Necessary and Sufficient Conditions for J-Spectral Factorizations With a J-Lossless Property for Infinite-Dimensional Systems in Continuous and Discrete Time

Ruth F. Curtain and Alejandro Rodríguez*

Department of Mathematics
University of Groningen
P. O. Box 800
9700 AV Groningen, the Netherlands

Submitted by M. L. J. Hautus

ABSTRACT

Necessary and sufficient conditions for J-spectral factorizations are given in terms of the existence of a self-adjoint, stabilizing solution of an appropriate Riccati equation in infinite dimensions. Furthermore, it is shown that a certain J-lossless property holds if and only if the stabilizing solution to the Riccati equation is nonnegative definite. We prove these results first for the continuous-time case and thereby derive analogous results for the discrete-time case by means of a bilinear transformation of the complex plane.

1. INTRODUCTION

In the papers by Green [9] and Green et al. [10] a theory for \( H_\infty \)-optimal control problems is developed in terms of J-spectral factorizations. In order to produce results in terms of state-space formulas they prove a key theorem in [10, Theorem 2.3] which relates J-spectral factorizations to certain Riccati equations. This theorem gives necessary and sufficient conditions for the existence of a J-spectral factorization \( G^* JG = W J W \), where \( W \) and \( W^{-1} \) are stable, in terms of the existence of a stabilizing, symmetric solution to a Riccati equation depending on the matrix parameters in the realization \( G(s) = D + C(sI - A)^{-1}B \). Furthermore, in [9, Theorem 1.1], Green proves

---

*Supported by a D.G.A.P.A. grant from the National University of Mexico.
that $GW^{-1}$ is $J$-lossless if and only if the stabilizing solution of the Riccati equation is nonnegative definite.

Here we prove an analogous result for the case of the transfer matrix $G(s) = D + C(sI - A)^{-1}B$, where $A$ is the infinitesimal generator of an exponentially stable semigroup on a Hilbert space $\mathcal{L}$, and $B \in \mathcal{L}(\mathcal{C}^{q+m}, \mathcal{L})$, $C \in \mathcal{L}(\mathcal{C}^{l+m}, \mathcal{L})$, $D \in \mathcal{L}(\mathcal{C}^{q+m}, \mathcal{C}^{l+m})$. As well as being an interesting new result in its own right, it is used in Curtain and Green [5] to derive a state-space solution for the $H_\omega$-optimal control problem in infinite dimensions. Finally, by using the Cayley transformation $A = (A_d + I)^{-1}(A_d - I)$ we associate the discrete-time operator $A_d$ with the bounded operator $A$ of a continuous-time system. We then associate with the discrete-time Riccati equation a continuous-time one which has the same solutions. In this way, we are able to derive analogous results on $J$-spectral factorization for the discrete-time case.

2. PRELIMINARIES AND NOTATION

In this section we set up the notation we will use throughout the paper and gather some known results we will need in subsequent sections.

Denote by $\sigma(A)$ the maximum singular value of the matrix $A$. Denote by $A^*$ its complex transpose. Let $\mathbb{C}^+ = \{s \in \mathbb{C} : \Re s > 0\}$ and $j\mathbb{R} = \{s \in \mathbb{C} : \Re s = 0\}$.

Let $L^p_{\infty} = \{F : j\mathbb{R} \to \mathbb{C}^{p \times q} : \|F\|_\infty = \sup_{\omega \in \mathbb{R}} \sigma(F(j\omega)) < \infty\}$. Denote by $H^p_{\infty \times q}$ the space of bounded holomorphic functions $F : \mathbb{C} \to \mathbb{C}^{p \times q}$ with the norm $\|F\|_\infty = \sup_{s \in \mathbb{C}} \sigma(F(s)) < \infty$. $M^- = [M(-\bar{s})]^*$. For a real $\gamma > 0$, define $J_{pq}(\gamma)$ by

$$J_{pq}(\gamma) = \begin{pmatrix} I_p & 0 \\ 0 & -\gamma^2 I_q \end{pmatrix}.$$ 

DEFINITION 2.1. A partitioned matrix function $M : \overline{\mathbb{C}}^+ \to \mathbb{C}^{(l+m) \times (q+m)}$ is $J$-lossless if

$$M(s)^* J_{lm}(\gamma) M(s) \preceq J_{qm}(\gamma) \quad \text{on } \mathbb{C}^+$$

and

$$M(j\omega)^* J_{lm}(\gamma) M(j\omega) = J_{qm}(\gamma) \quad \text{on } j\mathbb{R}.$$
We consider the following classes of stable transfer functions:

\[ \mathcal{A} = \left\{ f : \mathbb{R} \to \mathbb{C} : f(t) = \begin{cases} f_0 \delta(t) + f_a(t), & t \geq 0, \\ 0, & t < 0, \end{cases} \right\} \]

for some \( f_0 \in \mathbb{C} \) and \( \int_0^\infty |f_a(t)| \, dt < \infty \).

where \( \hat{f} \) denotes the Laplace transform;

\[ \mathcal{A}_- = \left\{ f : \mathbb{R} \to \mathbb{C} : f(t) = \begin{cases} f_0 \delta(t) + f_a(t), & t \geq 0, \\ 0, & t > 0, \end{cases} \right\} \]

where \( f_0 \in \mathbb{C} \) and \( \int_0^\infty e^{\varepsilon t} |f_a(t)| \, dt < \infty \) for some \( \varepsilon > 0 \).

We remark that \( \hat{f} \in \mathcal{A} \) has the limit \( f_0 \) at infinity. \( \mathcal{A}^{p \times q} \) denotes the class of \( p \times q \) matrices with components in \( \mathcal{A} \). We also need to consider the following classes of stable plus antistable transfer functions:

\[ \mathcal{W} = \left\{ \hat{f} = \hat{f}_1 + \hat{f}_2, \text{ where } \hat{f}_1 \text{ and } \hat{f}_2 \text{ belong to } \mathcal{A} \right\} \]

and

\[ \mathcal{W}_- = \left\{ \hat{f} = \hat{f}_1 + \hat{f}_2, \text{ where } \hat{f}_1 \text{ and } \hat{f}_2 \text{ belong to } \mathcal{A}_- \right\} \]

\( \mathcal{W} \) is the Wiener algebra (see Wiener and Hopf [16]), and \( \mathcal{W}_- \) is a subalgebra whose elements have holomorphic extensions in some vertical strip \( |\text{Re} \, s| < \mu \). \( \mathcal{A} \) and \( \mathcal{A}_- \) are subalgebras of stable transfer functions in \( \mathcal{W} \). These algebras are also closed under inversion, in the following sense.

**Lemma 2.2.**

(i) \( \hat{f} \in \mathcal{A}^{n \times n} \) (or \( \mathcal{A}_-^{n \times n} \)) is invertible over \( \mathcal{A}^{n \times n} \) (or \( \mathcal{A}_-^{n \times n} \)) if and only if

\[ \inf \{ |\det \hat{f}(s)| : s \in \mathbb{C}_+ \} > 0. \]  \hspace{1cm} (2.1)

(ii) \( \hat{f} \in \mathcal{H}_\infty^{n \times n} \) is invertible over \( \mathcal{H}_\infty^{n \times n} \) if and only if (2.1) holds.
(iii) $f \in \mathcal{H}^{n \times n}$ (or $\mathcal{H}_-^{n \times n}$) is invertible over $\mathcal{H}^{n \times n}$ (or $\mathcal{H}_-^{n \times n}$) if and only if

$$\det(f(j\omega)) \neq 0 \quad \text{for} \quad \omega \in \mathbb{R} \cup \{\infty\}.$$ 

**Proof.** See Callier and Desoer [2], Callier and Winkin [3], Wiener and Hopf [16], and Gohberg and Krein [8].

We now define certain factorizations of transfer matrices in $\mathcal{H}_-$ over $\mathcal{H}_-$. 

**Definition 2.3.** Suppose that $G \in \mathcal{H}_{-(1+m) \times (q+m)}$. A $J$-spectral factorization of $G$ is a factorization of the form

$$G^{-1}J_{lm}(\gamma)G = W^{-1}J_{qm}(\gamma)W$$

for $s \in j\mathbb{R}$, \quad (2.2)

for some $W$ and $W^{-1} \in \mathcal{H}_{-(q+m) \times (q+m)}$. 

In the case that $J_{qm}(\gamma)$ reduces to the identity, the factorization (2.2) is usually called a spectral factorization.

We could equally well have replaced $\mathcal{H}_-$ by $\mathcal{H}$ and $\mathcal{H}_-$ by $\mathcal{H}$ in the above definition, but in our applications the spectral factor $W(s) = D + C(sI - A)^{-1}B$, where $A$ is the infinitesimal generator of an exponentially stable $C_0$-semigroup and so is always in $\mathcal{H}_{(1+m) \times (q+m)}$. Similar remarks apply the following definition of the more general canonical Wiener-Hopf factorizations.

**Definition 2.4.** Consider $G \in \mathcal{H}_{-(m \times m)}$ with the limit $I_m$ at infinity. A canonical Wiener-Hopf factorization of $G$ is a factorization of the form

$$G = W_-W_+ \quad \text{on} \quad j\mathbb{R},$$

(2.3)

for some $W_+$ and $W_- \in \mathcal{H}_{m \times m}$, both having limits $I_m$ at infinity, and $W_+^{-1}, (W_-^{-1})^{-1}$ belonging to $\mathcal{H}_{m \times m}$.

It is known (Gohberg and Krein [8]) that if they exist, canonical Wiener-Hopf factorizations are unique. The following theorem gives explicit formulas for a canonical Wiener-Hopf factorization for a class of systems in $\mathcal{H}_{m \times m}$ with a state-space realization.

**Theorem 2.5** (Bart et al. [1]). Suppose that $\mathcal{A}$ is an exponentially dichotomous operator on the complex separable Hilbert space $\mathcal{X}$, $\mathcal{B} \in$
$\mathcal{L}(\mathbb{R}^n, \mathbb{L})$, and $\mathcal{C} \in \mathcal{L}(\mathbb{L}, \mathbb{R}^m)$. The transfer matrix $\mathcal{H}(s) = I + \mathcal{C}(sI - \mathcal{A})^{-1} \mathcal{B}$ has a canonical Wiener-Hopf factorization if and only if

$$\det\left[ I + \mathcal{C}(j\omega I - \mathcal{A})^{-1} \mathcal{B} \right] \neq 0 \quad \text{for } \omega \in \mathbb{R} \quad (2.4)$$

and

$$\mathcal{X} = \text{Ker} \Pi^* \oplus \text{Im} \Pi, \quad (2.5)$$

where $\Pi$ and $\Pi^*$ are the separating projections for $\mathcal{A}$ and $\mathcal{A}^* = \mathcal{A} - \mathcal{B} \mathcal{C}$, respectively. If $\tilde{\Pi}$ denotes the projection of $\mathcal{X}$ onto Ker $\Pi^*$ along $\text{Im} \Pi$ as in (2.5), then the unique canonical Wiener-Hopf factorization is given by (2.3), where

$$W_+(s) = I + \mathcal{C}\tilde{\Pi}(sI - \mathcal{A})^{-1} \mathcal{B}, \quad (2.6)$$

$$W_-(s) = I + \mathcal{C}(sI - \mathcal{A})^{-1}(I - \tilde{\Pi}) \mathcal{B}. \quad (2.7)$$

Furthermore, $\tilde{\Pi}$ maps $\mathcal{D}(\mathcal{A})$ into $\mathcal{D}(\mathcal{A})$, $\mathcal{A}^*$ is exponentially dichotomous, and $W_+$ and $W_- \in \mathcal{L}^{\infty \times m}$.

To understand this theorem we need to define the concept of exponentially dichotomous. It shows that Im $\Pi$ is closed.

**Definition 2.6.** Let $\mathcal{A}$ be a linear operator on the separable complex Hilbert space $\mathcal{X}$ with domain $D(\mathcal{A})$. We say that $\mathcal{A}$ is **exponentially dichotomous** if $\mathcal{A}$ is densely defined and $\mathcal{X}$ admits a topological direct-sum decomposition

$$\mathcal{X} = \mathcal{X}_- \oplus \mathcal{X}_+$$

such that this decomposition reduces $\mathcal{A}$, and $-\mathcal{A}|_{\mathcal{X}_-}$ and $\mathcal{A}|_{\mathcal{X}_+}$ are infinitesimal generators of exponentially stable $C_0$-semigroups on $\mathcal{X}_-$ and $\mathcal{X}_+$, respectively.

The projection of $\mathcal{X}$ onto $\mathcal{X}_-$ along $\mathcal{X}_+$ is called the **separating projection** for $\mathcal{A}$, and we denote it by $\Pi$.

We remark that a consequence of Definition 2.6 is that

$$\mathcal{A}: \text{Ker} \Pi \cap D(\mathcal{A}) \to \text{Ker} \Pi, \quad (2.8)$$

$$\mathcal{A}: \text{Im} \Pi \cap D(\mathcal{A}) \to \text{Im} \Pi. \quad (2.9)$$
If $\mathcal{A}$ is bounded, then $\mathcal{A}$ is exponentially dichotomous if and only if its spectrum does not meet the imaginary axis. Then $\Pi$ is the Riesz projection corresponding to the unstable part of the spectrum of $\mathcal{A}$. For unbounded operators it is more complicated (see Bart et al. [1]), but for our purposes the following characterization suffices.

**Lemma 2.7.** Let $A$ be the infinitesimal generator of an exponentially stable $C_0$-semigroup on the complex Hilbert space $\mathcal{H}$, and let $Q \in \mathcal{L}(\mathcal{H})$. Then

$$
\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix} \quad \text{and} \quad \mathcal{A}_Q = \begin{pmatrix} A & 0 \\ -Q & -A^* \end{pmatrix}
$$

are exponentially dichotomous with domain $D(A) \oplus D(A^*)$.

**Proof.** It is obvious that $\mathcal{A}$ is exponentially dichotomous. Recall that, since $A$ generates an exponentially stable $C_0$-semigroup, the following Lyapunov equation has a unique solution $N = N^* \in \mathcal{L}(\mathcal{H})$ (Curtain and Pritchard [6]):

$$NAx + A^*Nx + Qx = 0 \quad \text{for} \quad x \in D(A),$$

and furthermore, $N : D(A) \to D(A^*)$. Then

$$\mathcal{A}_Q = \begin{pmatrix} I & 0 \\ N & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix} \begin{pmatrix} I & 0 \\ -N & I \end{pmatrix}$$

shows that $\mathcal{A}_Q$ is similar to $\mathcal{A}$ and hence is also exponentially dichotomous. Note that

$$\begin{pmatrix} I & 0 \\ -N & I \end{pmatrix} : D(A) \oplus D(A^*) \to D(A) \oplus D(A^*).$$

We recall the definitions of stabilizability and detectability for infinite-dimensional systems.

**Definition 2.8.** Suppose that $A$ is the infinitesimal generator of the $C_0$-semigroup on the complex Hilbert space $\mathcal{H}$. Let $B \in \mathcal{L}(\mathbb{C}^m, \mathcal{H})$ and $C \in \mathcal{L}(\mathcal{H}, \mathbb{C}^p)$. We say that $(A, B)$ is **exponentially stabilizable** if there exists an $F \in \mathcal{L}(\mathcal{H}, \mathbb{C}^m)$ such that $A + BF$ generates an exponentially stable $C_0$-semigroup on $\mathcal{H}$. $(C, A)$ is **exponentially detectable** if there exists an
Finally, we introduce some notation concerning exponentially stabilizing solutions of Riccati equations of the following general form:

\[
A^* X x + X A x - (L + XB) R^{-1}(B^* X + L^*) x + S x = 0
\]

for \( x \in D(A) \), (2.10)

where \( A \) is the infinitesimal generator of a \( C_0 \)-semigroup \( T(t) \) on the separable complex Hilbert space \( \mathcal{Z} \), \( B \in \mathcal{L}(\mathbb{C}^m, \mathcal{Z}) \), \( S = S^* \in \mathcal{L}(\mathcal{Z}) \), \( R = R^* \in \mathcal{L}(\mathbb{C}^m) \), \( R^{-1} \in \mathcal{L}(\mathbb{C}^m) \), and \( L \in \mathcal{L}(\mathbb{C}^m, \mathcal{Z}) \). We do not assume that \( R \) is positive definite.

The Hamiltonian operator associated with (2.10) is

\[
\mathcal{H} = -BR^{-1} L^* - BR^{-1} B^* - S + LR^{-1} L^* - (A - BR^{-1} L^*)^* \quad (2.11)
\]

with domain \( D(\mathcal{H}) = D(A) \oplus D(A^*) \).

We say that \( \mathcal{H} \in \text{dom}(\text{Ric}) \) if there exists a self-adjoint operator \( X = X^* \in \mathcal{L}(\mathcal{Z}) \) which satisfies (2.10) and has the property that \( A - BR^{-1} L^* - BR^{-1} B^* X \) is the infinitesimal generator of an exponentially stable \( C_0 \)-semigroup. This solution is unique (see Lemma 8 in Weiss[15]), and we write \( X = \text{Ric}(\mathcal{H}) \) for this stabilizing solution.

3. THE CONTINUOUS-TIME CASE

Here we apply Theorem 2.5 on canonical Wiener-Hopf factorizations to obtain two results on necessary and sufficient conditions for the existence of spectral and \( J \)-spectral factorizations.

**Theorem 3.1.** Suppose that \( G(s) = D + C(sI - A)^{-1} B \), where \( A \) is the infinitesimal generator of an exponentially stable \( C_0 \)-semigroup on the separable complex Hilbert space \( \mathcal{Z} \), \( B \in \mathcal{L}(\mathbb{C}^{q+m}, \mathcal{Z}) \), \( C \in \mathcal{L}(\mathcal{Z}, \mathbb{C}^{l+m}) \), and \( D \in \mathcal{L}(\mathbb{C}^{q+m}, \mathbb{C}^{l+m}) \). Then \( G \) has a \( J \)-spectral factorization as in Definition 2.3 if and only if

(i) there exists a nonsingular constant matrix \( W_\alpha \) such that

\[
D^* J_{lm}(\gamma) D = W_\alpha^* J_{lm}(\gamma) W_\alpha \quad (3.1)
\]
\[ \mathcal{H} = \begin{pmatrix} A & 0 \\ -C^*JC & -A^* \end{pmatrix} - \begin{pmatrix} B \\ -C^*JD \end{pmatrix} (D^*JD)^{-1} \begin{pmatrix} D^*JC & B^* \end{pmatrix} \] (3.2)

and \( J := J_{lm}(\gamma) \).

In this case, \( W(s) \) satisfies (2.2) if and only if for some solution \( W_\infty \) of (3.1), \( W(s) \) satisfies

\[ W(s) = W_\infty + L(sI - A)^{-1}B, \] (3.3)

where \( L = [J_qm(\gamma)]^{-1}W_\infty^{-1}(D^*JC + B^*X) \), and \( X = \text{Ric}(\mathcal{H}) \).

**Proof.**

(a) **Sufficiency.** This can be verified by direct substitution, exactly as in the finite-dimensional case, noting that \( W(s)^{-1} = W_\infty^{-1} - W_\infty^{-1}L(sI - A + BW_\infty^{-1}L)^{-1}BW_\infty^{-1} \) and \( A - BW_\infty^{-1}L = A - B[D^*J_{lm}(\gamma)D]^{-1}(D^*JC + B^*X) \) generates an exponentially stable \( C_0 \)-semigroup, since \( \mathcal{H} \in \text{dom}(\text{Ric}) \).

(b) **Necessity.** (i): Suppose that there exists a \( J \)-spectral factorization (2.2). Then taking limits as \( |\omega| \to \infty \) (which is permissible, since \( A \) is exponentially stable and since \( W \in \mathcal{A}_{q+m} \times q+m \) possesses a well-defined limit at infinity), we obtain (3.1), where \( W_\infty \) is the limit of \( W(\cdot) \) at infinity.

(ii): It is readily verified that \( G^*JG \) has the following realization:

\[ G^*JG = D^*JD + (D^*JC \quad B^*) \begin{pmatrix} sI - \begin{pmatrix} A \\ -C^*JC \end{pmatrix} & 0 \\ -C^*JC & -A^* \end{pmatrix} \end{pmatrix}^{-1} \begin{pmatrix} B \\ -C^*JD \end{pmatrix}. \]

Define the following Hamiltonian operators:

\[ \mathcal{A} = \begin{pmatrix} A \\ -C^*JC \end{pmatrix} \]

and

\[ \mathcal{A}^\times = \mathcal{A} - \begin{pmatrix} B \\ -C^*JD \end{pmatrix} (D^*JD)^{-1} \begin{pmatrix} D^*JC & B^* \end{pmatrix}. \]

Consider the transfer matrix

\[ \mathcal{E}(s) = I + \mathcal{E}(sI - \mathcal{A})^{-1} \mathcal{A}, \]
where
\[
\mathcal{G} = \left[ J_{q_m}(\gamma) \right]^{-1} W_\infty^- (D^*JC \quad B^*) , \quad \mathcal{B} = \begin{pmatrix} B \\ -C^*JD \end{pmatrix} W_\infty^{-1} .
\]

It is readily verified that
\[
\mathcal{G} = \left[ J_{q_m}(\gamma) \right]^{-1} W_\infty^- G^- J_{q_m}(\gamma) G W_\infty^{-1}
\]
\[
= \left[ J_{q_m}(\gamma) \right]^{-1} W_\infty^- W^- J_{q_m}(\gamma) W W_\infty^{-1} ,
\]
since $G^- JG$ has a J-spectral factorization: $G^- JG = W^- J_{q_m}(\gamma) W$. So $\mathcal{G}$ has the canonical Wiener-Hopf factorization
\[
\mathcal{G} = W_- W_+ ,
\]
where $W_-$ and $W_+ := WW_\infty^{-1}$ belong to $\mathcal{A}_q^{(q+m)\times(q+m)}$ (here $W_- := [(J_{q_m}(\gamma))^{-1} W_\infty^- W^- J_{q_m}(\gamma)]^+$), and Theorem 2.5 applies. We shall obtain an explicit representation of $W_+(s)$ using (2.10), but first we analyze $\tilde{\Pi}$, the projection of $\mathcal{Z} = \mathcal{Z} \oplus \mathcal{Z} = \text{Ker} \Pi \oplus \text{Im} \Pi$ onto $\text{Ker} \Pi^\times$ along $\text{Im} \Pi$.

Recall that $\text{Im} \Pi$ corresponds to the unstable part of the spectrum of $\mathcal{A}$, and $\text{Ker} \Pi^\times$ corresponds to the stable part of the spectrum of $\mathcal{A}^\times$. Now
\[
\mathcal{A} = \begin{pmatrix} A \\ -C^*JC \\ -A^* \end{pmatrix} ,
\]
and with respect to these coordinates
\[
\Pi = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} .
\]

It is easy to see that with respect to these coordinates
\[
\tilde{\Pi} = \begin{pmatrix} \gamma & 0 \\ X & 0 \end{pmatrix} \quad \text{for some} \quad X, Y \in \mathcal{L}(\mathcal{Z}).
\]
and since $\Pi^2 = \Pi$, we see that
\[
\tilde{\Pi} = \begin{pmatrix} I & 0 \\ X & 0 \end{pmatrix}
\quad \text{for some} \quad X \in \mathcal{L}(\mathcal{X}),
\]
Moreover, since $\tilde{\Pi} : D(\mathcal{A}) \to D(\mathcal{A})$, we deduce that $X : D(A) \to D(A^*)$. Consider now
\[
\mathcal{A}^x \tilde{\Pi} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A - MD^*JC - MB^*X \\ F \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
\]
for
\[
\begin{pmatrix} x \\ y \end{pmatrix} \in D(A) \oplus D(A^*) = D(\mathcal{A}),
\]
where $M := B(D^*JD)^{-1}$ and $F := -A^*X - C^*JC + C^*JD(D^*JD)^{-1}(D^*JC + B^*X)$. $\mathcal{A}_s$ generates a $C_0$-semigroup of the form
\[
T_s(t) = \begin{pmatrix} T_+(t) & 0 \\ F(t) & I \end{pmatrix},
\]
where $T_+(t)$ is the $C_0$-semigroup with the infinitesimal generator $A_+ = A - B(D^*JD)^{-1}(D^*JC + B^*X)$ and $F(t)$ is bounded for all $t$.

Now $\tilde{\Pi}$ projects onto the stable part of the spectrum of $\mathcal{A}^x$, and since $\mathcal{A}^x$ is exponentially dichotomous, $\mathcal{A}^x |_{\ker \Pi^x}$ generates an exponentially stable $C_0$-semigroup $T^x(t)$ on $\ker \Pi^x$. Equation (3.4) shows that $T_+(t) = T^x(t)$ and so $A_+$ is the generator of an exponentially stable semigroup.

Next we show that $X$ is self-adjoint. From (3.4), we have
\[
\tilde{\Pi} \mathcal{A}^x \tilde{\Pi} z = \mathcal{A}^x \tilde{\Pi} z = \tilde{\Pi} \mathcal{A}_s^x z \quad \text{for} \quad z \in D(\mathcal{A}),
\]
where for the left equality we use the fact that $\mathcal{A}^x : \ker \Pi^x \cap D(\mathcal{A}) \to \ker \Pi^x$ for an exponentially dichotomous operator $\mathcal{A}^x$ with stable part of the spectrum $\ker \Pi^x$ [see (2.8)]. It is readily verified that $\mathcal{A}^x$ with
\[
\mathcal{Z} := \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},
\]
is self-adjoint, and so, premultiplying (3.5) by $\tilde{\Pi}^* \mathcal{A}$, we obtain

$$\tilde{\Pi}^* \mathcal{A} \tilde{\Pi} z = \tilde{\Pi}^* \mathcal{A} \tilde{\Pi} \mathcal{A} z \quad \text{for} \quad z \in D(\mathcal{A}).$$

This shows that the right-hand side is also self-adjoint, i.e.,

$$\mathcal{A}^* \tilde{\Pi}^* \mathcal{A} \tilde{\Pi} z = \tilde{\Pi}^* \mathcal{A} \tilde{\Pi} \mathcal{A} z \quad \text{for} \quad z \in D(\mathcal{A}),$$

and so

$$\mathcal{A}^* \begin{pmatrix} X - X^* & 0 \\ 0 & 0 \end{pmatrix} z = \begin{pmatrix} -X + X^* & 0 \\ 0 & 0 \end{pmatrix} \mathcal{A} z \quad \text{for} \quad z \in D(\mathcal{A}).$$

Writing this out, we obtain

$$-A^* (-X + X^*) z_1 = (-X + X^*) A^* z_1 \quad \text{for} \quad z_1 \in D(A), \quad (3.6)$$

where from the above $A^+$ generates an exponentially stable semigroup $T_+(t)$. Let $z_1 = T_+(t)z_0$, where $z_0 \in D(A)$, and take inner products with (3.6) to obtain

$$\langle T_+(t)z_0, A^* (-X + X^*) T_+(t)z_0 \rangle + \langle T_+(t)z_0, (-X + X^*) A^+ T_+(t)z_0 \rangle = 0$$

or equivalently

$$\frac{d}{dt} \langle T_+(t)z_0, (-X + X^*) T_+(t)z_0 \rangle = 0.$$

Integrating from 0 to $t$ gives

$$\langle T_+(t)z_0, (-X + X^*) T_+(t)z_0 \rangle = \langle z_0, (-X + X^*) z_0 \rangle,$$

and since $T_+(t)z_0 \to 0$ as $t \to \infty$, we see that $\langle z_0, (-X + X^*) z_0 \rangle = 0$ for all $z_0 \in D(A)$. But $D(A)$ is dense in $\mathcal{Z}$, and so $X = X^*$. To show that $X$ satisfies the Riccati equation associated with $\mathcal{A}^\times$, we use

$$(I - \tilde{\Pi}) \mathcal{A}^\times \tilde{\Pi} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \quad \text{for} \quad \begin{pmatrix} x \\ y \end{pmatrix} \in D(\mathcal{A}),$$
again using the fact that $\mathcal{A}^\times : \ker \Pi^\times \cap D(\mathcal{A}) \to \ker \Pi^\times$ [see (2.8)]. This yields the relevant Riccati equation, and so $\mathcal{A}^\times \in \text{dom}(\text{Ric})$ and $X \in \text{Ric}(\mathcal{A}^\times)$.

(c) To obtain the representation (3.3), we appeal to the representation (2.6) for $W(\cdot)$ in Theorem 2.5. This shows that

$$W_+(s) = I + \mathcal{C}_2 A(sI - A)^{-1}$$

$$= I + (\mathcal{C}_1 + \mathcal{C}_2 X)(sI - A)^{-1}$$

$$= I + [J_{qm}(\gamma)]^{-1} W_{\infty}^{-*} (D^* J+C + B^* X)(sI - A)^{-1} B W_{\infty}^{-1}$$

and

$$W(s) = W_+(s) W_{\infty}$$

agrees with (3.3).

As remarked earlier, this theorem includes the special case of spectral factorizations. In this case, this result can also be used to find an inner-outer factorization of $G(s) = G_i(s)G_o(s)$, with $G_o(s) = W(s)$ outer and $G_i(s) = G(s)W(s)^{-1}$ inner.

Our last main result in this section is to relate the nonnegative definiteness of $X = \text{Ric}(\mathcal{A})$ to the $J$-losslessness of $GW^{-1}$, but first we need two lemmas. The first lemma is an extension of Lemma 3.3 in Green [9].

**Lemma 3.2.** Suppose that $X \in \mathcal{A}^{(l+m)\times(q+m)}$. Then $X$ is $J$-lossless if and only if

$$X^* \text{Im}(\gamma) X = J_{qm}(\gamma) \quad \text{for } |\text{Re } s| < \epsilon$$

(3.7)

for some $\epsilon > 0$ and $X_{22}^{-1} \in \mathcal{A}_{-}^{m \times m}$.

**Proof.** Suppose that $X \in \mathcal{A}^{(l+m)\times(q+m)}$ is $J$-lossless. Then from Definition 2.1, we have

$$X_{12}^* X_{12} = \gamma X_{22}^* X_{22} \leq -\gamma^2 I \quad \text{in } \overline{C_+}.$$ 

Thus $|\text{det } X_{22}(s)| \geq 1$ for $s \in \overline{C_+}$, and by Lemma 2.2(i) $X_{22}^{-1} \in \mathcal{A}^{m \times m}$. Furthermore,

$$X(j\omega)^* \text{Im}(\gamma) X(j\omega) = J_{qm}(\gamma) \quad \text{for } \omega \in \mathbb{R}.$$
Define $f(\cdot s) := X^{-1}(s)J_{\text{im}}(\gamma)X(s) - J_{q_m}(\gamma)$. It is holomorphic on $|\Re s| < \mu$ for some $\mu > 0$, and $f(j\omega) = 0$ for $\omega \in \mathbb{R}$. So by [11, Theorem 8.1.3, p. 198], $f(s) = 0$ on $|\Re s| < \mu$.

Conversely, suppose that $X \in \mathcal{A}_{(q+m)}^{(l+m)}$ satisfies (3.7) and $X_{22}^{-1} \in \mathcal{A}_{m \times m}$. Introduce

$$Y_1 = \begin{pmatrix} X_{11} & X_{12} \\ 0 & \gamma I \end{pmatrix}, \quad Y_2 = \begin{pmatrix} I & 0 \\ \gamma X_{21} & \gamma X_{22} \end{pmatrix},$$

and $Y = Y_1Y_2^{-1} \in \mathcal{A}_{(l+m) \times (q+m)}$. Observe that $X^*JX - J = Y^*_2(Y^*Y - I)Y_2$, and that $Y_2^{-1} \in \mathcal{A}_{(l+m) \times (q+m)}$, since $X_{22}^{-1} \in \mathcal{A}_{m \times m}$. Using (3.7) we see that $Y^*Y = I$ for $s = j\omega$. Since $Y$ is continuous on $s = j\omega$ and is holomorphic in $\Re s > 0$, we may apply the maximum-modulus principle [11, p. 207] to conclude that $\|Y(s)\|_{\infty} \leq 1$. Thus $X^*(s)JX(s) - J = Y^*_2(s)[Y^*(s)Y(s) - I]Y_2(s) < 0$, and $X$ is $J$-lossless.

We shall use the following lemma.

**Lemma 3.3.** Let $A$ be the infinitesimal generator of a $C_0$-semigroup on a Hilbert space $\mathcal{Z}$, and let $C \in \mathcal{L}(\mathcal{Z}, Y)$, where $Y$ is another Hilbert space. If $(C, A)$ is exponentially detectable, then the Lyapunov equation

$$(3.8) \quad \langle z, A^*Xz \rangle + \langle z, XAz \rangle + \langle Cz, Cz \rangle = 0, \quad z \in D(A),$$

has a self-adjoint, nonnegative solution if and only if $A$ generates an exponentially stable semigroup.

**Proof.**

(a) The sufficiency is proved in Curtain and Rodman [7, Lemma 2.1].

(b) For necessity see Zabczyk [17, Lemma 3].

**Theorem 3.4.** Suppose that $G(s)$ is as in Theorem 3.1. Then there exists a $W \in \mathcal{A}_{(q+m) \times (q+m)}$ such that $W^{-1} \in \mathcal{A}_{(q+m) \times (q+m)}$ and $GW^{-1}$ is $J$-lossless if and only if

(i) there exists a nonsingular constant matrix $W_0$ such that (3.1) holds,

(ii) $\mathcal{R} \in \text{dom}(\text{Ric})$, where $\mathcal{R}$ is given by (3.2),

(iii) $X = \text{Ric}(\mathcal{R}) > 0$.

**Proof.** In Theorem 3.1 we showed that (i) and (ii) are necessary and sufficient for the existence of a $W$ and $W^{-1} \in \mathcal{A}_{(q+m) \times (q+m)}$ such that [using (2.2)]

$$(GW^{-1})^{-J}(GW^{-1}) = J_{q_m}(\gamma) \quad \text{on } j\mathbb{R}. \quad (3.9)$$
It remains to show (iii). Simple calculations verify that $GW^{-1}$ has the following realization:

$$Y := GW^{-1} = \overline{D} + \overline{C}(sI - \overline{A})^{-1} \overline{B},$$

where $\overline{A} = A - BW_{\infty}^{-1}L$, $\overline{B} = BW_{\infty}^{-1}$, $\overline{C} = C - \overline{D}L$, $\overline{D} = DW_{\infty}^{-1}$, and $L = \left[J_{q m}(\gamma)\right]^{-1}W_{\infty}^{\star}(D^{\star}J\overline{C} + B^{\star}X)$, $X = \text{Ric}(\mathscr{A})$. Thus

$$\overline{D}^{\star} \overline{J}\overline{D} = J_{q m}(\gamma),$$

(3.10)

$$\overline{D}^{\star} \overline{J}\overline{C} + \overline{B}^{\star}X = 0,$$

(3.11)

$$X\overline{A}z + \overline{A}^{\star}Xz + \overline{C}^{\star}J\overline{C}z = 0 \quad \text{for} \quad z \in D(A).$$

(3.12)

Applying Theorem 3.1 to $GW^{-1}$, we deduce that $\overline{A}$ generates an exponentially stable semigroup, and so $Y \in \mathscr{S}^{(l + m) \times (q + m)}$. By Lemma 3.2, $Y = GW^{-1}$ is $J$-lossless if and only if $Y_{22}$, $Y_{22}^{-1} \in \mathscr{S}^{m \times m}$ and $Y^{-1}JY = J_{q m}(\gamma)$ for $|\text{Re} \ s| < \varepsilon$. Equation (3.9) gives the latter for $s = j\omega$, and since $Y$ and $Y^{-1}$ are holomorphic on some strip $|\text{Re} \ s| < \varepsilon$, (3.9) extends to this strip (see Hille [11, Theorem 8.1.3, p. 198]). To examine $Y_{22}$, we partition $\overline{C}$, $\overline{B}$ and $D$ appropriately. $Y_{22} \in \mathscr{S}^{m \times m}$ since $Y \in \mathscr{S}^{(l + m) \times (q + m)}$, and

$$Y_{22}^{-1} = \overline{D}_{22}^{-1} - \overline{D}_{22}^{-1}\overline{C}_{2}\left(sI - \overline{A} + \overline{B}_{2}\overline{D}_{22}^{-1}\overline{C}_{2}\right)^{-1}\overline{B}_{2}\overline{D}_{22}^{-1} \in \mathscr{S}^{m \times m}$$

if and only if $\overline{D}_{22}$ is invertible and $\overline{A} - \overline{B}_{2}\overline{D}_{22}^{-1}\overline{C}_{2}$ generates an exponentially stable semigroup (Jacobson and Nett [12]). We now prove that $\overline{A} - \overline{B}_{2}\overline{D}_{22}^{-1}\overline{C}_{2}$ generates an exponentially stable $C_{0}$-semigroup if and only if $X > 0$.

From the $(2, 2)$ block of (3.10) we obtain

$$\overline{D}_{12}^{\star}\overline{D}_{12} - \gamma^{2}\overline{D}_{22}^{\star}\overline{D}_{22} = -\gamma^{2}I,$$

and so $\overline{D}_{22}$ is nonsingular and

$$\|\overline{D}_{12}^{\star}\overline{D}_{22}^{-1}\| < \gamma.$$

(3.13)

From the $(2, 1)$ block of (3.11) we obtain

$$\overline{D}_{12}^{\star}\overline{C}_{1} - \gamma^{2}\overline{D}_{22}^{\star}\overline{C}_{2} + \overline{B}_{2}^{\star}X = 0,$$

and substituting this in (3.12), we obtain

$$\overline{X}\overline{A}z + \overline{A}^{\star}Xz = -\overline{C}^{\star}R\overline{C}z \quad \text{for} \quad z \in D(A),$$

(3.14)
where \( \mathbf{A} = \mathbf{\bar{A}} - \mathbf{\bar{B}}_2 \mathbf{\bar{D}}^{-1}_{22} \mathbf{\bar{C}}_2 \),

\[
\mathbf{\bar{C}} = \begin{pmatrix} \mathbf{\bar{C}}_1 \\ \mathbf{\bar{C}}_2 \end{pmatrix}, \quad \text{and} \quad R = \begin{pmatrix} \mathbf{I} & -\mathbf{\bar{D}}_{12} \mathbf{\bar{D}}^{-1}_{22} \\ -\mathbf{\bar{D}}^{-1}_{22} \mathbf{\bar{D}}^*_{12} & \gamma^2 \mathbf{I} \end{pmatrix}.
\]

Notice that \( R > 0 \), since, from (3.13), \( \| \mathbf{\bar{D}}_{12} \mathbf{\bar{D}}^{-1}_{22} \| < \gamma \). Notice also that \( (R^{1/2} \mathbf{\bar{C}}, \mathbf{\bar{A}} - \mathbf{\bar{B}}_2 \mathbf{\bar{D}}^{-1}_{22} \mathbf{\bar{C}}_2) \) is exponentially detectable. Thus Lemma 3.3 shows that (3.14) has a solution \( X = X^* > 0 \) if and only if \( \mathbf{\bar{A}} - \mathbf{\bar{B}}_2 \mathbf{\bar{D}}^{-1}_{22} \mathbf{\bar{C}}_2 \) generates an exponentially stable semigroup. So we have shown that \( GW^{-1} \) is \( \mathbf{J} \)-lossless if and only if \( X = X^* > 0 \). 

### 4. THE DISCRETE-TIME CASE

In this section, we show how the discrete-time versions of Theorems 3.1 and 3.4 can be derived from the continuous-time case, using the familiar bilinear transformation \( z = (1 + s)/(1 - s) \) which transforms the open right half plane to the exterior of the unit disc.

For this section we will use the following notation and definitions. Let

\[ \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}, \quad \partial \mathbb{D} = \{ z \in \mathbb{C} : |z| = 1 \}, \quad \text{and} \quad \mathbb{D}_+ = \{ z \in \mathbb{C} : |z| \geq 1 \}. \]

Let \( M_\mathbf{z}(z) = [M(z)]^*, \ M_\mathbf{\bar{z}}^*(z) = [M(z)]^* \). Let \( \mathbf{L}_p^{\times q}(\partial \mathbb{D}) \) be the space of functions \( \mathbf{F} : (\partial \mathbb{D}) \rightarrow \mathbb{C}^{p \times q} \) for which \( \| \mathbf{F} \|_\mathbf{\infty} = \text{ess sup}_{0 < \theta < 2\pi} \mathbf{\sigma}(\mathbf{F}(e^{j\theta})) < \infty \). Let us denote by \( \mathbf{H}_p^{\times q}(\mathbb{D}_+) \) the space of bounded holomorphic functions \( \mathbf{F} : \mathbb{D}_+ \rightarrow \mathbb{C}^{p \times q} \) that satisfy \( \| \mathbf{F} \|_\mathbf{\infty} = \sup_{|z| > 1} \mathbf{\sigma}(\mathbf{F}(z)) < \infty \).

**Definition 4.1.** A partitioned matrix \( \mathbf{N} \in \mathbf{L}_e^{(l+m) \times (q+m)}(\partial \mathbb{D}) \) is \( \mathbf{J} \)-lossless if

\[
\mathbf{N}(z)^* J_{lm}(\gamma) \mathbf{N}(z) \preceq J_{qm}(\gamma) \quad \text{on} \quad \mathbb{D}_+
\]

and

\[
\mathbf{N}(e^{j\theta})^* J_{lm}(\gamma) \mathbf{N}(e^{j\theta}) = J_{qm}(\gamma) \quad \text{for} \quad \theta \in [0, 2\pi].
\]

Let us consider the following classes of functions:

\[
l_1 = \left\{ \mathbf{g} = (g_0, g_1, \ldots) \mid g_i \in \mathbb{C}, \sum_{n=0}^{\infty} |g_n| < \infty \right\}
\]
and

\[ l_{1-} := \left\{ g = (g_0, g_1, \ldots) \mid \exists \eta > 0 \Rightarrow \sum_{n=0}^{\infty} |g_n| \eta^{-n} < \infty \right\}. \]

Given a sequence, \( g \), in either \( l_1 \) or \( l_{1-} \), we define its \( z \)-transform by

\[ \tilde{g}(z) := \mathcal{Z}\{g\} = \sum_{n=0}^{\infty} g_n z^{-n} \]

whenever this series converges absolutely for some \( z \) in \( \mathbb{C} \).

Then we define \( \tilde{l}_1 := \{ \tilde{g} \mid g \in l_1 \} \) and \( \tilde{l}_{1-} := \{ \tilde{g} \mid g \in l_{1-} \} \), and \( \tilde{l}_{p \times q} \) denotes the class of \( p \times q \) matrices whose elements are all in \( \tilde{l}_{1-} \). We also define the discrete-time Wiener algebras

\[ \mathcal{W}_z = \left\{ \tilde{g} = \tilde{g}_1 + \tilde{g}_2 \mid \tilde{g}_1, \tilde{g}_2 \in \tilde{l}_1 \right\}, \]

and

\[ \mathcal{W}_{z-} = \left\{ \tilde{g} - \tilde{g}_1 + \tilde{g}_2 \mid \tilde{g}_1, \tilde{g}_2 \in \tilde{l}_{1-} \right\}. \]

The elements of \( \mathcal{W}_{z-} \) have holomorphic extensions in some annular region of the \( z \)-plane, \( a < |z| < b \), \( 0 < a < b \in \mathbb{R} \).

**Definition 4.2.** Suppose that \( G_d \in \mathcal{W}_{z}^{(l+m) \times (q+m)} \). A discrete-time \( J \)-spectral factorization for \( G_d \) is defined by

\[ G_d J_{lm} (\gamma) G_d = W_d J_{qm} (\gamma) W_d, \quad \text{for} \quad z = e^{j\theta}, \quad 0 \leq \theta \leq 2\pi. \quad (4.1) \]

and \( W_d, W_d^{-1} \in \tilde{l}_{(q+m) \times (q+m)} \).

The analogue of the exponential stability in discrete time is power stability.

**Definition 4.3.** Consider the discrete-time system \( z(k+1) = A_d z(k) \), where \( A_d \in \mathcal{L} (\mathcal{L}^r) \) and \( \mathcal{L}^r \) is a complex separable Hilbert space. We say that this system (or \( A_d \)) is power stable if there exist constants \( M \geq 1 \) and \( \gamma \in (0, 1) \) such that

\[ \| A_d^k \| \leq M \gamma^k \quad \forall k \in \mathbb{Z}. \]
Suppose that \( B_d \in \mathcal{L}(\mathbb{C}^n, \mathcal{Z}) \). Then \((A_d, B_d)\) is power stabilizable if there exists an \( F_d \in \mathcal{L}(\mathcal{Z}, \mathbb{C}^m) \) such that \( A_d + B_d F_d \) is power stable.

We now show that a power stable discrete-time operator \( A_d \in \mathcal{L}(\mathcal{Z}) \) is related by the Cayley transform to a bounded continuous-time operator \( A \in \mathcal{L}(\mathcal{Z}) \) which generates an exponentially stable \( C_0 \)-semigroup.

**Lemma 4.4.**

(a) For \( A_d \in \mathcal{L}(\mathcal{Z}) \) and power stable, where \( \mathcal{Z} \) is a complex separable Hilbert space, define the operator

\[
A := (I + A_d)^{-1} (A_d - I).
\]

Then \( A \in \mathcal{L}(\mathcal{Z}) \), its spectrum is contained in \( \text{Re } s \leq -\mu \) for some \( \mu > 0 \), and it is the infinitesimal generator of an exponentially stable strongly continuous semigroup.

(b) Let \( A \in \mathcal{L}(\mathcal{Z}) \) be the infinitesimal generator of an exponentially stable \( C_0 \)-semigroup. Then the operator

\[
A_d = (I + A)(I - A)^{-1}
\]

is in \( \mathcal{L}(\mathcal{Z}) \), and it is power stable.

**Proof.** (a): We examine the spectrum of \( A \) using

\[
(sI - A)^{-1} = \frac{1}{1 - s} \left( \frac{s + 1}{1 - s} I - A_d \right)^{-1} (I + A_d) \quad \text{for } s \neq 1.
\]

First, note that

\[
(I - A)^{-1} = \frac{1}{2} (I + A_d) \in \mathcal{L}(\mathcal{Z}).
\]

Since \( A_d \) is power stable, its spectral radius is less than or equal to \( \sqrt{1 - \epsilon} \) for some \( \epsilon > 0 \) (Przyluski [14]). Thus from (4.4) the spectrum of \( A \) is contained in

\[
\left| \frac{1 + s}{1 - s} \right| \leq \sqrt{1 - \epsilon} \quad \text{for some } \epsilon > 0.
\]
Substituting \( s = x + iy \), the latter is equivalent to
\[
\epsilon x^2 + x(4 + 2\epsilon) + \epsilon y^2 + \epsilon \leq 0 \quad (4.5)
\]
or
\[
\left[ x + \left( \frac{2}{\epsilon} - 1 \right) \right]^2 + y^2 \leq \frac{4}{\epsilon^2} (1 - \epsilon).
\]
Taking \( \epsilon = 0 \), we see that \( x = \text{Re} \ s \leq 0 \), and a more careful estimate for small, positive \( \epsilon \) shows that (4.5) implies that
\[
x = \text{Re} \ s \leq \frac{\epsilon - 2}{\epsilon} + \frac{2}{\epsilon} \sqrt{1 - \epsilon}
\]
\[
\sim -\frac{\epsilon}{4}
\]
to a second order estimate in \( \epsilon \). So the spectrum of \( A \) is contained in \( \text{Re} \ s \leq -\mu \) for some \( \mu > 0 \), and since \( A \) is bounded, this shows that it generates an exponentially stable semigroup.

(b): Since \( A \) is the generator of an exponentially stable semigroup, we know that the spectrum of \( A \) lies in \( \text{Re} \ s \leq -\epsilon \) for some \( \epsilon > 0 \) (Curtain and Pritchard [6]). Consider the spectrum of \( A_d \), using
\[
(zI - A_d)^{-1} = \frac{1}{z + 1} (I - A) \left( \frac{z - 1}{z + 1} I - A \right)^{-1} \quad \text{for } z \neq -1.
\]
First note that with \( z = -1 \) we have
\[
(-I - A_d)^{-1} = -\frac{1}{2} (I - A),
\]
and since \( A \) is bounded, \( -1 \) is in the resolvent set of \( A_d \). The inequality
\[
\text{Re} \left( \frac{z - 1}{z + 1} \right) \leq -\epsilon
\]
is equivalent to \( |z| \leq \sqrt{1 - \epsilon} \), and so the spectral radius of \( A_d \) is less than 1, and it is power stable. \( \blacksquare \)
Lemma 4.4 gives a one-to-one correspondence between discrete-time systems with power stable generators and continuous-time systems with bounded generators of exponentially stable semigroups. It is now straightforward to deduce the discrete-time results from continuous-time ones by relating the following two systems via the bilinear transformation \( z = (1 + s)/(1 - s) \), as has been done in finite dimensions (see Green [9] and Kondo and Hara [13]).

Let us consider the discrete-time system

\[
G_d(z) = D_d + C_d(zI - A_d)^{-1}B_d,
\]

where

\[
A_d \in \mathcal{L}(\mathbb{Z}), \quad B_d \in \mathcal{L}(\mathbb{C}^{q+m}, \mathbb{Z}), \\
C_d \in \mathcal{L}(\mathbb{Z}, \mathbb{C}^{l+m}), \quad D_d \in \mathcal{L}(\mathbb{C}^{q+m}, \mathbb{C}^{l+m}).
\]

We assume that \( A_d \) is power stable and define the continuous-time system

\[
G(s) = D + C(sI - A)^{-1}B,
\]

where \( A \in \mathcal{L}(\mathbb{Z}), \ B \in \mathcal{L}(\mathbb{C}^{q+m}, \mathbb{Z}), \ C \in \mathcal{L}(\mathbb{Z}, \mathbb{C}^{l+m}), \) and \( D \in \mathcal{L}(\mathbb{C}^{q+m}, \mathbb{C}^{l+m}) \) are defined by

\[
A = (I + A_d)^{-1}(A_d - I), \quad B = \sqrt{2}(I + A_d)^{-1}B_d, \\
C = \sqrt{2}C_d(I + A_d)^{-1}, \quad D = D_d - C_d(I + A_d)^{-1}B_d. \tag{4.8}
\]

By Lemma 4.4, \( A \in \mathcal{L}(\mathbb{Z}) \) generates an exponentially stable semigroup on \( \mathbb{Z} \). It is easily shown that

\[
G(s) = G_d\left(\frac{1 + s}{1 - s}\right), \quad G_d(z) = G\left(\frac{z - 1}{z + 1}\right) \tag{4.9}
\]

and

\[
A_d = (I + A)(I - A)^{-1}, \quad B_d = \sqrt{2}(I - A)^{-1}B, \\
C_d = \sqrt{2}C(I - A)^{-1}, \quad D_d = D + C(I - A)^{-1}B. \tag{4.10}
\]
Power stable discrete-time systems and exponentially stable continuous-time systems are in the algebras \( \mathcal{F}_1 \) and \( \mathcal{F}_\infty \), respectively, and a similar remark holds for the associated bilinearly transformed systems.

**Lemma 4.5.**

(a) Suppose that \( A_d \in \mathcal{L}(\mathcal{Z}) \) is power stable, \( B_d \in \mathcal{L}(\mathcal{C}^{q+m}, \mathcal{Z}) \), \( C_d \in \mathcal{L}(\mathcal{Z}, \mathcal{C}^{l+m}) \), and \( D_d \in \mathcal{L}(\mathcal{C}^{q+m}, \mathcal{C}^{l+m}) \). Then \( G_d(z) = D_d + C_d(zI - A_d)^{-1}B_d \in \bar{\mathcal{F}}_{1+q+m}^{(l+m)} \) and

\[
G(s) = G_d \left( \frac{1+s}{1-s} \right) \in \mathcal{F}_{1+q+m}^{(l+m)}.
\]

(b) Suppose that \( A \in \mathcal{L}(\mathcal{Z}) \) generates an exponentially stable \( C_0 \)-semigroup on \( \mathcal{Z} \), \( B \in \mathcal{L}(\mathcal{C}^{q+m}, \mathcal{Z}) \), \( C \in \mathcal{L}(\mathcal{Z}, \mathcal{C}^{l+m}) \), and \( D \in \mathcal{L}(\mathcal{C}^{q+m}, \mathcal{C}^{l+m}) \). Then \( G(s) = D + C(sI - A)^{-1}B \in \mathcal{F}_{1+q+m}^{(l+m)} \) and

\[
G_d(z) = G \left( \frac{z - 1}{z + 1} \right) \in \bar{\mathcal{F}}_{1+q+m}^{(l+m)}.
\]

**Proof.** (a): For \( G_d \) to be in \( \bar{\mathcal{F}}_{1+q+m}^{(l+m)} \) we must show that there exists a \( \eta > 0 \) such that \( \sum_{n=0}^{\infty} \|g_n\| \eta^{-n} < \infty \), where \( g_n = C_d A_n^d B_d \). Consider therefore

\[
\sum_{n=0}^{\infty} \|g_n\| \eta^{-n} = \sum_{n=0}^{\infty} \|C_d A_n^d B_d\| \eta^{-n} \\
\leq \sum_{n=0}^{\infty} \|C_d\| \|A_n^d\| \|B_d\| \eta^{-n} \\
\leq \sum_{n=0}^{\infty} \|C_d\| \|M\|^n \|B_d\| \eta^{-n} \\
< \infty,
\]

provided \( \eta > \gamma \).

From (4.9) we have that

\[
G_d \left( \frac{1+s}{1-s} \right) = D + C(sI - A)^{-1}B.
\]
where $A$, $B$, $C$, and $D$ are defined by (4.8). From Lemma 4.4 we have that $A$ is bounded and generates an exponentially stable $C_0$-semigroup.

We know that $G$ is the Laplace transform of $D\delta(t) + Ce^{At}B$ and so

$$
\int_0^\infty e^{\alpha t}\|Ce^{At}B\|\,dt \leq \int_0^\infty e^{\alpha t}\|CNe^{-\alpha t}\|\,dt < \infty \quad \text{for} \quad \alpha > \epsilon,
$$

since $A$ generates an exponentially stable $C_0$-semigroup (here $N$, $\alpha$, and $\epsilon$ are positive constants). Hence $G$ belongs to $\mathcal{A}^{(q+m)}_{(l+m)}$.

(b): From (a), since $A$ generates an exponentially stable $C_0$-semigroup, we have $G(s) \in \mathcal{A}^{(q+m)}_{(l+m)}$. Again, from (4.9) we have that

$$G_d(z) = G\left(\frac{z - 1}{z + 1}\right),$$

that is, $G_d(z) = D_d + C_d(zI - A_d)^{-1}B_d$, where $A_d$, $B_d$, $C_d$, and $D_d$ are defined by (4.10). Also, from Lemma 4.4 we have that $A_d$ is bounded and power stable. Thus, from part (a), $G_d \in \mathcal{A}^{(q+m)}_{(l+m)}$.

For systems related by a bilinear transformation we show that the concept of $J$-losslessness is equivalent.

**Lemma 4.6.** Let $G_d$ and $G$ be as in (4.6) and (4.7), respectively. Then $G$ is $J$-lossless according to Definition 2.1 if and only if $G_d$ is $J$-lossless according to Definition 4.1.

**Proof.** Suppose that $G$, as given by (4.7), is $J$-lossless according to Definition 2.1. Then

$$G(s)^* J_l m(\gamma) G(s) \leq J_q m(\gamma) \quad \text{on} \quad C^+$$

and

$$G(j\omega)^* J_l m(\gamma) G(j\omega) = J_q m(\gamma) \quad \text{on} \quad j\mathbb{R}.$$  

Applying the bilinear transformation $s = (z - 1)/(z + 1)$ in the latter expressions, we get, explicitly,

$$
\begin{bmatrix}
D^* + B^*\left(\frac{z - 1}{z + 1} I - A^*\right)^{-1} C^*
\end{bmatrix} J_l m
\begin{bmatrix}
D + C\left(\frac{z - 1}{z + 1} I - A\right)^{-1} B
\end{bmatrix} \leq J_q m.
$$
on $\mathbb{D}_+$, and
\[
\left[ D^* + B^* (\zeta I - A^*)^{-1} C^* \right] J_{im} \left[ D + C (\xi I - A)^{-\beta} \right] = J_{qm} \quad \text{on } \partial \mathbb{D},
\]
where $\zeta := (e^{j\theta} - 1)/(e^{j\theta} + 1)$. Next, substituting for $A$, $B$, $C$, and $D$ from (4.8) and performing the calculations, we end up with
\[
\left[ D^*_d + B^*_d (z I - A^*_d)^{-1} C^*_d \right] J_{im} \left[ D_d + C_d (z I - A_d)^{-1} B_d \right] = J_{qm} \quad \text{on } \mathbb{D}_+,
\]
and
\[
\left[ D^*_d + B^*_d (e^{-j\theta} I - A^*_d)^{-1} C^*_d \right] J_{im} \left[ D_d + C_d (e^{j\theta} I - A_d)^{-1} B_d \right] = J_{qm} \quad \text{on } \partial \mathbb{D}.
\]
Therefore we have
\[
G_d(z) J_{im}(\gamma) G_d(z) \leq J_{qm}(\gamma) \quad \text{on } \mathbb{D}_+.
\]
and
\[
G_d(e^{j\theta}) J_{im}(\gamma) G_d(e^{j\theta}) = J_{qm}(\gamma) \quad \text{on } \partial \mathbb{D}.
\]
Hence $G_d$ is J-lossless according to Definition 4.1.

For the converse, suppose $G_d$, given by (4.6), is J-lossless as in Definition 4.1. That is,
\[
G_d(z) J_{im}(\gamma) G_d(z) \leq J_{qm}(\gamma) \quad \text{on } \mathbb{D}_+.
\]
and
\[
G_d(e^{j\theta}) J_{im}(\gamma) G_d(e^{j\theta}) = J_{qm}(\gamma) \quad \text{on } \partial \mathbb{D}.
\]
Applying the inverse bilinear transformation $z = (1 + s)/(1 - s)$ to these expressions and reversing all the previous computations, using (4.10) where appropriate, we get that $G$ is J-lossless according to Definition 2.1.

The next lemma relates discrete- and continuous-time Riccati equations.
**Lemma 4.7.** Consider the following discrete-time Riccati equation on the Hilbert space $\mathcal{Z}$ with $A_d \in \mathcal{L}(\mathcal{Z})$ power stable:

$$\begin{align*}
A_d^*X_d - X + C_d^*JC_d - K^*\left(B_d^*XB_d + D_d^*JD_d\right)^{-1}K &= 0, \\
A_d^*X_d - X + C_d^*JC_d - K^*\left(B_d^*XB_d + D_d^*JD_d\right)^{-1}K &= 0,
\end{align*}
$$

where $K := B_d^*X_d + D_d^*JC_d$, and the continuous-time Riccati equation

$$\begin{align*}
XA + A^*X + C^*JC - (XB + C^*JD)(D^*JD)^{-1}(D^*JC + B^*X) &= 0,
\end{align*}
$$

where $A_d, B_d, C_d, D_d$ and $A, B, C, D$ are defined by (4.8) and (4.10), and $J = J_m(-\gamma)$. Then $X \in \mathcal{L}(\mathcal{Z})$ is a solution to (4.11) if and only if it is a solution to (4.12). Furthermore, $A_dX := A_d - B_d(D_d^*JD_d + B_d^*XB_d)^{-1}K$ is power stable if and only if $A_d := A - B(D_d^*JD)^{-1}(B^*X + D_d^*JC)$ is the infinitesimal generator of an exponentially stable strongly continuous semi-group.

**Proof.** Note that from Lemma 4.4, $A$ is bounded and so all the operators in (4.11) and (4.12) are bounded and the calculations are straightforward. First, we introduce an equivalent form for the continuous-time Riccati equation (4.12),

$$\begin{align*}
\begin{pmatrix}
XA + A^*X + C^*JC \\
D^*JC + B^*X
\end{pmatrix}
\begin{pmatrix}
X \\
D^*JD
\end{pmatrix}
= \begin{pmatrix}
L^* \\
V^*
\end{pmatrix}
J
\begin{pmatrix}
L \\
V
\end{pmatrix},
\end{align*}
$$

and for the discrete-time Riccati equation (4.11)

$$\begin{align*}
\begin{pmatrix}
A_d^* & C_d^* \\
B_d^* & D_d^*
\end{pmatrix}
\begin{pmatrix}
X & 0 \\
0 & J
\end{pmatrix}
\begin{pmatrix}
A_d & B_d \\
C_d & D_d
\end{pmatrix}
- \begin{pmatrix}
I & L_d^* \\
0 & V_d^*
\end{pmatrix}
\begin{pmatrix}
X & 0 \\
0 & J
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
L_d & V_d
\end{pmatrix},
\end{align*}
$$

where $I, I_d \in \mathcal{L}(\mathbb{C}^{q+m}, \mathcal{Z})$ and $V_d$, $V$, and $J = J^*$ are nonsingular $m \times m$ matrices.

The next step is to verify, using (4.8) and (4.10) when appropriate, that if

$$L_d = \sqrt{2}L(I - A)^{-1} \quad \text{and} \quad V_d = V + L(I - A)^{-1}B$$

and

$$L = \sqrt{2}L_d(I + A_d)^{-1} \quad \text{and} \quad V = V_d - L_d(I A_d)\dagger,$$

then (4.13) and (4.14) are equivalent.
The procedure is to verify sequentially the following, using results from
the preceding step where appropriate:

1. \((I - A^*) \times [(1,1) \text{ block of } (4.14)] \times (I - A)\) equals the \((1,1)\) block
   of \((4.13)\). Conversely, \((I + A^*) \times [(1,1) \text{ block of } (4.13)] \times (I + A)\)
   equals the \((1,1)\) block of \((4.14)\).

2. \((I - A^*) \times [(1,2) \text{ block of } (4.14)]\) equals the \((1,2)\) block of \((4.13)\).
   Conversely, \((I + A^*) \times [(1,2) \text{ block of } (4.13)]\) equals the \((1,2)\) block
   of \((4.14)\).

3. The \((2,2)\) block of \((4.14)\) equals the \((2,2)\) block of \((4.13)\), and vice
   versa.

Since \(A\) is exponentially stable, by Lemma 4.4, \((I - A)\) is boundedly
invertible and so \((4.13)\) holds if and only if \((4.14)\) holds. Thus we have shown
that \(X\) satisfies \((4.11)\) if and only if it satisfies \((4.12)\).

Let us rewrite \(A_{dX}\) and \(A_X\) in terms of \(L, L_d, V,\) and \(V_d\). Then

\[
A_{dX} = A_d - B_d V_d^{-1} L_d \quad \text{and} \quad A_X = A - BV^{-1}L. \tag{4.17}
\]

We calculate

\[
V_d L_d = \sqrt{2} V^{-1}L(I - A + BV^{-1}L)^{-1}. \tag{4.18}
\]

Then, using \((4.8), (4.10), (4.15),\) and \((4.18)\) we calculate

\[
I + A_{dX} = I + A_d - B_d V_d^{-1} L_d
\]

\[= 2(I - A + BV^{-1}L)^{-1}\]

and

\[
A_{dX} - I = A_d - I - B_d V_d^{-1} L_d
\]

\[= 2(I - A)^{-1}[A - BV^{-1}L(I - A + BV^{-1}L)^{-1}]
\]

\[= -2(I - A)^{-1}(I - A)(-A + BV^{-1}L)(I - A + BV^{-1}L)^{-1}
\]

\[= 2(A - BV^{-1}L)(I - A + BV^{-1}L)^{-1}.
\]
Thus
\[
( A_{dx} + I)^{-1}( A_{dx} - I) = ( A_{dx} - I)( A_{dx} + I)^{-1} = A - BV^{-1}L = A_x.
\]

Noting that in our situation $A_x$ and $A_{dx}$ are bounded, Lemma 4.4 completes the proof. \[\square\]

Now we show that the existence of J-spectral factorizations is preserved under the bilinear transformation mentioned above.

**Lemma 4.8.** Let $G_d$ and $G$ be as in (4.6) and (4.7), respectively. Then $G$ has a J-spectral factorization according to Definition 2.3 for a $W(s)$ satisfying (3.1) and (3.3) if and only if $G_d$ has a discrete-time one according to Definition 4.2 for a

\[
W_d(z) = U_d + L_d(zI - A_d)^{-1}B_d,
\]

where $U_d$ is a nonsingular constant matrix. Moreover,

\[
W_d(z) = W\left(\frac{z - 1}{z + 1}\right) \quad \text{and} \quad W(s) = W_d\left(\frac{1 + s}{1 - s}\right). \tag{4.20}
\]

**Proof.** Let us assume that $G$, given by (4.7), has a J spectral factorization according to Definition 2.3 for $W(s)$ satisfying (3.1) and (3.3). Then, explicitly,

\[
\begin{bmatrix} D* - B*(sI + A*)^{-1}C* \end{bmatrix} J_m(\gamma) \begin{bmatrix} D + C(sI - A)^{-1}B \\ \end{bmatrix} = \begin{bmatrix} W_\omega* - B*(sI + A*)^{-1}L* \end{bmatrix} J_q(\gamma) \begin{bmatrix} W_\omega + L(sI - A)^{-1}B \\ \end{bmatrix}
\]

for $s \in j\mathbb{R}$. Applying the bilinear transformation $s = (z - 1)/(z + 1)$ to this expression, and substituting for $A, B, C, D$ from (4.8), for $L$ from (4.16), and $W_\omega = U_d -$
\[ L_d(I + A_d)^{-1}B_d, \] one can obtain explicitly that
\[
\begin{bmatrix}
D^*_d + B^*_d \left( \frac{1}{z} I - A^*_d \right)^{-1} C^*_d \\
- L^*_d
\end{bmatrix}
J_m \left[ D_d + C_d(zI - A_d)^{-1}B_d \right]
\]
\[ = \begin{bmatrix} U^*_d + B^*_d \left( \frac{1}{z} I - A^*_d \right)^{-1} L^*_d \end{bmatrix} J_m \left[ U_d + L_d(zI - A_d)^{-1}(B_d) \right] \]
for \( z \in \partial \mathbb{D} \).

Hence
\[
G_d(z)^{-1} J_m(\gamma) G_d(z) = W_d(z)^{-1} J_m(\gamma) W_d(z)
\]
for \( z = e^{i\theta}, \ \theta \in [0, 2\pi] \), (4.21)

for \( W_d(z) := U_d + L_d(zI - A_d)^{-1}B_d \). Recall that \( A_d \) is bounded and power stable, so that from Lemma 4.5, \( W_d \in l^{(q+m) \times (q+m)} \).

Now, let us show that \( W_d^{-1} \) also belongs to \( l^{(q+m) \times (q+m)} \). To see this, first use (4.10) and (4.15) in \( W_d = U_d - L_d(zI - A_d)^{-1}B_d \), to get \( U_d = W_\infty + L(I - A)^{-1}B \). Then
\[
U_d^{-1} = W_\infty^{-1} - W_\infty^{-1}L(I - A + BW_\infty^{-1}L)^{-1}BW_\infty^{-1}
\]
exists if \( W_\infty^{-1} \) exists and \( I - A + BW_\infty^{-1}L \) is boundedly invertible. But from (3.1) we know that \( W_\infty \) is nonsingular. From Lemma 4.7 we have that
\[
A_X := A - B(D^*JD)^{-1}(B^*X + D^*JC)
\]
is the infinitesimal generator of an exponentially stable \( C_0 \)-semigroup. Replacing \( V \) by \( W_\infty \) in (4.13) we obtain
\[
D^*JC + B^*X = W_\infty^*JL
\]
and
\[
D^*JD = W_\infty^*fW_\infty.
\]
Thus
\[
A_X = A - B(W_\infty^*fW_\infty)^{-1}W_\infty^*JL
\]
\[ = A - BW_\infty^{-1}L. \]
Hence $A - BW_{\infty}^{-1}L$ generates an exponentially stable $C_0$-semigroup, and $I - A + BW_{\infty}^{-1}L$ is boundedly invertible. Therefore $U_d^{-1}$ exists.

Next, from (4.19) we compute the inverse of $W_d(z)$. That is,

$$W_d^{-1}(z) = U_d^{-1} - U_d^{-1}L_d\left(zI - A_dB_dU_d^{-1}L_d\right)^{-1}B_dU_d^{-1}.$$  

We already know that $U_d$ is nonsingular. Again, from Lemma 4.7 we have that

$$A_{dX} := A_d - B_d(D_d^*J_d + B_d^*XB_d)^{-1}K$$

is power stable. Then, replacing $V_d$ by $U_d$ in (4.14), we obtain

$$D_d^*JD_d + B_d^*XB_d = U_d^*JU_d$$

and

$$B_d^*XA_d + D_d^*JC_d = U_d^*JL_d.$$  

Thus

$$A_{dX} = A_dB_d(U_d^*JU_d)^{-1}U_d^*JL_d$$

$$= A_d - B_dU_d^{-1}L_d.$$  

Hence $A_d - B_dU_d^{-1}L_d$ is power stable. Using Lemma 4.5 we can conclude that $W_d^{-1} \in \tilde{H}^{a+m\times(l+m)}$. Thus (4.21) is a discrete-time J-spectral factorization for $G_d$ according to Definition 4.2.

Moreover, using (4.8), (4.10), (4.15), (4.16), and $U_d = W_{\infty} + L(I - A)^{-1}B$, it can be shown, by direct computations, that (4.20) holds.

Conversely, suppose that $G_d$, given by (4.61), has a discrete-time J-spectral factorization according to Definition 4.2 for $W_d$ given by (4.19). Then applying the inverse bilinear transformation $z = (1 + s)/(1 - s)$ to (4.1), with this particular $W_d$, we get explicitly

$$\left[D_d^* + B_d^*\left(\frac{1 - s}{1 + s}I - A_d^*\right)^{-1}C_d^*\right]J_{im}\left[D_d + C_d\left(\frac{1 + s}{1 - s}I - A_d\right)^{-1}B_d\right]$$

$$= \left[U_d^* + B_d^*\left(\frac{1 - s}{1 + s}I - A_d^*\right)^{-1}L_d^*\right]$$

$$\times J_{qm}\left[U_d + L_d\left(\frac{1 + s}{1 - s}I - A_d\right)^{-1}B_d\right]$$

for $s \in j\mathbb{R}$.  

Substituting for $A, B, C, D$ from (4.10), for $L_d$ from (4.15), and $U_d = W_\infty + L(I - A)^{-1}B$, one can get

$$[D^* - B^* (sI + A^*)^{-1}]J_{lm}(\gamma)[D + C(sI - A)^{-1}B]$$

$$= [W_*^* - B^* (sI + A^*)^{-1}L^*]J_{qm}(\gamma)[W_* + L(sI - A)^{-1}B]$$

for $s \in j\mathbb{R}$.

Hence

$$G^-(s)J_{lm}(\gamma)G(s) = W^-(s)J_{qm}(\gamma)W(s) \quad \text{for} \quad s \in j\mathbb{R},$$

where $W$ satisfies (3.3). Since $A$ generates an exponentially stable $C_0$-semigroup, we have that $W \in \mathcal{S}_{(q + m)\times (q + m)}$, according to Lemma 4.5.

Now to prove that $W^{-1} \in \mathcal{S}_{(q - m)\times (l + m)}$ we proceed as follows. Take the inverse of $W(s) = W_* + L(sI - A)^{-1}B$ to get

$$W^{-1}(s) = W_*^{-1} - W_*^{-1}L(sI - A + BW_*^{-1}L)^{-1}BW_*^{-1}.$$

$W_*$ is nonsingular, and we have already shown that $A - BW_*^{-1}L$ generates an exponentially stable $C_0$-semigroup. Then it follows from Lemma 4.5 that $W^{-1} \in \mathcal{S}_{(q + m)\times (l + m)}$.

Hence $G$ has a $J$-spectral factorization according to Definition 4.2, for this $W$. Again, it is straightforward to verify (4.20).

So we have shown that for these particular $G, W, G_d$, and $W_d$ the existence of a $J$-spectral factorization is invariant under the considered bilinear transformation.

We now prove the analogue of Theorem 3.1 on $J$-spectral factorizations.

**Theorem 4.9.** Suppose that $G_d(z) = D_d + C_d(zI - A_d)^{-1}B_d$ where $A_d \in \mathcal{L}(\mathcal{Z})$ is power stable, $\mathcal{Z}$ is a separable Hilbert space, $B_d \in \mathcal{L}(\mathbb{C}^{q + m}, \mathcal{Z}), C_d \in \mathcal{L}(\mathcal{Z}, \mathbb{C}^{l + m}),$ and $D_d \in \mathcal{L}(\mathbb{C}^{q + m}, \mathbb{C}^{l + m})$. There exists a discrete-time $J$-spectral factorization for $G_d$, as in Definition 4.2, if and only if

(i) there exists a nonsingular constant matrix $U_d$ such that

$$B_d^* X B_d + D_d^* J_{lm}(\gamma) D_d = U_d^* J_{qm}(\gamma) U_d$$

(4.22)
(ii) the Riccati equation (4.11) has a self-adjoint stabilizing solution $X$. In this case, $W_d(z)$ satisfies (4.1) if and only if for some solution $U_d$ of (4.22), $W_d(z)$ satisfies

$$W_d(z) = U_d + L_d(zI - A_d)^{-1}B_d$$  \hspace{1cm} (4.23)$$

where

$$L_d = [J_{qm}(\gamma)]^{-1}U_d^*\left[D_d^*J_{lm}(\gamma)C_d + B_d^*XA_d\right],$$

and $X = X^*$ is the solution to (4.11).

**Proof.** Sufficiency: Suppose $G_d$ has a discrete-time $J$-spectral factorization. Given the discrete-time transfer matrix $G_d(z) = D_d + C_d(zI - A_d)^{-1}B_d$ with $A_d$ bounded and power stable, and $B_d, C_d, D_d$ given as in the statement of the theorem, let us define the associated continuous-time system $G(s) = D + C(sI - A)^{-1}B$, where $A, B, C, D$ are defined by (4.8).

From Lemma 4.4, $A$ is bounded and generates an exponentially stable semigroup on $\mathcal{Z}$.

From Lemma 4.8 we know that for these $G$ and $G_d$, and $W$ and $W_d$ satisfying (3.3) and (4.23), respectively, the existence of a $J$-spectral factorization is maintained under the bilinear transformation. Moreover, the relationship between (3.3) and (4.23) is given by (4.20).

Thus we have a continuous-time system $G(s) = D + C(sI - A)^{-1}B$ which has a $J$-spectral factorization, and applying Theorem 3.1, we may conclude that (3.1) holds and $\mathcal{Z} \in \text{dom}(\text{Ric})$, where is given by (3.2). In other words, the continuous-time Riccati equation (4.12) has a self-adjoint stabilizing solution $X$. But Lemma 4.7 shows that this implies the existence of a self-adjoint stabilizing solution to the discrete-time Riccati equation (4.11). Moreover, that (3.1) and (4.22) are equivalent follows from Lemma 4.7 after substituting $W_d$ for $V$ in (4.13) and $U_d$ for $V_d$ in (4.14), respectively.

This proves the sufficiency, and the necessity can be proved by reversing the above arguments.

Equation (4.23) follows from its continuous-time analogue (3.3). $\blacksquare$

Now we do the same for the $J$-lossless property in the following theorem.

**Theorem 4.10.** Suppose $G_d(z)$ is as in Theorem 4.9. Then there exists a $W_d \in \mathcal{F}_{1+m}^{(q+m)\times(q+m)}$ such that $W_d^{-1} \in \mathcal{F}_{1+m}^{(q+m)\times(q+m)}$ and $G_dW_d^{-1}$ is $J$-lossless if
and only if

(i) there exists a nonsingular constant matrix $U_d$ such that (4.22) holds,
(ii) the Riccati equation (4.11) has a self-adjoint stabilizing solution $X$,
(iii) $X \geq 0$.

Proof. Suppose $G_d(z)$ is as in Theorem 4.9, and that there exists a $W \in \mathbb{R}^{(q + m) 	imes (q + m)}$ such that $W W^{-1} \in \mathbb{R}^{(q + m) 	imes (q + m)}$ and $G_d W^{-1}$ is $J$-lossless, where $W$ satisfies (4.23). (i) and (ii) have been established in Theorem 4.9, so we only need to establish (iii).

Then, simple calculations give that $G_d W^{-1}$ has the following realization:

$$Y_d(z) := G_d W^{-1} = D + C (zI - A)^{-1} B,$$

where

$$A = A_d - B_d U_d^{-1} L_d, \quad B = B_d U_d^{-1}, \quad C = C_d - D_d U_d^{-1} L_d, \quad D = D_d U_d^{-1},$$

and $L_d$ is given as in Theorem 4.9.

Let us define the associated continuous-time system

$$Y(s) = \bar{D} + \bar{C} (sI - \bar{A})^{-1} \bar{B},$$

where $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{D}$ are defined by

$$\bar{A} = (I + A)^{-1} (A - I), \quad \bar{B} = \sqrt{2} (I + A)^{-1} B,$$

$$\bar{C} = \sqrt{2} C (I + A)^{-1}, \quad \bar{D} = D - C (I + A)^{-1} B.$$ (4.25)

Using (4.9), it follows that

$$Y(s) = Y_d \left( \frac{1 + s}{1 - s} \right) \quad \text{and} \quad Y_d(z) = Y \left( \frac{z - 1}{z + 1} \right).$$

Moreover, using (4.8) and (4.13) in (4.24), we have that (1.25) can be expressed as follows:

$$\bar{A} - \Lambda - B W^{-1} L, \quad \bar{B} = B W^{-1},$$

$$\bar{C} = C - D L, \quad \bar{D} = D W^{-1},$$

and $L$ is given as in Theorem 3.1.
Substituting (4.26) in the expression for $Y(s)$, and factorizing properly, we get

$$Y(s) = \left[ D + C(sI - A)^{-1}B \right] \left[ W_x + L(sI - A)^{-1}B \right]^{-1}.$$ 

Hence we can write $Y = G(s)W^{-1}(s)$, with $G$ and $W$ as in Theorem 3.1. Theorem 3.4 tells us that $Y - GW^{-1}$ is $J$-lossless if and only if the stabilizing, self-adjoint solution to the continuous-time Riccati equation is nonnegative definite.

But from Lemma 4.7 and Theorem 4.9, the discrete-time Riccati equation (4.11) has a self-adjoint, nonnegative definite, stabilizing solution if and only if the continuous-time Riccati equation (4.12) has a self-adjoint, nonnegative definite, stabilizing solution, and these solutions are equal. Moreover, from Lemma 4.6 we have that $Y_d(z)$ is $J$-lossless if and only if $Y(s)$ is. This shows that $Y_d(z)$ is $J$-lossless if and only if the self-adjoint, stabilizing solution to the discrete-time Riccati equation is nonnegative definite.

The proof of Lemma 2.7 was communicated to the authors by both Professors Callier and Kaashoek. The secrets of the relationship between the discrete- and continuous-time $J$-spectral Riccati equations used in Lemma 4.7 were learned from Michael Green. We also thank the two reviewers for their careful reading of the manuscript and useful remarks.

REFERENCES


Received 15 December 1992; final manuscript accepted 3 August 1993