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# On the Structure of Averaging Operators

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## INTRODUCTION

Let $A$ be a commutative topological algebra over the real field. An averaging operator on $A$ is a linear continuous operator $T$ which satisfies the following identity: $T(xTy) = (Tx)(Ty)$. Such operators were first used (implicitly) by Reynolds [1] in connection with the theory of turbulence. In more recent times these operators have been discussed by Kampé de Fériet [2, 3] for various spaces of functions. A number of characterizations of these operators have appeared [2–7].

An important class of averaging operators used in turbulence theory is the class of averages over one portion of space-time of certain vectors fields. For example,

$$\bar{f}(\mathbf{x}, t) = \lim_{T\to\infty} \frac{1}{2T} \int_{t-T}^{t+T} f(\mathbf{x}, \tau)\, d\tau,$$

the time average of a real function $f$ defined on space-time is such an average. In case $f(\mathbf{x}, \tau)$ is an integrable function of $\tau$ which is bounded above or below by a constant, then $\bar{f}(\mathbf{x}, t)$ does not depend on $t$. In general one might ask the following question: Given an algebra $A$ which admits a representation as a ring of functions, and given an averaging operator $T$ on $A$, under what circumstances can $A$ be represented as functions on a product space such that $T$ is an integration over one factor of the product space? Part II of this paper is devoted to this question (Theorem 7.1).

Because of the relationship between integration theory and the theory of averaging operators as it is applied to turbulence and to probability [6], we consider the case where the averaging operator is defined on an abstract analogue of the ring of real valued measurable functions. The author has discussed such rings ($F$-rings) in previous papers [8, 9]. An *F-ring* is a com-

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mutative $\sigma$-complete lattice-ordered ring with an identity 1 which is positive and is a weak order unit, that is, $1 \wedge x = 0 \Rightarrow x = 0$. An $F$-ring $R$ is *bounded* if for each $x \in R$, there is a real number $\lambda$ such that $|x| \leqslant \lambda \cdot 1$.

An operator $T$ on a bounded $F$-ring $R$ is called a *Reynolds operator* if the following conditions are satisfied:

$T_1$. The operator $T$ is linear on $R$.
$T_2$. For $x \in R$, $x \geqslant 0 \Rightarrow Tx \geqslant 0$.
$T_3$. If $x_n \in R$ for $n \geqslant 1$ and $x_n \downarrow 0$, then $Tx_n \downarrow 0$.
$T_4$. For $x, y \in R$, $T(xTy) = (Tx)(Ty)$.

A Reynolds operator is a special case of the Reynolds endomorphism of Rota [10].

A Reynolds operator has a number of the properties of an integral and is in a certain sense a generalization of the integral. In Part I of this paper, using the theory of integration as a model, we given a method for extending a Reynolds operator from a bounded $F$-ring to the analogue of the $L^1$-space of integrable functions.

The relationship between Reynolds operators and conditional expectations on $L^2$-space is discussed in Part III.

In the sequel, the usual lattice theory notation is used. The symbol $R$ is used to denote a bounded $F$-ring with elements $x, y, z, \cdots$, and $T$ is used to denote a Reynolds operator on $R$. In [8, p. 675] it is shown that if $A$ is an arbitrary $F$-ring, the set $B(A) = \{x \in A \mid x = x^2\}$ is a $\sigma$-complete Boolean algebra with respect to the order relation of $A$, and by [9, p. 545] if $B$ is a $\sigma$-complete Boolean algebra, there is a regular $F$-ring $\hat{R}(B)$ which is unique up to isomorphism such that $B(\hat{R}(B)) = B$. In addition, if $B(A)$ is isomorphic to $B$, then $A$ can be embedded in $\hat{R}(B)$ in such a way that the embedding injection is an extension of the isomorphism of $B(A)$ onto $B$. In every $F$-ring $A$ there is a unique bounded $F$-subring composed of all the $x \in A$ for which a real number $\lambda$ exists such that $|x| \leqslant \lambda \cdot 1$. The unique bounded $F$-subring of $\hat{R}(B)$ is denoted $R(B)$. The notations $R(B)$, $\hat{R}(B)$, and $B(A)$ are used in the sequel. An *l-ideal* $L$ of an $F$-ring $R$ is a subgroup of $R$ with the property: $x \in L$ and $|y| \leqslant |x|$ imply $y \in L$.

A *$\sigma$-ideal $N$ of an $F$-ring $R$* is a set which is a ring ideal and an $l$-ideal, and which is closed with respect to countable suprema. A *$\sigma$-ideal of a Boolean algebra* is an ideal which is closed under countable suprema.

A subset $A$ of a $\sigma$-complete Boolean algebra $B$ is said to be a *$\sigma$-complete Boolean subalgebra* if $A$ forms a $\sigma$-complete Boolean algebra with respect to both the finite and the countable operations of $B$.

If $A$ is a subset of an $F$-ring, then $A^+$ stands for the set of nonnegative elements of $A$. If $x$ is an element of an $F$-ring, then $\bar{e}_x = \sup_n (n |x| \wedge 1)$ and $e_x = 1 - \bar{e}_x$. In [8] it is shown that for each $x$ in a regular $F$-ring $\hat{R}$,

$xe_x = 0$, $e_x = e_x^2$, and $(e_x + x)^{-1} \in R$. The element $e_x$ is uniquely determined by $x$.

In an $F$-ring $A$, $x_n \to x$ or $\lim_n x_n = x$ means that there exists a sequence $l_n \in A$ such that $l_n \downarrow 0$ and for each $n$, there exists an $N(n)$ with the property:

$$m \geqslant N(n) \Rightarrow | x\mu - x | \leqslant l_n.$$

A sequence $\{x_n\}$ is said to approach $x$ *uniformly* if in the above definitive $l_n = \lambda_n \cdot 1$ where $\{\lambda_n\}$ is a sequence of real numbers decreasing to zero.

A Boolean algebra $B$ satisfies the *countable chain condition* if every subset composed of disjoint elements of $B$ is at most countable. An $F$-ring $A$ is said to satisfy the countable chain condition if $B(A)$ satisfies it.

A function $\mu$ on a Boolean algebra $B$ into the real field is a *measure* if it is nonnegative, $\sigma$-additive, and if $\mu(1) = 1$. The function $\mu$ is said to be $\sigma$-additive if for a disjoint set $\{a_n\}_{n=1}^{\infty}$ with $\sup_n a_n \in B$,

$$\sum_{n=1}^{\infty} \mu(a_n) = \mu(\sup_n a_n).$$

All $F$-ring homomorphisms are assumed to preserve countable operations. The term $\sigma$-homomorphism is used, with regard to $\sigma$-complete Boolean algebras, for those homomorphisms which preserve countable operations.

If $\mu$ is a mapping of a set $Q$ into a set $M$ and if $J$ is a subset of $Q$, then $\mu_{|J}$ designates the restriction of $\mu$ to $J$.

## Part I

### 1. *General Structure of a Reynolds Operator*

Let $T$ be a Reynolds operator on a bounded $F$-ring $R$. Let $TR$ stand for the range of $T$, and let

$$E = \{x \in R \mid \forall_{z \in R}[T(xz) = xT(z)]\}.$$

In this section the relationship between $E$ and $TR$ is discussed. In particular it is shown that $T1 = 1$ if and only if $E = TR$ and $T$ is idempotent.

It is a matter of direct verification that (i) the sets $TR$ and $E$ are subalgebras of $R$, (ii) $TR$ is a subalgebra of $E$, (iii) the identity 1 of $R$ belongs to $E$, and (iv) $E = \{x \in R \mid \forall_{z \geqslant 0, z \in R}[T(xz) = xT(z)]\}$. Let $B(E)$ stand for the class of idempotents of $E$, and for $y \in R$ let $R_y$ denote the smallest $F$-subring of $R$ containing $y$. In order to show that $E$ is an $F$-subring of $R$, the following Lemmas are proved.

LEMMA 1.1. *The set $B(E)$ is a $\sigma$-complete Boolean subalgebra of $B(R)$.*

PROOF. Since for $e_1, e_2 \in B(R)$,

$$\inf(e_1, e_2) = e_1 e_2 \qquad \text{and} \qquad \sup(e_1, e_2) = e_1 + e_2 - e_1 e_2,$$

it follows that $B(E)$ is a Boolean subalgebra of $B(R)$. If $\{e_n\}$ is a sequence of elements of $B(E)$, then $\{a_n = \inf(e_1, \cdots, e_n) \mid n \geqslant 1\}$ is a non-increasing sequence of elements of $B(E)$. Thus, for $0 \leqslant x \in R$ and $a = \inf_n a_n$, the following statement follows from Condition $T_3$:

$$aTx = \inf_n a_n Tx = \inf_n Ta_n x = Tax.$$

Since $\inf_n a_n = \inf_n e_n$, it follows that $\inf_n e_n \in B(E)$. A similar argument shows that $\sup_n e_n \in B(E)$, and thus $B(E)$ is a $\sigma$-complete Boolean subalgebra of $B(R)$.

LEMMA 1.2. *If $y \in E$, then $R_y$ is a subset of $E$.*

PROOF. Since the proofs of Lemmas 1.2 and 1.3 of [6] can be rendered in the setting of this paper, it follows that if $x_n \to x$ uniformly, then $Tx_n \to Tx$. Thus $E$ is closed under uniform limits. By the Stone-Weierstrass Theorem, $R_y$ is composed of uniform limits of polynomials in $y$. Thus $R_y$ is a subset of $E$.

PROPOSITION 1.1. The set $E$ is an $F$-subring of $R$ and $E = R(B(E))$.

PROOF. As a corollary of [9, Theorem 5], it can be shown that every nonnegative element of a bounded $F$-ring $Q$ is the supremum of a nondecreasing sequence of finite linear combinations of idempotents of $Q$. Thus by Lemma 1.2, if $y \in E$, then $B(R_y) \subseteq R_y \subseteq E$, and hence $R_y \subseteq R(B(E))$. Therefore $E \subseteq R(B(E))$ because $y \in R_y$ for each $y \in R$.

Since the supremum of a nondecreasing sequence of elements of $E$ belongs to $E$ when it exists, $R(B(E)) \subseteq E$.

PROPOSITION 1.2. The following conditions are equivalent:

(1) $E = TR$.

(2) $(T1)^{-1}$ exists in $E$.

PROOF. If $E = TR$, then $1 \in TR$. Hence there is an element $x \in R$ such that $Tx = 1$. Since $x \leqslant |x| \leqslant \lambda \cdot 1$ for some real number $\lambda \geqslant 0$, $Tx = 1 \leqslant \lambda T(1)$. Thus by [8, p. 676], $(T1)^{-1}$ belongs to $E = R(B(E))$.

Conversely, if $x \in E$, then $xT1 = Tx$, and if $(T1)^{-1} \in E$, then

$$x = (Tx)(T1)^{-1} = T(x(T1)^{-1}) \in TR.$$

Hence $E \subseteq TR$. Therefore, by Statement (ii) at the beginning of the section, it follows that $E = TR$.

REMARK 1.1. Note that $(T1)^{-1} \in E$ if and only if $T1$ is a strong order unit of $E$, that is, if $x \in E$, then there is a $\lambda \geqslant 0$ such that $|x| \leqslant \lambda T1$.

PROPOSITION 1.3. The following conditions are equivalent:

(1) $T1 = 1$.

(2) $E = TR$ and $T = T^2$.

(3) For $x \in E$, $Tx = x$.

PROOF. If $T1 = 1$, then $T(Tx) = T(1 \cdot Tx) = T1 \cdot Tx = Tx$ and hence $T$ is idempotent. Since $1 = T1$, it follows from Proposition 1.2 that $TR = E$.

If (2) is valid, then for $x \in E$ there is a $y \in R$ such that $x = Ty$ and $Tx = T^2y = Ty = x$. Therefore (3) follows. From (3), (1) follows because $1 \in E$ by the definition of $E$.

*Example 1.1.* Let $R$ be the $F$-ring of bounded Lebesgue measurable functions on $(-\infty, +\infty)$. The following transformations on $R$ are Reynolds operators

(i) $f(x) \to \exp(-x^2) f(x)$,

(ii) $f(x) \to \frac{1}{2} f(x)$,

(iii) $f(x) \to f(x)$.

In case (i), $(T1)^{-1}$ does not belong to $E$; in case (ii), $(T1)^{-1}$ belongs to $E$ but $T1 \neq 1$; and in case (iii), $T1 = 1$.

If $(T1)^{-1} \in E$, then $T$ can be replaced without loss of generality by a new Reynolds operator $T_0$ for which $T_0 1 = 1$.

PROPOSITION 1.4. If $TR = E$, then the transformation

$$T_0 f = T[(T1)^{-1} f]$$

is a Reynolds operator on $R$, and $T_0 1 = 1$. In addition $Tf = T_0[(T1)f]$.

PROOF. By Proposition 1.2, $(T1)^{-1} \in E$. It is then a matter of direct verification to show that if $T$ is a Reynolds operator, then $T_0$ is as well. Since

$$T_0 1 = T[(T1)^{-1} 1] = (T1)^{-1} \cdot T1 = 1$$

and

$$T_0[(T1)f] = T[(T1)^{-1}(T1)f] = Tf,$$

the proposition is valid.

The following example exhibits another simplification which can be made on some occasions.

*Example 1.2.* Let $R$ be as in Example 1.1 and let the Reynolds operator be

$$f \to \int \chi_{[0,1]} f(x)\, dx$$

where $\chi_A$, represents the characteristic function on the set $A$. In this example not all the information in $f$ is utilized by the Reynolds operator. Only changes in $f$ on $[0, 1]$ will affect the values of $Tf$.

More generally, if $Tf = \bar{f}$, then for every $g \in f + K_T$, $Tg = \bar{f}$ where $K_T = \{x \in R \mid T \mid x \mid = 0\}$.

PROPOSITION 1.5. The set $K_T$ is a $\sigma$-ideal of $R$, and if $R$ satisfies the countable chain condition, then $R$ can be written as a direct sum of two $F$-rings $R_1$ and $R_2$ where $K_T = R_2$. For $x \in R_1$, $T \mid x \mid = 0$ if and only if $x = 0$.

PROOF. $K_T$ is clearly an $l$-ideal of $R$ which is closed with respect to countable suprema. It is only necessary to verify that $x \in K_T$ and $y \in R \Rightarrow xy \in K_T$. For $y \in R$ there is a real number $\lambda$ such that $\mid y \mid \leqslant \lambda \cdot 1$. Thus

$$0 \leqslant T \mid xy \mid = T(\mid x \mid \mid y \mid) \leqslant T(\lambda \mid x \mid) = \lambda T \mid x \mid = 0.$$

If $R$ satisfies the countable chain condition, then $\sup \{B(R) \cap K_T\} = e$ belongs to $B(R)$. Since, in the presence of the countable chain condition [11, p. 161], every supremum of a set $A$ can be replaced by the supremum of a countable subset of $A$, it follows that $e \in K_T$. If $x \in K_T$, then $\bar{e}_x \leqslant e$ and hence $K_T = Re$. Let $R_1$ be the $F$-ring $R(1 - e)$ with identity $(1 - e)$ and let $R_2 = Re$ with identity $e$. Then $R = R(1 - e) \oplus Re$ and the proposition follows.

Thus there is no less of generality if, when $R$ satisfies the countable chan condition, we assume $T1 = 1$ and $T \mid x \mid = 0 \Rightarrow x = 0$.

## 2. *Extension of Reynolds Operators*

As indicated in the Introduction, every bounded $F$-ring $R$ can be extended to a regular $F$-ring $\hat{R}$ which is unique up to isomorphism such that $B(R) = B(\hat{R})$. In general, a Reynolds operator $T$ on $R$ can be extended to an $F$-subspace $L_T$ of $\hat{R}$, that is, an $l$-ideal of $\hat{R}$ which contains 1. This extension enables us to analyse further the case where $TR$ is a proper subset of $E$. The extension is carried out for a general Reynolds operator $T$ on a bounded $F$-ring $R$ without invoking the countable chain condition.

Every element $f \in \hat{R}^+$ is the supremum of an nondecreasing sequence of elements of $R^+$: $f \wedge N \uparrow f$ where $N$ takes positive integer values. Let

$\hat{T}f = \sup_N T(f \wedge N)$ if it exists in $\hat{R}$. For a general element $f$ of $\hat{R}$, define

$$\hat{T}f = \hat{T}f^+ - \hat{T}f^-$$

if both $\hat{T}f^+$ and $\hat{T}f^-$ exist. Let $L_T$ be the set of all $f \in \hat{R}$ for which $\hat{T}f$ is defined. It is clear that for $f \in R$, $Tf = \hat{T}f$ and that $R \subseteq L_T$.

For $\hat{R}(B(E))$, the regular $F$-ring generated by $B(E)$, if $f \in \hat{R}(B(E))$, then $f^+ \wedge N$ and $f^- \wedge N$ both belong to $E$ by Proposition 1.1. Thus

$$(f^+ \wedge N)\, Tz = T[(f^+ \wedge N)\, z]$$

for all $z \geqslant 0$ in $R$. Since $\sup_N (f^+ \wedge N)\, T1$ exists in $\hat{R}$, it follows that $\hat{T}f^+$ exists. In a similar manner it can be shown that $\hat{T}f^-$ exists. Therefore, *$\hat{R}(B(E))$ is a subset of $L_T$.*

Lemma 2.1. *The mapping $\hat{T}$ is nonnegative and linear, and $L_T$ is an $F$-subspace of $\hat{R}$.*

Proof. By definition $\hat{T}$ is nonnegative.

To show that $\hat{T}$ is additive, consider $x, y \in L_T^+$. By using the measurable function representation of $\hat{R}$ (see [9]), we can show that

$$(x + y) \wedge N \leqslant x \wedge N + y \wedge N \leqslant x + y.$$

Thus

$$T[(x + y) \wedge N] \leqslant T(x \wedge N) + T(y \wedge N) \leqslant \hat{T}x + \hat{T}y$$

for each $N$, and so $\hat{T}(x + y)$ exists and is less than or equal to $\hat{T}x + \hat{T}y$.

Since $\hat{T}(x + y)$ exists and since $x \wedge N + y \wedge M \leqslant x + y$,

$$T(x \wedge N) + T(y \wedge M) \leqslant \hat{T}(x + y)$$

for all $N$ and $M$. Therefore

$$\hat{T}(x + y) = \hat{T}x + \hat{T}y.$$

If $x, y \in L_T$, then

$$\begin{aligned} \hat{T}x + \hat{T}y &= \hat{T}x^+ + \hat{T}y^+ - \hat{T}x^- - \hat{T}y^- \\ &= \hat{T}(x^+ + y^+) - \hat{T}(x^- + y^-). \end{aligned}$$

Since $(x + y)^\pm \leqslant x^\pm + y^\pm$, it follows that $x + y \in L_T$.

In general, if $f \leqslant 0 \leqslant g$, $f, g \in L_T$, and $f + g = h \in L_T$, then

$$h + (-f) = g \qquad \text{and} \qquad \hat{T}(h + (-f)) = \hat{T}g = \hat{T}h + \hat{T}(-f).$$

Since $f \leqslant 0$, $\hat{T}f = \hat{T}f^+ - \hat{T}f^- = -\hat{T}(-f)$. Therefore $\hat{T}g = \hat{T}h - \hat{T}f$ and hence $\hat{T}h = \hat{T}f + \hat{T}g$.

Since $-(x^- + y^-) \leqslant 0 \leqslant x^+ + y^+$, we can apply the results of the previous paragraph to yield

$$\hat{T}(x + y) = \hat{T}(x^+ + y^+) - \hat{T}(x^- + y^-) = \hat{T}x + \hat{T}y.$$

Thus $\hat{T}$ is additive and from the previous paragraph it can be deduced that $L_T$ is an additive subgroup of $\hat{R}$.

To show that $\hat{T}$ is homogeneous let $x \in L_T$ and let $\alpha \geqslant 0$ be a real number. Then $\alpha(x \wedge N/_\alpha) = \alpha x \wedge N$, $\alpha T(x \wedge N/_\alpha) = T(\alpha x \wedge N)$, and so $\alpha\hat{T}x = \hat{T}(\alpha X)$. If $x \in L_T$ is not assumed nonnegative, then $(\alpha x)^\pm = \alpha x^\pm$ and $\hat{T}\alpha x = \alpha\hat{T}x$. On the other hand, if $\alpha < 0$, then $(\alpha x)^+ = -\alpha x^-$ and $(\alpha x)^- = -\alpha x^+$. Thus $\hat{T}(-\alpha x^-) = -\alpha\hat{T}x^-$ and $\hat{T}(-\alpha x^+) = -\alpha\hat{T}x^+$. Since $-\alpha x^\pm \in L_T$, it follows that $\alpha x \in L_T$ and

$$\begin{aligned}\hat{T}\alpha x &= \hat{T}(\alpha x)^- - \hat{T}(\alpha x)^- \\ &= -\alpha\hat{T}x^- - (-\alpha)\,\hat{T}x^+ \\ &= \alpha(\hat{T}x^+ - \hat{T}x^-) \\ &= \alpha\hat{T}x.\end{aligned}$$

Thus $\hat{T}$ is homogeneous, and hence $L_T$ is a linear subspace of $\hat{R}$.

If $\hat{T}x$ exists, then $\hat{T}\,|\,x\,|$ exists; and for $|\,y\,| \leqslant |\,x\,|$, $\hat{T}y$ exists. Therefore $L_T$ is an $l$-ideal of $\hat{R}$ and hence is an $F$-subspace of $\hat{R}$.

Lemma 2.2. *If $x_n \uparrow x$ where $x \in L_T$ and $x_n \geqslant 0$, then $\hat{T}x_n \uparrow \hat{T}x$.*

Proof. Since $x \in L_T$, $\hat{T}x_n \leqslant \hat{T}x$ and $y = \sup_n \hat{T}x_n$ exists. Then

$$\begin{aligned}y &= \sup_n \sup_N T(x_n \wedge N) \\ &= \sup_N \sup_n T(x_n \wedge N) \\ &= \sup_N T(x \wedge N) \\ &= \hat{T}x\end{aligned}$$

by Condition $T_3$ and [12, Theorem 25].

Thus the mapping $\hat{T}$ from $L_T$ into $R$ satisfies Conditions $T_1$ through $T_3$. We now show $T_4$ is valid for $\hat{T}$ as well. First, however, we prove the following Proposition.

Proposition 2.1. Every element $x \in \hat{R}(B(E))$ satisfies the relation $x\hat{T}z = \hat{T}(xz)$ for all $z \in L_T$. In addition, if $y \in L_T$ and $x \in \hat{R}(B(E))$, then

$xy \in L_T$. That is, $L_T$ is a module over $\hat{R}(B(E))$, and $\hat{T}$ is an $\hat{R}(B(E))$-endomorphism.

PROOF. It has already been noted that $\hat{R}(B(E))$ is a subset of $L_T$. We begin by showing that $xy \in L_T$ if $x \in \hat{R}(B(E))$ and $y \in L_T$. Since

$$xy = x^+y^+ - x^+y^- - x^-y^+ + x^-y^-$$

and since $L_T$ is an $l$-ideal of $R$, it is sufficient to show that $xy \in L_T$ when $x \wedge y \geqslant 1$. In this case

$$xy \wedge N \leqslant (x \wedge N)(y \wedge N)$$

for $N = 1, 2, \cdots$, and

$$T(xy \wedge N) \leqslant T[(x \wedge N)(y \wedge N)].$$

Now

$$T[(x \wedge N)(y \wedge N)] = (x \wedge N)\, T(y \wedge N)$$

because $x \wedge N \in R(B(E)) = E$. Hence

$$T(xy \wedge N) \leqslant x\hat{T}y$$

for all $N$, and $\hat{T}xy$ exists. Thus $xy \in L_T$ and therefore $L_T$ is an $\hat{R}(B(E))$-module.

To show that $\hat{T}(xz) = x\hat{T}(z)$ for all $x \in \hat{R}(B(E))$ and $z \in L_T$, note that $TR \subseteq E = R(B(E))$, and hence for $y \in L_T^+$,

$$\hat{T}y = \sup_N T(y \wedge N)$$

belongs to $\hat{R}(B(E))$. This remark follows because $\hat{R}(B(E))$ is composed of the suprema of sequences of elements of $R(B(E))$. Thus $\hat{T}L_T \subseteq \hat{R}(B(E))$, and $\hat{T}$ maps $L_T$ into $\hat{R}(B(E)) \subseteq L_T$.

If $y \in L_T^+$ and $x \in \hat{R}(B(E))^+$, then by the previous paragraph $xy \in L_T^+$, and since $(x \wedge N)(y \wedge N) \uparrow xy$,

$$\hat{T}xy = \sup_N T[(x \wedge N)(y \wedge N)] = \sup_N (x \wedge N)\, T(y \wedge N)$$

and hence

$$\hat{T}xy = x\hat{T}y. \tag{2.1}$$

By a standard argument it follows that Eq. (2.1) is valid without the restriction that $x$ and $y$ be nonnegative. Therefore $\hat{T}$ is an $\hat{R}(B(E))$-endomorphism on $L_T$.

THEOREM 2.1. *The extension $\hat{T}$ of $T$ is an $\hat{R}(B(E))$-endomorphism of $L_T$ which satisfies Conditions $T_1$ through $T_4$ relative to $L_T$. In addition,*

$$\hat{T}L_T \subseteq \hat{R}(B(E)) \subseteq L_T.$$

PROOF. The final remark in the statement of the theorem follows from the proof of Proposition 2.1. It has already been proved in Lemmas 2.1 and 2.2 and in Proposition 2.1, that all the conclusions of the first part of the theorem are valid except for the one which asserts that Condition $T_4$ holds.

To prove $T_4$ for $\hat{T}$, consider $x, y \in L_T^+$. Then

$$\begin{aligned}\hat{T}x\hat{T}y &= \sup_N \hat{T}(x \wedge N) \sup_M \hat{T}(y \wedge M)\\ &= \sup_N \sup_M T[(x \wedge N)\, T(y \wedge M)]\\ &= \hat{T}(x\, \hat{T}y)\end{aligned}$$

by Lemma 2.2 and [12, Theorem 2.6]. Since

$$x\hat{T}y = x^+\hat{T}y^+ + x^-\hat{T}y^- - x^+\hat{T}y^- - x^-\hat{T}y^+,$$

it follows from the above remark that

$$\hat{T}(x\, \hat{T}y) = (\hat{T}x)\, (\hat{T}y).$$

Therefore $T_4$ is valid for $\hat{T}$ and the theorem is proved.

Analogous to Proposition 1.2 we have the following:

PROPOSITION 2.2. If $T$ is a Reynolds operator on $R$, then $\hat{T}L_T = \hat{R}(B(E))$ if and only if $T1$ is a weak unit of $R$.

PROOF. If $T1$ is a weak unit of $R$, then it is a weak unit of $\hat{R}$ and hence has an inverse in $\hat{R}$ by [8, Theorem 1]. Since $T1 \in E^+$, $(T1)^{-1}$ belongs to $\hat{R}(B(E))^+$. Therefore $\hat{T}[(T1)^{-1}] = (T1)^{-1}T1 = 1$. If $f \in \hat{R}(B(E))^+$, then

$$f(T1)^{-1} \in \hat{R}(B(E))^+ \qquad \text{and} \qquad \hat{T}[f(T1)^{-1}] = f\hat{T}[(T1)^{-1}] = f.$$

Thus $f \in \hat{T}L_T$. From Theorem 2.1 it follows that $\hat{T}L_T = \hat{R}(B(E))$.

Conversely, if $\hat{T}L_T = \hat{R}(B(E))$, then there is an $x \in L_T$ such that $\hat{T}x = 1$. Therefore if

$$1 - e_{T1} = \sup_n (n \mid T1 \mid \wedge 1),$$

then $\hat{T}(xe_{T1}) = e_{T1}$ and $T(e_{T1}) = 0$. Since both $x^+$ and $x^-$ are suprema of sequences of nonnegative finite linear combinations of elements of $B(R)$ and since $\hat{T}$ preserves order,

$$\hat{T}(x^+e_{T1}) = \hat{T}(x^-e_{T1}) = 0$$

by Lemma 2.2. Therefore $e_{T1} = 0$, $(T1)^{-1}$ exists in $R$, and $T1$ is a weak order unit of $\hat{R}$.

PROPOSITION 2.3. If $\hat{T}L_T = \hat{R}(B(E))$, then the operator

$$Qf = \hat{T}[(T1)^{-1}f]$$

has range $\hat{R}(B(E))$ and domain $L_T$, and satisfies the following conditions:

(i) $Q$ satisfies $T_1$ through $T_4$,
(ii) $Q1 = 1$,
(iii) $Tf = Q[(T1)f]$.

PROOF. If one observes that $0 \leqslant (T1)^{-1} \in \hat{R}(B(E))$ and $\hat{T}[(T1)^{-1}] = 1$, then the proposition can be verified directly.

PROPOSITION 2.4. $Q^2 = Q$.

PROOF. Follows directly from Proposition 2.3 parts (i) and (ii).

Thus the study of $\hat{T}$ can be reduced to the study of an operator $Q$ which has the properties of $\hat{T}$ and which in addition is idempotent and carries 1 into 1.

Finally we give a necessary and sufficient condition for $\hat{T}$ to be positive under the hypothesis that $\hat{T}L_T = \hat{R}(B(E))$. Remember that an operator $T$ is *positive* if $x > 0 \Rightarrow Tx > 0$.

PROPOSITION 2.5. If $\hat{T}L_T = \hat{R}(B(E))$, then $\hat{T}$ is positive if and only if $\bar{e}_{\hat{T}e} \geqslant e$ for all $e \in B(R)$.

PROOF. If $\hat{T}$ is positive, then $\hat{T}\,|f| = 0 \Rightarrow f = 0$. Hence,

$$e_{\hat{T}e}\hat{T}e = 0 \Rightarrow \hat{T}(e_{\hat{T}e} \cdot e) = 0 \Rightarrow e_{\hat{T}e} \cdot e = 0.$$

Thus $e \leqslant \bar{e}_{\hat{T}e}$.

Conversely, let $e \leqslant \bar{e}_{\hat{T}e}$ for all $e \in B(R)$. Since $\hat{T}\,|f| = 0$ implies $\hat{T}(n\,|f| \wedge 1) = 0$ and hence $\hat{T}\bar{e}_f = 0$, it follows that $\bar{e}_{Te_f} = 0$. Therefore $\bar{e}_f = 0$, and $f = f\bar{e}_f = 0$.

PROPOSITION 2.6. If $T$ is positive, then $\hat{T}L_T = \hat{R}(B(E))$.

PROOF. Since $T1 \cdot e_{T1} = 0$, it follows from Theorem 2.1, that $T(1 \cdot e_{T1}) = Te_{T1} = 0$ and hence $e_{T1} = 0$. Therefore $(T1)^{-1}$ is a weak order unit, and by Proposition 2.2, $\hat{T}L_T = \hat{R}(B(E))$.

On the other hand, $T$ positive does not imply $TR = E$:

*Example 2.1.* Let $R$ be the $F$-ring of bounded sequences and let $Ts = q$ where $q_n = s_n \exp(-n^2)$.

## Part II

### 3. *A Precise Statement of a Problem*

Let $R$ be a bounded $F$-ring and let $T$ be a positive Reynolds operator on $R$ for which $T1 = 1$. It is clear from the results of Section 2 that the only essential restriction made here over the general case is that which requires $T$ to be positive. Indeed if $T$ is positive, $\hat{T}$ is positive, and by Propositions 2.3, 2.4, and 2.6, there is an operator $Q$ associated with $\hat{T}$ such that $Q$ is idempotent, $Q1 = 1$, and $\hat{T}(x) = Q[(T1)\,x]$. The restriction of $Q$ to $R$ is still idempotent and carries 1 into 1.

It is well known [9] that $R$ can be represented as an $F$-ring of measurable functions modulo a $\sigma$-ideal. The construction goes roughly as follows: Associated with the $\sigma$-complete Boolean algebra $B = B(R)$ of idempotents of $R$ is its Stone representation $\langle \Omega, \tilde{B}, \sigma \rangle$ where $\Omega$ denotes the Stone space of $B$, $\tilde{B}$ the field of the open-closed subsets of $\Omega$, and $\sigma$ the isomorphism of $B$ onto $\tilde{B}$. The field $\tilde{B}$ generates a $\sigma$-field $\tilde{\mathfrak{L}}$ of subsets of $\Omega$, and if $\mathfrak{N}$ denotes the class of sets of first category in $\tilde{\mathfrak{L}}$, then there is a $\sigma$-homomorphism $\tilde{\sigma}$ of $\tilde{\mathfrak{L}}$ onto $B$ such that $\ker \tilde{\sigma} = \mathfrak{N}$ and $\tilde{\sigma}_{|\tilde{B}} = \sigma^{-1}$. If instead of $\tilde{\mathfrak{L}}$ we consider $\mathfrak{M}(\Omega, \tilde{\mathfrak{L}})$, the $F$-ring of bounded real valued $(\Omega, \tilde{\mathfrak{L}})$-measurable functions, then the set $\mathfrak{J}$ of functions in $\mathfrak{M}$ which are zero except on a set in $\mathfrak{N}$ forms a $\sigma$-ideal of $\mathfrak{M}$ and there is a $\sigma$-homomorphism $\rho$ of $\mathfrak{M}(\Omega, \tilde{\mathfrak{L}})$ onto $R$ such that: (i) $\ker \rho = \mathfrak{J}$; (ii) if $\iota$ denotes the mapping which carries a characteristic function onto its carrier, then[1] $\rho_{|B(\mathfrak{M})} = \tilde{\sigma}_0\iota$; (iii) if $C(\Omega)$ denotes the $F$-ring of continuous functions on $\Omega$, then $\rho_{|C(\Omega)}$ is an isomorphism of $C(\Omega)$ onto $R$.

The $F$-ring $\mathfrak{M}(\Omega, \tilde{\mathfrak{L}})$ together with the $\sigma$-homomorphism $\rho$ is called a representation of $R$. More generally, if $\mathfrak{M}$ is a ring of measurable functions and $\tau$ is a $\sigma$-homomorphism of $\mathfrak{M}$ onto $R$, then $\langle \mathfrak{M}, \tau \rangle$ is called a *representation* of $R$.

The following is a precise statement of the problem proposed in the Introduction: Under what circumstances does there exist a measurable product space $(\Omega_0 \times \Omega_1, \mathfrak{L}_0 \times \mathfrak{L}_1)$, a measure $\mu_1$ on $\mathfrak{L}_1$, and $\sigma$-homomorphism $\tau$ of $\mathfrak{M}(\Omega_0 \times \Omega_1, \mathfrak{L}_0 \times \mathfrak{L}_1)$, the $F$-ring of bounded real $(\Omega_0 \times \Omega_1, \mathfrak{L}_0 \times \mathfrak{L}_1)$-measurable functions, onto $R$ such that

(1) $\langle \mathfrak{M}(\Omega_0 \times \Omega_1, \mathfrak{L}_0 \times \mathfrak{L}_1), \tau \rangle$ is a representation of $R$,

[1] If $\varphi$ maps $A$ into $B$ and $\psi$ maps $B$ into $C$, then the composition of $\varphi$ followed by $\psi$ is denoted $\psi_0\varphi$.

(2) $\tau\mathfrak{M}(\Omega_0 \times \Omega_1, \mathfrak{L}_0 \times \Omega_1) = TR$,
(3) $\tau\mathfrak{M}(\Omega_0 \times \Omega_1, \Omega_0 \times \mathfrak{L}_1)$ is composed of elements $x$ such that $Tx$ is a real multiple of 1,
(4) if $Tf = \hat{f}$ and $\tau\hat{f} = f$, then $\tau(g) = \hat{f}$ where

$$g(\omega_0, \omega_1) = \int_{\Omega_1} \hat{f}(\omega_0, \gamma)\, d\mu_1(\gamma)\,?$$

Item (4) indicates, roughly speaking, that

$$Tf = \tau\left[\int_{\Omega_1} (\tau^{-1}f)\,(\omega_0, \omega_1)\, d\mu_1(\omega_1)\right]$$

or that $T$ is an integration over one variable of a product space.

This problem can be solved in a natural way by using a theorem of Birkhoff (see Section 5). In order to apply this theorem to our advantage we must consider the following two sets of idempotents of $B = B(R)$:

$$G = \{e \in B \mid Te \text{ is a real multiple of } 1\},$$
$$B_0 = \{e \in B \mid Te = e\}.$$

From the results of Section 1, it is clear that $TR = E = R(B(E))$, $B(E) = B_0$, and for $x \in TR$, $Tx = x$. The properties of $G$ are discussed in the next section.

## 4. *Elements with Constant Image under T*

Let $G = \{e \in B \mid Te \text{ is a real multiple of } 1\}$. $G$ must contain 0 and 1, and is closed with respect to complementation. In addition, if $e_n$ is a monotone sequence of elements of $G$, then by Condition $T_3$, $\sup e_n$ and $\inf e_n$ both belong to $G$. However, $G$ need not be a Boolean algebra. Indeed consider $L^\infty([0, 1] \times [0, 1])$, the $F$-ring of essentially bounded functions on the unit square modulo the ideal of null functions, and let

$$(Tf)\,(\omega_1, \omega_2) = \int_0^1 f(\omega_1, \gamma)\, d\gamma.$$

$G$, in this case, contains the characteristic functions of all sets which are almost everywhere of constant width. In this case $G$ is not a Boolean algebra, but it does contain a maximal subset $B_1$ which is a $\sigma$-complete Boolean subalgebra of $B(L^\infty)$. For example, $B_1$ could be the set of all characteristic functions independent of $\omega_1$.

In general the following theorem is valid.

THEOREM 4.1. *If $T$ is a positive Reynolds operator on $R$ with $T1 = 1$, then for any $\sigma$-complete Boolean subalgebra $A$ of $B$ which is a subset of $G$, there is a maximal subalgebra $M$ of $B$ relative to the condition: $A \subseteq M \subseteq G$. In addition, $M$ is a $\sigma$-complete subalgebra of $B$.*

PROOF. If $G = \{0, 1\}$, the theorem is trivial. Assume, then, that $A$ contains $\{0, 1\}$ properly.

By Zorn's Lemma it can be proved that there is a Boolean subalgebra $M$ (not necessarily $\sigma$-complete) which is maximal relative to the properties indicated in the statement of the theorem.

To establish that $M$ is a $\sigma$-complete Boolean subalgebra of $B$, note that there is a minimal monotone subclass $S$ of $G$ which contains $M$. Remember that a *monotone class* is one which is closed with respect to suprema and infima of monotone sequences. It is clear that $G$ is such a class and that $S$ is the intersection of all such monotone classes which contain $A$.

$S$ is a $\sigma$-complete Boolean subalgebra of $B$: Indeed, let

$$K(f) = \{e \in G \mid e - fe, f - ef, e \vee f \text{ all belong to } S\}$$

for each $f \in G$. Then for $f, g \in G$,

$$g \in K(f) \Leftrightarrow f \in K(g).$$

In addition, if $e_n \in K(f)$ and $e_n \uparrow e$, then

$$e_n - e_n f \uparrow e - ef, \qquad f - e_n f \downarrow f - ef, \qquad \text{and} \qquad e_n \vee f \uparrow e \vee f,$$

and hence since $S$ is a monotone class, $e - fe$, $f - ef$, and $e \vee f$ all belong to $S$. Thus $e \in K(f)$; by a similar argument, it follows that if $e_n \in K(f)$ and $e_n \downarrow e$, then $e \in K(f)$. Therefore $K(f)$ is a monotone class provided it is nonvoid. Let $f \in M$. Then $M \subseteq K(f)$ and hence $S \subseteq K(f)$. If $e \in S$, then $e \in K(f)$ and $f \in K(e)$. Since this is true for any $f \in M$, $S \subseteq K(e)$. Therefore $S$ is a Boolean subalgebra of $B$ which is a monotone classand for which

$$M \subseteq S \subseteq G.$$

It is clear that $S$ is a $\sigma$-complete Boolean subalgebra of $B$ and equals $M$.

REMARK 4.1. The proof of this theorem is an analogue of that given in [13, p. 27].

COROLLARY 4.1. *Every element of $G$ is contained in a Boolean subalgebra of $G$ maximal relative to the condition that it be contained in $G$.*

COROLLARY 4.2. *The maximal subalgebra $M$ of Theorem 1 is necessarily complete as a lattice and satisfies the countable chain condition. In addition, the mapping $\mu : e \to \lambda_e$ is a positive measure on $M$ where $\lambda_e$ is the real number such that $Te = \lambda_e \cdot 1$.*

PROOF. Clearly, $\mu$ is a positive measure on $M$. The remainder of the Corollary follows from some remarks of Maharam [11, § 1.8].

The measure $\mu$ can be extended to $B$ by using a result of Klee [14] as indicated in the following theorem.

THEOREM 4.2. *If $M \subseteq G$ is a $\sigma$-complete Boolean subalgebra of $B$ and if $\mu$ is the measure induced on $M$ by $T$ (Corollary 4.2), then there is a linear functional $\hat{\mu}$ on $R$ with the following properties:*

(1) If $f \in R(M)$, then $\hat{\mu}(f) = \int f\, d\mu$.

(2) $\hat{\mu}$ is nonnegative.

(3) $\hat{\mu}T = T\hat{\mu}$.

REMARK 4.2. The symbol $\int f\, d\mu$ stands for the integral induced on $R(M)$ by the measure $\mu$, and is defined in the standard manner by using a representation of $R(M)$.

PROOF OF THEOREM. Note that (i) $\mu(f) = \int f\, d\mu$ is a linear functional on $R(M)$, (ii) if $\|f\| = \inf\{\lambda \,|\, |f| \leqslant \lambda \cdot 1\}$, then $\|\cdot\|$ is a norm on $R$, (iii) $\{T\}$ is a semigroup of linear transformations on $R$, (iv) $\mu(f) \leqslant \|f\|$ for all $f \in R(M)$, (v) $TR(M) \subseteq R(M)$ and $\mu Tf = \mu f$ for $f \in R(M)$, and (vi) $\|Tf\| \leqslant \|f\|$. These six conditions insure [14, Corollary 3.1] that there is at least one linear functional $\hat{\mu}$ on $R$ which extends $\mu$ and for which (iv) and (v) are valid. Thus (1) and (3) are valid for $\hat{\mu}$.

Condition (2) follows from Corollary 2.3 of [15]. In fact, Corollary 2.3 in [15] is a specialization of the Hahn-Banach Theorem covering the extension of nonnegative linear functionals to nonnegative linear functionals. This specialized Hahn-Banach Theorem may be used in Klee's derivation to prove the existence of a nonnegative $\hat{\mu}$.

A pair of subsets $S_1$ and $S_2$ of a $\sigma$-complete Boolean algebra are *algebraically independent* if, for $b_i \in S_i$; $b_1 \wedge b_2 = 0$ implies that one of the factors is zero.

PROPOSITION 4.1. The set $G$ and the set $B_0$ are algebraically independent.

PROOF. If $b \in G$ and $e \in B_0$, then

$$T(be) = (Tb)(Te) = (Tb)\, e.$$

If $be = 0$, then $T(be) = (Tb)e = 0$; and since $T$ is positive, either $b = 0$ or $e = 0$.

## 5. *Birkhoff's Theorem*

The following theorem is necessary for our development. It was discovered by Birkhoff [5] and later rediscovered by Wright [16].

THEOREM 5.1 (Birkhoff). *If $X$ is a compact Hausdorff space, $C(X)$ the ring of continuous real functions on $X$, and $T$ a Reynolds operator on $C(X)$ such that $T1 = 1$, then there is a partition of $X$ into closed sets $\{X_\alpha \mid \alpha \in \Omega_0\}$ such that:*

(1) *$\omega_1$ and $\omega_2$ belong to the same $X_\alpha$ if and only if $f(\omega_1) = f(\omega_2)$ for every $f \in TC$.*

(2) *For $f \in C(X)$, $Tf$ is constant on $X_\alpha$ for each $\alpha \in \Omega_0$ and the value of $Tf$ on a particular $X_\alpha$ is uniquely determined by the values of $f$ on $X_\alpha$.*

(3) *On each $X_\alpha$, the value of $Tf$ is given as follows:*

$$(Tf)(\omega) = \int_{X_\alpha} f(\omega')\, d\mu_\alpha(\omega') \tag{5.1}$$

*for $\omega \in X_\alpha$ where $\mu_\alpha$ is a Borel measure on $X_\alpha$.*

Since there is an isomorphism $\rho_{|C}$ (defined in Section 3) of the $F$-ring $C(\Omega)$ onto $R$ where $\Omega$ is the Stone space of $B$, the above theorem can be applied to our problem.

It can be verified that the partition $\{X_\alpha \mid \alpha \in \Omega_0\}$ effected by the application of Theorem 5.1 to $C(\Omega)$ and $\rho^{-1}T\rho$ is identical with that effected by the equivalence relation:

$\omega_1 \equiv \omega_2$ if and only if for every idempotent $e \in TC$, $e(\omega_1) = e(\omega_2)$.

In addition, if $e \in G$, then for arbitrary $\alpha, \beta \in \Omega_0$

$$\mu(e) = \int_{X_\alpha} (\rho^{-1}e)(\omega)\, d\mu_\alpha(\omega) = \int_{X_\beta} (\rho^{-1}e)(\omega)\, d\mu_\beta(\omega). \tag{5.2}$$

## 6. *The Product Space*

Let $T$ be a Reynolds operator on the bounded $F$-ring $R$ with Boolean algebra $B = B(R)$ of idempotents. In this section, necessary and sufficient conditions are found for the existence of a measurable product space representation for $R$ which will satisfy the conditions of our problem. This programme is carried out in a setting which is slightly more general than that necessary for the problem.

The plan of attack is as follows: Let $B_0$ and $B_1$ be arbitrary $\sigma$-complete Boolean subalgebras of $B$. We find necessary and sufficient conditions for the existence of a product space $(\Omega_0 \times \Omega_1, \mathfrak{L}_0 \times \mathfrak{L}_1)$ and a $\sigma$-homomorphism $\xi$ such that

$$\xi : \mathfrak{L}_0 \times \mathfrak{L}_1 \xrightarrow{\text{onto}} B, \tag{6.1}$$

and

$$\xi : \Omega_0 \times \mathfrak{L}_1 \xrightarrow{\text{onto}} B_1, \tag{6.2}$$

$$\xi : \mathfrak{L}_0 \times \Omega_1 \xrightarrow{\text{onto}} B_0. \tag{6.3}$$

Then $\xi$ can be "raised" in a natural way to a $\sigma$-homomorphism $\hat{\xi}$ of $\mathfrak{M}(\Omega_0 \times \Omega_1, \mathfrak{L}_0 \times \mathfrak{L}_1)$ onto $R$ such that

$$\hat{\xi} : \mathfrak{M}(\Omega_0 \times \Omega_1, \mathfrak{L}_0 \times \Omega_1) \xrightarrow{\text{onto}} R(B_0)$$

and

$$\hat{\xi} : \mathfrak{M}(\Omega_0 \times \Omega_1, \Omega_0 \times \mathfrak{L}_1) \xrightarrow{\text{onto}} R(B_1).$$

Let $\langle \Omega, \tilde{B}, \sigma \rangle$ be the Stone representation of $B$, let $\tilde{\mathfrak{L}}_i$ stand for the $\sigma$-field generated by $\sigma B_i$ for $i = 0, 1$, and let $\mathfrak{L}$ be the $\sigma$-field generated by $\tilde{\mathfrak{L}}_0 \cup \tilde{\mathfrak{L}}_1$. $\mathfrak{L}$ is a $\sigma$-subfield of $\tilde{\mathfrak{L}}$ the $\sigma$-field generated by $\tilde{B}$. From the remarks of Section 3, it follows that every $S \in \tilde{\mathfrak{L}}$ is of the form

$$S = S_0 \dot{+} N$$

where $S_0 \in \tilde{B}$ and $N$ is a set of 1st Category.

First we find necessary and sufficient conditions for the $\sigma$-homomorphism $\tilde{\sigma}_{|\mathfrak{L}}$ to be a $\sigma$-homomorphism of $\mathfrak{L}$ *onto* $B$.

PROPOSITION 6.1. *If $B_0 \cup B_1$ generates the $\sigma$-complete Boolean algebra $B$, that is if $B$ is the smallest $\sigma$-complete Boolean subalgebra of $B$ which contains $B_0 \cup B_1$, then $\tilde{\sigma}_{|\mathfrak{L}}$ maps $\mathfrak{L}$ onto $B$ and $\tilde{\sigma}\tilde{\mathfrak{L}}_i = B_i$ $(i = 0, 1)$.*

PROOF. Every element $S_i \in \tilde{\mathfrak{L}}_i$ can be represented in the for $S_i = D_i \dot{+} N_i$ where $D_i \in \sigma B_i$ and $N_i$ is of 1st Category. Indeed it can easily be verified that the class $\{D_i \dot{+} N \mid \text{where } D_i \in \sigma B_i \text{ and } N \text{ is of 1st Category}\}$ is a $\sigma$-field and hence contains $\tilde{\mathfrak{L}}_i$. Thus

$$\tilde{\sigma} S_i = \sigma^{-1} D_i \qquad (i = 0, 1).$$

Therefore $\tilde{\sigma}$ carries $\tilde{\mathfrak{L}}_1$ onto $\sigma^{-1}\sigma B_i = B_i$. Now $\mathfrak{L}$ is generated by $\tilde{\mathfrak{L}}_0 \cup \tilde{\mathfrak{L}}_1$ so $\tilde{\sigma}\mathfrak{L}$ is generated by $\tilde{\sigma}\tilde{\mathfrak{L}}_0 \cup \tilde{\sigma}\tilde{\mathfrak{L}}_1$ or $B_0 \cup B_1$ and hence by hypothesis $\tilde{\sigma}\mathfrak{L} = B$.

REMARK 6.1. If $\tilde{\sigma}_{|\mathfrak{L}}$ maps $\mathfrak{L}$ onto $B$, then it is clear, since $\tilde{\sigma}\tilde{\mathfrak{L}}_i = B_i$, that $B_0 \cup B_1$ generates $B$.

Next we find necessary and sufficient conditions for the existence of a measurable product space and a $\sigma$-homomorphism of the $\sigma$-field of that product space onto $\mathfrak{L}$. Let $\{X_\alpha \mid \alpha \in \Omega_0\}$ be the partition of $\Omega$ effected by the equivalence relation:

$\omega_1 \equiv \omega_2$ if an only if $\chi_S(\omega_1) = \chi_S(\omega_2)$ for all $S \in \sigma B_0$ and let $\{Y_\gamma \mid \gamma \in \Omega_1\}$ be the partition of $\Omega$ effected by the analogous equivalence relation defined in terms of sets in $\sigma B_1$.

PROPOSITION 6.2. Every element of $\tilde{\mathfrak{L}}_0$ is a union of $X_\alpha$'s and every element of $\tilde{\mathfrak{L}}_1$ is a union of $Y_\gamma$'s.

PROOF. By symmetry it is sufficient to prove that every element of $\tilde{\mathfrak{L}}_0$ is a union of $X_\alpha$'s. Every set $S \in \sigma B_0$ is a union of $X_\alpha$'s. Let $\mathfrak{N}_0$ be the $\sigma$-ideal generated by elements $E = \cap_{k=0}^{\infty} S_k$ where $S_k \in \sigma B_0$ and $\sigma B_0$-inf $\{S_k\} = \phi$. $\mathfrak{N}_0$ is composed of sets of 1st category which are unions of $X_\alpha$'s. Consider the class $\mathfrak{Q}$ of sets of the form $S \dotplus N$ where $S \in \sigma B_0$ and $N \in \mathfrak{N}_0$. The set $\mathfrak{Q}$ can be shown to be a $\sigma$-field by direct verification. Therefore $\tilde{\mathfrak{L}}_0 \subseteq \mathfrak{Q}$ and the proposition follows.

PROPOSITION 6.3. If $B_0$ and $B_1$ are algebraically independent, then for each pair $(\alpha, \gamma) \in \Omega_0 \times \Omega_1$, the set $X_\alpha \cap Y_\gamma$ is nonvoid and

$$\mathfrak{A} = \{X_\alpha \cap Y_\alpha \mid (\alpha, \gamma) \in \Omega_0 \times \Omega_1\}$$

is a partition of $\Omega$.

PROOF. It is clear that $\mathfrak{A}$ contains a partition of $\Omega$. However, it must be proved that $X_\alpha \cap Y_\gamma \neq \phi$ for each choice of $(\alpha, \gamma)$. Since

$$X_\alpha = \cap \{D \in \sigma B_0 \mid X_\alpha \subseteq D\} \qquad \text{and} \qquad Y_\gamma = \cap \{D \in \sigma B_1 \mid Y_\gamma \subseteq D\},$$

$$X_\alpha \cap Y_\gamma = \cap \{D \cap E \mid D \in \sigma B_0 \text{ and } X_\alpha \subseteq D, \text{ and } E \in \sigma B_1 \text{ and } Y_\gamma \subseteq E\}. \quad (6.4)$$

Due to the algebraic independence of $B_0$ and $B_1$, the bracketed class of sets in expression (6.4) enjoys the finite intersection property, and hence it follows by the campactness of $\Omega$ that $X_\alpha \cap Y_\gamma$ is nonvoid.

REMARK 6.2. If for each $(\alpha, \gamma) \in \Omega_0 \times \Omega_1$, $X_\alpha \cap Y_\gamma$ is nonvoid, then $\sigma B_0$ and $\sigma B_1$ must be algebraically independent and therefore $B_0$ and $B_1$ are as well.

Assume for the remainder of this section that $B_0 \cup B_1$ generates $B$, and that $B_0$ and $B_1$ are algebraically independent. Consider the mapping $\varphi : \Omega \to \Omega_0 \times \Omega_1$ defined as follows: $\varphi(\omega) = (\alpha, \gamma)$ if $\omega \in X_\alpha \cap Y_\gamma$. The mapping is well defined and carries $\Omega$ onto $\Omega_0 \times \Omega_1$ by Proposition 6.3. By

Propositions 6.2 and 6.3, $\varphi$ is an injection (which preserves $\sigma$-operations) of $\tilde{\mathfrak{L}}_i$ into $\mathfrak{P}(\Omega_0 \times \Omega_1)$, the $\sigma$-field of all subsets of $\Omega_0 \times \Omega_1$, for $i = 0, 1$.

PROPOSITION 6.4. If $D \in \tilde{\mathfrak{L}}_0$, then $\varphi(D) = D_0 \times \Omega_1$ for some $D_0 \in \mathfrak{P}(\Omega_0)$, and similarly if $D \in \tilde{\mathfrak{L}}_1$, then $\varphi(D) = \Omega_0 \times D_1$ for some $D_1 \in \mathfrak{P}(\Omega_1)$.

PROOF. Since $D \in \tilde{\mathfrak{L}}_0$ is a union of $X_\alpha$'s

$$\varphi(D) = \cup\{(\alpha, \gamma) \mid X_\alpha \cap Y_\gamma \subseteq D\} = \cup\{\{\alpha\} \times \Omega_1 \mid X_\alpha \subseteq D\} = D_0 \times \Omega_1$$

where $D_0 = \varphi\{\alpha \mid X_\alpha \subseteq D\}$. A similar demonstration yields the other half of the proposition.

Let $\mathfrak{L}_0$ designate the $\sigma$-field composed of subsets of $\Omega_0$ of the form $D_0 = \{\alpha \mid X_\alpha \subseteq D\}$ for $D \in \tilde{\mathfrak{L}}_0$, and let $\mathfrak{L}_1$ be analogously defined. Then the mapping $\varphi_i : D \to D_i$ is an isomorphism of the $\sigma$-field $\tilde{\mathfrak{L}}_i$ onto $\mathfrak{L}_i$ for $(i = 0, 1)$.

THEOREM 6.1. *The mapping $\varphi$ carries $\mathfrak{L}$ onto $\mathfrak{L}_0 \times \mathfrak{L}_1$ while*

$$\varphi\tilde{\mathfrak{L}}_0 = \mathfrak{L}_0 \times \Omega_1 \qquad \text{and} \qquad \varphi\tilde{\mathfrak{L}}_1 = \Omega_0 \times \mathfrak{L}_1.$$

PROOF. From Proposition 6.4, $\varphi$ has the indicated property with respect to $\tilde{\mathfrak{L}}_0$ and $\tilde{\mathfrak{L}}_1$. Since (i) $\mathfrak{L}$ is generated by $\tilde{\mathfrak{L}}_0 \cup \tilde{\mathfrak{L}}_1$, (ii) $\mathfrak{L}_0 \times \mathfrak{L}_1$ is generated by $(\mathfrak{L}_0 \times \Omega_1) \cup (\Omega_0 \times \mathfrak{L}_1)$, and (iii) $\varphi$ is an isomorphism of the $\sigma$-structure of $\mathfrak{L}$, it follows that

$$\varphi\mathfrak{L} = \mathfrak{L}_0 \times \mathfrak{L}_1.$$

THEOREM 6.2. *The mapping $\tilde{\sigma} \circ \varphi^{-1}$ is a $\sigma$-homomorphism of $\mathfrak{L}_0 \times \mathfrak{L}_1$ onto $B$.*

PROOF. The mapping $\varphi^{-1}$ is an isomorphism of $\mathfrak{L}_0 \times \mathfrak{L}_1$ onto $\mathfrak{L}$ which preserves $\sigma$-operations, and $\tilde{\sigma}$ is a $\sigma$-homomorphism of $\mathfrak{L}$ onto $B$. Therefore the theorem follows.

By theorem 6.2 and Propositions 6.1 and 6.4, it follows that $\xi = \tilde{\sigma} \circ \varphi^{-1}$ has the properties (6.1), (6.2), and (6.3)—provided, of course, that $B_0 \cup B_1$ generates $B$ and that $B_0$ and $B_1$ are algebraically independent.

If, for $f \in \mathfrak{M}(\Omega_0 \times \Omega_1, \mathfrak{L}_0 \times \mathfrak{L}_1)$, $\Phi f(\omega) = f(\varphi(\omega))$, then $\Phi$ is an isomorphism of $\mathfrak{M}(\Omega_0 \times \Omega_1, \mathfrak{L}_0 \times \mathfrak{L}_1)$ onto $\mathfrak{M}(\Omega, \mathfrak{L})$ because $\varphi$ is a measurable [13, p. 164] transformation of $(\Omega, \mathfrak{L})$ onto $(\Omega_0 \times \Omega_1, \mathfrak{L}_0 \times \mathfrak{L}_1)$ which carries $\mathfrak{L}$ onto $\mathfrak{L}_0 \times \mathfrak{L}_1$ in an isomorphic manner.

Since, by Proposition 6.1, $\tilde{\sigma}$ maps $\mathfrak{L}$ onto $B$, it follows that $\rho$ (defined in Section 3) carries $B(\mathfrak{M}(\Omega, \mathfrak{L}))$ onto $B$ and hence that $\rho_{|\mathfrak{M}(\Omega,\mathfrak{L})}$ maps $\mathfrak{M}(\Omega, \mathfrak{L})$ onto $R$. In addition

$$\rho\mathfrak{M}(\Omega, \tilde{\mathfrak{L}}_i) = R(B_i)$$

for $i = 0, 1$ because $\rho_{|B(\mathfrak{M}(\Omega,\mathfrak{L}))} = \tilde{\sigma} \circ \iota$. Therefore the following Theorem is valid.

THEOREM 6.3. *If $B_0 \cup B_1$ generates $B$ and if $B_0$ and $B_1$ are algebraically independent, then the mapping $\xi = \rho \circ \Phi$ has the following properties:*

(1) *$\xi$ is a $\sigma$-homomorphism of $\mathfrak{M}(\Omega_0 \times \Omega_1, \mathfrak{L}_0 \times \mathfrak{L}_1)$ onto $R$.*

(2) $\xi \mathfrak{M}(\Omega_0 \times \Omega_1, \mathfrak{L}_0 \times \Omega_1) = R(B_0)$.

(3) $\xi \mathfrak{M}(\Omega_0 \times \Omega_1, \Omega_0 \times \mathfrak{L}_1) = R(B_1)$.

## 7. *The Main Result*

Let $B_0 = \{e \in B \mid Te = e\}$ and let $G$ be the set of elements $e$ of $B$ with $Te$ equal to a real multiple of 1. If $B_1$ is a $\sigma$-complete Boolean subalgebra of $B$ and $G \supseteq B_1$, then $B_1$ and $B_0$ are algebraically independent by Proposition 4.1. In addition $B_1$ can be chosen maximal in $G$ by Theorem 4.1. However, the maximality of $B_1$ in $G$ is necessary but not sufficient for $B_0 \cup B_1$ to generate $B$. See Section 11 in Part III. Assume for the remainder of this section that $B_0 \cup B_1$ generates $B$. Thus we can use the results of Section 6. By Corollary 4.2, there is a positive measure $\mu$ induced on $B_1$ by $T$. The mapping $\mu\sigma^{-1}$ is a positive measure on $\sigma B_1$; this measure can be extended to a ($\sigma$-additive nonnegative) measure $\bar{\mu}$ on the $\sigma$-field $\tilde{\mathfrak{L}}_1$; and finally $\nu = \bar{\mu}\varphi_1^{-1}$ is a measure on $\mathfrak{L}_1$ because $\varphi_1$ is an isomorphism of $\tilde{\mathfrak{L}}_1$ onto $\mathfrak{L}_1$, the $\sigma$-field of the measurable space $(\Omega_1, \mathfrak{L}_1)$. (The symbols $\mathfrak{L}_0, \mathfrak{L}_1, \Omega_0, \Omega_1$, etc. are defined in Section 6.) Let $J$ stand for the operator on $\mathfrak{M}(\Omega_0 \times \Omega_1, \mathfrak{L}_0 \times \mathfrak{L}_1)$ defined as follows: $Jf = \bar{f}$ where

$$\bar{f}(\omega_0, \omega_1) = \int_{\Omega_1} f(\omega_0, \gamma)\, d\nu(\gamma).$$

$J$ is clearly a Reynolds operator on $\mathfrak{M}(\Omega_0 \times \Omega_1, \mathfrak{L}_0 \times \mathfrak{L}_1)$ with $J1 = 1$. Our purpose in this section is to show that $\xi J \xi^{-1} x = Tx$ where $\xi^{-1}x$ stands for a member of the $\xi$-coset of preimages of $x \in R$.

From a remark at the end of Section 5, it follows that the decomposition $\{X_\alpha \mid \alpha \in \Omega_0\}$ effected by the Birkhoff Theorem (Theorem 5.1) is the same as that described in Section 6: $\omega_1$ and $\omega_2$ belong to the same $X_\alpha$ if and only if $\chi_S(\omega_1) = \chi_S(\omega_2)$ for each $S \in \sigma B_0$. Let $\{Y_\gamma \mid \gamma \in \Omega_1\}$ be the partition defined in an analogous fashion (in Section 6) with respect to $\sigma B_1$. By Proposition 6.3, $X_\alpha \cap Y_\gamma$ is nonvoid for each $(\alpha, \gamma) \in \Omega_0 \times \Omega_1$, and hence the mapping $S \to S \cap X_\alpha$ is an isomorphims of the $\sigma$-complete Boolean algebra $\sigma B_1$ onto $\sigma B_1 \cap X_\alpha$. In addition, since every element of $\tilde{\mathfrak{L}}_1$ is a union of $Y_\gamma$'s, it follows that $S \to S \cap X_\alpha$ is also an isomorphism of $\tilde{\mathfrak{L}}_1$ onto $\tilde{\mathfrak{L}}_1 \cap X_\alpha$. The smallest $\sigma$-field of subsets of $X_\alpha$ generated by $\sigma B_1 \cap X_\alpha$ is $\tilde{\mathfrak{L}}_1 \cap X_\alpha$. Indeed, every element of $\tilde{\mathfrak{L}}_1 \cap X_\alpha$ is of the form $S = S_1 \dotplus N$ where $S_1 \in \sigma B_1 \cap X_\alpha$, and $N \in \mathfrak{N}_1 \cap X_\alpha$ where the definition of $\mathfrak{N}_1$ is analogous to that given for $\mathfrak{N}_0$

in the proof of Proposition 6.2. Since $\mathfrak{N}_1 \cap X_\alpha$ is a subset of the $\sigma$-field generated by $\sigma B_1 \cap X_\alpha$, it follows that $S$ is an element of that $\sigma$-field and $\tilde{\mathfrak{L}}_1 \cap X_\alpha$ coincides with it. By a similar argument it follows that the field $\tilde{B} \cap X_\alpha$ of subsets of $X_\alpha$ generates the $\sigma$-field $\tilde{\tilde{\mathfrak{L}}} \cap X_\alpha$.

Since for each $\alpha$, a measure $\mu_\alpha$ is defined on $\tilde{B} \cap X_\alpha$ by Theorem 5.1, it follows that $\mu_\alpha$ extends at least to $\tilde{\mathfrak{L}}_1 \cap X_\alpha$. This extended measure can then be transferred to $\tilde{\mathfrak{L}}_1$ by the inverse of the isomorphism $\zeta : S \to S \cap X_\alpha$. It can yet again be tranferred to $\mathfrak{L}_1$ by the isomorphism $\varphi_1$. Thus if $\bar{\mu}_\alpha$ is the extension of $\mu_\alpha$ to $\tilde{\mathfrak{L}}_1 \cap X_\alpha$, then the function $\bar{\mu}_\alpha \zeta \varphi_1^{-1}$ from $\mathfrak{L}_1$ into the real numbers is a measure on $\mathfrak{L}_1$.

Proposition 7.1. For each $\alpha \in \Omega_0$, the measure $\bar{\mu}_\alpha \zeta$ coincides with $\bar{\mu}$ on $\tilde{\mathfrak{L}}_1$, and hence $\bar{\mu}_\alpha \zeta \varphi_1^{-1}$ coincides with $\nu$ on $\mathfrak{L}_1$.

Proof. The measure $\mu_\alpha \zeta$ coincides with $\mu \sigma^{-1}$ on $\sigma B_1$. Indeed, from Eq. (5.2),

$$\begin{aligned} \mu \sigma^{-1} S &= \int_{X_\alpha} \rho^{-1} \sigma^{-1} S(\omega)\, d\mu_\alpha(\omega) \\ &= \int_{X_\alpha} \chi_S(\omega)\, d\mu_\alpha(\omega) = \mu_\alpha(S \cap X_\alpha) \\ &= \bar{\mu}_\alpha \zeta S. \end{aligned}$$

Thus by definition of $\nu$, $\mu_\alpha \zeta \varphi_1^{-1} = \nu$ on $\varphi_1 \sigma B_1$. Therefore they coincide on $\mathfrak{L}_1$.

Proposition 7.2. The $\sigma$-field $\mathfrak{L} \cap X_\alpha$ of subsets of $X_\alpha$ coincides with $\tilde{\tilde{\mathfrak{L}}}_1 \cap X_\alpha$.

Proof. The mapping $\varphi$ defined in Section 6 carries $\Omega$ into $\Omega_0 \times \Omega_1$ and by Theorem 6.1 it carries $\mathfrak{L}$ onto $\mathfrak{L}_0 \times \mathfrak{L}_1$. In addition, $\mathfrak{L} \cap X_\alpha$ is mapped onto the class of $\alpha$-sections of sets in $\mathfrak{L}_0 \times \mathfrak{L}_1$. By [13, Theorem A p. 141], this last class is $\{\alpha\} \times \mathfrak{L}_1$. Therefore $\mathfrak{L} \cap X_\alpha = \tilde{\mathfrak{L}}_1 \cap X_\alpha$.

Let $\Pi_i$ be defined as follows: if $\varphi(\omega) = (\alpha, \gamma)$ then $\Pi_1(\omega) = \gamma$ and $\Pi_0(\omega) = \alpha$. $\Pi_i$ is a single-valued mapping of $\Omega$ onto $\Omega_i$ and $\Pi_i^{-1}$ is an isomorphism of $\mathfrak{L}_i$ onto $\tilde{\mathfrak{L}}_i$ for $(i = 0, 1)$. In addition, $\varphi_i^{-1} = \Pi_i^{-1}$ for $(i = 1, 0)$. Finally $\Pi_1$ is a measure preserving transformation of $(\Omega, \tilde{\mathfrak{L}}_1, \bar{\mu})$ onto $(\Omega_1, \mathfrak{L}_1, \nu)$.

Let $\Phi$ be the mapping defined in Section 6.

Proposition 7.3. If $g \in \mathfrak{M}(\Omega, \mathfrak{L})$, then

$$(\Phi J \Phi^{-1} g)\,(\omega) = \int_{X_\alpha} g(\gamma)\, d\mu_\alpha(\gamma)$$

for $\omega \in X_\alpha$.

PROOF. If $\omega \in X_\alpha$ and $f \in \mathfrak{M}(\Omega_0 \times \Omega_1, \mathfrak{L}_0 \times \mathfrak{L}_1)$, then

$$(\Phi f)(\omega) = f(\Pi_0(\omega), \Pi_1(\omega)) = f(\alpha, \Pi_1(\omega)).$$

If we let $f_{\omega_0}(\omega_1) = f(\omega_0, \omega_1)$ where $\omega_0$ is considered fixed, then

$$(\Phi f)(\omega) = f_\alpha(\Pi_1(\omega)) \in \mathfrak{M}(\Omega, \mathfrak{L}).$$

Since $\Pi_1^{-1} = \varphi_1^{-1}$, it follows that

$$(\Pi_{1|X_\alpha})^{-1} S = (\Pi_1^{-1} S) \cap X_\alpha = \zeta \varphi^{-1} S$$

and hence $\Pi_1 \mid X_\alpha$ is a measure preserving transformation from $(X_\alpha, \tilde{\mathfrak{L}}_1 \cap X_\alpha, \mu_\alpha)$ onto $(\Omega_1, \mathfrak{L}_1, \nu)$. Therefore for $\omega \in X_\alpha$ and $f \in \mathfrak{M}(\Omega, \mathfrak{L})$,

$$\int_{\Omega_1} f(\alpha, \gamma)\, d\nu(\gamma) = \int_{X_\alpha} f_\alpha(\Pi_1(\omega))\, d\mu_\alpha(\omega)$$

$$= \int_{X_\alpha} \Phi f(\omega)\, d\mu_\alpha(\omega).$$

Let $\Phi^{-1} = f$ and the proposition follows.

From the remarks of Section 3 it follows that for every element $g \in \mathfrak{M}(\Omega, \mathfrak{L})$ there is a $\hat{g} \in C(\Omega)$ and $n \in \mathfrak{M}(\Omega, \tilde{\mathfrak{L}})$ which is nonzero on a set of first category such that

$$g = \hat{g} + n.$$

The functions $\hat{g}$ and $n$ are uniquely determined by $g$.

PROPOSITION 7.4. Let $g = \hat{g} + n$ be the representation of $g \in \mathfrak{M}(\Omega, \mathfrak{L})$ by a function in $C(\Omega)$ and a function in $\mathfrak{M}(\Omega, \tilde{\mathfrak{L}})$ which is nonzero on a set of 1st category. Then for each $\alpha \in \Omega_0$.

$$\int_{X_\alpha} g(\omega)\, d\mu_\alpha(\omega) = \int_{X_\alpha} \hat{g}(\omega)\, d\mu_\alpha(\omega).$$

PROOF. The support of $n$ is a set $S$ which is a union of sets $S_n$ each of which is the intersection of a nonincreasing sequence $\{S_{in}\}_i$ of elements of $\tilde{B}$ such that $\tilde{B}\text{-inf}_i\, S_{in} = \phi$. If $S_n \cap X_\alpha \neq \phi$, then $S_{in} \cap X_\alpha$ is nonvoid and belongs to $\tilde{\mathfrak{L}} \cap X_\alpha$. Now

$$0 \leqslant \mu_\alpha(S_n \cap X_\alpha) \leqslant \mu_\alpha(S_{in} \cap X_\alpha) = [\rho T \rho^{-1}(\chi_{S_{in}})]\,(\omega)$$

for $\omega \in X_\alpha$ by Theorem 5.1.

Since $[\rho T \rho^{-1} \chi_{S_{in}}] \downarrow 0$ as $i \to \infty$, it follows that $\mu_\alpha(S_n \cap X_\alpha) = 0$. Therefore the support of $n$ has $\mu_\alpha$-measure zero and the proposition is valid.

Now we are ready to prove the representation theorem.

THEOREM 7.1. *If $T$ is a positive Reynolds operator on $R$ such that $T1 = 1$ and $G \neq \{0, 1\}$, then there is a $\sigma$-complete Boolean subalgebra $B_1$ of $B$ which is maximal relative to the condition $B_1 \subseteq G$. If $B_0 \cup B_1$ generates $B$, then there is a product space $(\Omega_0 \times \Omega_1, \mathfrak{L}_0 \times \mathfrak{L}_1)$, a measure $\nu$ on $\mathfrak{L}_1$, and a $\sigma$-homomorphism $\xi$ of $\mathfrak{M}(\Omega_0 \times \Omega_1, \mathfrak{L}_0 \times \mathfrak{L}_1)$ onto $R$ such that*

(1) $\xi\mathfrak{M}(\Omega_0 \times \Omega_1 \mathfrak{L}_0 \times \Omega_1) = R(B_0)$,

(2) $\xi\mathfrak{M}(\Omega_0 \times \Omega_1\Omega_0 \times \mathfrak{L}_1) = R(B_1)$,

(3) if $Tf = f$ and $\xi(\hat{f}) = f$, *then* $\xi(g) = f$ *where*

$$g(\omega_0, \omega_1) = \int_{\Omega_1} \hat{f}(\omega_0, \gamma)\, d\nu(\gamma).$$

PROOF. Except for statement (3) the theorem is a recapitulation of previous results. We prove statement (3) here.

By Proposition 7.3,

$$\int_{\Omega_1} \hat{f}(\alpha, \gamma)\, d\nu(\gamma) = \int_{X_\alpha} \Phi\hat{f}(\omega)\, d\mu_\alpha(\omega) \tag{7.1}$$

for $\omega \in X_\alpha$. From Proposition 7.4, there is a continuous function $h$ on $\Omega$ which differs from $\Phi f$ on a set of 1st category, and

$$\int_{\Omega_1} \hat{f}(\omega_0, \gamma)\, d\nu(\gamma) = \int_{X_\alpha} h(\omega)\, d\mu_\alpha(\omega). \tag{7.2}$$

Therefore by Theorem 5.1,

$$\int_{\Omega_1} \hat{f}(\omega_0, \gamma)\, d\nu(\gamma) = (\rho^{-1}T\rho h)(\omega) \tag{7.3}$$

for $\omega \in X_{\omega_0}$. In addition, if $\omega \in X_\alpha$,

$$(\Phi g)(\omega) = g(\alpha, \Pi_1(\omega)) = g(\alpha, \gamma)$$

for any $\gamma \in \Omega_1$ because $g(\alpha, \gamma)$ is independent of $\gamma$. Thus

$$(\Phi g)(\omega) = (\rho^{-1}T\rho)\, h(\omega)$$

for $\omega \in X_\alpha$. Since $\rho h = \rho\Phi f$, we have

$$\Phi g = \rho^{-1}T\rho\Phi\hat{f},$$

and hence by Eqs. (7.1)–(7.3),

$$\rho\Phi g = \xi g = T\xi\hat{f} = Tf = f.$$

## Part III

### 8. *Introduction*

In Parts I and II we have discussed the action of a Reynolds operator $T$ on a bounded $F$-ring $R$ and the extension of $T$ to $L_T$ an $F$-subspace of $\hat{R}$, the regular $F$-ring associated with $R$ [8, 9]. In Part III we discuss the case where $R$ is provided with a positive linear functional which can be interpreted as an integral. It is shown that under certain circumstances a Reynolds operator on $R$ can be extended to a conditional expectation on an $L^2$-space containing $R$; the results of Theorem 7.1 are shown to hold for conditional expectations; and finally conditional expectations are classified in terms of the results of Part II.

Let $B$ be a $\sigma$-complete Boolean algebra which supports a positive measure $\mu$. The pair $(B, \mu)$ is called a measure algebra. Maharam [11] has noted that:

(1) $B$ is complete as a lattice.

(2) $B$ satisfies the countable chain condition.

(3) There is a measure space $(\Omega, \tilde{\mathfrak{L}}, \bar{\mu})$ such that $B \simeq \tilde{\mathfrak{L}}/\mathfrak{N}$ where $\mathfrak{N}$ is the $\sigma$-ideal of sets of $\tilde{\mathfrak{L}}$ with $\bar{\mu}$-measure zero. $\Omega$ can be chosen as the Stone space of $B$ and $\tilde{\mathfrak{L}}$ as the $\sigma$-field generated by $\tilde{B}$, the Boolean algebra of open-closed sets of $\Omega$, and $\mu(S/\mathfrak{N}) = \bar{\mu}(S)$.

As indicated in Section 3, there is a $\sigma$-homomorphism $\rho$ of $\mathfrak{M}(\Omega, \tilde{\mathfrak{L}})$ onto $R(B)$ such that $\langle \mathfrak{M}, \rho \rangle$ is a representation of $R$. It is easy to verify that $R(B)$ is isomorphic to $L^\infty(\Omega, \tilde{\mathfrak{L}}, \bar{\mu}) = L^\infty(B, \mu)$. In addition, it can be shown that up to isomorphisms

$$B \subseteq R(B) = L^\infty(B, \mu) \subseteq L^2(B, \mu) \subseteq \hat{R}(B).$$

If $B_0$ is a $\sigma$-complete Boolean subalgebra of $B$, then, when $\tilde{\sigma}$ designates the $\sigma$-homomorphism of $\tilde{\mathfrak{L}}$ onto $B$ defined in 3, $\tilde{\sigma}^{-1}B_0 = \tilde{\mathfrak{L}}_0$ is a $\sigma$-subfield of $\tilde{\mathfrak{L}}$ and $\tilde{\sigma}\tilde{\mathfrak{L}}_0 = B_0$. Let $\hat{\rho}$ be the natural extension of $\rho$ which maps $\hat{\mathfrak{M}}(\Omega, \tilde{\mathfrak{L}})$, the regular $F$-ring of all $(\Omega, \tilde{\mathfrak{L}})$-measurable functions, onto $\hat{R}(B)$. If $(\Omega, \mathfrak{A}, \bar{\mu})$ is a measure space, let $\mathscr{L}^2(\Omega, \mathfrak{A}, \bar{\mu})$ designate the square $\bar{\mu}$-integrable functions in $\hat{\mathfrak{M}}(\Omega, \mathfrak{A})$. Then $\hat{\rho}\mathscr{L}^2(\Omega, \tilde{\mathfrak{L}}_0, \bar{\mu}) \simeq L^2(B_0, \mu_{|B_0})$.

Let $T$ be a positive Reynolds operator on $R(B)$ for which $T1 = 1$. $T$ is said to be *compatible with* $\mu$ if

$$\mu(Te) = \mu(e)$$

for each $e \in B$. Note that for $f \in L^1(B, \mu)$, $\mu(f)$ is defined equal to $\int \bar{f} d\bar{\mu}$ where $\bar{f}$ stands for a representative of $f$ in $\hat{\mathfrak{M}}(\Omega, \tilde{\mathfrak{L}})$.

We will show that if $T$ is compatible with $\mu$, then $\hat{T}$, the extension of $T$ discussed in Part I, is a conditional expectation when restricted to $L^2(B, \mu)$.

## 9. *Reynolds Operators and Conditional Expectation*

For the remainder of Part III, let $T$ be a positive Reynolds operator on $R(B)$ for which $T1 = 1$. In Reynolds operator on $R(B)$ for which $T1 = 1$. In Section 2 it was shown that $T$ can be extended to an operator $\hat{T}$ on an $F$-subspace $L_T$ of $\hat{R}(B)$ which satisfies Conditions $T_1$ thru $T_4$. Since $\hat{T}$ is an extension of $T$, $\hat{T}1 = 1$; and since $T$ is positive, it follows that $\hat{T}$ is also positive. From Propositions 2.6 and 2.4, it also follows that $\hat{T} = (\hat{T})^2$ and $\hat{T}L_T = \hat{R}(B(E))$.

PROPOSITION 9.1. *If $T$ is compatible with $\mu$, then $L_T \supseteq L^2(B, \mu)$, $\hat{T}L^2(B, \mu) \subseteq L^2(B, \mu)$, and $\mu[(\hat{T}f)^2] \leqslant \mu(f^2)$.*

PROOF. If $f \in L^2(B, \mu)$ and $f \geqslant 0$, then $f$ is integrable. For each $N \geqslant 1$, $T(f \wedge N)$ is defined. Let $h_N$ be in the coset in $\mathscr{L}^2(\Omega, \tilde{\mathfrak{L}}, \bar{\mu})$ determined by $T(f \wedge N)$. Then $0 \leqslant h_1 \leqslant h_2 \cdots \leqslant h_N \leqslant \cdots$ almost everywhere. Therefore $h_N \uparrow h$ a.e. where $h$ may be infinite-valued. By the monotone convergence theorem

$$\int h_N d\bar{\mu} \uparrow \int h d\bar{\mu} = \bar{\mu}(h)$$

while

$$\int h_n d\bar{\mu} = \mu(T(f \wedge N)) \uparrow \mu(f).$$

Thus $\mu(h) < \infty$ and so $h$ is integrable. Since $T(f \wedge N) \leqslant \rho h$ for all $N \geqslant 1$, it follows that $\sup_N T(f \wedge N)$ exists, $f \in L_T$, and $\hat{T}f$ is integrable. In addition, $\mu\ \hat{T}f) = \mu(f)$.

To complete the proof, note by Schwarz' inequality that

$$[\mu(fT(f \wedge N))]^2 \leqslant \mu(f^2)\, \mu(fT(f \wedge N)) < \infty$$

because $f \in L^2(B, \mu)$ and $T(f \wedge N)$ is bounded. If $f \neq 0$, then

$$\mu(fT(f \wedge N)) \neq 0$$

and hence

$$\mu(fT(f \wedge N)) \leqslant \mu(f^2)$$

for all $N \geqslant 1$. Therefore since

$$\mu[T(f \wedge N)^2] \leqslant \mu(f^2)$$

for all $N$, it follows that $\mu[(\hat{T}f)^2]$ exists when $\mu(f^2)$ exists, and

$$\mu[(\hat{T}f)^2] \leqslant \mu(f^2).$$

Thus the restriction of $\hat{T}$ to $L^2(B, \mu)$ is a linear operator on $L^2(B, \mu)$ with the following properties:

(1) $(\hat{T})^2 = \hat{T}$.
(2) $\hat{T}$ is a self-adjoint operator on $L^2(B, \mu)$.
(3) $\hat{T}(\lambda 1) = \lambda \cdot 1$ where $\lambda$ is a real number.
(4) $\hat{T}$ is positive.

Indeed (1), (3), (4) have already been mentioned. To show (2) consider

$$\begin{aligned}(\hat{T}f, g) &= \mu[(\hat{T}f)\, g] = \mu[\hat{T}((\hat{T}f)\, g)] = \mu[(\hat{T}f)\,(\hat{T}g)]\\ &= \mu[\hat{T}(f\hat{T}g)] = \mu(f\hat{T}g)\\ &(f, \hat{T}g).\end{aligned}$$

Therefore a theorem of Bahadur [4] implies that for $f \in L^2(B, \mu)$, $\hat{T}f = E(f \mid \tilde{\mathfrak{L}}_0)$ where $E(\ \mid \cdot\ \tilde{\mathfrak{L}}_0)$ designates the conditional expectation operator associated with $\tilde{\mathfrak{L}}_0 = \tilde{\sigma}^{-1}B(E)$. Remember that $B(E)$ designates the Boolean algebra of idempotents of $E = \{f \in R(B) \mid Tf = f\}$. It is a matter of direct verification to show that $E(\cdot \mid \tilde{\mathfrak{L}}_0)$ depends only on $\tilde{\sigma}\tilde{\mathfrak{L}}_0 = B_0$, $B$, and $\mu$. Thus we deviate from the standard notation and designate the conditional expectation associated with $B_0$, a $\sigma$-complete Boolean subalgebra of $B$, by the symbol $E_{B_0}$. Hence we have the following theorem.

THEOREM 9.1. *If $T$ is a Reynolds operator on $R(B)$, then there is a uniquely determined extension $E_{B_0}$ of $T$ which is the conditional expectation on $L^2(B, \mu)$ associated with $B_0$, the idempotent algebra of the range of $T$. The mapping $T \to E_{B_0}$ is a one to one mapping of the class of Reynolds operators on $R(B)$ onto the class of conditional expectations on $L^2(B, \mu)$.*

## 10. *Product Representation for Conditional Expectation*

In this section an analogue of Theorem 7.1 is derived for conditional expectations on $L^2(B, \mu)$. In order to facilitate this derivation, we note that for a given $\sigma$-complete Boolean subalgebra $A$ of $B$ the conditional expectation $E_A$ is a projection of $L^2(B, \mu)$ onto the $L^2$-subspace $L^2(A, \mu_{|A})$ of $L^2(B, \mu)$. If $\mathscr{E}$ designates the class of conditional expectations on $L^2(B, \mu)$ ordered by the relation: $E_{A_1} \leqslant E_{A_2}$ if and only if $E_{A_1}E_{A_2} = E_{A_1}$ and if $\mathfrak{A}$ is the class of $\sigma$-complete Boolean subalgebras of $B$ ordered by inclusion, then the mapping $A \to E_A$ is an isomorphism of the partially ordered set $\mathfrak{A}$ onto $\mathscr{E}$. An element $E_A \in \mathscr{E}$ is said to be *complemented* if there is an $A' \in \mathfrak{A}$ such that

(i) $E_A E_{A'} = \mu = E_{\{0,1\}}$,
(ii) $A \cup A'$ generates $B$.

LEMMA 10.1. *If $A_1$, $A_2 \in \mathfrak{A}$ and $E_{A_1}E_{A_2} = E_{A_2}E_{A_1}$, then and only then $E_{A_1}E_{A_2} = E_{A_1 \cap A_2}$.*

PROOF. If $E_{A_1}$ and $E_{A_2}$ commute, then by Bahadur's result [4], $E_{A_1}E_{A_2}$ is a conditional expectation. Therefore there is a $D \in \mathfrak{A}$ such that $E_{A_1}E_{A_2} = E_D$. Thus $E_D \leqslant E_{A_1}$ and $E_D \leqslant E_{A_2}$ and hence $D \subseteq A_1 \cap A_2$. In addition, $E_{A_1 \cap A_2}E_D = E_{A_1 \cap A_2}E_{A_1}E_{A_2} = E_{A_1 \cap A_2}$, so $A_1 \cap A_2 \subseteq D$. Therefore $A_1 \cap A_2 = D$.

Let $G_A = \{e \in B \mid E_A e \text{ is a real multiple of } 1\}$.

PROPOSITION 10.1. If $E_{A_1}E_{A_2} = \mu$, then $A_1 \cap A_2 = \{0, 1\}$. In addition, the following conditions are equivalent:

(1) $E_{A_1}E_{A_2} = \mu$.
(2) $A_1 \subseteq G_{A_2}$.
(3) $A_2 \subseteq G_{A_1}$.

PROOF. Apply Lemma 10.1.

THEOREM 10.1. *If $E_{B_0}$ has complement $E_{B_1}$, then there is a measure space $(\Omega_0 \times \Omega_1, \mathfrak{L}_0 \times \mathfrak{L}_1, \mu_0 \times \mu_1)$ and a $\sigma$-homomorphism $\xi$ of*

$$\mathfrak{L}^2(\Omega_0 \times \Omega_1, \mathfrak{L}_0 \times \mathfrak{L}_1, \mu_0 \times \mu_1)$$

*onto $L^2(B, \mu)$ such that*

(1) *$\xi\mathfrak{L}^2(\Omega_0 \times \Omega_1, \mathfrak{L}_0 \times \Omega_1, \mu_0 \times \mu_1) = L^2(B_0, \mu)$,*
(2) *$\xi\mathfrak{L}^2(\Omega_0 \times \Omega_1, \Omega_0 \times \mathfrak{L}_1, \mu_0 \times \mu_1) = \mathfrak{L}^2(B_1, \mu)$ where $B_1$ has the properties: $B_0 \cup B_1$ generates $B$ and $B_0 \cap B_1 = \{0, 1\}$,*
(3) *if $f \in L^2(B, \mu)$ then $E_{B_0} f = \xi g$ where*

$$g(\omega_0, \omega_1) = \int_{\Omega_1} \xi^{-1} f(\omega_0, \gamma)\, d\mu_1(\gamma)$$

*and $\xi^{-1}f$ stands for a member of the $\xi$-coset of preimages of $f$.*

REMARK 10.1. In Theorem 10.1, $L^2(B_i, \mu)$ stands for the $L^2$-subspace of $L^2(B, \mu)$ generated by $B_i \subseteq L^2(B, \mu)$.

PROOF. By Theorem 7.1, it follows that a $\sigma$-homomorphism $\xi$ exists which maps $\mathfrak{M}(\Omega_0 \times \Omega_1, \mathfrak{L}_0 \times \mathfrak{L}_1)$ onto $L^\infty(B, \mu)$ in such a manner that (1), (2), and (3) are satisfied if $L^2$ and $\mathscr{L}^2$ are replaced by $L^\infty$ and $\mathscr{L}^\infty$ respectively and if $\mu_1 = \nu$. Since $E_{B_0}$ is compatible with $\mu$, $\mu$ is equal to the measure defined in terms of $E_{B_0}$ on $B_1$ and $\nu = \mu\varphi_1^{-1}$. A measure $\mu_0$ can be defined on $(\Omega_0, \mathfrak{L}_0)$ as follows: $\mu_0 = \mu\varphi_0^{-1}$. Then $(\Omega_0 \times \Omega_1, \mathfrak{L}_0 \times \mathfrak{L}_1, \mu_0 \times \mu_1)$ becomes a measure space, and if $S \in \mathfrak{L}_0 \times \mathfrak{L}_1$, then $\mu_0 \times \mu_1(S) = \mu[\xi(\chi_S)]$. Since $\xi$ can be extended to $\mathscr{L}^2(\Omega_0 \times \Omega_1, \mathfrak{L}_0 \times \mathfrak{L}_1, \mu_0 \times \mu_1)$ in a natural way, the theorem is valid.

## 11. *Classification of Conditional Expectations*

A conditional expectation $E_A$ on $L^2(B, \mu)$ will fall into one of three classes:

(1) where $G_A = \{0, 1\}$,

(2) where $G_A \neq \{0, 1\}$ but $E_A$ is not complemented,

(3) where $E_A$ is complemented.

By Proposition 10.1, it follows that these classes are disjoint. However, it is possible that class (2) is void. The following example shows this is not the case.

*Example* 11.1. Let $L^2(B, \mu)$ be the $F$-ring of all ordered 6-tupples of real numbers where

$$\mu(1, 0, 0, 0, 0, 0) = \mu(0, 1, 0, 0, 0, 0) = \cdots = \mu(0, 0, 0, 0, 0, 1) = \tfrac{1}{6},$$

and let

$$T = \begin{pmatrix} \frac{1}{4} I_4 & 0 \\ 0 & \frac{1}{2} I_2 \end{pmatrix}$$

where $I_k$ stands for the $k \times k$ matrix of 1's. The atoms of

$$B_0 = \{e \in B \mid Te = e\}$$

are $(1, 1, 1, 1, 0, 0)$ and $(0, 0, 0, 0, 1, 1)$ while the atoms of $G_{B_0}$ are the following twelve: $(1, 1, 0, 0, 1, 0)$, $(1, 1, 0, 0, 0, 1)$, $(1, 0, 1, 0, 1, 0)$, $\cdots$, $(0, 0, 1, 1, 0, 1)$. The maximum number of elements in a Boolean subalgebra $B_1$ of $B$ contained in $G_{B_0}$ is four. Therefore the Boolean subalgebra of $B$ generated by $B_0 \cup B_1$ contains at most $2^4$ elements while $B$ contains $2^6$ elements. Therefore $T = E_{B_0}$ cannot be complemented.

If $E_A$ belongs to class (1), it is called *purely conditional.*

PROPOSITION 11.1. The following statements are equivalent:

(1) $E_A$ is purely conditional.

(2) If $A'$ is a $\sigma$-complete Boolean subalgebra of $B$, then

$$E_A E_{A'} = \mu \Rightarrow E_{A'} = \mu.$$

(3) For each $f \in B$, there is an $e \in A - \{0, 1\}$ such that $\mu(ef) \neq \mu(e)\,\mu(f)$.

PROOF. (1) $\Leftrightarrow$ (2) follows by Proposition 10.1 because $G_A = \{0, 1\} \Leftrightarrow \{0, 1\}$ is the only $\sigma$-complete Boolean subalgebra of $B$ which is a subset of $G_A$.

(1) $\Rightarrow$ (3): If $\mu(ef) = \mu(e)\,\mu(f)$ for all $e \in A$, then

$$\mu(E_A\, x\, E_F y) = \mu(E_A x)\,\mu(E_F y)$$

for all $x, y \in B$ where $F = \{0, f, 1 - f, 1\}$. It follows from [18, p. 351] that $E_A E_F y = \mu(E_F y) = \mu(y)$. Thus $F \subseteq G_A$ which contradicts the assumption of pure conditionality.

(3) $\Rightarrow$ (1): If $E_A$ is not purely conditional then $G_A \neq \{0, 1\}$. Let $f \in G_A$ where $f \neq 0$, 1. Then for $e \in A$, $E_A(ef) = eE_A f = [\mu(f)]\, e$, and hence

$$\mu(ef) = \mu(E_A(ef)] = \mu(e)\mu(f).$$

PROPOSITION 11.2. If $E_{A^c}$ is a complement of $E_A$, then $A^c$ is a $\sigma$-complete Boolean subalgebra of $B$ which is maximal relative to the condition $A^c \subseteq G_A$.

PROOF. If $A^c$ is not maximal in $G_A$, then there exists $b \in G_A - A^c$ such that $A^c \cup \{b\}$ generates a $\sigma$-complete Boolean subalgebra $D$ of $B$ which is contained in $G_A$. Each $v \in D$ has the form

$$v = v_1 b + v_2(1 - b)$$

where $v_i \in A^c$. By Theorem 10.1, there are elements $\hat{b}$, $\hat{v}_i$ of

$$B(\mathscr{L}^2(\Omega_0 \times \Omega_1, \mathfrak{L}_0 \times \mathfrak{L}_1, \mu_0 \times \mu_1))$$

such that $\xi\hat{b} = b$ and $\xi\hat{v}_i = v_i$. Thus for $\hat{v} = \hat{v}_1\hat{b} + \hat{v}_2(1 - \hat{b})$, $\xi\hat{v} = v$ and

$$\int_{\Omega_1} \hat{v}(\omega_0, \omega_1)\, d\mu_1 = \int_{\Omega_1} \hat{v}_1\hat{b}d\mu_1 + \int \hat{v}_2\hat{b}d\mu_1$$

is a constant. By construction, $\hat{v}_i$ is independent of $\omega_0$. Let $v_i(\omega_0, \omega_1) = \hat{u}_i(\omega_1)$. Then

$$\int_{\Omega_1} \hat{v}d\mu_1 = \int_{\Omega_1} (\hat{u}_1 - \hat{u}_2)\, \hat{b}d\mu_1 + \int_{\Omega_1} \hat{u}_2 d\mu_1,$$

and hence

$$\int_{\Omega_1} (\hat{u}_1 - \hat{u}_2)\, \hat{b}d\mu_1$$

is a constant for any choice of the $\hat{u}_i$'s in $B(\mathfrak{M}(\Omega_1, \mathfrak{L}_1))$. Assume that for some pair $(\omega_0, \alpha_0)$,

$$\hat{b}(\omega_0, \omega_1) \neq \hat{b}(\alpha_0, \omega_1)$$

on some $\omega_1$-set of positive measure. Define $\omega_1$-sets $S_0$, $S_1$ and $S_{-1}$ as follows:

$$\omega_1 \in S_k \qquad \text{if} \qquad \hat{b}(\omega_0, \omega_1) - \hat{b}(\alpha_0, \omega_1) = k$$

for $k = 0, 1, -1$. The functions $\chi_{\Omega_0 \times S_k} \in \Omega_0 \times \mathfrak{L}_1$ for $k = 0, 1, -1$ and are admissible $\hat{v}_i$'s. Thus

$$\int_{\Omega_1} \chi_{\Omega_0 \times S_k}[\hat{b}(\omega_0, \omega_1) - \hat{b}(\alpha_0 \omega_1)]\, d\mu_1 = 0$$

for $i = 0, -1, 1$. Therefore $S_1$ and $S_{-1}$ have zero measure contrary to the assumption. Thus $\hat{b}(\omega_0, \omega_1) = \hat{b}(\alpha_0, \omega_1)$ a.e. for every pair $(\alpha_0, \omega_0)$, and hence $b \in A^c$. Since $b$ is assumed not to be in $A^c$, it follows that $A^c$ is maximal in $G_A$.

Proposition 11.3. Let $A'$ be maximal among the $\sigma$-complete Boolean subalgebras of $B$ which are contained in $G_A$. If $D$ is the $\sigma$-complete Boolean subalgebra generated by $A \cup A'$, then $D$ is purely conditional.

Proof. If $E_D$ is not purely conditional, then by Propositions 10.1 and 11.1, there is an $f \in B$ such that for $F = \{0, 1, f, 1 - f\}$, $E_D E_F = \mu$ and hence $F \subseteq G_D$. Consider $A^* = \{fa_0 + (1-f)\, a_1 \mid a_i \in A'\}$. If $g = fa_0 + (1-f)\, a_1$, then

$$E_A g = E_A E_D g = E_A\, (a_0 \mu(f) + a_1\, (1 - \mu(f)))$$
$$= \text{constant multiple of } 1.$$

Thus $A^* \subseteq G_A$ and hence $A'$ is not maximal relative to the condition $A' \subseteq G$. This contradicts the hypothesis of nonpure conditionality and hence the proposition is valid.

The converse of Proposition 11.3 is not valid, that is, the algebra generated by $A \cup A'$ may be purely conditional when $A'$ is not maximal.

*Example* 11.2. *Let* $L^2(B, \mu)$ be as in Example 11.1 and let

$$T = \begin{pmatrix} \frac{1}{3} I_3 & 0 \\ 0 & \frac{1}{3} I_3 \end{pmatrix}$$

If $A$ is the subalgebra of $B$ generated by $(1, 1, 1, 0, 0, 0)$ and $A'$ is the subalgebra generated by $(0, 1, 1, 0, 1, 1)$, then $A \cup A'$ generates a subalgebra $D$ of $B$ where $E_D$ is purely conditional, while $A'$ is a proper $\sigma$-subalgebra of the algebra $A_0$ generated by $(1, 0, 0, 1, 0, 0)$ and $(0, 1, 0, 0, 1, 0)$ and contained in $G_A$.

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