downarrow C_m^n\) are \(G(m, p, n)\)-conjugate, and

(b) \(Q \cong P^g\) as representations of \(\overline{F}(\psi)\), where \(P^g(x) = P(gxg^{-1})\) for all \(x \in \overline{F}(\psi)\).

**Proof.** (i) follows from Lemmas 11 and 12. To prove (ii), we use a result of Karkar and Green ([6, p. 132]) which implies that (ii) holds if and only if there exists a one-dimensional representation \(\eta\) of \(G(m, 1, n)\) which is the identity on \(G(m, p, n)\), and is such that \(\left( (\chi' \otimes P) \uparrow G(m, 1, n) \right) \sim \left( (\psi' \otimes Q) \uparrow G(m, 1, n) \right) \eta\). Assume such an \(\eta\) exists.

Let \(\eta_t\) be the restriction of \(\eta\) to \(C_m^n\). Then \(\eta_t = (\eta_1)_1 \eta_2 \cdots (\eta_n)_n\) satisfies \((\eta_a)_i = \xi^{a_1}(a \in \{1, \ldots, p\})\) for all \(i = 1, \ldots, n\) (in the notation of Lemma 5). Thus \(\overline{F}(\eta_t \psi) = \overline{F}(\psi),\) and \(\psi(\eta_1, \ldots, \eta_n) = (\psi \eta_t)'\).

Now \(\left( (\psi' \otimes Q) \uparrow G(m, 1, n) \right) \eta \cong \left( (\psi' \otimes Q)(\eta \downarrow C_m^n F(\chi)) \right) \uparrow G(m, 1, n)\) (since \(\eta\) is a character of \(G(m, 1, n)\)), and thus (ii) holds if and only if there exists an \(\eta\) satisfying the above, such that \(\left( (\chi' \otimes P) \uparrow G(m, 1, n) \right) \sim \left( (\psi \eta_t)' \otimes Q \right) \uparrow G(m, 1, n)\).

By Theorem 4(iii) this is true if and only if there exists such an \(\eta\), and \(g \in S_n\) such that

(a) \(\psi \eta_t = \chi^g\) (as representations of \(C_m^n\)), and

(b) \(Q \cong P^g\).

However, it is clear that (a) holds if and only if \(\psi = \chi^g\) on \(C_m^n\), which proves the result.
We now apply these results to the groups $G(2,1,n)$ and $G(2,2,n)$, which are respectively isomorphic to the Weyl groups of types $B_n$ and $D_n$ (see [2, p. 12]).

We shall use the notation developed above.

**Definition 3.** Let $\theta$ be a cycle in $G(2,1,n)$. We say that $\theta$ is **positive** if $f(\theta)$ is even, and **negative** if $f(\theta)$ is odd.

**Theorem 11.** Let $\sigma = \theta_1 \ldots \theta_i \in G(2,2,n)$. Then $\text{Ccl}_{G(2,2,n)}(\sigma)$ forms a complete conjugacy class in $G(2,2,n)$, except in the case where all $\theta_i$ are positive and of even length, when it is the union of two conjugacy classes in $G(2,2,n)$ which have representative elements $\sigma$ and $\rho \sigma \rho^{-1}$, where $\rho = (1)$.

**Proof.** The proof follows from Theorem 1 and the fact that $d(\sigma) = (f(\theta_1), \ldots, f(\theta_i), 1, \ldots, 1, J, 2) = 1$ except when all $\theta_i$ are positive and of even length, when $d(\sigma) = 2$. See [1, p. 26] for an alternative derivation of this result.

The irreducible representations of $G(2,2,n)$ are also easily determined.

**Lemma 13.** Let $\chi = \chi_1 \ldots \chi_n$ and $\psi = \psi_1 \ldots \psi_n$ be irreducible representations of $C_{2n}$ of type $[t_1, t_2]$ and $[t_1', t_2']$, respectively. Then

(i) $\chi$ and $\psi$ are $G(2,2,n)$-conjugate representations of $C_{2n}$ if and only if either $t_1 = t_1'$ and $t_2 = t_2'$ or $t_1 = t_2'$ and $t_2 = t_1'$.

(ii) $\chi$ and $\psi$ are $G(2,1,n)$-conjugate representations of $C_{2n}$ if and only if $t_1 = t_1'$ and $t_2 = t_2'$.

**Proof.** (i) Define a representation $\chi^*$ of $C_{2n}$ by setting $\chi^*_i = -\chi_i$ for $i = 1, \ldots, n$. Then type $\chi^* = [t_1', t_2']$, and as $E_{\infty}(\chi) = \{\chi, \chi^*\}$, the result follows from Lemma 5. (ii) follows immediately from Lemma 10.

Let $i \in \{1, \ldots, n\}$. Define a representation $\chi^{(i)} = \chi_1^{(i)} \ldots \chi_n^{(i)}$ of $C_{2n}$ by setting $\chi_j^{(i)} = -1$ if $j \in \{1, \ldots, i\}$, and $\chi_j^{(i)} = -1$ if $j \in \{i+1, \ldots, n\}$. Then $F(\chi^{(i)}) = S_{[1,\ldots,i]} \times S_{[i+1,\ldots,n]}$, where $S_{\Omega}$ denotes the symmetric group on the set $\Omega$. Further, it is easy to see from Lemma 13(ii) that $\{\chi^{(i)} \mid i = 1, \ldots, n\}$ is a full set of representative elements from each class of $G(2,1,n)$-conjugate representations of $C_{2n}$.

Let $n$ be odd. Then it is also easy to see from Lemma 13(i) that $\{\chi^{(i)} \mid i = 1, \ldots, \frac{1}{2}(n+1)\}$ is a full set of representative elements from each class of $G(2,2,n)$-conjugate representations of $C_{2n}$. Further, since $n$ is odd, $k$ must $=2$, and thus $F_2(\chi^{(i)}) = F(\chi^{(i)})$, $i = 1, \ldots, \frac{1}{2}(n+1)$. Hence, the irreducible representations of $F_2(\chi^{(i)})$ are $\{P = (P_1, P_2)\}$, where $P_1$ and $P_2$ are irreducible representations of $S_{[1,\ldots,i]}$ and $S_{[i+1,\ldots,n]}$, respectively, $i = 1, \ldots, \frac{1}{2}(n+1)$. 


Theorem 12. Let \( n \) be odd, and let \( P \) be an irreducible representation of \( \bar{F}(\chi^{(i)}) \) and \( Q \) an irreducible representation of \( \bar{F}(\chi^{(j)}) \), where \( i < j \in \{1, \ldots, n\} \). Then

(i) \( ((\chi^{(i)}') \otimes P) \uparrow G(2, 1, n) \downarrow G(2, 2, n) \) is irreducible, and

(ii) \( ((\chi^{(j)}') \otimes Q) \uparrow G(2, 1, n) \downarrow G(2, 2, n) \) if and only if \( j = n - i \); and \( P^* \simeq Q \), where

\[
g = \begin{pmatrix}
1 & 2 & \cdots & i & i+1 & i+2 & \cdots & n \\
 j+1 & j+2 & \cdots & j & i+1 & i+2 & \cdots & j \\
\end{pmatrix}.
\]

Proof. (i) follows immediately from Theorem 10, and (ii) from Theorem 10 and Lemma 13(i). It is easy to check that the conditions of Theorem 10(ii) hold in this case when \( g \) is of the above form.

Let \( n \) be even. In this case, Lemma 13(i) implies that \( \{\chi^{(i)} | i = 1, \ldots, \frac{1}{2} n\} \) is a full set of representative elements from each class of \( G(2, 2, n) \)-conjugate representations of \( C_{2n} \).

Theorem 13. Let \( n \) be even. Then

(i) If \( i \neq \frac{1}{2} n \), \( F_d(\chi^{(i)}) = \bar{F}(\chi^{(i)}) \), and if \( P = (P_1, P_2) \) is an irreducible representation of \( \bar{F}(\chi^{(i)}) \) (here \( k = 2 \)), then

(a) \( ((\chi^{(i)}') \otimes P) \uparrow G(2, 1, n) \downarrow G(2, 2, n) \) is irreducible, and

(b) If \( j > i \), and \( Q \) is an irreducible representation of \( \bar{F}(\chi^{(j)}) \), then \( ((\chi^{(j)}') \otimes Q) \uparrow G(2, 1, n) \downarrow G(2, 2, n) \) if and only if \( j = n - 1 \) and \( P^* \simeq Q \), where \( g \) is defined as in Theorem 12(ii).

(ii) Let \( P = (P_1, P_2) \) be an irreducible representation of \( \bar{F}(\chi^{(1/2)n}) \) (here \( k = 1 \)).

Then (a) If \( P_1 \neq P_2 \), \( I(P) = \bar{F}(\chi^{(1/2)n}) \) and thus \( ((\chi^{(1/2)n})' \otimes P) \uparrow G(2, 1, n) \downarrow G(2, 2, n) \) is irreducible, and

(b) If \( P_1 = P_2 \), \( I(P) = F_d(\chi^{(1/2)n}) = \bar{F}(\chi^{(1/2)n}) \cup \bar{F}(\chi^{(1/2)n}) h \), where \( h = (1 + 1)(2 + 2) \cdots (\frac{1}{2} n n) \), and

\[
((\chi^{(1/2)n})' \otimes P) \uparrow G(2, 1, n) \downarrow G(2, 2, n) \simeq \begin{pmatrix}
\chi^{(1/2)n} \otimes P_0 & 0 \\
0 & \chi^{(1/2)n} \otimes P_{-1} \\
\end{pmatrix}.
\]

Proof. The proof follows along similar lines to the proof of Theorem 12, and is omitted.


References