Decoding Cardano’s Liber de Ludo Aleae

David Bellhouse

Department of Statistical and Actuarial Sciences, University of Western Ontario, London, Ontario, Canada N6A 5B7
Available online 4 June 2004

Abstract

Written in the 16th century, Cardano’s Liber de Ludo Aleae was, in its time, an advanced treatment of the probability calculus. At the same time it could be viewed as a gambling manual. Several commentators on the book have concluded that it is a mishmash of several, sometimes contradictory, results and statements written over an approximate 40-year period. In the current paper the Liber de Ludo Aleae is examined as a Renaissance text written in the intellectual milieu of humanism. A close examination of the book shows that it was heavily influenced by Aristotle’s Ethics, in particular Aristotle’s concept of justice. By reading the Liber de Ludo Aleae in this way, it is shown that there is an internal consistency to the work with a common thread of justice (ius) and knowledge (scientia) running through it. These themes are examined in detail. It is also argued that some of Cardano’s probability calculations related to dice might have been taken from a printed version of the late medieval poem De Vetula.

© 2004 Elsevier Inc. All rights reserved.

Résumé

Écrit au seizième siècle, le Liber de Ludo Aleae de Cardano était à l’époque un traitement avancé des calculs de probabilités. De plus, ce livre pouvait être vu comme un manuel sur les jeux de chance. Plusieurs commentateurs ont conclu que ce livre était un pot-pourri de plusieurs résultats, parfois contradictoires, et constatations écrits sur une période de près de quarante ans. Dans cet article, le Liber de Ludo Aleae est examiné comme un texte de la Renaissance, écrit dans le milieu intellectuel de l’humanisme. En examinant de près le livre, on voit qu’il était fortement influencé par l’Ethics d’Aristote, en particulier du concept de justice d’Aristote. Une telle lecture de Liber de Ludo Aleae révèle une consistance interne avec l’œuvre tenu ensemble par des thèmes courants de justice (ius) et connaissance (scientia). Ces thèmes sont examinés en détail. Des arguments sont aussi présentés qui suggèrent que certains calculs de probabilités de Cardano au sujet de dés auraient été tirés d’une version écrite du poème médiéval De Vetula.

© 2004 Elsevier Inc. All rights reserved.

MSC: 01A40; 60-03

E-mail address: bellhouse@stats.uwo.ca.

0315-0860/$ – see front matter © 2004 Elsevier Inc. All rights reserved.
doi:10.1016/j.hm.2004.04.001
1. Introduction

Until the publication in the mid-17th century of Christiaan Huygens’s highly influential work on probability *De Ratiociniis in Ludo Aleae* [Huygens, 1657], Girolamo Cardano’s *Liber de Ludo Aleae*, written in the 16th century [Cardano, 1953, 1966], was the most complete treatment of the probability calculus. Among all the aleatoric calculations prior to the 1650s, it is only in Cardano’s work that we find a discussion of the actual calculations and the assumptions behind them. Cardano calculated probabilities for the sum of the faces that show on two and three dice as well as some simple calculations related to card games of the time. He also provided a simple version of the multiplication rule for probabilities using dice. In one sense Cardano went beyond Huygens, who devoted himself only to the calculation of chances, although he did use new concepts such as expectation. The *Liber de Ludo Aleae* contains rules for games as well as advice on how to protect oneself against cheating. Cardano discussed methods of cheating that include false dice, marked cards, palming cards, tilted gaming tables, and the use of kibitzers. Although it was written in the 16th century (there is an internal reference that yields completion of the work in 1564 or later), it is unclear why the manuscript was unpublished in Cardano’s lifetime, reaching print only in 1663 [Cardano, 1966]. Since it was unpublished, it is also unclear what audience Cardano intended to reach.

Cardano not only anticipated the development of the probability calculus, but also was the forerunner of a genre of publications that could be described as gambling manuals, or manuals of games. Like the probability calculus, these did not appear until the second half of the 17th century, and like the probability calculus, they developed much more fully in the 18th century. For example, in the English literature the first complete gambling manual was Cotton’s [1674] *The Compleat Gamber*, and the apex of this literature in 18th-century England was Hoyles’s [1743] *A Short Treatise on the Game of Whist*. Consequently the *Liber de Ludo Aleae* has been viewed as a probability text, albeit a primitive one, or as a gambling manual with some helpful probability calculations in it.

A translation of the *Liber de Ludo Aleae* into English [Cardano, 1953] revived the interests of historians of probability in this work. These historians have analyzed the book mainly for its treatment of the probability calculus and have ignored or downplayed most of the other material in it. One such example is Mora Charles [1981], who translated extracts from the *Liber de Ludo Aleae* into Spanish and for the most part provided a mathematical analysis of the work. Typically, historians of probability have asked questions such as: Are the calculations correct? And to what extent did Cardano anticipate later writers on the subject? The other material gets brief mention and is sometimes treated in a pejorative fashion, such as Ore’s [1953] discussion of Cardano’s approach to luck. Franklin [2001], for example, has provided a good summary of the overall impact of the *Liber de Ludo Aleae* on modern readers:

*It is a confusing work; it is often not revised well enough to make the author’s intention clear, and there remain in it sections explicitly contradicted by later ones.*

Certainly, Cardano was not particularly concerned about consistency in his writings. Jensen [1994], for example, has examined the inconsistency in some other of Cardano’s writings, in particular the *De Subilitate* and the *De Rerum Varietate*. 
Aristotle’s influence on Cardano’s writings has been known for some time. Margolin [1976], for example, has looked at Cardano’s interpretation of Aristotle in Cardano’s works such as *De Subilitate* and *De Rerum Varietate*. Some recent efforts have been made to connect the *Liber de Ludo Aleae* to its Renaissance context. Tamborini [1999] discussed Cardano’s approach to luck or *fortuna* in the *Liber de Ludo Aleae* and related it to some Aristotelian concepts. Despite these kinds of efforts, much more can be done to understand the *Liber de Ludo Aleae* in context. By not reading the *Liber de Ludo Aleae* as a Renaissance text, many analysts of the work have overlooked the basic structure of the book and have not fully appreciated the source material that Cardano relied upon.

A secondary benefit to putting the *Liber de Ludo Aleae* in its Renaissance context is the discovery of likely connections to earlier probability calculations. The typical interpretation of the calculations prior to the Pascal–Fermat correspondence of 1654 has been that these calculations were disconnected, almost random, occurrences. There is very strong evidence that Cardano’s initial dicing calculations were based on a reading of *De Vetula*, a medieval poem first written circa 1250.

2. Girolamo Cardano (1501–1576) and his educational background

Before proceeding with an analysis of the *Liber de Ludo Aleae* it is useful to present a brief biography of Cardano and to describe some of the mathematical milieu in which he worked and studied. Besides his work in mathematics, Cardano was widely known in his day for his work in medicine and astrology. (Extensive biographical material on Cardano may be found in Fierz [1983], Ore [1953], and Rose [1975], as well as Cardano’s own autobiography [Cardano, 1930].) The combination of these three fields is, in a sense, natural to this time period. Medieval university professorships in mathematics were actually professorships in mathematics, astronomy, and astrology. Astronomy and astrology were seen as more important, with a background in mathematics serving as a preparation for the study of these subjects. Professorships in mathematics alone emerged by the end of the 16th century and astrology was very much on the decline by that time, although serious study of it continued well into the 19th century. Astrology and medicine were also closely connected, with astrology playing the role of servant to medicine. The positions of celestial objects were thought not only to have an influence on the lives of people in general, but also specifically to have an influence on the progress of disease. Knowledge of astrology could then be used in a variety of cures [Grendler, 2002, pp. 408–409; Grafton, 1999, p. 42]. Some of the mathematical, astrological, and medical elements of Cardano’s career are seen in the *Liber de Ludo Aleae*.

There are three strands that contributed to the development of mathematics during the Renaissance. The first is the development of commercial arithmetic, which led to an increasingly numerate population. The second is the development of other branches of mathematics, particularly geometry and related areas, through the recovery of classical mathematical manuscripts. The final strand is the teaching of mathematics at the universities. Cardano’s mathematical work has ties with all three of these strands.

The study of commercial arithmetic in Italy grew substantially between the 13th and 16th centuries. The need for this arithmetic and the concomitant introduction to the West of the Hindu–Arabic number system resulted from increased trade in the Mediterranean area with the Muslim world [Lieber, 1968; Mack, 2002]. Out of this interaction between Muslim and Italian traders, Italian merchants learned Muslim commercial practices such as bills of exchange and the recording of business transactions. In order to train merchants in these trading practices, schools of arithmetic or abbaco schools were started in many Italian city-states [Grendler, 1989]. Associated with these schools were arithmetic or abbaco books.
Van Egmond [1981] has provided an extensive list of these books up to the year 1600, with Leonardo of Pisa’s Liber Abaci [Fibonacci, 2002] the earliest on the list. The abbaco books were usually meant as manuals for teachers or for merchants already working in trade, rather than as student textbooks. These books contain discussions of the basic arithmetical operations of addition, subtraction, multiplication, and division, as well as discussions of fractions and the extraction of square and cube roots. To this point the abbaco books can viewed as strictly derivative of Arabic arithmetical books. The difference between the two is that the arithmetics in the Arabic books is followed by a development of mathematics for astronomy. (See, for example, a 10th-century Arabic arithmetical by Uqlidisi, Ahmad ibn Ibrahim [1978].) The Italian abbaco books take a different direction, often going well beyond the basic arithmetical operations by including, for example, business problems and recreational mathematics problems. These problems were often, but not always, accompanied by a discussion of elementary geometry and algebra, as well as miscellaneous material such as calendars and astrology [Van Egmond, 1981]. The geometry in the abbaco books is mostly arithmetical, dealing with lengths, areas, and volumes, rather than Euclidean in the sense of abstract mathematical proofs of geometrical relationships [Peterson, 1997]. The rise of Venice as a center of trade is tied to the publication of abbaco books. According to the data in Van Egmond [1981], between the earliest printed abbaco book in 1477 and 1600, the terminal date for data collection, 55% of all editions of abbaco books were printed in Venice; the next highest percentage was Naples, which had less than 8% of the total.

Mathematics was taught in the medieval universities as part of the quadrivium, composed of arithmetic, geometry, astronomy, and music. The leading Italian universities in mathematics were Bologna and Pavia, in that order. Following on the centuries-old curriculum of the church and cathedral schools, the writings of Boethius (ca. 480–524) were central to the teaching of arithmetic and geometry in the medieval university. The arithmetic of Boethius is not what we would call arithmetic today. His arithmetic involved little or no calculation and instead was confined to the study of the properties of numbers including ratio, proportion, and fractions [Kline, 1972; Masi, 1983; Schrader, 1967]. In the 12th and 13th centuries the curriculum changed slightly, as some major Greek mathematical texts coming from Arabic sources were translated into Latin, Adelard of Bath’s translation of Euclid being the prime example. The material in the university curriculum was transformed even more during the Renaissance as more mathematics manuscripts from antiquity were discovered and translated. The recovery of antiquity with respect to mathematical manuscripts is described in detail in Rose [1975]. The major mathematical works in their original Greek were by Euclid, Archimedes, and Apollonius. The work of Archimedes had a distinct impact on the applications of mathematics in the Renaissance but not on the university curriculum [Laird, 1991]. Taken over four years, the typical Renaissance university curriculum in mathematics was Euclidean geometry and Ptolemaic astronomy. In the first year there was a treatment of arithmetic and algebra as well as the introductory topics in geometry and astronomy. Subsequent years covered more advanced topics in geometry (later books of Euclid, for example) and astronomy. Also within this curriculum were topics in astrology because of its perceived relationship to medicine. Grendler [2002] has described the typical early Renaissance curriculum in mathematics and the changes that occurred during the Renaissance. On completion of the quadrivium, students could pursue higher studies in law, theology, or medicine as doctoral degrees.

Cardano was, in part, a product of the university system. Cardano, himself, has described his education in mathematics [Cardano, 1930]. Initially, Cardano’s father Fazio, a lawyer, had taught the son arithmetic and the first six books of Euclid at home. Fazio Cardano was himself an able mathematician. In addition to his legal work, Fazio lectured on geometry at the University of Pavia and at the Piatti foundation.
in Milan. At the age of 18 (about 1520), the son Girolamo entered the University of Pavia, where he would have followed the quadrivium. Three years after entering Pavia, Cardano gave public lectures in Euclidean geometry. Lecturing and disputation were part of the normal learning process for a university student of the time; students were required to give lectures and public disputations on a variety of topics and questions prior to their examinations for their degrees [Grendler, 2002]. Cardano left Pavia because of war in the district and stayed at home with his father for about two years. He entered the University of Padua in 1524 where he completed his studies in medicine.

Cardano’s first publication in mathematics, the *Practica Arithmetice* [Cardano, 1539], shows Cardano straddling both worlds of the abbaco school and the university. The work is listed in Van Egmond’s [1981] extensive catalogue of abbaco books and manuscripts, and so the *Practica Arithmetice* may be classified, in part, as an abbaco book. Cardano’s book is both typical and atypical of the genre. Cardano had never been a teacher in an abbaco school, nor had he learned his mathematics in an abbaco school. What we have is an individual, who had no experience as a merchant and who had not been formally trained in commercial arithmetic, writing a book that contains commercial arithmetic. To a certain extent it shows. The book was written in Latin rather than the vernacular, as the vast majority of Italian abbaco books were written. It was also written at a “higher level” than the normal abbaco book. Smith [1970] has described the *Practica Arithmetice* as “one of the most pretentious arithmetics of the sixteenth century,” but one that “did much to influence the advanced teaching of the subject.” The *Practica Arithmetice* shows up in the Bodleian library catalogue of 1620 [James, 1620] and so probably became part of the new arithmetic curriculum at some universities. Cardano probably had a different market in mind than a reference manual for abbaco teachers and merchants; he probably wanted to advertise his mathematical skills to a wider and more mathematically adept audience across Europe. Indeed, Maclean [1994] has asserted that Cardano’s reason for writing the *Practica Arithmetice* was a mixture of self-promotion and money. The 10 crowns Cardano received from the publisher was probably used to finance the publication of some of his works in astrology, which in the long run might have been more lucrative to him. He may also have used the 10 crowns to obtain a protective license from the Holy Roman Emperor to publish certain books he had written, in particular the books given on a list printed in the *Practica Arithmetice*.

Sixth on Cardano’s list of 34 books that he was ready to publish was one on games, entitled *De Ludis*. Cardano had been gambling seriously from about 1525. (A quotation describing his early gambling in his own words appears in Ore [1953]; David [1962] has provided a reference to the quotation, placing it in the 1551 edition of *De Subtilitate*.) At the same time that he started gambling seriously, Cardano began collecting facts about games. Later he expanded the collection of facts into a book written in the vernacular, the *De Ludis* listed in the *Practica Arithmetice*. The book was divided into four parts, of which the second was about games of chance. Tamborini [1999] has listed several references to this work throughout Cardano’s writings. Since we have only the 1663 printing [Cardano, 1966] of the *De Ludo Aleae* manuscript, it is impossible to establish the exact relationship between the two books.

In the past and, for some, to the present day, many have viewed Cardano as a superstitious charlatan. This view originates with Gabriel Naudé, who wrote the preface to the first publication of Cardano’s autobiography *De Propria Vita Liber* in 1643. Jean Stoner [Cardano, 1930, p. xiii], who translated the autobiography into English, has summed up Naudé’s position:

…Gabriel Naudé edited the book with a prefatory judicium which has long influenced every estimate and every picture posterity has held of the Milanese, for he implies that Cardan was a moral monster in general, and in particular superstition ridden and careless of the truth.
The charge of superstition probably came from Cardano’s work in astrology and from his interpretations of dreams. The label of charlatan comes from both his astrological and mathematical work. He cast a very positive horoscope for the young King Edward VI of England; Edward died shortly thereafter. In mathematics he has come out badly in his dispute with Tartaglia over Cardano’s publication of the solution for the roots of the cubic equation. Cardano’s solution was published in his major mathematical work, the *Ars Magna* [Cardano, 1968]. Feldmann [1961] has documented the dispute and has put Cardano in a much better light than earlier interpretations. In the past few years, general opinion on Cardano has become more positive and several scholars have now studied different aspects of Cardano’s career. (See Grafton [1999] on Cardano and astrology and Siraisi [1997] on Cardano and Renaissance medicine, as well as a collection of articles on a wide range of Cardano’s work edited by Kessler [1994].)

3. The *Liber de Ludo Aleae* as an argument

Beginning with Ore [1953] the *Liber de Ludo Aleae* has been analyzed carefully, and in detail, as a mathematical work. It was examined in less mathematical detail by Todhunter [1865], who was critical and dismissive of the work. One of the major issues that has never been addressed by historians is that Cardano himself did not view the *Liber de Ludo Aleae* as a mathematical work. In his autobiography Cardano [1930] listed his work on games of chance under the heading “Various Arguments” rather than among his mathematical works. If Cardano considered his *Liber de Ludo Aleae* to be an argument of some sort rather than a work of mathematics, what argument was he making and on what subject? A variety of arguments are given in the book, but the question he addresses is not clearly stated in the text. Since the book is about games of chance and the gambling associated with these games, it is useful to look at what other arguments were made about such games during the Italian Renaissance.

In general, there was no blanket condemnation of gambling; rather, the treatment of this issue was mixed.

Prior to the Renaissance, Thomas Aquinas (1225–1274) took up the question of gambling during his discussion of almsgiving in his *Summa Theologæ* (2a2æ. 32, 7: Thomas Aquinas [1975]). He made a distinction between civil and divine law. With respect to the Church, Aquinas wrote:

> For in the first place, some things are forbidden by divine law; for instance, winnings at the expense of minors and those out of their minds, who have no power to alienate their property; or out of sheer greed to induce someone else to gamble; or again, to win by cheating.

He noted that in some cases civil law also prohibits gambling. Aquinas went on to say that not everyone is subject to civil law and that the law may become outdated and changed. Later, in a discussion of avarice, Aquinas (2a2æ. 118, 8: Thomas Aquinas [1972]) referred to the connection Aristotle made between dice players and this vice. The appropriate quotation is from the *Ethics* (Book IV, i, 43: Aristotle [1955]):

> But the dicer and the pickpocket belong to the illiberal class, because they are sordidly avaricious: it is for gain that both types follow their profession and submit to a bad reputation, the one [pickpocket] accepting the severest risks for the sake of their pilferings, the other [dicer] profiting at the expense of their friends, to whom they ought to give; so both are sordidly avaricious, because they want to make gain from a wrong source.
This quotation was an important impediment to Cardano in his argument about games of chance and gambling. As will be seen in Section 4 Cardano used Aristotle to support his argument about games of chance; by contrast the quotation above is distinctly negative.

Near the beginning of the Italian Renaissance, Petrarch (1304–1374), in his *De Remediis Utriusque Fortune* (Book I, 26 and 27, and Book II, 16: Petrarca [1991]), made several arguments against gambling and games of chance. Among these arguments is one that comes close to Aristotle’s reasoning: Petrarch claimed that the winnings from gambling are illicit gains. The gains to be made from gambling are also unstable. Petrarch, putting words into the mouth of “Reason,” stated:

“There is no profit in gambling, only evil and misery; because he who loses suffers and he who wins is tempted and lured into the trap.”

Pietro Aretino (1492–1556) made a similar point in a diametrically opposed context. In his pornographic *I Regionamenti* [Aretino, 1971, pp. 222–223], the character Nanna tells her daughter Pippa how to be a good prostitute. She advises Pippa to stay away from gambling and to advise her men to stay away as well; men who lose their money in gambling would be unable to shower money on her. Petrarch also admitted to some of the benefits of dice games but immediately downplayed these benefits.

Others played up the positive aspects of games of chance. The Roman humanist and Vatican librarian Bartolomeo Sacchi (1421–1481), writing under the name Platina, praised games of chance in the context of a meal. Unless cheating is involved, playing games of chance after supper aids in digestion [Platina, 1998, p. 109].

The individual who came closest to the question that Cardano asked is Baldesar Castiglione (1478–1529), an Italian courtier, soldier, and diplomat. Based on his experience in the court of the Duke of Urbino, Castiglione wrote the highly influential *Il Cortegiano* or *The Courtier*, a Renaissance courtesy book whose perfect courtier became a model for the educated classes of Europe. Castiglione was concerned with maintaining the facade developed by the courtier. The following dialogue discussing whether or not a courtier should play at games of chance is taken from *The Courtier* [Castiglione, 1967, p. 140]:

“‘It seems to me,’ replied Federico, ‘that we have given the courtier a knowledge of so many subjects that he can readily vary his conversation a great deal and adapt himself to the qualities of those with whom he has dealings, assuming that he possesses good judgement and allows himself to be ruled by that, and, depending on the circumstances, attends sometimes to grave matters and sometimes to festivities and games.’

‘And which games?’ asked signor Gaspare.

Federico answered with a laugh: ‘For this let us go for advice to Fra Serafino, who invents new ones every day.’

‘Joking apart,’ answered signor Gaspare, ‘does it seem to you that it is wrong for the courtier to play at cards and dice?’

‘To me, no,’’ said Federico, ‘unless he does so too assiduously, and in consequence neglects things of greater importance, or indeed for no other reason than to win money and cheat his partner, and then, when he loses, is so dismayed and angry as to prove his avarice.’”

Cardano, an avid gambler for much of his life, asked a question similar to signor Gaspare’s. Rather than the question being specific to the courtier, Cardano asked for different reasons, “Does it seem to you that it is wrong for a man to play at cards and dice?” Another way of phrasing the question is in terms of the Renaissance concept of justice, “Under what conditions can the act of gambling at cards and dice be considered a just act?” Castiglione gave his own answer in the final sentence of the quotation; Cardano took several pages to make his arguments, addressing all the points made by Castiglione and many more.
Chief among Cardano’s additional points is that he used mathematics to argue when it is not wrong to play at games of chance.

That the question addressed in Cardano’s argument concerns justice is evident in his approach to games of chance. At the beginning of Chapter 6 of the *Liber de Ludo Aleae*, Cardano [1953] stated his basic assumptions:

The most fundamental principle of all in gambling is simply equal conditions, e.g., of opponents, of bystanders, of money, of situation, of the dice box, and of the die itself. To the extent to which you depart from that equality, if it is in your opponent’s favor, you are a fool, and if in your own, you are unjust.

The key word is “unjust.” In the previous chapter Cardano had given his reasons for writing the *Liber de Ludo Aleae*. First he said that gambling has useful features and that it has some advantages. The word that Cardano used is a form of *utilitas*, which also translates to utility. The ancient and later Renaissance concept of utility was tied to the concept of justice. For example, Cicero, a popular author in the Renaissance said in his *De Officiis* (Book II, 10 and 20; Cicero [2000]) that what is just is also useful and what is useful is also honorable. A useful act is also just so that when Cardano was looking to the useful features of gambling he was also looking into the parts of gambling that are just.

It is impossible to say whether or not Cardano took the question he was arguing directly from Castiglione’s *The Courtier*. The existing evidence is circumstantial. First, Cardano claimed an indefinite family connection with Castiglione. In his autobiography he [Cardano, 1930, p. 1] wrote of his ancient and noble lineage, hinting that the Cardano family was really a branch of the Castiglione family. Second, there is a connection of opposites between Cardano and Castiglione in terms or their approach to courtiers. Castiglione was himself a courtier, while Cardano appears to have despised courtiers. Early on in the *Liber de Ludo Aleae*, Cardano advised that a prince should not gamble. On this point Castiglione and Cardano are in basic agreement. Cardano’s discussion shows his attitude to courtiers.

Writing some years later in his autobiography Cardano [1930, p. 124] made another revealing statement about his attitude to courtiers. He claimed that he never searched out honors saying that the search usually brought grief:

Again, a zeal for honors urges us to the verge of death itself by ways too numerous to recall—duels, wars, quarrels, disgraceful litigation, attendance upon the favor of princes, …

The initial form of Cardano’s argument concerning the justice of gambling and games of chance is part of the source of confusion about the work. After a brief introductory chapter in the *Liber de Ludo Aleae* that describes games in general, Cardano opened his argument in Chapter 2, but did not explicitly say what argument he was making or what it was about. Rather, he began by listing some of the pros and cons of gambling. In Chapter 2 the pros and cons alternate one by one. Gambling is permitted at funeral banquets, but is also condemned by the Titian and Cornelian laws from ancient republican Rome, Cardano said. Play at cards and dice is beneficial during times of grief, stress, and anxiety; on the other hand, one’s time is better spent doing more worthwhile activities. This method, giving both sides of opposing positions, is the method of arguing *in utramque partem*. It was a method of argumentation that is
based on Ciceronian rhetorical methods and principles; the method was very popular among Renaissance humanists. Cardano returned to this method of argumentation in Chapter 4. In this short chapter the pros were grouped together followed by a group of cons.

Franklin [2001] has called attention to contradictions in the Liber de Ludo Aleae, some of which result from the argument in utramque partem. For example, in Chapter 2, Cardano wrote that gambling “arouses anger and disturbs the mind” and then in Chapter 4 asserted that gambling can provide the opposite—relaxation from anxiety. In a similar vein, Cardano advised playing only for small stakes in Chapter 3, but in the following chapter stated that the beauty of large stakes is that it can provide insight into the character of an opponent. It may seem confusing to us today; for his time Cardano was using a standard form of argumentation.

4. Aristotle’s concept of justice and Cardano’s probability calculations

Justice is the major theme of the Liber de Ludo Aleae. Not only did it motivate how Cardano approached all his probability calculations, but also it guided his approach to subjects such as cheating in games of chance. Cardano’s fundamental principle of gambling in the Liber de Ludo Aleae rests on equality, and hence on justice. Usually, equality is taken to mean equal chances for the players in a game of chance. Cardano, on the other hand, took equality well beyond equal chances in the probabilistic sense. The idea for this fundamental principle comes directly from Aristotle’s Ethics (Book V, iii, 5–6: Aristotle [1955, pp. 177–178]). Aristotle defined what is unjust as what is unequal and what is just is what is equal. He went on to say that

… a just act necessarily involves at least four terms: two persons for whom it is in fact just, and two shares in which its justice is exhibited. And there will be the same equality between the shares as between the persons, because the shares will be in the same ratio to one another as the persons; for if the persons are not equal, they will not have equal shares; and it is when equals have or are assigned unequal shares, or people who are not equal, equal shares, that quarrels and complaints break out.

That Cardano relied on Aristotle for his definition of the fundamental principle of gambling is confirmed by his discussion of games in which the participants have unequal chances to win. In this discussion he used the term “circuit” to describe the “sample space” or the set of possible outcomes of the throw of the dice. He wrote [Cardano, 1953, p. 18]:

Other questions must be considered more subtly, since mathematicians also may be deceived, but in a different way. I have wished this matter not to lie hidden because many people, not understanding Aristotle, have been deceived, and with loss. So there is one general rule, namely, that we should consider the whole circuit, and the number of those casts which represent in how many ways the favorable result can occur, and compare to that number to the remainder of the circuit, and according to that proportion should the mutual wagers be laid so that one may contend on equal terms.

This is a direct application of Aristotle’s rule for a just act. In modern terms, suppose that one player is wagering an amount $x$ against another player who is wagering $y$. The probability of the first player winning the wager is $p$ and so the probability of winning for the second is $1 - p$. In the modern context, the game is fair or just if the expected gains of both players are the same, i.e.,

$$y \cdot p - x(1 - p) = x(1 - p) - y \cdot p.$$
This can be rewritten as

\[
\frac{y}{x} = \frac{1 - p}{p},
\]

which is the same as saying that the ratio of the wagers must be the same as the ratio of the chances of winning, or Cardano’s rule. This follows Aristotle’s prescription that “the shares will be in the same ratio to one another as the persons,” where the shares are equivalent to the stakes and the measure of “justness” of the individual is the probability of winning.

Historians of probability have looked to Aristotle when searching out the genesis of probabilistic ideas. (See, for example, Sambursky [1956], Sheynin [1974], Hacking [1975], and Styan [1998].) Typically, they have concentrated on Aristotle’s ideas of the meaning of a chance event and the subsequent Scholastic interpretations of chance. They have also looked to the evolution in the meaning of probability as it applies to an argument that is probable or has reasonable grounds for acceptance. Cardano’s use of Aristotle is entirely different, relying on the definition of justice rather than chance.

The mathematical discussion of games of chance begins in Chapter 9 of the *Liber de Ludo Aleae*, where a basic description of dice is given. Cardano described two kinds of dice, the regular die of six sides and the *talus* or *astragalus*, a die with four sides typically made from the knucklebones taken from the hind legs of sheep or goats. At this point in the book only a passing reference is given to the *talus*; the main discussion is concerned with the regular die of six sides. Central to Cardano’s initial mathematical argument is the circuit or the modern day sample space as it applies to dice. There are six sides to the die and Cardano reasoned that the die should complete the circuit of all six possible faces after six throws of the die. In a later chapter, Chapter 15, Cardano was very explicit about the concept of the circuit, saying,

… the magnitude of the circuit is the length of time which shows forth all forms.

This is a rather odd definition in view of the strictness of it. Cardano also knew that this definition did not hold empirically. At the beginning of Chapter 9 he stated,

… the die has six [faces]; in six casts each point should turn up once; but since some will be repeated, it follows that others will not turn up.

Based on this initial definition of the circuit, Cardano described the number of possible outcomes in the circuits for two and three dice, 36 and 216, respectively. After some discussion about some of the outcomes in the throws of these dice, he calculated in Chapters 10, 11, and 12 the probabilities of the points in a game called *Sors* and another called *Fritillus*. The game of *Sors* is straightforward. The points are the sum of the faces that show in the throw of the die, be they two or three dice. Cardano’s calculations for *Sors* are correct, and appear in Table 1 for three dice. The point system for *Fritillus* is more complicated; Ore [1953, p. 161] has given a reconstruction of the rules of the system.
Ore [1953] has given a detailed analysis of Cardano’s mathematical probability calculations on dice. One of Ore’s major insights into Cardano’s mathematical arguments is a method purportedly used by Cardano’s that Ore called “reasoning on the mean.” For a single die Ore’s reasoning on the mean goes as follows. The probability that any particular face shows in the throw of a single die is one in six since the length of the circuit is six. If the die is thrown three times then the expected or mean number of times the face shows is $\frac{1}{2}$. From this Cardano concluded that there are equal chances for one particular face to show at least once in three tosses of the same die. In translation, Cardano [1953, Chapter 9] expressed his reasoning as follows:

One-half of the total number of faces always represents equality; thus the chances are equal that a given point will turn up in three throws, for the total circuit is completed in six, or again that one of three points will turn up in one throw.

Cardano’s reasoning is incorrect; the probability of obtaining an ace, two, or three in one throw of a single die is $\frac{1}{2}$, for example, while the chance of obtaining at least one ace in three throws of a single die is $\frac{91}{216}$ or 0.42. Ore’s comment on this particular derivation is:

This value is fairly satisfactory since the correct figure [the number of tosses of the die required to obtain a probability of $\frac{1}{2}$ that a certain face shows at least once] lies between 3 and 4. It may well have been this result, conforming to his gambling experience, which brought Cardano to place so much faith in his reasoning on the mean; also in other instances one can see that he attempts to generalize on evidence which is very slim indeed.

From a modern perspective Ore’s approach is a very nice interpretation of what Cardano was trying to do. The problem with the analysis is that it is based on a well-developed concept of mathematical expectation, a concept that Ore has perhaps read into Cardano’s writings rather than extracted from them. I would contend that Cardano’s initial, and incorrect, arguments based on the circuit and Ore’s “reasoning to the mean” is tied to Aristotle’s concept of justice, which Cardano interpreted as equiprobable outcomes under equal stakes. There are six faces on a single die, of which three, for example the ace, two, and three, are of interest. The ratio of the total number of faces to the faces of interest is $\frac{6}{3}$ or 2. There are also six possible casts of the die. In order to maintain the same Aristotelian ratio $\frac{6}{3}$ for justice Cardano assumed that the number of casts of interest (for example, containing at least one ace) should be 3. Cardano has incorrectly made the correspondence between the three different faces in a single toss and a single face in three tosses. At this point in the Liber de Ludo Aleae Cardano wanted his mathematics to force events to be equiprobable or just, rather than having the mathematics show where the justice lies. It was not until later in the book after these initial incorrect calculations have been given that Cardano realized his error in mathematical reasoning and calculated the chances correctly. As Ore has shown, these incorrect calculations, whatever their philosophical origins, provide insight into some of Cardano’s analyses of the point systems in certain dice games. Ore has also noticed that Cardano has made some attempts to reconcile the correct calculations with calculations based on reasoning on the mean. If we interpret the reasoning as toward justice instead, we can see why Cardano’s more obscure statements in the Liber de Ludo Aleae would, in Ore’s words,

… be concerned precisely with this problem of bringing the two points of view [the correct method and reasoning on the mean] into harmony.

Cardano was concerned that the correct calculations show a just act. The approximate agreement between the two approaches may have led Cardano [1930, p. 195] to comment in his autobiography,
It is all like trying to calculate one’s chances in gambling: the system comes to naught or is ambiguous.

Cardano made some minor contributions to the calculation of chances in card games, in particular, the game primero. Cardano treated a type of division-of-stakes problem for primero, but one that is simpler than the classic problem of points, a problem solved by Pascal and Fermat that led to the formal development of the probability calculus. In primero, when two players remained in the game with a card draw each left to be made, the player with the lower number of points in his hand could ask for a *fare a salvare*. At that point in the game the pot could be divided into two parts if the player asked for it. One part of the pot was split, usually evenly, between the two players; and the other part was played for and taken by the winner. To maintain justice in the game, Cardano stated that the *fare a salvare* should be decided upon before the game begins since the actual play of the game may provide information on what cards are outstanding so that the underdog may find it sometimes advantageous and sometimes disadvantageous to invoke this rule when the decision is left entirely to him. Using his mathematical criterion of justice and basing his calculations on the chances that the outstanding cards to be drawn will lead to a win, Cardano showed the division rule favors the underdog.

There are no calculations regarding the chances of each type of hand in primero. These can be obtained by some relatively simple combinatorial calculations similar to poker hand probabilities. Upward of six years after the completion of the *Liber de Ludo Aleae*, Cardano had developed the necessary mathematical theory to make these calculations. As part of a larger work Cardano [1570] published techniques using an arithmetical triangle to calculate the number of combinations of objects taken from a group of dissimilar objects. The chances of card hands can be calculated by extending and using Cardano’s triangle. Neither card games nor any other games of chance were among the examples that he used; Boyer [1950] has incorrectly stated that Cardano applied the results of his arithmetical triangle to games of chance. Instead, Cardano’s used as his example the selection or arrangements of ten men sitting to dinner at a table, which was a typical problem appearing in a number of previously published mathematics books. For example, Edwards [1987] referenced three others who dealt with counting the number of ways 10 men can sit down at a table.

How did Cardano make the connection between Aristotle’s definition of justice and the calculation of chances? One possible explanation might be found in Cicero’s *De Officiis* [Cicero, 2000], where Cicero provided the solution to what is now known in philosophy and law as the lifeboat problem. The modern problem may be stated simply as follows. There are several people in a lifeboat that will sink unless one person is thrown overboard. Who should be picked? Broome [1984] has described the original historical situation in which the ship’s mate made the decision to throw several men overboard while saving women and children along with the crew. Cicero’s scenario was different, but the question essentially the same. Cicero [2000] described the problem and solution as:

> Another question: assuming that there is one plank and two ship-wrecked passengers, both of them wise men, should each try to grab it for himself, or should one yield to the other? “One should give way, yielding to the one whose life is more important whether intrinsically or to the state.” “But supposing the balance is equal on both sides?” “Then there will be no contest. One will yield to the other as if in a lottery or a game of chance.”

When the two wise men are equal, then the decision is left to pure chance or an equiprobable event. In a trial arising from the original lifeboat problem that occurred in 1841, the judge also decided that the choice should have been by lot.
5. *Scientia* and its relation to justice

Since Cardano was relying on Aristotle’s concept of justice (Book V: *Aristotle [1955]*) to support his arguments in favor of gambling, Aristotle’s negative attitude to gambling as quoted in Section 3 was something that Cardano had to deal with. The issue, as Cardano set it out in Chapter 10, is the nature of gain. Cardano categorized various kinds of gain from gambling and said that certain kinds of gain through gambling are acceptable. Gould, the translator of the *Liber de Ludo Aleae*, expressed Cardano’s version of the best kind of gain as “from those who are willing and aware.” For some the meaning of this translation can be a little obscure. The Latin phrase used by Cardano is *Lucrum enim a volentibus, atque scientibus optimum est*. Gould translated *volentia* as willing; it can also mean “inclination”; likewise *scientia* might be more commonly translated as “knowledgeable” or “skilled.” Consequently, another way of translating the passage is that the best kind of gain in a gambling situation is from those who have the inclination to gamble and who are skilled or knowledgeable in the game. Cardano ran through the various kinds of gain within his categorization, describing unacceptable gain as that which is taken from those who are unwilling or disinclined to play and at the same time are unskilled in play. This categorization of the various types of gain to be made from gambling handles the various objections to gambling, including those of Petrarch. After defining gain in gambling, Cardano returned to the method of *in utramque partem* to continue his argument. Citing Aristotle, Cardano stated that gain from gambling is base gain and is therefore sordid and unacceptable. Then he finished his argument by taking the other side, giving an argument in favor of gambling; Cardano claimed that the Church did not condemn gambling as such, being concerned mainly with the blasphemy that might accompany the act of gambling. The early Church had generally condemned gambling and games of chance. This had changed as canon law in the medieval Church developed. In addition to the restrictions noted by Aquinas in Section 3, the only general prohibition on gambling in Cardano’s day came from the decisions of the Fourth Lateran Council of 1215 in which the clergy were prohibited from playing games of chance or to be present at them. (See, for example, the article entitled “Gambling” in the *Catholic Encyclopedia* [Herbermann, 1907].)

The key to responding to Aristotle’s objections to gambling is *scientia*. Knowledge, skill, or *scientia* is the second theme of the *Liber de Ludo Aleae*. Cardano provided the reader with knowledge of games so that the reader with *volentia* will have the *scientia* to play. Whether the gain goes to the reader or to his opponent in a just game, it can be considered the best kind of gain.

The promotion of *scientia* and its relation to justice is apparent in Cardano’s first treatment of methods of cheating. The first mention of cheating in the *Liber de Ludo Aleae* is related mainly to cards games and appears in Chapter 6, entitled “The Fundamental Principle of Gambling.” The discussion in this chapter begins with the fundamental principle as quoted in Section 3. Then follows a description of kibitzers and other types of onlookers at games of chance. These people can provide information to an opponent or cause distractions making a player lose his concentration on the game. Cardano set the tone for his view of cheating when he made his first comments about kibitzers.

… and so it happens that, if you play in a large crowd of people, you can scarcely avoid folly if they are against you, or else injustice if they are for you.

The connection to justice is explicit here. In his later descriptions of cheating at cards and dice in Chapters 17 and 7, respectively, the connection to justice is not made explicit. What Cardano did in all these situations was to provide information about how cheating occurs so that the player can be aware
(or have scientia) of these methods in order to protect himself and to avoid folly. At the same time the player should not exercise this knowledge in such a way as to lead to injustice.

Here is one example of how Cardano approached the subject of cheating. Fair dice are perfect cubes. Cardano described three alterations that result in biased dice. Two result from altering a fair die. Through alteration one of the corners or edges of a cubical die can be rounded off, thus giving an advantage to one of the faces on the opposite side of the die. The other alteration is to apply pressure to opposite faces of the die. This will tend to make these two sides flatter in comparison to the remaining four, thereby giving advantage to the two flattened sides. The third alteration is to construct a noncubical die that has two opposite faces square in shape and the remaining four faces rectangular in shape, leading to a bias away from the square faces. To combat these kinds of cheating with dice, Cardano advised looking at the three sets of the opposite sides of the die to determine whether or not it is a true die. Curiously, Cardano did not mention loaded dice.

Cheating at games of chance was not unique to Cardano’s circle or to Italy in general. With the exception of the dice with rounded edges all the methods of cheating at dice, including loaded dice, are mentioned in the sixteenth century English literature of roguery. (See Bellhouse [1993], for a description of this literature and its possible relationship to the history of probability.) This literature also mentions methods of cheating at cards, including stacking the deck and various methods of legerdemain to bring forward a desired card. Cardano has added to this list rings with mirrors on them to see the cards as they are dealt face down. He also included marked cards and the soaping of cards to make the cards slide better over one another. In line with the promotion of scientia, Cardano gave advice about examining the deck of cards in order to guard against the typical methods of cheating.

The use of legerdemain at cards is linked to Cardano’s work as an astrologer. In Chapter 17 of the Liber de Ludo Aleae there is a description of cheating at cards through legerdemain. A more detailed description of this type of card sharping is given in Book 18 of Cardano’s De Subtilitate. (A translation of the appropriate passage is given in Maxwell-Stuart [1998].) What is of interest in the De Subtilitate passage is that Cardano watched the card sharper very carefully to see if he could figure out how the trick was done. He could not work out the trick, but still concluded that the sharping was due to legerdemain rather than to spirits or to magic. It is apparent that Cardano did not believe in magic and divination through a randomizing device. Rather, he was very careful to distinguish between true divination (through astrology) and tricks, and between true divination and pure chance. (Cardano’s work in astrology and the astrological milieu in which he worked is described in Grafton [1999].)

According to Cardano, the best type of gain a person can made in a game of chance is from those who are both willing to gamble and skilled at the game. Cardano promoted legitimate skill and did not equate highly skilled players with cheaters. Among the skills he saw as legitimate was the use of memory. In Chapter 23 Cardano recognized the importance of remembering what cards have been played. Earlier in Chapter 17 he stated explicitly that

Those, however, who know merely by close attention what cards they are to expect are not usually called cheats, but are reckoned to be prudent men.

Cardano condemned some of the obvious skills, already mentioned, of cheating at cards and dice. Some of the skills he condemned were not necessarily universally recognized as illegitimate. For example, in Book IV of The Courtier, Castiglione wrote:
For it is impossible to govern either oneself or others well without the help of God, who to the good sometimes sends good fortune as His minister, to protect them against grave dangers, and sometimes adverse fortune to prevent their being so lulled by prosperity that they forget Him or human prudence, which often offsets ill fortune as a good player remedies bad throws of the dice by the way he places the board.

In Chapter 7 of the Liber de Ludo Aleae, Cardano condemned the practice of manipulating the gaming board as a method of cheating.

6. The Liber de Ludo Aleae as a humanist document

Humanism was a major intellectual movement during the Renaissance. It was based on the belief that a study of the ancient classical texts, in particular the Greek and Roman literature, could provide a cultural rebirth [Nauert, 1995]. Humanists of the Renaissance were active in the recovery, annotation and publication of these ancient texts, as well as the translation of them, typically from Greek to Latin. Aristotle was a favorite classical author among the humanists and his Ethics was widely read to the point that Celenza [1999, p. 48] has commented:

If one were a humanist, then, one way to achieve a connection with one’s audience would have been to use terminology from the Nicomachean Ethics, which at that point would have been fashionably familiar to the reading public.

It is then not surprising that Cardano relied on the Ethics to justify his probability calculations and that Cardano had to come to terms with Aristotle’s comments on dicers in order to put forward his own arguments. We have already seen this and other humanist influences in the Liber de Ludo Aleae. With the exception of a passing reference to humanism by David [1962], no one has looked at the Liber de Ludo Aleae in the context of Renaissance humanism.

Cardano’s humanism shows through from the very beginning of the Liber de Ludo Aleae, where there are several references to classical sources and a description of the ancient dice called tali or astragali. At this point in the book there is no mention of actual ancient games of chance. These games are treated in more detail near the very end of the book in Chapters 30 and 31 with some follow-up discussion in the concluding chapter, Chapter 32. The information about ancient games here is taken from Celio Calcagnini’s De Talorum ac Tesserarum et Calculorum [Calcagnini, 1544, pp. 286–300]. The talus, as previously mentioned, is a four-sided die made from the anklebones of sheep or goats; a tessera is a regular six-sided die and a calculus is a stone. Cardano dealt with tali and tesserae only, ignoring any games with stones. His discussion of ancient dice not only underlines his humanist learning, but also reveals an attempt to relate the interpretation of dice games to Aristotle’s Ethics through Aristotle’s doctrine of the mean.

In Chapter 30 of the Liber de Ludo Aleae, Cardano described tesserae and made reference to many ancient sources that were all taken from Calcagnini. Within the context of the ancients, Cardano continued with the themes of ius and scientia. He noted that the numbers on opposite faces of tesserae always sum to seven, which is still the case today. Hence it was easier to detect cheating with false dice that have some of the numbers one through six missing and other numbers repeated. Cardano went on to describe three other methods of cheating with dice that require legerdemain and skill at throwing the dice to get a desired result.
Chapter 31 is devoted to *tali*. With this four-sided die, the opposite sides are 1 and 6, and 3 and 4, again both summing to 7. In a typical game, four *tali* or *astragali* were thrown. After a brief description of *tali*, Cardano counted the number of ways in which each of the possible throws of four *tali* can occur. For example, there are 4 ways that the *tali* can be thrown such that the faces are all the same, such as (3, 3, 3, 3), and there are 24 ways to get a throw with all different faces, such as (1, 3, 4, 6). What Cardano did was to enumerate the number of chances of each kind of throw. This enumeration does not immediately lead to the probability of various throws, and Cardano did not recognize this problem. The reason for the difference between the enumeration and the probability is that the faces of the *tali* are not equally likely to show. The flat sides show with a frequency of about 4 in 10 throws and the narrow sides show about 1 in 10 throws. (Hagstroem [1932], who had his daughters throw the *tali* several hundred times, obtained these numbers empirically.) Cardano finished this chapter by naming and trying to describe various kinds of throws with the *tali*. The Venus throw, which was considered lucky, was a throw with all different faces; Cardano noted in this chapter that it was the throw with the highest number of chances, 24 out of 256. The dogs was an unlucky throw. What exactly constituted this throw is uncertain; the throw involves throwing at least one face showing a one.

Cardano tried to clear up the little mystery of the dogs in Chapter 32, probably by using Aristotle’s doctrine of the mean outlined in Book II of the *Ethics* [Aristotle, 1955]. Aristotle defined the mean for a thing as that which is equidistant from the extremes; he also defined the mean for an individual as something that is neither excessive nor deficient. The two means are not necessarily the same and Aristotle provided an example; the range of amounts of food, with a specific mean, available to an athlete in training and the mean amount of food that is appropriate to the athlete’s needs can be different. The mean should be striven for and virtue for individuals lies in striving for the mean. Cardano initially defined his mean (he used the word *mediocris* meaning literally “in a middle state between too much and too little”) for a set of six dice in which only one face on each die is numbered and all the numbers one through six appear for the six dice. He obtained the arithmetical mean in the usual way by adding up the first six integers and then dividing by six to obtain 3½. In this case he seems to have been concerned that there will be a tendency to numbers below the mean; taking the blank faces on the dice to be 0, I have calculated that the probability that the sum of the faces that show is less than the mean is 0.548 and the probability that the sum is 0 is 0.342. From a modern viewpoint Cardano’s concerns are to be expected; the distribution he has constructed has a long tail on the right, so that the mean of the distribution is greater than the median. After this initial description Cardano returned to the throw of four *tali*. The calculation of the mean in this case follows Aristotle’s prescription for the calculation of the mean of things, the average of the two extreme numbers. Cardano may have Aristotle in mind when he concluded that the throw of a dog must have more than one die with its face showing a one. The smallest sum for the throw of four *tali* is 4 and the largest is 24, so that the Aristotelian mean is 14. If two 1s and two 6s show in the throw of four *tali*, then the sum is also the mean of 14, so that a player can never go above the mean with a throw of two 1s. Although Cardano made no mention of it, the *Ethics* may also explain to Cardano why the Venus throw was a lucky throw. Not only does this throw have the highest number of chances, but also the sum of the faces that show is always 14, the Aristotelian mean.

Cardano may also have been trying to use Aristotle’s doctrine of the mean to justify the cut points between high and low bids in different primero hands; this discussion appears in Chapter 19. The mean point (or *mediocris*) that Cardano used in any type of primero hand that he discussed is the Aristotelian mean based on the point scoring assignment given for primero: face cards are worth 10 points, an ace is worth 16, 6s and 7s get three times their value, and 2 through 5 get their value plus 10 points. A hand
of four kings would be worth 40 points and a hand of four 7s would be worth 84 points, so that the
Aristotelian mean for four of a kind (or a chorus hand in primero terminology) would be 62. In his
discussion, Cardano tried to tie a cutpoint to the Aristotelian mean obtained from the range of total
points in various hands; when this simple mean did not work as the cutpoint, he tried to obtain the mean,
again using Aristotelian principles, but based on the structure of the hand.

Another attribute of humanist writing is the use of classical exempla, and the Liber de Ludo Aleae is
liberally sprinkled with classical exempla. A detailed examination of the three reasons Cardano gave in
favor of gambling, listed in Chapter 4, provides a good example of Cardano’s exempla.

The first exemplum used by Cardano to support gambling that is discussed here is actually the third
one that appears in Chapter 4. It is a twist on an earlier condemnation of gambling. It is also the only one
that is accompanied by an explicit classical reference. In support of gambling Cardano stated:

It is also a means of gaining friendship, and many have arisen from obscurity because of the friendship of princes formed in play.

Then he quoted from Cicero’s Philippi:

This is what Cicero meant in his Philippi by the words “that fellow-player of yours, condemned for gambling.”

The actual passage in the Second Philippic (2.56: Cicero [1986, p. 67]) is distinctly anti-gambling. In this passage, Cicero condemned Mark Antony for bringing back to Rome one Licinius Lenticula, someone whom Cicero considered a scoundrel, someone who actually gambled in the Roman Forum and was convicted of the offense. Cardano put the only possible positive spin on the episode. By gambling Lenticula had made friends with the powerful Antony and through Antony’s influence was able to come back to Rome.

Another reason in favor of gambling is the relaxation it provides. Cardano wrote,

As advantages from well-managed play we obtain relaxation from anxiety and a pleasure from which we arise ready and eager for serious business.

One likely source for this sentiment is Cicero’s De Oratore. Specific reference to Cicero and a quotation of his comments on dicing to pass the time during periods of bad weather are made in the final chapter of the Liber de Ludo Aleae rather than Chapter 4. The quotation as given by Cardano in the final chapter is,

Men who are accustomed to hard daily toil, when by reason of the weather they are kept from their work, betake themselves to playing with a ball, or with knucklebones or with dice, or they may also contrive for themselves some new game in their leisure.

Cardano wrote that the quotation is from Book 2 of De Oratore; the quotation is actually from Book 3
(3.58: Cicero [2001, p. 240]). A second exemplum from De Oratore related directly to relaxation, rather
than passing the time during bad weather, is the example of Publius Scaevola, a Roman jurist and consul
in the Republic, who, as Cicero claimed, obtained his relaxation from work by playing at ball games
and at a gaming board known at Twelve Lines [Cicero, 2001, 1.217, p. 110]. On the opposite side of
the interpretation is Petrarch in his De Remeduis Utriusque Fortune (Book I, 26: Petrarca [1991, vol. I,
p. 79]). Petrarch, writing 200 years before Cardano put pen to paper, condemned dice games. Petrarch
started his condemnation with classical references to those who enjoyed dice and board games and said,
putting words into the mouth of the character Reason,
Scaevola chose these games as a relaxation from the cult and ceremonies of the gods and the laws of men, in both of which he was enormously experienced, and Augustus to refresh himself from the cares of his great empire, which he governed long and well.

This quotation appears to favor Cardano’s position and could well have been his source regarding dicing as a form of relaxation. The quotation is taken out of context of the entire dialogue. In the words of Reason, Petrarch went on to say that one should not always try to imitate the “peculiar preferences” of “learned and prominent men,” since it can lead to disaster. One of the classical exempla on which Petrarch made his arguments is Suetonius’s *Divus Augustus* (Book LXXI: Suetonius [1914, p. 235]) in which Suetonius was highly critical of Augustus’s gambling habits. The comments of Cicero, in this case, were more tolerant of gaming.

A final point given in favor of gambling in Chapter 4 of the *Ludo de Ludo Aleae* runs counter to the theme of dissimulation in Castiglione’s *The Courtier*. Related to this point, Cardano wrote:

> . . . knowledge of the character of our fellow-citizens is, as it were, a rack on which anger, greed, and honesty or dishonesty are made clear. For play both produces important evidence and is an actual torturer if the stakes are large.

Gambling removes the facade constructed by the courtier and reveals the true nature of the individual behind the mask. The theme of dissimulation and the danger that gambling presents through destroying a carefully constructed facade appear elsewhere in Renaissance literature. In his *De Remediis Utriusque Fortune* (Book I, 27: Petrarca [1991, vol. I, p. 83]), Petrarch wrote,

> You will remember that Ovid, in the book about which he teaches an indecent and unnecessary art, inserting, however, something useful now and then, admonishes ladies in love to abstain from gambling and similar activities, in order to disguise the vices of their souls, lest they displease their lovers who see them “swollen with anger” or devoured by greed. This advice is even more appropriate for men, who should avoid giving offense, not only to the eyes of others, but to the eyes of God, Who sees all and Who loves good minds and decent manners.

Petrarch has misread Ovid in that the “admonishment” is for the men. He has also put his own distinct interpretation on Ovid. In the *Ars Amatoria* (Book III, 367–380: Ovid [1852, p. 448]) Ovid advised men to devise games for their lovers since this activity can encourage love. He then cautioned the men to be careful since the heat of the game may reveal aspects of their own characters that should not be shown to their lovers, aspects such as greed and quarrelsomeness.

Many more humanist influences can be found in the *Liber de Ludo Aleae*. The examples that have been given here illustrate the richness of the discussions of these topics in the Italian Renaissance and the breadth of the classical sources on which the discussion was based.

### 7. Relationship to De Vetula

Cardano’s description of the throw of three dice is very similar to, but much more concise than, the description given in the pseudo-Ovidian poem *De Vetula*. Written in Latin in about 1250 and purportedly an autobiographical work of Ovid, *De Vetula* is divided into three books. The first book of *De Vetula* describes Ovid’s youth, his love affairs, and some of his amusements and pastimes. The second book details a tragicomic love affair. Ovid becomes disillusioned with the pleasures of love and devotes himself to philosophical pursuits. The third book is about Ovid’s conversion to Christianity. Robathan [1968] and Klopsch [1967] have provided a transcription of the poem based on various manuscript sources, as well
as some textual commentary. The calculation of the chances that the various sums on the faces that show in the throw of three dice appears in the first book. Bellhouse [2000] has described these calculations, as well as some relevant marginalia in one of the manuscript sources, and has given an English translation of the relevant passage on dicing.

Both Cardano and the author of De Vetula approached their analyses of the dice throws by stating that there are 6 different throws when the faces are all alike (a triplet throw such as (1, 1, 1)), 30 throws with two faces alike and one different (a throw of a doublet and another face such as (1, 1, 2)) and 20 throws with the three faces all different (a throw such as (1, 2, 3)). They both argued that the 30 throws are obtained from the product $6 \times 5$ since there are six ways to obtain the doublet and five ways to obtain the third face different from the other two. Only the author of De Vetula described how he obtained 20 as the number of distinct throws when all the faces are different. The description of how 20 is obtained is somewhat obscure and at least one manuscript of De Vetula has marginalia explaining the description. Using more modern combinatorial mathematical arguments, the 20 throws are determined as the number of combinations obtained from choosing three objects (three different faces) from six different objects (the six faces of the die). Cardano could have easily obtained this number from an arithmetical triangle, such as one given by Tartaglia [1556]. (Edwards [1987] has an extensive description of arithmetical triangles and their derivations and uses prior to Pascal’s triangle.) Both Cardano and the author of De Vetula stated that there are three ways to get the same throw with a doublet and a different face (the three throws (1, 1, 2), (1, 2, 1), and (2, 1, 1), for example), and that there are six ways to obtain the same throw with three different faces. The main difference between the two analyses, other than some excess verbiage in De Vetula, is that the author of De Vetula provided a table to show how each of the different sums of the faces that show, given in Table 1, are obtained, and Cardano did not. Another difference is in the use of language. When referring to the number of chances in the throw of the dice, Cardano used the word sortes, which relates to the number of lots, and pseudo-Ovid used the word cadentia, which relates to the number of ways the dice can fall.

The relative closeness of these two approaches may be contrasted to one given by Galileo [1952]. (David [1962] has given an English translation of this work.) Galileo arrived at the same answer as Cardano and pseudo-Ovid, Table 1, but by a different route. Galileo went to great lengths to show that there is only one way to obtain a triplet, three ways to obtain a doublet and one other face, and six ways to obtain three different faces on the dice. Like De Vetula, Galileo made a table in order to calculate the various chances for the sums on the faces that show. In its form and layout Galileo’s table is quite different from the one in De Vetula and it seems clear that he did not rely on De Vetula to solve the dicing problem.

I would put forward the interpretation that Cardano took his calculations for three dice from De Vetula and applied the same approach given in De Vetula to the discussion of a single die and then two dice. Krischer [1994] has taken the opposite view that Cardano’s calculations are not derived from De Vetula by noting, for example, that Cardano did not quote from the poem itself. Kendall [1956] implicitly has taken a similar view, asserting that the probability results in De Vetula were “rediscovered” in Cardano’s Liber de Ludo Aleae. Nevertheless, the two approaches are very close and the interpretation of dependence cannot be rejected. As a humanist, it is quite possible that Cardano read De Vetula, most likely in printed rather than manuscript form. There are two early printed publications of De Vetula, one circa 1475 in Perugia and the other four years later in Cologne [pseudo-Ovid, 1475?, 1479]. The Perugia edition does not contain any of the numerical tables, the main difference in the treatments of the problem by Cardano and the author of De Vetula. Further, the tables appear in the part of the book where the word cadentia
occurs. The 15th-century publications of De Vetula were undoubtedly part of the recovery of antiquity that characterized the Renaissance and are likely the results of printers wanting to get an “ancient” work into print. Ovid was a popular author; the British Library's Short-Title Catalogue of Books Printed in Italy shows over 40 editions of various works of Ovid prior to 1500. The two editions of De Vetula are also typical of the printers of the time; the earlier edition contained many errors and in the next edition the printer stated that corrections have been diligently made [Robathan, 1968].

8. Discussion and conclusions

Many commentators on Cardano’s Liber de Ludo Aleae have concluded that the book is a mishmash of several, and sometimes contradictory, results and statements. Rather than the mishmash it is purported to be, I would argue that, however badly written, there is an internal consistency in the text and a logical progression to the whole work. The internal consistency results from Cardano’s attempt to show the situations in which gambling could be considered a just act (ius). Moreover, Cardano provides knowledge (scientia) of various aspects of games in order both to protect oneself against injustice and to provide a situation in which the gain from gambling can be considered to be of the best type. The way in which the Liber de Ludo Aleae has been structured is an attempt to show that, no matter what the classical authors concluded about gambling and games of chance, justice has always been available in these activities. At the beginning of the text, classical and modern exempla were used through the method of in utramque partem to raise the question of justice in gambling. As the discussion proceeded, Cardano demonstrated justice in gambling mathematically for the games of his day. Justice can be maintained, in part, through scientia and he provided the necessary information. At the end of the Liber de Ludo Aleae, Cardano returned to the ancient texts. By providing a mathematical, though in modern terms probabilistically incorrect, treatment of ancient games, we are meant to conclude that the potential for justice in games of chance has always been present.

Initially, Cardano assumed equal stakes and equal chances. This may have been a prevalent assumption in the time between the composition of De Vetula and Cardano’s own time. There is precedent for this going back to antiquity. Mention has already been made of Cicero’s solution to a variation of the lifeboat problem. In that case, equal value of the individuals implied equal chances for selection. On the other hand, equal values of items can be constructed and then chosen with equal probability. An example from antiquity that illustrates this situation is the division of property from an inheritance. In ancient Greece inherited property was divided into portions of equal value. Then lots were cast to distribute the portions. (See, for example, Thalmann [1978].) By the 16th century this method of property division had even found its way in English law to settle disputed estates. (Gataker [1619] has a reference to the English system of division of property by lot.) This possible desire to construct equiprobable events may explain a passage in the Pardoner’s Tale from the Canterbury Tales. At one point in this particular tale, the pardoner says [Chaucer, 1977, p. 227],

“By the blood of Crist that is in Hayles,
Sevene in my chaunce and thyn is cynk and treye.”

The use of the word “chaunce” in the above quotation has the following interpretation: if two people are playing at dice, the outcome (or outcomes) of the throw that leads to a player winning is known as his
chance. Though Chaucer made no statement of probability, the two chances given in the quotation have equal probability: a sum of 7 showing on the faces of two dice has probability $1/6$, likewise for the event 5 or 3.

At some point there was an intellectual transition from simple lots with equal chances to groups of lots or outcomes which as a group had equal chances. In the Liber de Ludo Aleae Cardano was trying to take this process one step further. Using Aristotle’s concept of justice, Cardano tried to generalize gambling problems beyond equal stakes. His generalization was limited in that he did not go beyond the calculation of the number of outcomes of an event. This can be seen in Chapter 32 where Cardano counted the number of chances in the throw of the tali rather than calculating the probability of the throw. Within this framework of chance, it is not necessary to account for later concepts of probability such as long-term frequencies, degrees of belief, or even expectation that Cardano came close to hitting upon. Prior to Cardano’s work it was only necessary to rely on justice to set up equitable initial conditions, in particular, equal stakes and equal chances, or lots of equal size, for all players. In the same context Cardano relied on justice to handle the situation of inequitable conditions.

Acknowledgments

I thank Professor Bill Acres of Huron University College for his comments and encouragement. I also thank the students (particularly Pat Giles, Sean Roche, Mike Snowdon, and Lynne Thompson) of History 460 at the University of Western Ontario for many insightful questions and comments. I also thank the editor, Craig Fraser, and the referees for many helpful comments.

References


De Mora Charles, M.S., 1981. La teoria de la probabilidad: los primeros calculi. Llull 4, 123–141.


James, T., 1620. Catalogus universalis librorum Bibliothecae publicae quam T. Bodleianus in academia Oxoniensi nuper institut.


