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An Algebraic Characterization of Some Principal Regulated Rational Cones

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The aim of this paper is to deal with formal power series over a commutative semiring A. Generalizing Wechler's pushdown automata and pushdown transition matrices yields a characterization of the A-semi-algebraic power series in terms of acceptance by pushdown automata. Principal regulated rational cones generated by cone generators of a certain form are characterized by algebraic systems given in certain matrix form. This yields a characterization of some principal full semi-AFL's in terms of context-free grammars. As an application of the theory, the principal regulated rational cone of one-counter "languages" is considered.

INTRODUCTION

Berstel [1, p. 267] states that "up to now, no characterization of the family of context-free grammars generating the languages of a cone is known." In this paper, we will solve this problem for principal full semi-AFLs that have a generator generated by a simple deterministic context-free grammar in the sense of Korenjak and Hopcroft [6], such that all leftmost terminal symbols of the right sides of the productions are distinct.

The paper is divided into six sections. Since we work within the theory of formal power series, Section 1 is devoted to the basic definitions and results of this theory as given by Salomaa and Soittola [10]. Section 2 gives some definitions and results on infinite matrices, infinite linear systems, and infinite automata. They are generalizations of the work of Kuich and Urbanek [9]. Section 3 introduces pushdown transition matrices and pushdown automata as special case of infinite automata. The classical characterization result that the context-free languages are exactly those languages accepted by quasi-real time pushdown automata is generalized to semi-algebraic formal power series and cycle-free pushdown automata.

In Section 4, we introduce regulated rational transductions and types of pushdown automata. The main result of this section states that if the power series s is the behavior of a cycle-free pushdown automaton of type t and a regulated rational transduction τ is applied to s, then $\tau(s)$ is the behavior of a cycle-free pushdown automaton of type t, too. In Section 5, sets $\mathscr{C}^t_A$ of power series, A a commutative semiring, are considered containing exactly all the power series that are the behavior

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of cycle-free pushdown automata of type t. It turns out that $\mathscr{C}^t_A$ is a principal regulated rational cone. The power series in $\mathscr{C}^t_A$ can be characterized as the first components of the unique solutions of certain algebraic systems given in matrix form. If the language-theoretic equivalent $\mathscr{C}^t$ of $\mathscr{C}^t_{\mathbb{B}}$ is considered, then $\mathscr{C}^t$ is a principal full semi-AFL or a principal semi-AFL, depending on whether the generator of $\mathscr{C}^t$ is erasable or not.

The last section considers and characterizes those power series that are the behavior of cycle-free pushdown automata with just a single pushdown symbol, i.e., the "one-counter languages" are considered.

1. Formal Power Series in Noncommuting Variables

A *monoid* consists of a set, an associative binary operation on this set, and a neutral element with respect to this binary operation. A monoid is called *commutative* iff the binary operation is commutative.

A *semiring* $\langle A, +, \cdot, 0, 1\rangle$ consists of a set A and of two binary operations, called addition and multiplication, denoted by $+$ and $\cdot$, respectively, and two constant elements 0 and 1 such that

(i) $\langle A, +, 0\rangle$ is a commutative monoid,

(ii) $\langle A, \cdot, 1\rangle$ is a monoid,

(iii) the distribution laws $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$ hold for all $a, b, c \in A$,

(iv) $0 \cdot a = a \cdot 0 = 0$ for all $a \in A$.

Usually multiplication is denoted by juxtaposition and the same notation is used for a semiring and its underlying set.

A semiring A is called *commutative* iff $ab = ba$ for all $a, b \in A$. The two most important semirings are

(i) The commutative semiring $\mathbb{B} = \{0, 1\}$ with $1 + 1 = 1$. The remaining rules for addition and multiplication are forced by the axioms.

(ii) The commutative semiring of nonnegative integers $\mathbb{N}$ with the usual addition and multiplication.

In the sequel, let A be a semiring, Σ be an alphabet, and Σ^* be the free monoid generated by Σ with the neutral element ε.

Mappings r of Σ^* into A are called *formal power series*. The value of r at $w \in \Sigma^*$ is denoted by (r, w) and r itself is written as a formal sum $r = \sum_{w \in \Sigma^*} (r, w)w$. The values (r, w) are also referred to as the *coefficients* of the formal power series r.

The collection of all power series r as defined above is denoted by $A\langle\langle\Sigma^*\rangle\rangle$. Given $r \in A\langle\langle\Sigma^*\rangle\rangle$, the subset of Σ^* defined by $\{w \mid (r, w) \neq 0\}$ is called the *support* of r and denoted by $\mathrm{supp}(r)$. The subset of $A\langle\langle\Sigma^*\rangle\rangle$ consisting of all series with a finite support is denoted by $A\langle\Sigma^*\rangle$. Its series are called polynomials.

Let $r_1, r_2 \in A\langle\langle\Sigma^*\rangle\rangle$. Then $r_1 + r_2 \in A\langle\langle\Sigma^*\rangle\rangle$ is defined by $(r_1 + r_2, w) = (r_1, w) + (r_2, w)$ and $r_1 \cdot r_2 \in A\langle\langle\Sigma^*\rangle\rangle$ is defined by $(r_1 \cdot r_2, w) = \sum_{w = w_1 w_2} (r_1, w_1)(r_2, w_2)$, $w \in \Sigma^*$. The sets $A\langle\langle\Sigma^*\rangle\rangle$ and $A\langle\Sigma^*\rangle$ together with the operations of addition $+$ and (Cauchy) product $\cdot$ and the constant power series 0 and ε are again semirings.

Let $k \geqslant 0$. Then the *restriction* $R_k(r)$ of a series $r \in A\langle\langle\Sigma^*\rangle\rangle$ is defined by $R_k(r) = \sum_{|w| \leqslant k} (r, w)w$, where $|w|$, $w \in \Sigma^*$, denotes the length of w.

A sequence $r_1, r_2, \ldots, r_n, \ldots$ of elements of $A\langle\langle\Sigma^*\rangle\rangle$ *converges* to the *limit* $r \in A\langle\langle\Sigma^*\rangle\rangle$, in symbols, $\lim_{n\to\infty} r_n = r$, iff for all $k \geqslant 0$ there exists an $m(k) \geqslant 1$ such that for all $j \geqslant 0$ $R_k(r_{m(k)+j}) = R_k(r_{m(k)}) = R_k(r)$.

An element $r \in A\langle\langle\Sigma^*\rangle\rangle$ is called *quasiregular* iff $(r, \varepsilon) = 0$. If r is quasiregular, then $\lim_{n\to\infty} \sum_{j=1}^{n} r^j$ exists. It is called the *quasi-inverse* of r and denoted by r^+. A subsemiring of $A\langle\langle\Sigma^*\rangle\rangle$ is called *rationally closed* iff it contains the quasi-inverse of every quasiregular element. The smallest rationally closed subsemiring of $A\langle\langle\Sigma^*\rangle\rangle$ that contains all polynomials is denoted by $A^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle$.

For the rest of this section, let A be a commutative semiring and let $Z = \{z_1, \ldots, z_n\}$, $n \geqslant 1$, be an alphabet with $Z \cap \Sigma = \emptyset$, the empty set. An algebraic system over $A\langle\langle\Sigma^*\rangle\rangle$ (with variables in Z) is a set of equations of the form

$$z_i = p_i, \qquad p_i \in A\langle(Z \cup \Sigma)^*\rangle, \quad 1 \leqslant i \leqslant n.$$

It is called *strict* iff $\mathrm{supp}(p_i) \subseteq (Z \cup \Sigma)^* \Sigma (Z \cup \Sigma)^* \cup \{\varepsilon\}$, $1 \leqslant i \leqslant n$.

Let $\sigma = (\sigma_1, \ldots, \sigma_n)$, $\sigma_i \in A\langle\langle(Z \cup \Sigma)^*\rangle\rangle$, $1 \leqslant i \leqslant n$, and $p \in A\langle(Z \cup \Sigma)^*\rangle$. Then $\sigma(p) \in A\langle\langle(Z \cup \Sigma)^*\rangle\rangle$ denotes the series obtained from p by replacing simultaneously each occurrence of z_i by σ_i, $1 \leqslant i \leqslant n$. Hence σ is a substitution homomorphism σ: $A\langle(Z \cup \Sigma)^*\rangle \to A\langle\langle(Z \cup \Sigma)^*\rangle\rangle$, defined by $\sigma(z_i) = \sigma_i$, $1 \leqslant i \leqslant n$, and $\sigma(x) = x$, $x \in \Sigma$.

An n-tuple $\sigma = (\sigma_1, \ldots, \sigma_n)$ of elements of $A\langle\langle\Sigma^*\rangle\rangle$ is called a *solution* of $z_i = p_i$, $1 \leqslant i \leqslant n$, iff $\sigma_i = \sigma(p_i)$, $1 \leqslant i \leqslant n$.

A slight generalization of Berstel [1, Theorem 1.5] yields that each strict algebraic system has a unique solution (see [8, Theorem 34.3]).

A series in $A\langle\langle\Sigma^*\rangle\rangle$ is called *A-semi-algebraic* iff it is a component of the solution of a strict algebraic system over $A\langle\langle\Sigma^*\rangle\rangle$. The collection of all A-semi-algebraic series in $A\langle\langle\Sigma^*\rangle\rangle$ is denoted by $A^{\mathrm{semi\text{-}alg}}\langle\langle\Sigma^*\rangle\rangle$. This definition of $A^{\mathrm{semi\text{-}alg}}\langle\langle\Sigma^*\rangle\rangle$ is equivalent to that of Salomaa and Soittola [10]. The collection of all quasiregular series in $A^{\mathrm{semi\text{-}alg}}\langle\langle\Sigma^*\rangle\rangle$ is denoted by $A^{\mathrm{alg}}\langle\langle\Sigma^*\rangle\rangle$.

The classical theory of context-free and right-linear languages can be thought of as the theory of power series in $\mathbb{B}^{\mathrm{semi\text{-}alg}}\langle\langle\Sigma^*\rangle\rangle$ and $\mathbb{B}^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle$, respectively (see [8, 10]).

Let $z_i = p_i$, $1 \leqslant i \leqslant n$, be an algebraic system over $\mathbb{B}\langle\langle\Sigma^*\rangle\rangle$ with variables in $Z = \{z_1, \ldots, z_n\}$. Then this algebraic system *induces* a context-free grammar $G = (Z, \Sigma, P, z_1)$ with $z_i \to \alpha \in P$ iff $(p_i, \alpha) = 1$, $1 \leqslant i \leqslant n$. By [8, Theorem 37.1], $L(G)$ equals the support of the first component of the minimal solution of $z_i = p_i$, $1 \leqslant i \leqslant n$.

2. Matrices and Automata

Throughout this section, A is a semiring, Σ is an alphabet and I with or without indices is an at most countable index set. The next definitions and results are slight generalizations of Kuich and Urbanek [9].

A matrix M with entries in the semiring $A\langle\langle\Sigma^*\rangle\rangle$ is a mapping $M: I_1 \times I_2 \to A\langle\langle\Sigma^*\rangle\rangle$. Hence the rows of M are indexed by I_1 and the columns of M are indexed by I_2. The (i_1, i_2) entry of M is denoted by M_{i_1,i_2}, $i_1 \in I_1$, $i_2 \in I_2$. The collection of all matrices as defined above is denoted by $(A\langle\langle\Sigma^*\rangle\rangle)^{I_1\times I_2}$. If I_1 or I_2, respectively, is a singleton, M is called *row* or *column vector*, respectively. If I_1 and I_2 are finite, M is called a *finite matrix*. If $I_1 = \{1,..., n\}$ or $I_2 = \{1,..., m\}$, $n, m \geqslant 1$, respectively, then $(A\langle\langle\Sigma^*\rangle\rangle)^{I_1\times I_2}$ is also denoted by $(A\langle\langle\Sigma^*\rangle\rangle)^{n\times I_2}$ or $(A\langle\langle\Sigma^*\rangle\rangle)^{I_1\times m}$, respectively.

A matrix M is called *row finite* (*column finite*, respectively) iff for each $i_1 \in I_1 (i_2 \in I_2$, respectively) the set $\{i_2 \mid M_{i_1,i_2} \neq 0\}$ ($\{i_1 \mid M_{i_1,i_2} \neq 0\}$, respectively) is finite. The *null matrix* $0 \in (A\langle\langle\Sigma^*\rangle\rangle)^{I_1\times I_2}$ is defined by $0_{i_1,i_2} = 0$ for all $i_1 \in I_1$, $i_2 \in I_2$. The *matrix of unity* $E \in (A\langle\langle\Sigma^*\rangle\rangle)^{I\times I}$ is defined by $E_{i_1,i_2} = \delta_{i_1,i_2}\varepsilon$, where δ_{i_1,i_2} is the Kronecker symbol, i.e., $\delta_{i_1,i_2} = 1$ if $i_1 = i_2$ and $\delta_{i_1,i_2} = 0$ if $i_1 \neq i_2$, $i_1, i_2 \in I$.

Let M_1, $M_2 \in (A\langle\langle\Sigma^*\rangle\rangle)^{I_1\times I_2}$. Then $M_1 + M_2 \in (A\langle\langle\Sigma^*\rangle\rangle)^{I_1\times I_2}$ is defined by $(M_1 + M_2)_{i_1,i_2} = (M_1)_{i_1,i_2} + (M_2)_{i_1,i_2}$, $i_1 \in I_1$, $i_2 \in I_2$. Let $M_1 \in (A\langle\langle\Sigma^*\rangle\rangle)^{I_1\times I_2}$, $M_2 \in (A\langle\langle\Sigma^*\rangle\rangle)^{I_2\times I_3}$ and let M_1 be row finite or M_2 be column finite. Then $M_1 \cdot M_2 \in (A\langle\langle\Sigma^*\rangle\rangle)^{I_1\times I_3}$ is defined by

$$(M_1 \cdot M_2)_{i_1,i_3} = \sum_{i_2\in I_2} (M_1)_{i_1,i_2}(M_2)_{i_2,i_3}, \qquad i_1 \in I_1, \quad i_3 \in I_3.$$

Since by the extra assumption of M_1 being row finite or M_2 being column finite there are only finitely many $i_2 \in I_2$ such that $(M_1)_{i_1,i_2}(M_2)_{i_2,i_3} \neq 0$, $(M_1M_2)_{i_1,i_3}$, $i_1 \in I_1$, $i_3 \in I_3$ is well defined.

Let $(A\langle\langle\Sigma^*\rangle\rangle)_J^{I_1\times I_2}$ denote the collection of all row and column finite matrices of $(A\langle\langle\Sigma^*\rangle\rangle)^{I_1\times I_2}$. Then it is easily seen that $\langle(A\langle\langle\Sigma^*\rangle\rangle)^{I_1\times I_2}, +, 0\rangle$ is a commutative monoid and $\langle(A\langle\langle\Sigma^*\rangle\rangle)_J^{I\times I}, \cdot, E\rangle$ is a monoid. Since the distribution laws hold true and 0 is a multiplicative zero, $\langle(A\langle\langle\Sigma^*\rangle\rangle)_J^{I\times I}, +, \cdot, 0, E\rangle$ is a semiring.

If matrices are partitioned into blocks with suitable index sets, then sum and product can be defined in terms of these blocks in the usual manner.

Let $M \in (A\langle\langle\Sigma^*\rangle\rangle)^{I_1\times I_2}$ and $k \geqslant 0$. Then $R_k(M) \in (A\langle\langle\Sigma^*\rangle\rangle)^{I_1\times I_2}$ is defined by $(R_k(M))_{i_1,i_2} = R_k(M_{i_1,i_2})$, $i_1 \in I_1$, $i_2 \in I_2$. A *sequence* $M_1, M_2,..., M_n,...,$ of *matrices* in $(A\langle\langle\Sigma^*\rangle\rangle)^{I_1\times I_2}$ *converges* to the *limit* $M \in (A\langle\langle\Sigma^*\rangle\rangle)^{I_1\times I_2}$, in symbols $\lim_{n\to\infty} M_n = M$, iff for all $k \geqslant 0$ there exist $m(k) \geqslant 1$ such that for all $j \geqslant 0$, $R_k(M_{m(k)+j}) = R_k(M_{m(k)}) = R_k(M)$.

For $M \in (A\langle\langle\Sigma^*\rangle\rangle)_J^{I\times I}$, the *powers* M^k, $k \geqslant 0$, are defined by $M^0 = E$ and $M^{k+1} = MM^k = M^kM$.

The *quasi-inverse* M^+ of M, when it exists, is defined to be the matrix $M^+ = \lim_{k\to\infty} \sum_{j=1}^k M^j$. If M^+ exists, then $M^* = E + M^+$.

Given $M \in (A\langle\langle\Sigma^*\rangle\rangle)^{I_1\times I_2}$, the matrix (M, w), $w \in \Sigma^*$, is defined by $(M, w)_{i_1,i_2} = (M_{i_1,i_2}, w)$, $i_1 \in I_1$, $i_2 \in I_2$. The matrix M is called *quasiregular* iff $(M, \varepsilon) = 0$.

A matrix $M \in (A\langle\langle\Sigma^*\rangle\rangle)_J^{I \times I}$ is called *nilpotent of order n*, $n \geqslant 1$, iff $M^n = 0$ and it is called *cycle-free of order n* iff (M, ε) is nilpotent of order n. It is called *nilpotent* or *cycle-free*, respectively, iff there exists an $n \geqslant 1$ such that M is nilpotent or cycle-free, respectively, of order n.

We now want to show that the quasi-inverse of a cycle-free matrix does exist.

LEMMA 1. *Let $M \in (A\langle\langle\Sigma^*\rangle\rangle)_J^{I \times I}$ be cycle-free of order $n \geqslant 1$. Then for all $w \in \Sigma^*$ with $|w| = k$, $(M^{(k+1)n+j}, w) = 0$ for all $j \geqslant 0$.*

Proof. Let $M = M_0 + M_1$, $M_0 = (M, \varepsilon)\varepsilon$, and M_1 quasiregular. Then $(M^{n+j}, \varepsilon) = ((M_0 + M_1)^{n+j}, \varepsilon) = (M_0^{n+j}, \varepsilon) = 0$ for all $j \geqslant 0$. Now let $w \in \Sigma^*$ with $|w| = k + 1$, $k \geqslant 0$. Then

$$(M^{(k+2)n+j}, w) = \sum_{uv = w} (M^n, u)(M^{(k+1)n+j}, v) = 0 \qquad \text{for all} \quad j \geqslant 0. \quad \blacksquare$$

COROLLARY 1. *Let $P \in (A\langle\langle\Sigma^*\rangle\rangle)_J^{I_1 \times I_2}$ and let $M \in (A\langle\langle\Sigma^*\rangle\rangle)_J^{I_1 \times I_1}$ be cycle-free of order $n \geqslant 1$. Let $S_k \in (A\langle\langle\Sigma^*\rangle\rangle)^{I_1 \times I_2}$, $k \geqslant 0$, be defined by $S_0 = 0$, $S_{k+1} = P + MS_k$, $k \geqslant 0$. Then $S_k = (\sum_{j=0}^{k-1} M^j)P$ and $R_k(S_{(k+1)n+j}) = R_k(S_{(k+1)n})$ for $k \geqslant 0$, $j \geqslant 0$, and $\lim_{k \to \infty} S_k = S$ exists.*

THEOREM 1. *Let $M \in (A\langle\langle\Sigma^*\rangle\rangle)_J^{I \times I}$ be cycle-free. Then M^+ and M^* do exist.*

Proof. In Corollary 1, let $P = E$. $\blacksquare$

An $A\langle\langle\Sigma^*\rangle\rangle$-linear system is defined to be of the form $Y = P + MY$, where $P \in (A\langle\langle\Sigma^*\rangle\rangle)_J^{I_1 \times I_2}$, $M \in (A\langle\langle\Sigma^*\rangle\rangle)_J^{I_1 \times I_1}$, and Y is a matrix of distinct variables whose rows and columns are indexed by I_1 and I_2, respectively. An $A\langle\langle\Sigma^*\rangle\rangle$-linear system is called *cycle-free of order n* iff $M \in (A\langle\langle\Sigma^*\rangle\rangle)_J^{I_1 \times I_1}$ is cycle-free or order n. It is called *cycle-free* iff there exists an $n \geqslant 1$ such that it is cycle-free of order n.

A matrix $S \in (A\langle\langle\Sigma^*\rangle\rangle)^{I_1 \times I_2}$ is called a *solution* of an $A\langle\langle\Sigma^*\rangle\rangle$-linear system $Y = P + MY$ iff $S = P + MS$. Let $S_k \in (A\langle\langle\Sigma^*\rangle\rangle)^{I_1 \times I_2}$, $k \geqslant 0$, be the sequence defined by $S_0 = 0$, $S_{k+1} = P + MS_k$, $k \geqslant 0$. If this sequence converges to a matrix $S \in (A\langle\langle\Sigma^*\rangle\rangle)^{I_1 \times I_2}$, then S is referred to as the *strong solution* of $Y = P + MY$.

LEMMA 2. *Let $Y = P + MY$ be an $A\langle\langle\Sigma^*\rangle\rangle$-linear system. Let $S_0 = 0$, $S_{k+1} = P + MS_k$, $k \geqslant 0$, and assume the existence of $\lim_{k \to \infty} S_k = S$. Then the strong solution S is unique and S is a solution.*

Proof. By definition, the strong solution is unique. Since $\lim_{k \to \infty} S_k$ does exist, for any $k \geqslant 0$ there exists an integer $m(k) \geqslant 0$ such that $R_k(S_{m(k)+j}) = R_k(S_{m(k)})$ for all $j \geqslant 0$. Hence $R_k(P + MS) = R_k(P + MS_{m(k)}) = R_k(S_{m(k)+1}) = R_k(S)$ for all $k \geqslant 0$. This implies that S is a solution of $Y = P + MY$. $\blacksquare$

The next theorem shows that cycle-free systems have the nice property of a unique solution.

THEOREM 2. *Let $Y = P + MY$ be a cycle-free $A\langle\langle\Sigma^*\rangle\rangle$-linear system. Then its strong solution is its unique solution.*

Proof. Let $S_0 = 0$, $S_{k+1} = P + MS_k$, $k \geqslant 0$, and $\lim_{k\to\infty} S_k = S$. By Corollary 1, the strong solution does exist and

$$R_k(S) = R_k(S_{(k+1)n}) = R_k\left(\left(\sum_{j=0}^{(k+1)n-1} M^j\right) P\right).$$

Let T be a solution of $Y = P + MY$, i.e., $T = P + MT$. Then by substitution, $T = (\sum_{j=0}^{l-1} M^j)P + M^l T$ for arbitrary $l \geqslant 1$. Hence $R_k(T) = ((\sum_{j=0}^{(k+1)n-1} M^j)P) + R_k(M^{(k+1)n}T) = R_k(S)$ for all $k \geqslant 0$, i.e., $T = S$. ▮

COROLLARY 2. *Let $Y = P + MY$ be a cycle-free $A\langle\langle\Sigma^*\rangle\rangle$-linear system with unique solution S. Let $M = M_0 + M_1$, $M_0 = (M, \varepsilon)\varepsilon$, and M_1 quasiregular. Then S is the unique solution of $Y = M_0^* P + M_0^* M_1 Y$.*

Proof. Let $M_0^n = 0$ for some $n \geqslant 1$. Then

$$\begin{aligned} S &= P + M_0 S + M_1 S = P + M_0 P + M_0^2 S + M_0 M_1 S + M_1 S \\ &= \cdots = P + M_0 P + \cdots + M_0^{n-1} P + M_0^n S + M_0^{n-1} M_1 S + \cdots + M_1 S \\ &= M_0^* P + M_0^* M_1 S. \end{aligned}$$

The solution S is unique, since $M_0^* M_1$ is quasiregular. ▮

COROLLARY 3. *The cycle-free $A\langle\langle\Sigma^*\rangle\rangle$-linear system $Y = P + MY$ has the unique solution $S = M^*P$.*

Proof. $P + MM^*P = M^*P$. ▮

We now turn to automata theory. The first definitions are based on Eilenberg [2]. An *$A\langle\langle\Sigma^*\rangle\rangle$-automaton* $\mathfrak{A} = (I, M, q_0, P)$ is given by:

(i) an at most countable set I of elements called *states*,

(ii) a matrix $M \in (A\langle\langle\Sigma^*\rangle\rangle)_J^{I\times I}$ called the *transition matrix*,

(iii) $q_0 \in I$ called the *initial state*,

(iv) a column vector $P \in (A\langle\langle\Sigma^*\rangle\rangle)_J^{I\times 1}$ called the *final state vector*.

If $M_{p,q} = r \neq 0$, $p, q \in I$, then we say that the *edge* (p, q) with *label* r is in $\mathfrak{A}$. A *path* c from p to q in $\mathfrak{A}$ is a finite sequence of edges (p_0, p_1), (p_1, p_2),..., (p_{k-1}, p_k), $p = p_0$, $q = p_k$, $k > 0$. It is written c: $p \to q$. The integer k is called the *length* of the path and is denoted by $|c|$. If r_i is the label of (p_{i-1}, p_i), $1 \leqslant i \leqslant k$, then the *label* $\|c\|$ of the path c is defined to be $\|c\| = r_1 r_2 \cdots r_k$. For each state $q \in I$ we introduce the null path λ_q from q to q with $|\lambda_q| = 0$ and $\|\lambda_q\| = \varepsilon$.

Let c: $p \to q$ and d: $q \to r$ be paths, then the *composition* cd: $p \to r$ is defined by concatenation. We have $|cd| = |c| + |d|$ and $\|cd\| = \|c\|\,\|d\|$.

When it exists, the *behavior* $\|\mathfrak{A}\| \in A\langle\langle\Sigma^*\rangle\rangle$ of an $A\langle\langle\Sigma^*\rangle\rangle$-automaton $\mathfrak{A} = (I, M, q_0, P)$ is defined by

$$\|\mathfrak{A}\| = \sum_{q \in I} \sum_{c: q_0 \to q} \|c\| P_q.$$

LEMMA 3. *Let $\mathfrak{A} = (I, M, q_0, P)$ be an $A\langle\langle\Sigma^*\rangle\rangle$-automaton. Then for all $k \geqslant 0$, $p, q \in I$, the formal power series $(M^k)_{p,q}$ is the sum of all labels of paths $c: p \to q$ of length k, i.e.,*

$$(M^k)_{p,q} = \sum_{c: p \to q, |c| = k} \|c\|.$$

Proof. The proof is straightforward. ∎

An $A\langle\langle\Sigma^*\rangle\rangle$-automaton $\mathfrak{A} = (I, M, q_0, P)$ is called *cycle-free* iff M is cycle-free. We note that cycle-free $\mathbb{B}\langle\langle\Sigma^*\rangle\rangle$-automata correspond to quasi-realtime automata.

THEOREM 3. *Let $\mathfrak{A} = (I, M, q_0, P)$ be a cycle-free $A\langle\langle\Sigma^*\rangle\rangle$-automaton and let S be the unique solution of $Y = P + MY$. Then $\|\mathfrak{A}\|$ is well defined and $\|\mathfrak{A}\| = (M^*P)_{q_0} = S_{q_0}$.*

Proof. $\|\mathfrak{A}\| = \sum_{q \in I} \sum_{c: q_0 \to q} \|c\| P_q = \sum_{q \in I} \sum_{k=0}^{\infty} \sum_{c: q_0 \to q, |c| = k} \|c\| P_q = \sum_{q \in I} \sum_{k=0}^{\infty} (M^k)_{q_0, q} P_q = (M^*P)_{q_0} = S_{q_0}$, by Lemma 3 and Corollary 3. ∎

3. PUSHDOWN AUTOMATA AND SEMI-ALGEBRAIC SERIES

We introduce the pushdown automaton as a special type of automaton as defined in Section 2. Throughout the rest of this paper, A is a commutative semiring, Σ is an alphabet, Γ is an alphabet of pushdown symbols, and Q is a finite index set. The items Σ, Γ, and Q may be indexed.

A matrix M whose entries are in the semiring $(A\langle\langle\Sigma^*\rangle\rangle)^{Q \times Q}$ and are indexed by the elements of $\Gamma^* \times \Gamma^*$ is a mapping $M: \Gamma^* \times \Gamma^* \to (A\langle\langle\Sigma^*\rangle\rangle)^{Q \times Q}$. All definitions and results concerning the matrices of Section 2 are easily transferred to the matrices defined above.

The collection of all matrices as defined above is denoted by $((A\langle\langle\Sigma^*\rangle\rangle)^{Q \times Q})^{\Gamma^* \times \Gamma^*}$. It is obvious that $\langle((A\langle\langle\Sigma^*\rangle\rangle)^{Q \times Q})_J^{\Gamma^* \times \Gamma^*}, +, \cdot, 0, E\rangle$ is a semiring that is isomorphic to the semiring $\langle(A\langle\langle\Sigma^*\rangle\rangle)_J^{(\Gamma^* \times Q) \times (\Gamma^* \times Q)}, +, \cdot, 0, E\rangle$.

The next definitions and results are generalizations of the work of Wechler [11]. Let $\mathscr{P} = (Q, \Sigma, \Gamma, \delta, q_0, p_0, F)$ be a pushdown automaton as defined by Harrison [5] and let $\vdash$ be the move relation between instantaneous descriptions of $\mathscr{P}$. Let T be the transpose operator [5, p. 4]. The move relation $\vdash$ defines a so-called pushdown transition matrix $M \in ((\mathbb{B}\langle\langle\Sigma^*\rangle\rangle)^{Q \times Q})_J^{\Gamma^* \times \Gamma^*}$ by

$$((M_{\pi_1, \pi_2})_{q_1, q_2}, x) = 1 \quad \text{iff} \quad (q_1, x, \pi_1^T) \vdash (q_2, \varepsilon, \pi_2^T), \quad \pi_1, \pi_2 \in \Gamma^*,$$
$$q_1, q_2 \in Q, \quad x \in \Sigma \cup \{\varepsilon\}.$$

Since $(q_1, x, \pi_1^T) \vdash (q_2, \varepsilon, \pi_2^T)$ implies $\pi_1 = p\pi_4$, $\pi_2 = \pi_3\pi_4$ for some $\pi_4 \in \Gamma^*$ and

$(q_2, \pi_3^T) \in \delta(q_1, x, p)$, M is of a certain regular structure. Furthermore, it is obvious that for $w \in \Sigma^*$ and $k \geqslant 0$, $(q_1, w, \pi_1^T) \vdash^k (q_2, \varepsilon, \pi_2^T)$ iff $(((M^k)_{\pi_1,\pi_2})_{q_1,q_2}, w) = 1$ and $(q_1, w, \pi_1^T) \vdash^* (q_2, \varepsilon, \pi_2^T)$ iff $((M^*)_{\pi_1,\pi_2})_{q_1,q_2}, w) = 1$.

Let $R \in (\mathbb{B}\langle\langle\Sigma^*\rangle\rangle)^{Q\times 1}$, $R = (R, \varepsilon)\varepsilon$ be defined by $R_q = \varepsilon$ iff $q \in F$. If $\mathscr{P}$ accepts by both final state and empty store, then $L(\mathscr{P}) = \text{supp}(((M^*)_{p_0,\varepsilon} R)_{q_0})$.

Example 1 (Wechler [11]). Let $Ł$ be the Lukasiewicz language over $\Sigma = \{a, b\}$, i.e., $Ł$ is the language generated by the context-free grammar $G = (\{S\}, \{a, b\}, \{S \to aSS, S \to b\}, S)$ (see [5, p. 323, Problem 16; or 1, p. 47]). Let $\mathscr{P} = (\{q\}, \Sigma, \{p\}, \delta, p, \{q\})$, $\delta(q, \varepsilon, p) = \varnothing$, $\delta(q, a, p) = \{(q, p^2)\}$, $\delta(q, b, p) = \{(q, \varepsilon)\}$, be a pushdown automaton. Then $\mathscr{P}$ accepts $Ł$ by both final state and empty store.

The pushdown transition matrix $M \in (\mathbb{B}\langle\langle\Sigma^*\rangle\rangle)_J^{p^*\times p^*}$ of $\mathscr{P}$ is defined by $M_{p^n,p^{n+1}} = a$, $M_{p^n,p^{n-1}} = b$, $n \geqslant 1$, while all other entries of M are equal to 0. Since $\mathscr{P}$ accepts $Ł$, $Ł = \text{supp}((M^*)_{p_0,\varepsilon})$. ∎

The concepts of pushdown automata and pushdown transition matrices are now generalized. A matrix $M \in ((A\langle\langle\Sigma^*\rangle\rangle)^{Q\times Q})_J^{\Gamma^*\times\Gamma^*}$ is called a *pushdown transition matrix* iff for $\pi_1, \pi_2 \in \Gamma^*$

$$M_{\pi_1,\pi_2} = M_{p,\pi_3}, \quad \text{if there exists } p \in \Gamma,\ \pi_4 \in \Gamma^* \text{ with } \pi_1 = p\pi_4 \text{ and } \pi_2 = \pi_3\pi_4,$$
$$= 0, \quad \text{otherwise.}$$

Hence a pushdown transition matrix is uniquely determined by the finite number of its blocks $M_{p,\pi} \neq 0$, $p \in \Gamma$, $\pi \in \Gamma^*$.

An $A\langle\langle\Sigma^*\rangle\rangle$-*pushdown automaton* $\mathscr{P} = (Q, \Gamma, \delta, q_0, p_0, R)$ is given by:

(i) a finite set Q of elements called *states*,

(ii) an alphabet Γ of pushdown symbols,

(iii) a mapping $\delta: Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \to A\langle Q \times \Gamma^*\rangle$,

(iv) $q_0 \in Q$ called the *initial state*,

(v) $p_0 \in \Gamma$,

(vi) $R \in (A\langle\langle\Sigma^*\rangle\rangle)^{Q\times 1}$, $R = (R, \varepsilon)\varepsilon$, called the *final state vector*.

An $A\langle\langle\Sigma^*\rangle\rangle$-pushdown automaton $\mathscr{P} = (Q, \Gamma, \delta, q_0, p_0, R)$ *defines* a pushdown transition matrix $M \in (A\langle\langle\Sigma^*\rangle\rangle)^{Q\times Q})_J^{\Gamma^*\times\Gamma^*}$ as follows:

$$(M_{\pi_1,\pi_2})_{q_1,q_2} = \sum_{x\in\Sigma\cup\{\varepsilon\}} (\delta(q_1, x, p), (q_2, \pi_3))x, \quad \text{if there exist } p \in \Gamma,\ \pi_4 \in \Gamma^* \text{ with } \pi_1 = p\pi_4 \text{ and } \pi_2 = \pi_3\pi_4,$$
$$= 0, \quad \text{otherwise,}$$

$M_{\pi_1,\pi_2} \in (A\langle\langle\Sigma^*\rangle\rangle)^{Q\times Q}$, $\pi_1, \pi_2 \in \Gamma^*$, $q_1, q_2 \in Q$.

The matrix M is row finite, since for any $p \in \Gamma$ there exist only a finite number of words π such that $(\delta(q_1, x, p), (q_2, \pi)) \neq 0$. The matrix M is column finite since any $\pi_2 \in \Gamma^*$ has only a finite number of factorizations $\pi_2 = \pi_3 \pi_4$.

An $A\langle\langle\Sigma^*\rangle\rangle$-pushdown automaton is called *cycle-free* iff its pushdown transition matrix is cycle-free. The *behavior* $\|\mathscr{P}\| \in A\langle\langle\Sigma^*\rangle\rangle$ of a cycle-free $A\langle\langle\Sigma^*\rangle\rangle$-pushdown automaton $\mathscr{P}$ is defined by

$$\|\mathscr{P}\| = \sum_{q \in Q} ((M^+)_{p_0, \varepsilon})_{q_0, q} R_q = ((M^+)_{p_0, \varepsilon} R)_{q_0}.$$

By Theorem 1, the behavior of a cycle-free $A\langle\langle\Sigma^*\rangle\rangle$-pushdown automaton is well defined.

THEOREM 4. *Let $\mathscr{P} = (Q, \Gamma, \delta, q_0, p_0, R)$ be a cycle-free $A\langle\langle\Sigma^*\rangle\rangle$-pushdown automaton with pushdown transition matrix M. Let $\mathfrak{A} = (\Gamma^* \times Q, M', (p_0, q_0), R')$ be the cycle-free $A\langle\langle\Sigma^*\rangle\rangle$-automaton with*

$$M'_{(\pi_1, q_1), (\pi_2, q_2)} = (M_{\pi_1, \pi_2})_{q_1, q_2}, \qquad R'_{(\varepsilon, q)} = R_q,$$
$$R'_{(\pi, q)} = 0, \qquad q, q_1, q_2 \in Q, \quad \pi_1, \pi_2 \in \Gamma^*, \quad \pi \in \Gamma^+.$$

Then $\|\mathfrak{A}\| = \|\mathscr{P}\|$.

Proof.

$$\|\mathfrak{A}\| = \sum_{(\pi, q) \in \Gamma^* \times Q} (M'^*)_{(p_0, q_0), (\pi, q)} \cdot R'_{(\pi, q)}$$
$$= \sum_{q \in Q} ((M^+)_{p_0, \varepsilon})_{q_0, q} R_q = \|\mathscr{P}\|. \quad \blacksquare$$

Hence cycle-free $A\langle\langle\Sigma^*\rangle\rangle$-pushdown automata are cycle-free $A\langle\langle\Sigma^*\rangle\rangle$-automata as defined in Section 2.

THEOREM 5. *Let $M \in ((A\langle\langle\Sigma^*\rangle\rangle)^{Q \times Q})_J^{\Gamma^* \times \Gamma^*}$ be a cycle-free pushdown transition matrix. Then for all $p_i \in \Gamma$, $1 \leqslant i \leqslant m$, $m \geqslant 1$,*

$$(M^*)_{p_1 \cdots p_m, \varepsilon} = (M^*)_{p_1, \varepsilon} \cdots (M^*)_{p_m, \varepsilon}.$$

Proof. Let $P \in ((A\langle\langle\Sigma^*\rangle\rangle)^{Q \times Q})_J^{\Gamma^* \times 1}$ with $P_\varepsilon = E$, $P_\pi = 0$, $\pi \in \Gamma^+$. If S is the unique solution of $Y = P + MY$, then $S_\pi = (M^*)_{\pi, \varepsilon}$, $\pi \in \Gamma^*$ by Corollary 3.

We claim $S_{p_1 \cdots p_m} = (M^*)_{p_1, \varepsilon} \cdots (M^*)_{p_m, \varepsilon}$, $p_i \in \Gamma$, $1 \leqslant i \leqslant m$, $m \geqslant 1$, and $S_\varepsilon = E$, or equivalently $S_{\pi_1 \pi_2} = (M^*)_{\pi_1, \varepsilon} (M^*)_{\pi_2, \varepsilon}$ for π_1, $\pi_2 \in \Gamma^*$. The claim holds true for $\pi_1 = \varepsilon$. Hence assume $\pi_1 \neq \varepsilon$.

$$S_{\pi_1 \pi_2} = P_{\pi_1 \pi_2} + (MS)_{\pi_1 \pi_2} = \sum_{\pi \in \Gamma^*} M_{\pi_1 \pi_2, \pi} S_\pi$$
$$= \sum_{\pi_3 \in \Gamma^*} M_{\pi_1 \pi_2, \pi_3 \pi_2} S_{\pi_3 \pi_2} = \sum_{\pi_3 \in \Gamma^*} M_{\pi_1, \pi_3} (M^*)_{\pi_3, \varepsilon} (M^*)_{\pi_2, \varepsilon}$$
$$= (M^+)_{\pi_1, \varepsilon} (M^*)_{\pi_2, \varepsilon} = (M^*)_{\pi_1, \varepsilon} (M^*)_{\pi_2, \varepsilon}. \quad \blacksquare$$

Our next goal is to show that $r \in A^{\text{semi-alg}}\langle\langle \Sigma^* \rangle\rangle$ iff r is the behavior of some cycle-free $A\langle\langle \Sigma^* \rangle\rangle$-pushdown automaton. Given Γ and Q, let $\Gamma' = \{ y^p_{q_1,q_2} \mid p \in \Gamma, q_1, q_2 \in Q\}$ be a set of variables. Let $Y(p) \in (A\langle \Gamma'^* \rangle)^{Q \times Q}$ with $(Y(p))_{q_1,q_2} = y^p_{q_1,q_2}$, $p \in \Gamma$, $q_1, q_2 \in Q$. Furthermore, let $Y(\varepsilon) = E$ and $Y(p_1 \cdots p_m) = Y(p_1) \cdots Y(p_m)$, $p_i \in \Gamma$, $1 \leqslant i \leqslant m$, $m \geqslant 1$.

LEMMA 4. *Let* $M = (M, \varepsilon)\varepsilon + \sum_{x \in \Sigma} (M, x)x \in ((A\langle\langle \Sigma^* \rangle\rangle)^{Q \times Q})_J^{\Gamma^* \times \Gamma^*}$ *be a cycle-free pushdown transition matrix. Let* $M_0 = (M, \varepsilon)\varepsilon$ *and* $M_1 = \sum_{x \in \Sigma} (M, x)x$. *Then* $(M^*)_{p,\varepsilon}$, $p \in \Gamma$, *is the unique solution of*:

(i) $Y(p) = \sum_{\pi \in \Gamma^*} M_{p,\pi} Y(\pi)$, $p \in \Gamma$, *and*

(ii) $Y(p) = \sum_{\pi \in \Gamma^*} (M_0^* M_1)_{p,\pi} Y(\pi) + (M_0^*)_{p,\varepsilon}$, $p \in \Gamma$.

Proof. Consider the $A\langle\langle \Sigma^* \rangle\rangle$-linear system $Y = P + MY$ with $P \in ((A\langle\langle \Sigma^* \rangle\rangle)^{Q \times Q})_J^{\Gamma^* \times 1}$, $P_\varepsilon = E$, $P_\pi = 0$, $\pi \in \Gamma^+$, and let S be its unique solution. Then by Corollary 3, $S_\pi = (M^*)_{\pi,\varepsilon}$. Hence $S_p = \sum_{\pi \in \Gamma^*} M_{p,\pi} S_\pi$ and Theorem 5 implies that S_p, $p \in \Gamma$, is the solution of (i). By Corollary 2, S is the unique solution of the $A\langle\langle \Sigma^* \rangle\rangle$-linear system $Y = M_0^* P + M_0^* M_1 Y$. Hence $S_p = (M_0^* P)_p + (M_0^* M_1 S)_p$ for $p \in \Gamma$, i.e., $S_p = (M_0^*)_{p,\varepsilon} + \sum_{\pi \in \Gamma^*} (M_0^* M_1)_{p,\pi} S_\pi$. Then Theorem 5 implies that S_p, $p \in \Gamma$, is the solution of (ii). Since the algebraic system (ii) is a strict one, S_p, $p \in \Gamma$, is the unique solution of (ii).

We now show that S_p, $p \in \Gamma$, is the unique solution of (i), too. Assume $T(p) \in (A\langle\langle \Sigma^* \rangle\rangle)^{Q \times Q}$, $p \in \Gamma$, is solution of (i). Let $T(p_1 \cdots p_m) = T(p_1) \cdots T(p_m)$, $p_i \in \Gamma$, $1 \leqslant i \leqslant m$, $m \geqslant 1$, and $T(\varepsilon) = E$. Let $T \in ((A\langle\langle \Sigma^* \rangle\rangle)^{Q \times Q})_J^{\Gamma^* \times 1}$ with $T_\pi = T(\pi)$, $\pi \in \Gamma^*$. Then $(P + MT)_\varepsilon = E = T_\varepsilon$ and for $p \in \Gamma$, $\pi \in \Gamma^*$,

$$(P + MT)_{p\pi} = \sum_{\pi_1 \in \Gamma^*} M_{p\pi,\pi_1} T_{\pi_1} = \sum_{\pi_2 \in \Gamma^*} M_{p,\pi_2} T_{\pi_2 \pi}$$
$$= \sum_{\pi_2 \in \Gamma^*} M_{p,\pi_2} T(\pi_2)\, T(\pi) = T(p)\, T(\pi) = T_{p\pi}.$$

Hence T is unique solution of $Y = P + MT$ and $T = S$. ∎

LEMMA 5. *Let* $\mathscr{P}$ *be a cycle-free* $A\langle\langle \Sigma^* \rangle\rangle$*-pushdown automaton and let* $a \in A$. *Then there exists a cycle-free* $A\langle\langle \Sigma^* \rangle\rangle$*-pushdown automaton* $\mathscr{P}'$ *such that* $\|\mathscr{P}'\| = a\varepsilon + \|\mathscr{P}\|$.

Proof. Let $\mathscr{P} = (Q, \Gamma, \delta, q_0, p_0, R)$ define the pushdown transition matrix M. Let $Q' = Q \cup \{i, t\}$, $\{i, t\} \cap Q = \varnothing$. Let $e_{q_0} \in (A\langle\langle \Sigma^* \rangle\rangle)^{Q \times 1}$, $(e_{q_0})_q = \delta_{q_0,q}\varepsilon$, and $M' \in ((A\langle\langle \Sigma^* \rangle\rangle)^{Q' \times Q'})_J^{\Gamma^* \times \Gamma^*}$ be a pushdown transition matrix defined by

$$M'_{p,\pi} = \begin{pmatrix} 0 & e_{q_0} M_{p,\pi} & e_{q_0} M_{p,\pi} R + a\varepsilon \\ 0 & M_{p,\pi} & M_{p,\pi} R \\ 0 & 0 & 0 \end{pmatrix},$$

$p \in \Gamma$, $\pi \in \Gamma^*$, where the blocks of $M'_{p,\pi}$ are indexed by i, Q, t. An easy proof by induction shows that for $k \geqslant 2$, $\pi_1 \in \Gamma^*$, $\pi_2 \in \Gamma^*$,

$$(M'^k)_{\pi_1,\pi_2} = \begin{pmatrix} 0 & e_{q_0}(M^k)_{\pi_1,\pi_2} & e_{q_0}(M^k)_{\pi_1,\pi_2} R \\ 0 & (M^k)_{\pi_1,\pi_2} & (M^k)_{\pi_1,\pi_2} R \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence M' is cycle-free and for $p \in \Gamma$

$$(M'^+)_{p,\varepsilon} = \begin{pmatrix} 0 & e_{q_0}(M^+)_{p,\varepsilon} & e_{q_0}(M^+)_{p,\varepsilon} R + a\varepsilon \\ 0 & (M^+)_{p,\varepsilon} & (M^+)_{p,\varepsilon} R \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $\mathscr{P}' = (Q', \Gamma, \delta', i, p_0, R')$, where δ' is defined by M' and $R'_i = 0$, $R'_q = 0$ for $q \in Q$ and $R'_t = \varepsilon$. Then

$$\|\mathscr{P}'\| = ((M'^+)_{p_0,\varepsilon} R')_i = e_{q_0}(M^+)_{p,\varepsilon} R + a\varepsilon = a\varepsilon + \|\mathscr{P}\|. \quad \blacksquare$$

An algebraic system over $A\langle\langle \Sigma^* \rangle\rangle$ with variables in $Z = \{z_1, ..., z_n\}$, $z_i = p_i$, $1 \leqslant i \leqslant n$, is in *Greibach normal form* iff $\operatorname{supp}(p_i) \subseteq \Sigma \cup \Sigma Z \cup \Sigma Z^2$, $1 \leqslant i \leqslant n$. Salomaa and Soittola [10, Theorem IV.2.3] state that for all $r \in A^{\mathrm{alg}}\langle\langle \Sigma^* \rangle\rangle$ there exists an algebraic system in Greibach normal form such that r is the first component of its unique solution. The analogous language-theoretic result is well known.

LEMMA 6. *Let $r \in A^{\mathrm{alg}}\langle\langle \Sigma^* \rangle\rangle$. Then there exists a cycle-free $A\langle\langle \Sigma^* \rangle\rangle$-pushdown automaton $\mathscr{P} = (Q, \Gamma, \delta, q_0, p_0, R)$ with $\delta(q, \varepsilon, p) = 0$ for all $q \in Q$, $p \in \Gamma$ such that $\|\mathscr{P}\| = r$.*

Proof. Without loss of generality, we may assume that r is the first component of the unique solution of an algebraic system $z_i = p_i$, $1 \leqslant i \leqslant n$, in Greibach normal form. Let $Z = \{z_1, ..., z_n\}$. Let $\mathscr{P} = (\{1\}, Z, \delta, 1, z_1, (\varepsilon))$ be an $A\langle\langle \Sigma^* \rangle\rangle$-pushdown automaton with $\delta(1, x, z_i) = \sum_{\pi \in Z^*} (p_i, x\pi)(1, \pi)$, $x \in \Sigma$, $1 \leqslant i \leqslant n$. Then the pushdown transition matrix $M \in (A\langle\langle \Sigma^* \rangle\rangle)^{Z^* \times Z^*}$ defined by $\mathscr{P}$ has entries $M_{z_i,\pi} = \sum_{x \in \Sigma} (p_i, x\pi)x$, $\pi \in Z^*$, $1 \leqslant i \leqslant n$, and $\|\mathscr{P}\| = (M^+)_{z_1,\varepsilon}$.

We claim that $\sigma = (\sigma_1, ..., \sigma_n)$ with $\sigma_i = (M^*)_{z_i,\varepsilon}$, $1 \leqslant i \leqslant n$, is the unique solution of $z_i = p_i$, $1 \leqslant i \leqslant n$.

$$\begin{aligned} \sigma(p_i) &= \sigma\left(\sum_{x \in \Sigma} \sum_{\pi \in Z^*} (p_i, x\pi)\, x\pi\right) = \sum_{\pi \in Z^*} M_{z_i,\pi}(M^*)_{\pi,\varepsilon} \\ &= (M^+)_{z_i,\varepsilon} = (M^*)_{z_i,\varepsilon} = \sigma_i, \qquad 1 \leqslant i \leqslant n. \end{aligned}$$

Hence $\|\mathscr{P}\| = \sigma_1 = r$. $\blacksquare$

THEOREM 6. *A power series r is in* $A^{\text{semi-alg}}\langle\langle\Sigma^*\rangle\rangle$ *iff r is the behavior of some cycle-free* $A\langle\langle\Sigma^*\rangle\rangle$*-pushdown automaton.*

Proof. (i) Let $r \in A^{\text{semi-alg}}\langle\langle\Sigma^*\rangle\rangle$, $r = (r, \varepsilon)\varepsilon + r_1$, $r_1 \in A^{\text{alg}}\langle\langle\Sigma^*\rangle\rangle$. Apply Lemma 6 on r_1, then apply Lemma 5 with $a = (r, \varepsilon)$.

(ii) If r is the behavior of a cycle-free $A\langle\langle\Sigma^*\rangle\rangle$-pushdown automaton $\mathscr{P} = (Q, \Gamma, \delta, q_0, p_0, R)$ with pushdown transition matrix M, then $r = ((M^+)_{p_0,\varepsilon} R)_{q_0}$. Since $(M^+)_{p_0,\varepsilon} = (M^*)_{p_0,\varepsilon}$ is a component of the unique solution of a strict algebraic system by Lemma 4 (ii), the entries of $(M^+)_{p_0,\varepsilon}$ are in $A^{\text{semi-alg}}\langle\langle\Sigma^*\rangle\rangle$. Hence $r \in A^{\text{semi-alg}}\langle\langle\Sigma^*\rangle\rangle$. ∎

Theorem 6 constitutes a generalization of the theorem of Wechler [11], which characterizes the power series of $A^{\text{alg}}\langle\langle\Sigma^*\rangle\rangle$ by $A\langle\langle\Sigma^*\rangle\rangle$-pushdown automata with one state only.

4. REGULATED RATIONAL TRANSDUCTIONS

We introduce in this section regulated representations, regulated rational transductions, types of pushdown transition matrices, and types of pushdown automata. The major result of this section is that if the semi-algebraic power series r is the behavior of a cycle-free pushdown automaton of type t, then $\tau(r)$ is again the behavior of a cycle-free pushdown automaton of type t, τ a regulated rational transduction.

Let $\mu: \Sigma_1 \to (A\langle\langle\Sigma_2^*\rangle\rangle)^{m\times m}$, $m \geq 1$, be a mapping. If μ is uniquely extended to a monoid homomorphism $\mu: \Sigma_1^* \to (A\langle\langle\Sigma_2^*\rangle\rangle)^{m\times m}$, then it is called a *representation*. By Salomaa and Soittola [10], a representation $\mu: \Sigma_1^* \to (A\langle\langle\Sigma_2^*\rangle\rangle)^{m\times m}$, $m \geq 1$, is called *regulated* iff there exists a natural number $k \geq 1$ such that the matrices $\mu(w)$, $|w| \geq k$, are quasiregular. The notion of a regulated representation is introduced to allow an extension of μ to a semiring homomorphism $\mu: A\langle\langle\Sigma_1^*\rangle\rangle \to (A\langle\langle\Sigma_2^*\rangle\rangle)^{m\times m}$ by $\mu(r) = \sum_{w\in\Sigma_1^*} (r, w)\mu(w)$, $r \in A\langle\langle\Sigma_1^*\rangle\rangle$.

Let $Q \times m = Q \times \{1,..., m\}$. By a slight generalization of [9, Lemma 5], further extensions of μ yield semiring homomorphisms:

(i) μ: $(A\langle\langle\Sigma_1^*\rangle\rangle)^{Q\times Q} \to (A\langle\langle\Sigma_2^*\rangle\rangle)^{(Q\times m)\times(Q\times m)}$ by $(\mu(M))_{(q_1,i_1),(q_2,i_2)} = (\mu(M_{q_1,q_2}))_{i_1,i_2}$, $M \in (A\langle\langle\Sigma_1^*\rangle\rangle)^{Q\times Q}$, $q_1, q_2 \in Q$, $1 \leq i_1, i_2 \leq m$, and

(ii) μ: $((A\langle\langle\Sigma_1^*\rangle\rangle)^{Q\times Q})_J^{\Gamma^*\times\Gamma^*} \to ((A\langle\langle\Sigma_2^*\rangle\rangle)^{(Q\times m)\times(Q\times m)})_J^{\Gamma^*\times\Gamma^*}$ by $(\mu(M))_{\pi_1,\pi_2} = \mu(M_{\pi_1,\pi_2})$, $M \in ((A\langle\langle\Sigma_1^*\rangle\rangle)^{Q\times Q})_J^{\Gamma^*\times\Gamma^*}$, $\pi_1, \pi_2 \in \Gamma^*$.

By Salomaa and Soittola [10], a mapping $\tau: A\langle\langle\Sigma_1^*\rangle\rangle \to A\langle\langle\Sigma_2^*\rangle\rangle$ is called a *regulated rational transduction* iff $\tau(r) = (r, \varepsilon) r_0 + \sum_{w\in\Sigma_1^+} (r, w)\mu(w)_{1,m}$, where $r_0 \in A^{\text{rat}}\langle\langle\Sigma_2^*\rangle\rangle$ and $\mu: \Sigma_1^* \to (A^{\text{rat}}\langle\langle\Sigma_2^*\rangle\rangle)^{m\times m}$, $m \geq 1$, is a regulated representation. In case of the Boolean semiring $\mathbb{B}$, a mapping $\tau: \mathbb{B}\langle\langle\Sigma_1^*\rangle\rangle \to \mathbb{B}\langle\langle\Sigma_2^*\rangle\rangle$ is called a *rational transduction* iff $\tau(r) = (r, \varepsilon) r_0 + \sum_{w\in\Sigma_1^+} (r, w)\mu(w)_{1,m}$, where $r_0 \in \mathbb{B}^{\text{rat}}\langle\langle\Sigma_2^*\rangle\rangle$ and μ: $\Sigma_1^* \to (\mathbb{B}^{\text{rat}}\langle\langle\Sigma_2^*\rangle\rangle)^{m\times m}$, $m \geq 1$, is a representation.

By [10, Theorem III.1.3], any rational transduction $\tau: \mathbb{B}\langle\langle\Sigma_1^*\rangle\rangle \to \mathbb{B}\langle\langle\Sigma_2^*\rangle\rangle$ has a factorization $\tau = h \circ \tau_1$, where $\tau_1: \mathbb{B}\langle\langle\Sigma_1^*\rangle\rangle \to \mathbb{B}\langle\langle\Sigma_2^*\rangle\rangle$ is a regulated rational transduction and $h: \Sigma_2 \to \Sigma_2 \cup \{\varepsilon\}$ extended to $h: \mathbb{B}\langle\langle\Sigma_2^*\rangle\rangle \to \mathbb{B}\langle\langle\Sigma_2^*\rangle\rangle$ is a semiring homomorphism, i.e., $\tau(r) = h(\tau_1(r))$ for $r \in \mathbb{B}\langle\langle\Sigma_1^*\rangle\rangle$.

The rational transductions are extensively treated in Berstel [1]; they are also called a-transductions by Ginsburg [3]. The regulated rational transductions correspond to the ε-output bounded a-transductions of Ginsburg [3, p. 35]. The definition of a rational transduction as given above is easily deduced from Berstel [1].

Let $\mu: \Sigma_1^* \to (A\langle\langle\Sigma_2^*\rangle\rangle)^{1\times 1}$ be a regulated representation and let $\tau: A\langle\langle\Sigma_1^*\rangle\rangle \to A\langle\langle\Sigma_2^*\rangle\rangle$ be a regulated rational transduction defined by $\tau(r) = (r, \varepsilon)\, r_0 + \sum_{w \in \Sigma_1^+} (r, w)\, \mu(w)_{1,1}$, $r \in A\langle\langle\Sigma_1^*\rangle\rangle$. Let $\rho: \Sigma_1^* \to (A\langle\langle\Sigma_2^*\rangle\rangle)^{2\times 2}$ be the regulated representation defined by

$$\rho(x) = \begin{pmatrix} 0 & \mu(x) \\ 0 & \mu(x) \end{pmatrix}, \qquad x \in \Sigma.$$

Then $\tau(r) = (r, \varepsilon)\, r_0 + \sum_{w \in \Sigma_1^*} (r, w)\, \rho(w)_{1,2} = (r, \varepsilon)\, r_0 + \rho(r)_{1,2}$, $r \in A\langle\langle\Sigma_1^*\rangle\rangle$. Hence without loss of generality, we may assume $m \geqslant 2$ and $\tau(r) = (r, \varepsilon)\, r_0 + \mu(r)_{1,m}$ in the definition of a regulated rational transduction. This will be needed in the proof of Theorem 7.

If I is a finite index set, let $e_i \in (A\langle\langle\Sigma^*\rangle\rangle)^{1\times I}$ be defined by $(e_i)_j = \delta_{i,j}\varepsilon$, $i, j \in I$. Given an alphabet Γ of pushdown symbols, a *type* t is a finite subset of $\Gamma \times \Gamma^*$ such that $t \supseteq \{(p, p) \mid p \in \Gamma\}$. A pushdown transition matrix $M \in ((A\langle\langle\Sigma^*\rangle\rangle)^{Q\times Q})_J^{\Gamma^*\times\Gamma^*}$ is of *type* $t \subset \Gamma \times \Gamma^*$ iff $M_{p,\pi} \neq 0$ implies $(p, \pi) \in t$, $p \in \Gamma$, $\pi \in \Gamma^*$. The collection of all pushdown transition matrices of type t as defined above is denoted by $((A\langle\langle\Sigma^*\rangle\rangle)^{Q\times Q})_t^{\Gamma^*\times\Gamma^*}$.

LEMMA 7. *Let $M \in ((A\langle\langle\Sigma_1^*\rangle\rangle)^{Q\times Q})_t^{\Gamma^*\times\Gamma^*}$ be a cycle-free pushdown transition matrix of type t. Let $\mu: \Sigma_1^* \to (A^{\mathrm{rat}}\langle\langle\Sigma_2^*\rangle\rangle)^{m\times m}$, $m \geqslant 1$, be a regulated representation and $r_0 \in A^{\mathrm{rat}}\langle\langle\Sigma_2^*\rangle\rangle$. Then there exists a cycle-free pushdown transition matrix $M' \in ((A\langle\langle\Sigma_2^*\rangle\rangle)^{Q'\times Q'})_t^{\Gamma^*\times\Gamma^*}$ of type t, $Q' = \{1\} \cup Q \cup Q \times m$, such that for some $q \in Q$,*

$$(M'^{+})_{p,\varepsilon} = \begin{pmatrix} 0 & e_q((M^{+})_{p,\varepsilon}, \varepsilon)\, r_0 & e_{(q,1)}\mu((M^{+})_{p,\varepsilon}) \\ 0 & ((M^{+})_{p,\varepsilon}, \varepsilon) & 0 \\ 0 & 0 & \mu((M^{+})_{p,\varepsilon}) \end{pmatrix},$$

$p \in \Gamma$, where the blocks are indexed by 1, Q, $Q \times m$.

Proof. Let

$$M'_{p,p} = \begin{pmatrix} 0 & e_q r_0 & e_{(q,1)} \\ 0 & (M_{p,p}, \varepsilon) & 0 \\ 0 & 0 & \mu(M_{p,p}) \end{pmatrix}, \qquad p \in \Gamma,$$

$$M'_{p,\pi} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & (M_{p,\pi}, \varepsilon) & 0 \\ 0 & 0 & \mu(M_{p,\pi}) \end{pmatrix}, \qquad p \in \Gamma, \quad \pi \in \Gamma^*, \quad \pi \neq p.$$

Obviously, M' is of type t. We claim that for $k \geqslant 1$,

$$(M'^k)_{\pi_1,\pi_2} = \begin{pmatrix} 0 & e_q((M^{k-1})_{\pi_1,\pi_2}, \varepsilon)\, r_0 & e_{(q,1)}\mu((M^{k-1})_{\pi_1,\pi_2}) \\ 0 & ((M^k)_{\pi_1,\pi_2}, \varepsilon) & 0 \\ 0 & 0 & \mu((M^k)_{\pi_1,\pi_2}) \end{pmatrix}.$$

The claim is easily shown by induction on k. The proof is omitted. Hence M' is cycle-free. Since $(M'^+)_{p,\varepsilon} = \sum_{k=1}^{\infty} (M'^k)_{p,\varepsilon}$, the lemma is proven. ∎

Note 1. If in Lemma 7, $r_0 = 0$, then it is sufficient to let $M'_{p,\pi} = \mu(M_{p,\pi})$, $p \in \Gamma$, $\pi \in \Gamma^*$. Then $(M'^+)_{p,\varepsilon} = \mu((M^+)_{p,\varepsilon})$.

Let P be a set and $M \in (A\langle\langle\Sigma^*\rangle\rangle)^{P \times P}$. Let $P = \bigcup_{i \in I} P_i$, I an index set, be a disjoint partition of P. Then M is said to be *partitioned* into *blocks* $M(P_i, P_j) \in (A\langle\langle\Sigma^*\rangle\rangle)^{P_i \times P_j}$, $i, j \in I$. The same definition holds for $R \in (A\langle\langle\Sigma^*\rangle\rangle)^{P \times 1}$.

If matrices or vectors are partitioned into blocks with suitable dimensions, then the sum and product can be defined in terms of these blocks in the usual manner.

We give an intuitive description of the contents of the next lemma. Given $M \in ((A^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle)^{Q \times Q})_t^{\Gamma^* \times \Gamma^*}$, by Theorem 4 we may assume that M is the pushdown transition matrix of a generalized pushdown automaton $\mathscr{P} = (\Gamma^* \times Q, M, (p_0, q_0), P)$ whose edges are labelled by rational power series. Since $(M_{p,\pi})_{q_1,q_2}$, $p \in \Gamma$, $\pi \in \Gamma^*$, $q_1, q_2 \in Q$, is in $A^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle$, there exists a finite automaton $\mathfrak{A}^{p,\pi}_{q_1,q_2} = (Q^{p,\pi}_{q_1,q_2}, M^{p,\pi}_{q_1,q_2}, q^{p,\pi}_{q_1,q_2}, R^{p,\pi}_{q_1,q_2})$ with $\|\mathfrak{A}^{p,\pi}_{q_1,q_2}\| = (M_{p,\pi})_{q_1,q_2}$.

For each $\pi_1 \in \Gamma^*$, let $\mathfrak{A}^{p\pi_1,\pi\pi_1}_{q_1,q_2} = (Q^{p,\pi}_{q_1,q_2} \times \{p\pi_1\}, M^{p,\pi}_{q_1,q_2}, (q^{p,\pi}_{q_1,q_2}, p\pi_1), R^{p,\pi}_{q_1,q_2})$ be a copy of $\mathfrak{A}^{p,\pi}_{q_1,q_2}$. The automata $\mathscr{P}$ and $\mathfrak{A}^{p\pi_1,\pi\pi_1}_{q_1,q_2}$, $p \in \Gamma$, π, $\pi_1 \in \Gamma^*$, q_1, $q_2 \in Q$, with $M_{p,\pi} \neq 0$, are "combined" and yield an $A\langle\langle\Sigma^*\rangle\rangle$-automaton $\mathscr{P}' = (\Gamma^* \times Q', M', (p_0, q_0), P')$ with $Q' = Q \cup \bigcup Q^{p,\pi}_{q_1,q_2}$, $P'(Q) = P$, $P'(Q^{p,\pi}_{q_1,q_2}) = 0$, and $\|\mathscr{P}'\| = \|\mathscr{P}\|$.

The edges of $\mathscr{P}'$ are given as follows: The edges of $\mathscr{P}$ are deleted, the edges of $\mathfrak{A}^{p\pi_1,\pi\pi_1}_{q_1,q_2}$ remain unchanged. For any $\pi_1 \in \Gamma^*$, an edge labelled by ε leads from $(q_1, p\pi_1)$ to the initial state $(q^{p,\pi}_{q_1,q_2}, p\pi_1)$ of $\mathfrak{A}^{p\pi_1,\pi\pi_1}_{q_1,q_2}$. An edge labelled by $(R^{p,\pi}_{q_1,q_2})_q$, $q \in Q^{p,\pi}_{q_1,q_2}$ leads from $(q, p\pi_1)$ to $(q_2, \pi\pi_1)$, i.e., an edge whose label is the "weight" of q leads from a final state of $\mathfrak{A}^{p\pi_1,\pi\pi_1}_{q_1,q_2}$ to $(q_2, \pi\pi_1)$. The transition matrix M' is given by the edges of $\mathscr{P}'$. It turns out that M' is a pushdown transition matrix and that $\mathscr{P}'$ is a pushdown automaton.

The next lemma states that $(M'^+)_{p,\varepsilon}(Q, Q) = (M^+)_{p,\varepsilon}$ and that the type of M' equals the type of M.

LEMMA 8. *Let $M \in ((A^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle)^{Q \times Q})_t^{\Gamma^* \times \Gamma^*}$ be a cycle-free pushdown transition matrix of type t. Then there exist a finite set Q', $Q' \supseteq Q$, and a cycle-free pushdown* *transition matrix of type t* $M' \in ((A\langle\langle\Sigma^*\rangle\rangle)^{Q'\times Q'})_t^{\Gamma^*\times\Gamma^*}$, $M' = (M', \varepsilon)\varepsilon + \sum_{x\in\Sigma}(M', x)x$, *such that for* $p \in \Gamma$, $((M'^+)_{p,\varepsilon})(Q, Q) = (M^+)_{p,\varepsilon}$, *where* $((M'^+)_{p,\varepsilon})(Q, Q)$ *is the finite block of* $(M'^+)_{p,\varepsilon}$ *indexed by* (Q, Q).

Proof. Since $(M_{p,\pi})_{q_1,q_2} \in A^{\text{rat}}\langle\langle\Sigma^*\rangle\rangle$, $p \in \Gamma$, $\pi \in \Gamma^*$, q_1, $q_2 \in Q$, there exist by [10, Theorem II.1.4] finite linear systems $Z^{p,\pi}_{q_1,q_2} = R^{p,\pi}_{q_1,q_2} + M^{p,\pi}_{q_1,q_2} Z^{p,\pi}_{q_1,q_2}$,

$$M^{p,\pi}_{q_1,q_2} = \sum_{x\in\Sigma} (M^{p,\pi}_{q_1,q_2}, x)x \in (A\langle\langle\Sigma^*\rangle\rangle)^{Q^{p,\pi}_{q_1,q_2}\times Q^{p,\pi}_{q_1,q_2}}$$

quasiregular, $R^{p,\pi}_{q_1,q_2} = (R^{p,\pi}_{q_1,q_2}, \varepsilon)\varepsilon \in (A\langle\langle\Sigma^*\rangle\rangle)^{Q^{p,\pi}_{q_1,q_2}\times 1}$, for some finite sets $Q^{p,\pi}_{q_1,q_2}$, with unique solutions $S^{p,\pi}_{q_1,q_2} = (M^{p,\pi}_{q_1,q_2})^* R^{p,\pi}_{q_1,q_2}$, such that for some $q^{p,\pi}_{q_1,q_2} \in Q^{p,\pi}_{q_1,q_2}$, the $q^{p,\pi}_{q_1,q_2}$-component of $S^{p,\pi}_{q_1,q_2}$ equals $(M_{p,\pi})_{q_1,q_2}$. Let $e^{p,\pi}_{q_1,q_2} = e_{q^{p,\pi}_{q_1,q_2}} \in (A\langle\langle\Sigma^*\rangle\rangle)^{1\times Q^{p,\pi}_{q_1,q_2}}$, then $e^{p,\pi}_{q_1,q_2} S^{p,\pi}_{q_1,q_2} = (M_{p,\pi})_{q_1,q_2}$. Let Q and $Q^{p,\pi}_{q_1,q_2}$ be pairwise disjoint for all $p \in \Gamma$, $\pi \in \Gamma^*$, such that $M_{p,\pi} \neq 0$ and q_1, $q_2 \in Q$, and let $Q' = Q \cup \bigcup Q^{p,\pi}_{q_1,q_2}$. Let

$$(E^{p,\pi}_{q_1,q_2})_{q_3,q_4} = \delta_{q_1,q_3}(e^{p,\pi}_{q_1,q_2})_{q_4}, \qquad E^{p,\pi}_{q_1,q_2} \in (A\langle\langle\Sigma^*\rangle\rangle)^{Q\times Q^{p,\pi}_{q_1,q_2}},$$

$$(F^{p,\pi}_{q_1,q_2})_{q_3,q_4} = \delta_{q_2,q_4}(R^{p,\pi}_{q_1,q_2})_{q_3}, \qquad F^{p,\pi}_{q_1,q_2} \in (A\langle\langle\Sigma^*\rangle\rangle)^{Q^{p,\pi}_{q_1,q_2}\times Q}.$$

Then $\sum_{q_1,q_2\in Q} E^{p,\pi}_{q_1,q_2}(M^{p,\pi}_{q_1,q_2})^* F^{p,\pi}_{q_1,q_2} = M_{p,\pi}$.

The blocks of $M'_{p,\pi} \in (A\langle\langle\Sigma^*\rangle\rangle)^{Q'\times Q'}$, $p \in \Gamma$, $\pi \in \Gamma^*$ are defined as follows:

(i) $M'_{p,\pi}(Q, Q) = 0$,

(ii) $M'_{p,p}(Q, Q^{p,\pi}_{q_1,q_2}) = E^{p,\pi}_{q_1,q_2}$,

(iii) $M'_{p,\pi}(Q^{p,\pi}_{q_1,q_2}, Q) = F^{p,\pi}_{q_1,q_2}$,

(iv) $M'_{p,p}(Q^{p,\pi}_{q_1,q_2}, Q^{p,\pi}_{q_1,q_2}) = M^{p,\pi}_{q_1,q_2}$.

All other blocks of $M'_{p,\pi}$, $p \in \Gamma$, $\pi \in \Gamma^*$ are equal to 0. Obviously, M' is of type t.

We claim that for $k \geqslant 0$:

(i) $((M'^{2k+1})_{p_1\pi_1,\pi_2}(Q, Q), \varepsilon) = 0$,

(ii)

$$\begin{aligned}((M'^{2k+1})_{p_1\pi_1,\pi_2}(Q, Q^{p,\pi}_{q_1,q_2}), \varepsilon) &= ((M^k)_{p_1\pi_1,\pi_2}, \varepsilon)\, E^{p,\pi}_{q_1,q_2}, &&\text{if } \pi_2 = p\pi_3,\\ &= 0, &&\text{otherwise;}\end{aligned}$$

(iii) $((M'^{2k+1})_{p_1\pi_1,\pi_2}(Q^{p,\pi}_{q_1,q_2}, Q), \varepsilon) = \delta_{p_1,p} F^{p,\pi}_{q_1,q_2}((M^k)_{\pi\pi_1,\pi_2}, \varepsilon)$;

(iv) $((M'^{2k+1})_{p_1\pi_1,\pi_2}(Q^{p_2,\pi_3}_{q_1,q_2}, Q^{p_3,\pi_4}_{q_3,q_4}), \varepsilon) = 0$, $p, p_1, p_2, p_3 \in \Gamma$,

$\pi, \pi_1, \pi_2, \pi_3, \pi_4 \in \Gamma^*$, $q_1, q_2, q_3, q_4 \in Q$.

This claim is proved by a tedious but straightforward proof by induction on k. It is omitted. The form of M'^{2k+1} implies at once that M' is cycle-free.

Our last claim is that

(i) $(M'^+)_{p\pi,\varepsilon}(Q, Q) = (M^+)_{p\pi,\varepsilon}$,

(ii) $(M'^+)_{p\pi,\varepsilon}(Q, Q^{p_1,\pi_1}_{q_1,q_2}) = 0$,

(iii) $(M'^+)_{p\pi,\varepsilon}(Q^{p_1,\pi_1}_{q_1,q_2}, Q) = \delta_{p,p_1}(M^{p_1,\pi_1}_{q_1,q_2})^* \cdot F^{p_1,\pi_1}_{q_1,q_2}(M^*)_{\pi_1\pi,\varepsilon}$,

(iv) $(M'^+)_{p\pi,\varepsilon}(Q^{p_1,\pi_1}_{q_1,q_2}, Q^{p_2,\pi_2}_{q_3,q_4}) = 0$, for $p, p_1, p_2 \in \Gamma$, $\pi, \pi_1, \pi_2 \in \Gamma^*$,

$q_1, q_2, q_3, q_4 \in Q$.

Let $Y = P + M'Y$, with $P \in ((A\langle\langle\Sigma^*\rangle\rangle)^{Q'\times Q'})_J^{\Gamma^*\times 1}$ and $P_\pi = M'_{\pi,\varepsilon}$, $\pi \in \Gamma^*$. Then the unique solution S of this cycle-free $A\langle\langle\Sigma^*\rangle\rangle$-linear system is given by Corollary 3 to be $S = M'^*P$, i.e.,

$$S_\pi = (M'^*P)_\pi = \sum_{\pi_1\in\Gamma^*} (M'^*)_{\pi,\pi_1} M'_{\pi_1,\varepsilon} = (M'^+)_{\pi,\varepsilon}.$$

We prove our claim by showing that (i)–(iv) is the solution of $Y = P + M'Y$.

The claims (ii) and (iv) obviously hold true.

(i)

$$\begin{aligned} &M'_{p\pi,\varepsilon}(Q,Q) + (M'M'^+)_{p\pi,\varepsilon}(Q,Q) \\ &= \sum_{\pi_3\in\Gamma^*}\sum_{p_4\in\Gamma}\sum_{\pi_4\in\Gamma^*}\sum_{q_5,q_6\in Q} M'_{p\pi,\pi_3}(Q, Q^{p_4,\pi_4}_{q_5,q_6})(M'^+)_{\pi_3,\varepsilon}(Q^{p_4,\pi_4}_{q_5,q_6}, Q) \\ &= \sum_{\pi_4\in\Gamma^*}\sum_{q_5,q_6\in Q} E^{p,\pi_4}_{q_5,q_6}(M^{p,\pi_4}_{q_5,q_6})^* F^{p,\pi_4}_{q_5,q_6}(M^*)_{\pi_4\pi,\varepsilon} \\ &= \sum_{\pi_4\in\Gamma^*} M_{p,\pi_4}(M^*)_{\pi_4\pi,\varepsilon} = (M^+)_{p\pi,\varepsilon}. \end{aligned}$$

(iii)

$$\begin{aligned} &M'_{p\pi,\varepsilon}(Q^{p_1,\pi_1}_{q_1,q_2}, Q) + (M'M'^+)_{p\pi,\varepsilon}(Q^{p_1,\pi_1}_{q_1,q_2}, Q) \\ &= \delta_{p,p_1}\delta_{\pi_1\pi,\varepsilon}F^{p_1,\pi_1}_{q_1,q_2} + \sum_{\pi_3\in\Gamma^*} M'_{p\pi,\pi_3}(Q^{p_1,\pi_1}_{q_1,q_2}, Q)(M'^+)_{\pi_3,\varepsilon}(Q,Q) \\ &\quad + \sum_{\pi_3\in\Gamma^*}\sum_{p_4\in\Gamma}\sum_{\pi_4\in\Gamma^*}\sum_{q_5,q_6\in Q} M'_{p\pi,\pi_3}(Q^{p_1,\pi_1}_{q_1,q_2}, Q^{p_4,\pi_4}_{q_5,q_6}) \\ &\quad \times (M'^+)_{\pi_3,\varepsilon}(Q^{p_4,\pi_4}_{q_5,q_6}, Q) \\ &= \delta_{p,p_1}\delta_{\pi_1\pi,\varepsilon}F^{p_1,\pi_1}_{q_1,q_2} + \delta_{p,p_1}M'_{p\pi,\pi_1\pi}(Q^{p_1,\pi_1}_{q_1,q_2}, Q)(M'^+)_{\pi_1\pi,\varepsilon}(Q,Q) \\ &\quad + \delta_{p,p_1}M'_{p\pi,p\pi}(Q^{p_1,\pi_1}_{q_1,q_2}, Q^{p_1,\pi_1}_{q_1,q_2})(M'^+)_{p\pi,\varepsilon}(Q^{p_1,\pi_1}_{q_1,q_2}, Q) \\ &= \delta_{p,p_1}\delta_{\pi_1\pi,\varepsilon}F^{p_1,\pi_1}_{q_1,q_2} + \delta_{p,p_1}F^{p_1,\pi_1}_{q_1,q_2}(M^+)_{\pi_1\pi,\varepsilon} \\ &\quad + \delta_{p,p_1}M^{p_1,\pi_1}_{q_1,q_2}(M^{p_1,\pi_1}_{q_1,q_2})^* F^{p_1,\pi_1}_{q_1,q_2}(M^*)_{\pi_1\pi,\varepsilon} \\ &= \delta_{p,p_1}(M^{p_1,\pi_1}_{q_1,q_2})^* F^{p_1,\pi_1}_{q_1,q_2}(M^*)_{\pi_1\pi,\varepsilon}. \quad \blacksquare \end{aligned}$$

Let $t \subseteq \Gamma \times \Gamma^*$ be a type and $\mathscr{P} = (Q, \Gamma, \delta, q_0, p_0, R)$ be an $A\langle\langle\Sigma^*\rangle\rangle$-pushdown automaton. Then $\mathscr{P}$ is said to have *type t* iff the pushdown transition matrix defined by $\mathscr{P}$ is of type t.

THEOREM 7. *Let* $\mathscr{P} = (Q, \Gamma, \delta, q_0, p_0, R)$ *be a cycle-free* $A\langle\langle\Sigma_1^*\rangle\rangle$*-pushdown automaton of type t and let* $\tau: A\langle\langle\Sigma_1^*\rangle\rangle \to A\langle\langle\Sigma_2^*\rangle\rangle$ *be a regulated rational transduction. Then there exists a cycle-free* $A\langle\langle\Sigma_2^*\rangle\rangle$*-pushdown automaton* $\mathscr{P}'' = (Q'', \Gamma, \delta'', 1, p_0, R'')$, $Q'' \supseteq Q \cup \{1\}$ *of type t such that* $\|\mathscr{P}''\| = \tau(\|\mathscr{P}\|)$.

Proof. Without loss of generality, let $\tau(r) = (r, \varepsilon)\, r_0 + \mu(r)_{1,m}$, $r \in A\langle\langle\Sigma_1^*\rangle\rangle$, $\mu: \Sigma_1^* \to (A^{\mathrm{rat}}\langle\langle\Sigma_2^*\rangle\rangle)^{m\times m}$, $m \geqslant 2$, be a regulated representation and $r_0 \in A^{\mathrm{rat}}\langle\langle\Sigma_2^*\rangle\rangle$. Let $M \in ((A\langle\langle\Sigma_1^*\rangle\rangle)^{Q\times Q})_t^{\Gamma^*\times\Gamma^*}$ be the cycle-free pushdown transition matrix defined by $\mathscr{P}$.

Let $Q' = \{1\} \cup Q \cup Q \times m$ and let $M' \in ((A\langle\langle\Sigma_2^*\rangle\rangle)^{Q'\times Q'})_t^{\Gamma^*\times\Gamma^*}$ be the cycle-free pushdown transition matrix constructed by Lemma 7 with $q = q_0$. Let $R' \in (A\langle\langle\Sigma_2^*\rangle\rangle)^{Q'\times 1}$ be defined by $R'(1) = 0$, $R'(Q) = R$, $R'(Q \times m) = \mu(R) f$ with $f \in (A\langle\langle\Sigma_2^*\rangle\rangle)^{m\times 1}$, $f_i = \delta_{i,m}\varepsilon$, $1 \leqslant i \leqslant m$.

Then

$$\begin{aligned}((M'^{+})_{p_0,\varepsilon} R')_1 &= e_{q_0}((M^+)_{p_0,\varepsilon}, \varepsilon)\, R r_0 + e_{(q_0,1)}\mu((M^+)_{p_0,\varepsilon})\, \mu(R) f \\ &= (\|\mathscr{P}\|, \varepsilon)\, r_0 + e_{(q_0,1)}\mu((M^+)_{p_0,\varepsilon} R) f \\ &= (\|\mathscr{P}\|, \varepsilon)\, r_0 + \mu(\|\mathscr{P}\|)_{1,m} = \tau(\|\mathscr{P}\|).\end{aligned}$$

Now apply the construction of Lemma 8 to the cycle-free pushdown matrix M' of type t, yielding $M'' \in ((A\langle\langle\Sigma_2^*\rangle\rangle)^{Q''\times Q''})_t^{\Gamma^*\times\Gamma^*}$, with $M'' = \sum_{x\in\Sigma_2}(M'', x)x + (M'', \varepsilon)\varepsilon$ and $(M''^{+})_{p_0,\varepsilon}(Q', Q') = (M'^{+})_{p_0,\varepsilon}$, $Q'' \supseteq Q' \supseteq Q \cup \{1\}$. Let $R'' \in (A\langle\langle\Sigma_2^*\rangle\rangle)^{Q''\times 1}$ with $R''(Q') = R'$ and $R''(Q'' - Q') = 0$. Then

$$((M''^{+})_{p_0,\varepsilon} R'')_1 = ((M''^{+})_{p_0,\varepsilon}(Q', Q')\, R''(Q'))_1 = ((M'^{+})_{p_0,\varepsilon} R')_1.$$

Let δ'' be defined by M''. Then $\|\mathscr{P}''\| = \tau(\|\mathscr{P}\|)$ and $\mathscr{P}''$ is cycle-free and of type t. ∎

LEMMA 9. *Let* $M_i \in ((A\langle\langle\Sigma^*\rangle\rangle)^{Q_i\times Q_i})_t^{\Gamma^*\times\Gamma^*}$ *be cycle-free pushdown transition matrices of type t,* $i = 1, 2$. *Then there exists a cycle-free pushdown transition matrix* $M \in ((A\langle\langle\Sigma^*\rangle\rangle)^{Q\times Q})_t^{\Gamma^*\times\Gamma^*}$, $Q = \{1\} \cup Q_1 \cup Q_2$ *such that for* $p \in \Gamma$, $q_i \in Q_i$, $i = 1, 2$,

$$(M^+)_{p,\varepsilon} = \begin{pmatrix} 0 & e_{q_1}(M_1^+)_{p,\varepsilon} & e_{q_2}(M_2^+)_{p,\varepsilon} \\ 0 & (M_1^+)_{p,\varepsilon} & 0 \\ 0 & 0 & (M_2^+)_{p,\varepsilon} \end{pmatrix},$$

where the blocks are indexed by 1, Q_1, Q_2.

Proof. Let

$$M_{p,p} = \begin{pmatrix} 0 & e_{q_1} & e_{q_2} \\ 0 & (M_1)_{p,p} & 0 \\ 0 & 0 & (M_2)_{p,p} \end{pmatrix}, \qquad p \in \Gamma,$$

$$M_{p,\pi} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & (M_1)_{p,\pi} & 0 \\ 0 & 0 & (M_2)_{p,\pi} \end{pmatrix}, \quad p \in \Gamma, \quad \pi \in \Gamma^*, \quad \pi \neq p.$$

Then the lemma is proven along the lines of the proof of Lemma 7. ∎

THEOREM 8. *Let $\mathscr{P}_i$ be cycle-free $A\langle\langle\Sigma^*\rangle\rangle$-pushdown automata of type t, $i = 1, 2$. Then there exists a cycle-free $A\langle\langle\Sigma^*\rangle\rangle$-pushdown automaton $\mathscr{P}$ of type t such that $\|\mathscr{P}\| = \|\mathscr{P}_1\| + \|\mathscr{P}_2\|$.*

Proof. By Lemma 9. ∎

5. Principal Regulated Rational Cones

Throughout this section, A is a commutative semiring, $\bar{\Sigma}$ is an infinite set of symbols, $\Sigma \subset \bar{\Sigma}$ is an alphabet, $Z = \{z_1, ..., z_n\}$, $n \geqslant 1$, is an alphabet with $Z \cap \Sigma = \varnothing$ and Q is a finite set. The item Σ may be indexed.

A nonempty set $\mathscr{S}_A$ of pairs (Σ, s), where $\Sigma \subset \bar{\Sigma}$ is an alphabet and $s \in A\langle\langle\Sigma^*\rangle\rangle$, is called a *family of series*. A family of series $\mathscr{C}_A$ is called a *regulated rational cone* (respectively, *rational cone*) iff for every pair $(\Sigma_1, s) \in \mathscr{C}_A$ and every regulated rational (respectively, rational) transduction $\tau: A\langle\langle\Sigma_1^*\rangle\rangle \to A\langle\langle\Sigma_2^*\rangle\rangle$, Σ_1, $\Sigma_2 \subset \bar{\Sigma}$, the pair $(\Sigma_2, \tau(s))$ belongs to $\mathscr{C}_A$. A (regulated) rational cone $\mathscr{C}_A$ is called *principal* iff it is generated by a single pair (Σ, s), $\Sigma \subset \bar{\Sigma}$, via (regulated) rational transductions. The pair (Σ, s) is called the *cone generator* of $\mathscr{C}_A$.

We now specialize to the Boolean semiring $\mathbb{B}$. Let $\mathscr{C}_\mathbb{B}$ be a regulated rational cone (respectively, rational cone) and let $\mathscr{C} = \{L \mid L = \operatorname{supp}(s),\ (\Sigma, s) \in \mathscr{C}_\mathbb{B}$ for some $\Sigma \subset \bar{\Sigma}\}$ be its *language-theoretic equivalent*. If $L \neq \varnothing$ for some L in $\mathscr{C}$, then in language theory, $\mathscr{C}$ is a trio (respectively, a full trio or rational cone). These families of languages are extensively treated in Berstel [1] and Ginsburg [3]. If $\mathscr{C}_\mathbb{B}$ is principal, then $\mathscr{C}$ is principal.

Let $\mathscr{C}_\mathbb{B}$ be a principal regulated rational cone generated by (Σ, s), $s \in \mathbb{B}\langle\langle\Sigma^*\rangle\rangle$, $s \neq 0$, and let $\mathscr{C}$ be its language theoretic equivalent. Then by [10, Theorem III.1.4], the principal rational cone generated by (Σ, s) is the closure of $\mathscr{C}_\mathbb{B}$ under homomorphism and by [3, Theorem 3.4.3], the language-theoretic principal rational cone generated by supp(s) is the closure of $\mathscr{C}$ under homomorphism.

If $\mathscr{C}_\mathbb{B}$ is a regulated rational cone (respectively, rational cone) closed under addition, then its language theoretic equivalent is called semi-AFL (respectively, full semi-AFL).

Let $Z = \{z_1, ..., z_n\}$ be an alphabet of pushdown symbols, and let $t \subset Z \times Z^*$ be a type. Then the collection of power series $A^t\langle\langle\Sigma^*\rangle\rangle$ is defined as follows: A power series $r \in A\langle\langle\Sigma^*\rangle\rangle$ is in $A^t\langle\langle\Sigma^*\rangle\rangle$ iff it is the behavior of a cycle-free $A\langle\langle\Sigma^*\rangle\rangle$-pushdown automaton $\mathscr{P} = (Q, Z, \delta, q_0, z_1, R)$ of type t.

Let $\bar{\Sigma}$ be a fixed infinite set of symbols. The family of series $\mathscr{C}_A^t$ is defined to be the set of pairs (Σ, s), $\Sigma \subset \bar{\Sigma}$ an alphabet, such that $s \in A^t\langle\langle\Sigma^*\rangle\rangle$. We show that $\mathscr{C}_A^t$ is a principal regulated rational cone that is characterized in the following way: A pair (Σ, s) is in $\mathscr{C}_A^t$ iff s is the first component of the solution of an algebraic system over $A\langle\langle\Sigma^*\rangle\rangle$ given in a certain matrix form (depending on t).

Throughout this section, let $t \subset Z \times Z^*$ be a type and $\Sigma_t = \{a_{z_i,\pi} \mid (z_i, \pi) \in t, 1 \leqslant i \leqslant n\}$ be an alphabet. The algebraic system over $A\langle\langle\Sigma_t^*\rangle\rangle$

$$z_i = \sum_{(z_i,\pi)\in t} a_{z_i,\pi}\pi, \qquad 1 \leqslant i \leqslant n,$$

is called the *cone generator system for type t*. Hence the cone generator system for type t is strict and the context-free grammar induced by it is simple deterministic in the sense of Korenjak and Hopcroft [6].

Thus the cone generator system for type t has a unique solution $\sigma = (\sigma_1, ..., \sigma_n)$. The pair (Σ_t, σ_1) is called the *cone generator for type t*.

For r_1, $r_2 \in A\langle\langle\Sigma^*\rangle\rangle$, let $r_1 \odot r_2 \in A\langle\langle\Sigma^*\rangle\rangle$ be defined by $(r_1 \odot r_2, w) = (r_1, w)(r_2, w)$. The series $r_1 \odot r_2$ is called the *Hadamard product* of r_1 and r_2. If r_1, $r_2 \in \mathbb{B}\langle\langle\Sigma^*\rangle\rangle$, then $\mathrm{supp}(r_1 \odot r_2) = \mathrm{supp}(r_1) \cap \mathrm{supp}(r_2)$, i.e., the Hadamard product represents intersection.

Let $w \in \Sigma^*$. Then $v \in \Sigma^*$ is a *factor* of w if $w = w_1 v w_2$ for some $w_1, w_2 \in \Sigma^*$. Let c be a symbol, $c \notin \Sigma$. Then $r \in A\langle\langle(\Sigma \cup \{c\})^*\rangle\rangle$ is termed *c-limited* iff there exists a $k \geqslant 1$ such that $(r, w) \neq 0$, $w \in (\Sigma \cup \{c\})^*$, and c^l, $l \geqslant 0$, a factor of w, imply $l < k$.

Let $h: (\Sigma \cup \{c\})^* \to \Sigma^*$ be the homomorphism defined by $h(c) = \varepsilon$, $h(x) = x$, $x \in \Sigma$. Let $r \in A^{\mathrm{rat}}\langle\langle(\Sigma \cup \{c\})^*\rangle\rangle$ be c-limited and let $\tau: A\langle\langle(\Sigma \cup \{c\})^*\rangle\rangle \to A\langle\langle\Sigma^*\rangle\rangle$ be a mapping defined by $\tau(s) = h(s \odot r)$, $s \in A\langle\langle\Sigma^*\rangle\rangle$. According to [10, Exercise IV.7.4], the mapping τ is a regulated rational transduction.

THEOREM 9. *Let $t \subset Z \times Z^*$ be a type. Then $\mathscr{C}_A^t$ is a principal regulated rational cone closed under addition. It is generated by (Σ_t, σ_1), the cone generator for type t.*

Proof. Let $z_i = \sum_{(z_i,\pi)\in t} a_{z_i,\pi}\pi$, $1 \leqslant i \leqslant n$, be the cone generator system for type t with unique solution $\sigma = (\sigma_1, ..., \sigma_n)$. Define the cycle-free $A\langle\langle\Sigma_t^*\rangle\rangle$-pushdown automaton $\mathscr{P} = (\{1\}, Z, \delta, 1, z_1, (\varepsilon))$ by

$$\begin{aligned} \delta(1, a_{z_i,\pi}, z_i) &= (1, \pi) && \text{if} \quad (z_i, \pi) \in t, \quad 1 \leqslant i \leqslant n, \\ \delta(1, a, z) &= 0, && a \in \Sigma_t, \quad z \in Z, \quad \text{otherwise}. \end{aligned}$$

Automaton $\mathscr{P}$ defines the cycle-free pushdown transition matrix $\bar{M} \in (A\langle\langle\Sigma_t^*\rangle\rangle)_t^{Z^* \times Z^*}$ with $\bar{M}_{z_i,\pi} = a_{z_i,\pi}$ if $(z_i, \pi) \in t$ and $\bar{M}_{z_i,\pi} = 0$ if $(z_i, \pi) \notin t$, $1 \leqslant i \leqslant n$. Hence $\mathscr{P}$ is of type t and $\sigma_1 = \|\mathscr{P}\| = (\bar{M}^+)_{z_1,\varepsilon}$ by Lemma 6.

(i) Let $\tau: A\langle\langle\Sigma_t^*\rangle\rangle \to A\langle\langle\Sigma^*\rangle\rangle$ be a regulated rational transduction. Then by Theorem 7, there exists a cycle-free $A\langle\langle\Sigma^*\rangle\rangle$-pushdown automaton $\mathscr{P}'$ of type t such that $\tau(\sigma_1) = \|\mathscr{P}'\|$. Hence $\tau(\sigma_1) \in A^t\langle\langle\Sigma^*\rangle\rangle$, $(\Sigma, \tau(\sigma_1)) \in \mathscr{C}_A^t$, and the principal regulated rational cone generated by (Σ_t, σ_1) is a subcone of $\mathscr{C}_A^t$.

(ii) Conversely, let $(\Sigma, s) \in \mathscr{C}_A^t$. Then $s \in A^t\langle\langle\Sigma^*\rangle\rangle$ and there exists an $A\langle\langle\Sigma^*\rangle\rangle$-pushdown automaton $\mathscr{P}' = (\{1,..., m\}, Z, \delta, 1, z_1, R)$, $m \geqslant 1$, of type t and cycle-free of order k, $k \geqslant 1$, such that $\|\mathscr{P}'\| = s$. Let $M \in ((A\langle\langle\Sigma^*\rangle\rangle)^{m \times m})_t^{Z^* \times Z^*}$ be the cycle-free pushdown transition matrix defined by $\mathscr{P}'$. Then $s = ((M^+)_{z_1,\varepsilon} R)_1$.

Define the pushdown transition matrix $M' \in ((A\langle\langle\Sigma^*\rangle\rangle)^{(m+1)\times(m+1)})_t^{Z^* \times Z^*}$ by

$$M'_{z_i,\varepsilon} = \begin{pmatrix} M_{z_i,\varepsilon} & M_{z_i,\varepsilon} R \\ 0 & 0 \end{pmatrix}, \qquad M'_{z_i,\pi} = \begin{pmatrix} M_{z_i,\pi} & 0 \\ 0 & 0 \end{pmatrix},$$

$1 \leqslant i \leqslant n$, $\pi \in \Gamma^+$. Obviously, M' is cycle-free of order k and

$$(M'^+)_{\pi,\varepsilon} = \begin{pmatrix} (M^+)_{\pi,\varepsilon} & (M^+)_{\pi,\varepsilon} R \\ 0 & 0 \end{pmatrix}, \qquad \pi \in \Gamma^*.$$

Let c be a symbol with $c \notin \Sigma$ and let

$$M'' = \sum_{x \in \Sigma_t} (M', x)x + (M', \varepsilon)c \in ((A\langle\langle(\Sigma \cup \{c\})^*\rangle\rangle)^{(m+1)\times(m+1)})_t^{Z^* \times Z^*}.$$

Let $w \in (\Sigma \cup \{c\})^*$, $w = w_1 c^l w_2$, $l \geqslant 0$, w_1, $w_2 \in (\Sigma \cup \{c\})^*$, $|w_1| = j_1 \geqslant 0$, $|w_2| = j_2 \geqslant 0$. Since M'' is quasiregular,

$$(M''^{j_1+l+j_2}, w) = (M''^{j_1}, w_1)(M''^l, c^l)(M''^{j_2}, w_2) = (M''^{j_1}, w_1)(M'^l, \varepsilon)(M''^{j_2}, w_2).$$

Hence if $(M''^{j_1+l+j_2}, w) \neq 0$, then $l \leqslant k-1$ and $((M''^+)_{\pi_1,\pi_2})_{l_1,l_2}$ is c-limited for π_1, $\pi_2 \in \Gamma^*$, $1 \leqslant l_1, l_2 \leqslant m+1$. Let $r_k \in A\langle\langle(\Sigma \cup \{c\})^*\rangle\rangle$ be defined as follows: $(r_k, w) = 1$ if $w \in ((\{\varepsilon\} \cup \{c\} \cup \cdots \cup \{c^{k-1}\})\Sigma)^*$ and $(r_k, w) = 0$, otherwise. Then $r_k \in A^{\text{rat}}\langle\langle(\Sigma \cup \{c\})^*\rangle\rangle$ is c-limited and $((M''^+)_{\pi_1,\pi_2})_{l_1,l_2} \odot r_k = ((M''^+)_{\pi_1,\pi_2})_{l_1,l_2}$ for π_1, $\pi_2 \in \Gamma^*$, $1 \leqslant l_1$, $l_2 \leqslant m+1$.

Let $\rho: \Sigma_t^* \to (A\langle\langle(\Sigma \cup \{c\})^*\rangle\rangle)^{(m+1)\times(m+1)}$ be the regulated representation defined by

$$\rho(a_{z_i,\pi}) = M''_{z_i,\pi}, \qquad (z_i, \pi) \in t, \qquad 1 \leqslant i \leqslant n.$$

Let $h: (\Sigma \cup \{c\})^* \to \Sigma^*$ be the homomorphism defined by $h(c) = \varepsilon$, $h(x) = x$, $x \in \Sigma$. Let $\tau: A\langle\langle\Sigma_t^*\rangle\rangle \to A\langle\langle\Sigma^*\rangle\rangle$ be the mapping defined by $\tau(r) = h((\rho(r))_{1,m+1} \odot r_k)$, $r \in A\langle\langle\Sigma_t^*\rangle\rangle$.

Since regulated rational transductions are closed under functional composition, τ is a regulated rational transduction. Since

$$\begin{aligned} \tau(\sigma_1) &= h((\rho(\sigma_1))_{1,m+1} \odot r_k) = h((\rho(\tilde{M}^+)_{z_1,\varepsilon}))_{1,m+1} \odot r_k) \\ &= h(((M''^+)_{z_1,\varepsilon})_{1,m+1} \odot r_k) = h(((M''^+)_{z_1,\varepsilon})_{1,m+1}) \\ &= ((M'^+)_{z_1,\varepsilon})_{1,m+1} = ((M^+)_{z_1,\varepsilon} R)_1 = s, \end{aligned}$$

(Σ, s) is generated by (Σ_t, σ_1) via the regulated rational transduction τ.

By Theoem 8, $\mathscr{C}_A^t$ is closed under addition. ∎

Let $Y_m = \{y^i_{j,k} \mid 1 \leqslant j,\ k \leqslant m,\ 1 \leqslant i \leqslant n\}$, $m \geqslant 1$, be an alphabet of variables. Let $Y(z_i) \in (A\langle Y_m^* \rangle)^{m \times m}$, $1 \leqslant i \leqslant n$, be defined by $(Y(z_i))_{j,k} = y^i_{j,k}$, $1 \leqslant j,\ k \leqslant m$. Let $Y(\varepsilon) = E$ and $Y(z\pi) = Y(z)\,Y(\pi)$, $z \in Z$, $\pi \in Z^*$.

An algebraic system over $A\langle\langle \Sigma^* \rangle\rangle$ with variables in $\{y\} \cup Y_m$, $m \geqslant 1$,

$$y = (Y(z_1)R)_1, \qquad Y(z_i) = \sum_{(z_i, \pi) \in t} M_{z_i, \pi} Y(\pi), \qquad 1 \leqslant i \leqslant n,$$

is called a *regulated rational cone system for type t* iff

$$M = \sum_{x \in \Sigma} (M, x)x + (M, \varepsilon)\varepsilon \in ((A\langle\langle \Sigma^* \rangle\rangle)^{m \times m})_t^{Z^* \times Z^*}$$

is a cycle-free pushdown transition matrix of type t and $R = (R, \varepsilon)\varepsilon \in (A\langle\langle \Sigma^* \rangle\rangle)^{m \times 1}$.

By Lemma 4, the regulated rational cone system for type t as defined above has the unique solution $(((M^*)_{z_1,\varepsilon} R)_1, (M^*)_{z_1,\varepsilon}, ..., (M^*)_{z_n,\varepsilon})$. The component $((M^*)_{z_1,\varepsilon} R)_1$ of the unique solution is called the first component of the solution.

THEOREM 10. *Let $t \subset Z \times Z^*$ be a type. Then $(\Sigma, s) \in \mathscr{C}^t_A$ iff there exists a regulated rational cone system for type t such that s is the first component of its unique solution.*

Proof. (i) By definition, s is the behavior of a cycle-free $A\langle\langle \Sigma^* \rangle\rangle$-pushdown automaton $\mathscr{P} = (\{1,..., m\}, Z, \delta, 1, z_1, R)$ of type t. Let $M \in ((A\langle\langle \Sigma^* \rangle\rangle)^{m \times m})_t^{Z^* \times Z^*}$ be the cycle-free pushdown transition matrix defined by $\mathscr{P}$. Then by Lemma 4, $((M^*)_{z_1,\varepsilon}, ..., (M^*)_{z_n,\varepsilon})$ is the unique solution of

$$Y(z_i) = \sum_{(z_i, \pi) \in t} M_{z_i, \pi} Y(\pi), \qquad 1 \leqslant i \leqslant n.$$

Hence $((M^*)_{z_1,\varepsilon} R)_1 = s$ is the first component of the unique solution of the regulated rational cone system for type t

$$y = (Y(z_1)R)_1, \qquad Y(z_i) = \sum_{(z_i, \pi) \in t} M_{z_i, \pi} Y(\pi), \qquad 1 \leqslant i \leqslant n.$$

(ii) By Lemma 4, the regulated rational cone system for type t

$$y = (Y(z_1)R)_1, \qquad Y(z_i) = \sum_{(z_i, \pi) \in t} M_{z_i, \pi} Y(\pi), \qquad 1 \leqslant i \leqslant n,$$

has the unique solution $(((M^*)_{z_1,\varepsilon} R)_1, (M^*)_{z_1,\varepsilon}, ..., (M^*)_{z_n,\varepsilon})$. Hence there exists a cycle-free $A\langle\langle \Sigma^* \rangle\rangle$-pushdown automaton $\mathscr{P} = (\{1,..., m\}, Z, \delta, 1, z_1, R)$ defining M such that $\|\mathscr{P}\| = ((M^+)_{z_1,\varepsilon} R)_1 = ((M^*)_{z_1,\varepsilon} R)_1$. Hence the first component of the unique solution is in $A^t\langle\langle \Sigma^* \rangle\rangle$. ∎

We now turn to language theory, i.e., we consider the Boolean semiring $\mathbb{B}$. Let $\mathscr{C}^t = \{\mathrm{supp}(s) \mid (\Sigma, s) \in \mathscr{C}^t_{\mathbb{B}}$ for some $\Sigma \subset \bar{\Sigma}\}$. Then either $\mathscr{C}^t = \{\varnothing\}$ or $\mathscr{C}^t$ is a semi-AFL containing $\{\varepsilon\}$. A set $\mathscr{L}$ of languages is *erasable* iff the smallest semi-AFL containing $\mathscr{L} \cup \{\{\varepsilon\}\}$ equals the smallest full semi-AFL containing $\mathscr{L}$ [4]. Let (Σ_t, σ_1), $\sigma_1 \in \mathbb{B}\langle\langle \Sigma_t^* \rangle\rangle$, $\sigma_1 \neq 0$, be the cone generator for type t. If $\{\text{supp}(\sigma_1)\}$ is erasable, then $\mathscr{C}^t$ is the principal full semi-AFL generated by $\text{supp}(\sigma_1)$. Hence we have proved

THEOREM 11. *Let $t \subset Z \times Z^*$ be a type and let (Σ_t, σ_1), $\sigma_1 \in B\langle\langle \Sigma_t^* \rangle\rangle$, $\sigma_1 \neq 0$, be the cone generator for type t. Let $\{\text{supp}(\sigma_1)\}$ be erasable. Then $\mathscr{C}^t$ is the principal full semi-AFL generated by* $\text{supp}(\sigma_1)$.

In this case Theorem 10 gives a characterization of $\mathscr{C}^t$ in terms of context-free grammars. If $\{\text{supp}(\sigma_1)\}$ is not erasable, then $\mathscr{C}^t$ is the principal semi-AFL generated by $\text{supp}(\sigma_1)$. If we modify Theorem 10, then we get a characterization of the principal full semi-AFL generated by $\text{supp}(\sigma_1)$. A context-free grammar $G = (\{y\} \cup Y_m, \Sigma, P, y)$, $m \geqslant 1$, is called a *full semi-*AFL *context-free grammar for type t* iff it is induced by an algebraic system over $\mathbb{B}\langle\langle \Sigma^* \rangle\rangle$ with variables in $y \cup \{Y_m\}$

$$y = (Y(z_1)R)_1, \qquad Y(z_i) = \sum_{(z_i, \pi) \in t} M_{z_i, \pi} Y(\pi), \qquad 1 \leqslant i \leqslant n,$$

where

$$M_{z_i, \pi} = \sum_{x \in \Sigma} (M_{z_i, \pi}, x)x + (M_{z_i, \pi}, \varepsilon)\varepsilon \in (\mathbb{B}\langle\langle \Sigma^* \rangle\rangle)^{m \times m},$$

$(z_i, \pi) \in t$, $1 \leqslant i \leqslant n$, and $R = (R, \varepsilon)\varepsilon \in (\mathbb{B}\langle\langle \Sigma^* \rangle\rangle)^{m \times 1}$.

THEOREM 12. *Let $t \subset Z \times Z^*$ be a type and let (Σ_t, σ_1), $\sigma_1 \in B\langle\langle \Sigma_t^* \rangle\rangle$, $\sigma_1 \neq 0$, be the cone generator for type t. Then the language L is in the principal full semi-*AFL *generated by* $\text{supp}(\sigma_1)$ *iff L is generated by a full semi-*AFL *context-free grammar for type t.*

Proof. By [3, Theorem 3.4.3] and Theorem 10. ∎

6. ONE-COUNTER LANGUAGES

In language theory, a language L is called a *one-counter language* iff it is accepted (by empty tape) by a pushdown automaton with a single pushdown symbol. The goal of this last section is to characterize those power series in $A\langle\langle \Sigma^* \rangle\rangle$, A a commutative semiring, that are the behavior of an $A\langle\langle \Sigma^* \rangle\rangle$-pushdown automaton with a single pushdown symbol.

Let $Z = \{z\}$ and $\text{rocl} \subseteq Z \times Z^*$ be the type $\text{rocl} = \{(z, \varepsilon), (z, z), (z, z^2)\}$.

LEMMA 10. *The power series s is the behavior of a cycle-free $A\langle\langle \Sigma^* \rangle\rangle$-pushdown automaton with a single pushdown symbol iff $(\Sigma, s) \in \mathscr{C}_A^{\text{rocl}}$.*

Proof. Let $\mathscr{P} = (Q, Z, \delta, q_0, z, R)$ be a cycle-free $A\langle\langle \Sigma^* \rangle\rangle$-pushdown automaton which defines the cycle-free pushdown matrix $M \in ((A\langle\langle\Sigma^*\rangle\rangle)^{Q \times Q})_J^{Z^* \times Z^*}$. Let $s = \|\mathscr{P}\| = ((M^+)_{z,\varepsilon} R)_{q_0}$. Let $m = \max\{i \mid M_{z,z^i} \neq 0\}$ and let $M' \in ((A\langle\langle\Sigma^*\rangle\rangle)^{(Q \times (m-1)) \times (Q \times (m-1))})_{\text{rocl}}^{Z^* \times Z^*}$ be an $A\langle\langle\Sigma^*\rangle\rangle$-pushdown transition matrix defined by $M'_{z,\varepsilon}, M'_{z,z}, M'_{z,z^2} \in (A\langle\langle\Sigma^*\rangle\rangle)^{(Q \times (m-1)) \times (Q \times m-1)}$ as follows:

$$M'_{z,\varepsilon} = \begin{pmatrix} 0 & 0 & \cdots & M_{z,\varepsilon} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$M'_{z,z} = \begin{pmatrix} M_{z,z} & M_{z,z^2} & \cdots & M_{z,z^{m-1}} \\ M_{z,\varepsilon} & M_{z,z} & \cdots & M_{z,z^{m-2}} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & M_{z,z} \end{pmatrix},$$

$$M'_{z,z^2} = \begin{pmatrix} M_{z,z^m} & 0 & \cdots & 0 \\ M_{z,z^{m-1}} & M_{z,z^m} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ M_{z,z^2} & M_{z,z^3} & \cdots & M_{z,z^m} \end{pmatrix},$$

where the blocks are indexed by $Q \times \{1\}$, $Q \times \{2\}$,..., $Q \times \{m-1\}$. It is obvious that for $k \geqslant 1$:

(i) $(M'^k)_{\varepsilon, z^{j_2}} = 0$ for $j_2 \geqslant 0$,

(ii) $(M'^k)_{z^{j_1},\varepsilon}(Q \times \{i_1\}, Q \times \{i_2\}) = 0$ for $j_1 > 0$, $1 \leqslant i_1 \leqslant m-1$, $1 \leqslant i_2 \leqslant m-2$,

and

(iii) $(M'^k)_{z^{j_1}, z^{j_2}}(Q \times \{i_1\}, Q \times \{i_2\}) = (M^k)_{z^{(j_1-1)(m-1)+i_1}, z^{(j_2-1)(m-1)+i_2}}$ for $j_1 > 0$, $j_2 > 0$, $1 \leqslant i_1 \leqslant m-1$, $1 \leqslant i_2 \leqslant m-1$ and for $j_1 > 0$, $j_2 = 0$, $1 \leqslant i_1 \leqslant m-1$, $i_2 = m-1$.

Hence M' is cycle-free and $(M'^+)_{z,\varepsilon}(Q \times \{1\}, Q \times \{m-1\}) = (M^+)_{z,\varepsilon}$.

Let $\mathscr{P}' = (Q \times (m-1), Z, \delta', (q_0, 1), z, R')$, be the cycle-free $A\langle\langle\Sigma^*\rangle\rangle$-pushdown automaton of type rocl, with δ' given by M' and $R'(Q \times \{i\}) = 0$, $1 \leqslant i \leqslant m-2$, $R'(Q \times \{m-1\}) = R$. Then

$$\begin{aligned} \|\mathscr{P}'\| &= ((M'^+)_{z,\varepsilon} R')_{(q_0,1)} \\ &= \left(\sum_{i=1}^{m-1} (M'^+)_{z,\varepsilon}(Q \times \{1\}, Q \times \{i\})\, R'(Q \times \{i\})\right)_{(q_0,1)} \\ &= ((M'^+)_{z,\varepsilon}(Q \times \{1\}, Q \times \{m-1\})\, R'(Q \times \{m-1\}))_{(q_0,1)} \\ &= ((M^+)_{z,\varepsilon} R)_{q_0} = \|\mathscr{P}\| = s. \end{aligned}$$

The converse obviously holds true. ∎

LEMMA 11. *Let s_1 and s_2, respectively, be the unique solutions of the algebraic systems $z = a_2 z^2 + a_0$ and $z = a_2 z^2 + a_1 z + a_0$, respectively. Then there exists a regulated rational transduction $\tau: A\langle\langle\{a_2, a_0\}^*\rangle\rangle \to A\langle\langle\{a_2, a_1, a_0\}^*\rangle\rangle$, such that $\tau(s_1) = s_2$.*

Proof. Define the regulated rational transduction τ by $\tau(r) = (\mu(r))_{1,2}$, $r \in A\langle\langle\{a_2, a_0\}^*\rangle\rangle$, and

$$\mu(a_0) = \begin{pmatrix} 0 & a_1^* a_0 \\ 0 & a_1^* a_0 \end{pmatrix}, \qquad \mu(a_2) = \begin{pmatrix} 0 & a_1^* a_2 \\ 0 & a_1^* a_2 \end{pmatrix}.$$

Let l and r be substitutions defined by $l(a_i) = a_1^* a_i$, $r(a_i) = a_i a_1^*$, $i = 0, 2$. Then $\tau(s_1) = l(s_1)$. Since

$$l(xw) = a_1^* r(w) x, \qquad \tau(s_1) = \sum_{x \in \Sigma} \sum_{w \in \Sigma^*} (s_1, wx)\, a_1^* r(w) x, \qquad \Sigma = \{a_0, a_2\}.$$

We claim that $\tau(s_1) = s_2$.

$$\tau(s_1) = (a_2 s_1 s_1 + a_0) = l(a_2)\, l(s_1)\, l(s_1) + l(a_0)$$
$$= a_1^* \left(\sum_{x_1} \sum_{x_2} \sum_{w_1} \sum_{w_2} (s_1, w_1 x_1)(s_1, w_2 x_2)\, r(a_2 w_1 x_1 w_2)\, x_2 \right) + a_1^* a_0,$$

where $x_1, x_2 \in \Sigma$, $w_1, w_2 \in \Sigma^*$. Since $(s_1, w) \in \{0, 1\}$ for $w \in \Sigma^*$, $(s_1, a_2 wx) = 1$ iff $wx = w_1 x_1 w_2 x_2$, and $(s_1, w_1 x_1) = (s_1, w_2 x_2) = 1$, we have

$$\tau(s_1) = \sum_{x_1, x_2 \in \Sigma} \sum_{w_1, w_2 \in \Sigma^*} (s_1, w_1 x_1)(s_1, w_2 x_2)\, r(a_2 w_1 x_1 w_2)\, x_2$$
$$+ a_0 + a_1^+ \left(\sum_{x \in \Sigma} \sum_{w \in \Sigma^+} (s_1, a_2 wx)\, r(a_2 w) x \right) + a_1^+ a_0$$
$$= a_2 l(s_1)\, l(s_1) + a_0 + a_1 \left(\sum_{x \in \Sigma} \sum_{w \in \Sigma^*} (s_1, wx)\, a_1^* r(w) x \right)$$
$$= a_2 l(s_1)\, l(s_1) + a_1 l(s_1) + a_0 = a_2 \tau(s_1)\, \tau(s_1) + a_1 \tau(s_1) + a_0.$$

Hence $\tau(s_1)$ is the unique solution of $z = a_2 z^2 + a_1 z + a_0$ and $\tau(s_1) = s_2$. ∎

THEOREM 13. *Let $s \in A\langle\langle\Sigma^*\rangle\rangle$. Then the following statements are equivalent:*

(i) $(\Sigma, s) \in \mathscr{C}_A^{\mathrm{rocl}}$,

(ii) *(Σ, s) is in the principal regulated rational cone generated by the unique solution of $z = a_2 z^2 + a_0$,*

(iii) *s is the first component of the unique solution of a regulated rational cone system for type* rocl,

(iv) *s is the behavior of a cycle-free $A\langle\langle\Sigma^*\rangle\rangle$-pushdown automaton $\mathscr{P} = (Q, \Gamma, \delta, q_0, z, R)$ with $\Gamma = \{z\}$.*

The equivalent language-theoretic results to Theorem 13(i,ii,iv) are well known.

The work of Kuich and Urbanek [9] yields two more characterizations of s with $(\Sigma, s) \in \mathscr{C}_A^{\text{rocl}}$.

COROLLARY 4. *The cone* $\mathscr{C}_A^{\text{rocl}}$ *is a principal rational cone closed under addition. Its language-theoretic equivalent* $\mathscr{C}^{\text{rocl}}$ *is a principal full semi-*AFL.

Proof. The family of one-counter languages is uniformly erasable by [4]. Apply Theorem 11. ∎

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