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Note on a theorem of Bousfield and Friedlander

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ABSTRACT

We examine the proof of a classical localization theorem of Bousfield and Friedlander and we remove the assumption that the underlying model category be right proper. The key to the argument is a lemma about factoring in morphisms in the arrow category of a model category.

1. Introduction

Let $\mathcal{C}$ be a (Quillen) model category. A (*left*) *Bousfield localization of* $\mathcal{C}$ is another model category structure on $\mathcal{C}$ having the same class of cofibrations as the given one and a bigger class of weak equivalences. There are several methods for constructing left Bousfield localizations for (some classes of model categories) $\mathcal{C}$, see e.g. [4] and the references therein.

In their work on the construction of the stable homotopy category, Bousfield and Friedlander introduced [3, Theorem A.7] a method of localization involving an endofunctor $Q : \mathcal{C} \to \mathcal{C}$ with good enough properties. Later on, Bousfield [2, Theorem 9.3 and Remark 9.5] improved the result by weakening the hypotheses on $\mathcal{C}$ and refining the axioms that Q has to satisfy.

The purpose of this note is to further remove one of the hypotheses of the Bousfield's version of the original Bousfield and Friedlander theorem. The details are as follows. Let $\mathcal{C}$ be a model category together with a functor $Q : \mathcal{C} \to \mathcal{C}$. We say that a map f of $\mathcal{C}$ is a Q*-equivalence* if $Q(f)$ is a weak equivalence, and we say that a map is a Q*-fibration* if it has the right lifting property with respect to all the cofibrations of $\mathcal{C}$ which are Q-equivalences. An object X of $\mathcal{C}$ is Q*-fibrant* if the map $X \to 1$ is a Q-fibration. Here 1 denotes the terminal object of $\mathcal{C}$. We prove

Theorem 1.1. *Let $\mathcal{C}$ be a model category and let $\gamma : \mathcal{C} \to Ho(\mathcal{C})$ be the localization functor. Suppose that there are a functor $Q : \mathcal{C} \to \mathcal{C}$ and a natural transformation $\alpha : Id \Rightarrow Q$ satisfying the following properties*:

(A1) *the functor Q preserves weak equivalences*;
(A2) *for each $X \in \mathcal{C}$, the map $Q(\alpha_X)$ is a weak equivalence and the map $\gamma(\alpha_{Q(X)})$ is a monomorphism*;
(A3) *Q-equivalences are stable under pullbacks along fibrations between fibrant objects $f : X \to Y$ such that α_X and α_Y are weak equivalences.*

Then $\mathcal{C}$ admits a left Bousfield localization with the class of Q-equivalences as weak equivalences.

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The theorem differs from [2, Theorem 9.3] to the amount that it does not require $\mathcal{C}$ to be right proper. (The resulting model structure will be right proper because of (A3).) Its proof is a modification of the proofs given in [4, Theorem X.4.1] and [2, Theorem 9.3]. It will be given is Section 2 after few lemmas.

2. Proof of Theorem 1.1

The setting in which we shall work for the next lemmas is the following. $\mathcal{C}$ is a model category with localization functor $\gamma : \mathcal{C} \to Ho(\mathcal{C})$. We are given a functor $Q : \mathcal{C} \to \mathcal{C}$ and a natural transformation $\alpha : Id \Rightarrow Q$ satisfying the following properties:

(A1) the functor Q preserves weak equivalences;
(A2) for each $X \in \mathcal{C}$, the map $Q(\alpha_X)$ is a weak equivalence and the map $\gamma(\alpha_{Q(X)})$ is a monomorphism.

Lemma 2.1. *Let $\mathcal{K} := \{X \in \mathcal{C} \mid \alpha_X$ is an isomorphism in $Ho(\mathcal{C})\}$. We view $\mathcal{K}$ as a full subcategory of $Ho(\mathcal{C})$. Then*

(i) $Q(X) \in \mathcal{K}$ *for all* $X \in \mathcal{C}$;
(ii) $1 \in \mathcal{K}$;
(iii) $\mathcal{K}$ *is replete in* $Ho(\mathcal{C})$;
(iv) *the maps* $\gamma(Q(\alpha_X))$ *and* $\gamma(\alpha_{Q(X)})$ *are equal.*

Proof. (i) and (iii) are clear. For (ii), a result of Lambek [5, Lemma 1.1.4] shows that α_1 is actually an isomorphism in $\mathcal{C}$.

We prove now (iv). By general theory there are: (a) a functor $\hat{Q} : Ho(\mathcal{C}) \to Ho(\mathcal{C})$ such that $\hat{Q}\gamma = \gamma Q$, and (b) a natural transformation $\hat{\alpha} : Id \Rightarrow \hat{Q}$ such that $\hat{\alpha}\gamma = \gamma\alpha$. Let X be an object of $\mathcal{C}$. We have a commutative diagram

$$\begin{array}{ccc} \gamma X & \xrightarrow{\gamma\alpha_X} & \gamma Q(X) \\ {\scriptstyle \gamma\alpha_X}\downarrow & & \downarrow{\scriptstyle \gamma Q(\alpha_X)} \\ \gamma Q(X) & \xrightarrow[\gamma(\alpha_{Q(X)})]{} & \gamma Q(Q(X)). \end{array}$$

Let $g := \gamma(\alpha_{Q(X)})$, $f := \gamma Q(\alpha_X)$ and $u := f^{-1}g$. Then $u\hat{\alpha}_{\gamma X} = \hat{\alpha}_{\gamma X}$, hence $\hat{Q}(u)\hat{Q}(\hat{\alpha}_{\gamma X}) = \hat{Q}(\hat{\alpha}_{\gamma X})$, which implies that $\hat{Q}(u)$ is the identity map. The commutative diagram

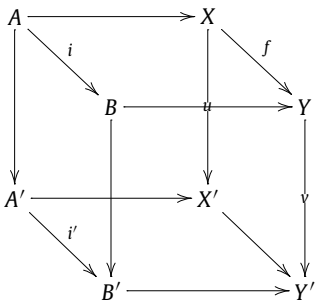
$$\begin{array}{ccc} \hat{Q}(\gamma X) & \xrightarrow{\hat{\alpha}_{\hat{Q}\gamma X}} & \hat{Q}^2(\gamma X) \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle \hat{Q}(u)} \\ \hat{Q}(\gamma X) & \xrightarrow[\hat{\alpha}_{\hat{Q}\gamma X}]{} & \hat{Q}^2(\gamma X) \end{array}$$

implies then that u is the identity, and therefore the maps $\gamma(Q(\alpha_X))$ and $\gamma(\alpha_{Q(X)})$ are equal. □

Lemma 2.2. *A map of $\mathcal{C}$ is a trivial fibration iff it is a Q-fibration and a Q-equivalence.*

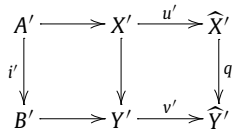
Proof. This is [4, Lemma X.4.3]. □

Lemma 2.3. *Let*

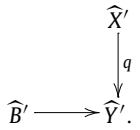


be a (commutative) cube diagram in a model category $\mathcal{E}$. Suppose that i is a cofibration, f is a fibration between fibrant objects and i', u and v are weak equivalences. Then the top face of the cube has a diagonal filler.

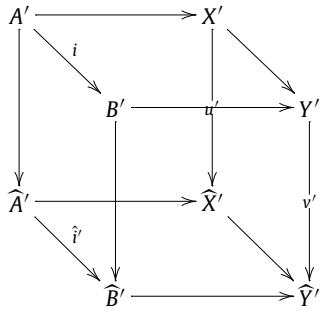
Proof. Consider the diagram



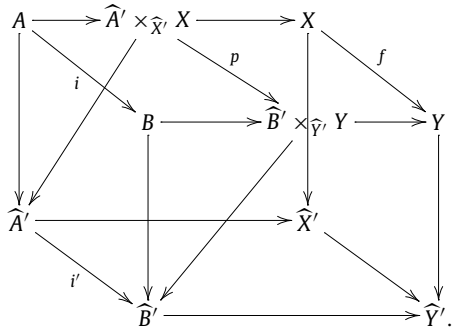
where u' and v' are trivial cofibrations and q is a fibration between fibrant objects. We factor the composite map $B' \to \widehat{Y}'$ as a trivial cofibration $B' \to \widehat{B}'$ followed by a fibration $\widehat{B}' \to \widehat{Y}'$ and then take the pullback P of the diagram



We factor the canonical map $A' \to P$ as a trivial cofibration $A' \to \widehat{A}'$ followed by a fibration $\widehat{A}' \to P$ and we obtain a commutative cube



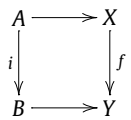
in which the maps $\widehat{A}' \to \widehat{X}'$ and $\widehat{B}' \to \widehat{Y}'$ are fibrations between fibrant objects and the map $\widehat{i'}$ is a weak equivalence. Composing the above cubes and then taking the pullbacks of the front and back new faces results in a commutative diagram



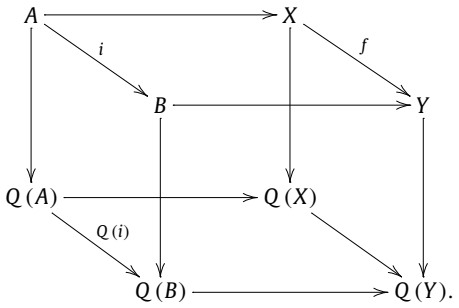
It follows that the map p is a weak equivalence. As such, p has a factorization qj, where j is a trivial cofibration and q is a trivial fibration. Since i was a cofibration and f a fibration, the top face of the original cube diagram has a diagonal filler. □

Lemma 2.4. *A cofibration of $\mathcal{C}$ is a Q-equivalence iff it has the left lifting property with respect to every fibration between fibrant objects which belong to $\mathcal{K}$.*

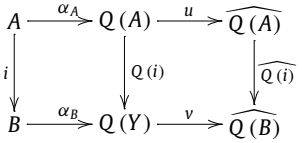
Proof. ($\Rightarrow$) Let



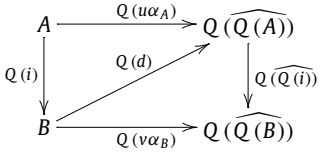
be a commutative diagram with i a cofibration Q-equivalence and f a fibration between fibrant objects which belong to $\mathcal{K}$. Apply the previous lemma to the cube diagram



($\Leftarrow$) Let $i : A \to B$ be a cofibration of $\mathcal{C}$ which has the left lifting property with respect to every fibration between fibrant objects which belong to $\mathcal{K}$. Consider the diagram



where u and v are trivial cofibrations and $\widehat{Q(i)}$ is a fibration between fibrant objects. By hypothesis the outer diagram has a diagonal filler d. Applying Q to the previous diagram we obtain a diagram

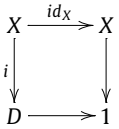


in which both horizontal arrows are weak equivalences. By the two out of six property of weak equivalences it follows that $Q(d)$ is a weak equivalence, hence i is a Q-equivalence. □

Lemma 2.5. (i) *An object X of $\mathcal{C}$ is Q-fibrant iff X is fibrant and $X \in \mathcal{K}$.*

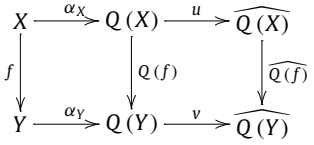
(ii) *A map between Q-fibrant objects is a Q-fibration iff it is a fibration.*

Proof. (i) If X is fibrant and α_X is a weak equivalence then by Lemmas 2.1 and 2.4 we conclude that X is Q-fibrant. Conversely, let X be Q-fibrant. We factor the map α_X as pi, where $i : X \to D$ is a cofibration and $p : D \to Q(X)$ is a trivial fibration. Then i is a Q-equivalence, so the diagram



has a diagonal filler. Consequently, α_X is a retract of α_D. But $D \in \mathcal{K}$ by Lemma 2.1. Part (ii) follows from (i) and Lemma 2.4. □

Proof of Theorem 1.1. Since we have Lemma 2.2 it only remains to show that every arrow $f : X \to Y$ of $\mathcal{C}$ can be factored into a cofibration Q-equivalence followed by a Q-fibration. The proof follows exactly the proof of [2, Theorem 9.3] with the difference that we appeal to Lemma 2.5. To make things clear we repeat the argument. Consider the diagram



where u and v are trivial cofibrations and $\widehat{Q(f)}$ is a fibration between fibrant objects. The map $\widehat{Q(f)}$ is a Q-fibration by Lemma 2.5(ii). We pull it back along the Q-equivalence $v\alpha_Y$ to obtain a Q-fibration $g : E \to Y$ such that the map $E \to \widehat{Q(X)}$ is a Q-equivalence by (A3). Therefore the canonical map $X \to E$ is a Q-equivalence. We factor it into a cofibration j followed by a trivial fibration p, and then $f = (gp)j$ is the desired factorization of f. □

Remark 2.6. If $\mathcal{C}$ is a combinatorial model category and Q is an accessible functor, then it follows from Smith's theorem [1, Theorem 1.7] that the conclusion of Theorem 1.1 remains valid without imposing the axiom (A3).

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