On the coefficients of Bazilevič functions and circularly symmetric functions

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ABSTRACT

In this work, we study the coefficients of Bazilevič functions and circularly symmetric functions, and obtain exact estimates.

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1. Introduction

Throughout this work, $\mathcal{A}$ denotes the class of analytic functions $f(z)$ in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ normalized such that $f(0) = 0$ and $f'(0) = 1$.

Let $\mathcal{S}$, $\mathcal{K}$, $\mathcal{S}^*$, $\mathcal{S}_c$ and $\mathcal{B}_\alpha$ denote the subclasses of $\mathcal{A}$ of functions that are univalent, convex, starlike, close-to-convex and Bazilevič, respectively. We also denote by $\mathcal{P}$ the class of analytic functions $p$ with $p(0) = 1$ and $\text{Re}\{p(z)\} > 0$ in $U$. Note that $\mathcal{P}$ is known as the Carathéodory class. Also, it is well known that the inclusion relations

$$\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{S}_c \subset \mathcal{B}_\alpha \subset \mathcal{S}$$

are valid. See [1–3] for further information.

Following [3,4], for $f(z) \in \mathcal{A}$, we see that $f(z) \in \mathcal{B}_\alpha$ if and only if

$$zf'(z) \left( \frac{f(z)}{g(z)} \right)^\alpha \in \mathcal{P}, \quad z \in U$$

for some $g(z) \in \mathcal{S}^*, \alpha \geq 0$.

Let $D$ be a domain in $\mathbb{C}$ with $0 \in D$. We shall say that $D$ is circularly symmetric if, for every $R$ with $0 < R < +\infty$, $D \cap \{ |z| = R \}$, is either empty, the whole circle $|z| = R$, or a single arc on $|z| = R$ which contains $z = R$ and is symmetric with respect to the real axis. Following [5,6], we shall denote by $Y$ the class of those functions $f(z)$ in $\mathcal{S}$ which map $U$ onto a circularly symmetric domain. The elements of $Y$ will be called circularly symmetric functions.

Associated with each $f(z)$ in $\mathcal{S}$ is a well defined logarithmic function

$$\log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n z^n, \quad z \in U. \quad (1.2)$$

The numbers $\gamma_n$ are called the logarithmic coefficients of $f$. Thus the Koebe function $k(z) = z(1-z)^{-2}$ has logarithmic coefficients $\gamma_n = \frac{1}{n}, n \in \mathbb{N} = \{1, 2, 3, \ldots\}$. 

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The inequality $|γ_n| \leq \frac{1}{n}$ holds for functions $f(z)$ in $S^*$, but is false for the full class $S$, even in order of magnitude. Indeed (see Theorem 8.4 on p. 242 of [1]) there exists a bounded function $f(z) \in S$ with logarithmic coefficients $γ_n \neq O(n^{-0.83})$.

In the paper [5], it is shown that the inequality $|γ_n| \leq \frac{1}{n}$ is false for functions $f(z)$ in $S_r$.

Let $f(z) \in S$, $ψ(z) = \left[\frac{1}{z}\right]^2 = 1 + \sum_{n=1}^{∞} D_n(γ)z^n$, $0 < γ < 1$. For the estimation of coefficients $|D_n(γ)|$, when $γ > \frac{1}{4}$, Milin (see [7]) gives a sharp estimate. But when $0 < γ < \frac{1}{4}$, we only obtain $|D_n(γ)| = O(n^{-1/2} \log n)$. Estimating the sharp order of $|D_n(γ)|$ is a difficult problem which is still unresolved. In this work, for when $f(z)$ is a Bazilevič function $B_γ$, we study the coefficients $|D_n|$ and obtain an exact estimate for the order of $|D_n(γ)|$. Furthermore, by using the Milin–Lebejew method, we investigate the coefficients $|D_n|$ and obtain a sharp estimate for the order of $|D_n(γ)|$ when $f(z) \in Y$. Throughout the work, let $A$ denote the absolute constant, whose value varies in different places.

2. On the coefficients of Bazilevič functions

We use the symbols $[f]_n$ to denote the coefficients of $z^n$ in the Taylor expansion of a function $f(z)$. In order to obtain Theorem 1, we need the following lemmas.

**Lemma 1** ([2]). Let $f(z) \in S$. Then, for $z = re^{iθ}$, $\frac{1}{2} \leq r < 1$,

$$\int_0^{2π} \frac{|f'(z)|^2}{f(z)} \, dθ ≤ A(1 - r)^{-1} \log \frac{1}{1 - r}. \quad (2.1)$$

**Lemma 2** ([8]). Let $p(z) \in P$. Then, for $z = re^{iθ}$, $\frac{1}{2} \leq r < 1$,

$$\int_0^{2π} \left| \frac{p'(z)}{p(z)} \right| \, dθ ≤ 4 \log \frac{1 + r}{1 - r}. \quad (2.2)$$

**Lemma 3** ([2]). Let $f(z) \in S$. Then, for $z = re^{iθ}$, $0 < r < 1$,

$$|f(z)| ≤ \frac{r}{(1 - r)^2}. \quad (2.3)$$

**Theorem 1.** Let $f(z) \in B_γ$, $ψ(z) = \left[\frac{1}{z}\right]^2 = 1 + \sum_{n=1}^{∞} D_n(γ)z^n$, $0 < γ < 1$. Then

$$|D_n| ≤ An^{2γ - 1}(\log n)^{\frac{3}{2}}. \quad (2.4)$$

The exponent $2γ - 1$ is the best possible.

**Proof.** Assume without loss of generality that $|f(r)| = \max_{|z|=r} |f(z)|$; otherwise we consider the function $e^{-\varrho_0}f(re^{\varrho_0})$. It is clear that

$$nD_n = \left\{ z \left[ \left( \frac{f(z)}{z} \right)^{1-1} \right] \right\}_n = \{ λ[f(z)]^{1-1}f'(z)z^{2-λ} - λ[f(z)]^{1-λ}z^{-λ} \}_n \quad (2.5)$$

where $h_1(z) = [f(z)]^{1-1}f'(z)z^{2-λ}$, $h_2(z) = [f(z)]^{1-λ}z^{-λ}$.

By means of Lemma 3, we get that, for $r < 1$,

$$|h_2(z)|_n ≤ \frac{r^n}{2π} \int_0^{2π} |f(z)|^{1-γ} |z|^{-λ} \, dθ ≤ \frac{r^n}{2π} \frac{1}{(1 - r)^{2λ}} \cdot 2π = r^n(1 - r)^{-2λ}. \quad (2.6)$$

Let $r = 1 - \frac{1}{n}$; we obtain from (2.6) that, for $n = 2, 3, \ldots$,

$$|h_2(z)|_n ≤ An^{2λ}. \quad (2.7)$$

A short calculation shows that

$$zh'_1(z) = h_1(z) \left[ (λ - 1) \frac{f''(z)}{f(z)} + \frac{zf''(z)}{f(z)} + (1 - λ) \right]. \quad (2.8)$$

Because $f(z) \in B_γ$, there exists a starlike function $g(z) \in S^*$ such that $Re\left\{ \frac{f'(z)}{f(z)} \left( \frac{f(z)}{g(z)} \right)^α \right\} > 0$, $α ≥ 0$. Write $p(z) = \frac{f'(z)}{f(z)} \left( \frac{f(z)}{g(z)} \right)^α$; then

$$\frac{zf''(z)}{f'(z)} = \frac{zp'(z)}{p(z)} - (α - 1) \frac{zf'(z)}{f(z)} + α \frac{zp''(z)}{p(z)}. \quad (2.9)$$
By applying (2.9), from (2.8) we obtain

\[ zh'_1(z) = (\lambda - \alpha) \frac{zf'(z)}{f(z)} h_1(z) + \frac{zp'(z)}{p(z)} h_1(z) + \alpha \frac{zg'(z)}{g(z)} h_1(z) - \lambda h_1(z) \]  

(2.10)

By means of Lemmas 1 and 3, we obtain

\[ \sum_{m=0}^{n} |[h_1(z)]_m|^2 \leq r^{-2n} \sum_{m=0}^{\infty} |[h_1(z)]_m|^2 r^{2m} \leq r^{-2n} \int_{0}^{2\pi} |h_1(z)|^2 d\theta \]

\[ = \frac{r^{-2n+2-2\lambda}}{2\pi} \int_{0}^{2\pi} |f(z)|^2 |\frac{f'(z)}{f(z)}|^2 d\theta \leq Ar^{-2n+2}(1 - r)^{-2\lambda - 1} \log \frac{1}{1 - r}. \]

(2.11)

Let \( r = 1 - \frac{1}{n} \), we obtain from (2.11) that, for \( n = 2, 3, \ldots \),

\[ \sum_{m=0}^{n} |[h_1(z)]_m|^2 \leq An^{4\lambda + 1} \log n. \]

(2.12)

From (2.12) we obtain

\[ |[h_1(z)]_n| \leq \frac{n}{n} \left( \sum_{m=0}^{n} |[h_1(z)]_m|^2 \right)^{\frac{1}{2}} \leq An^{2\lambda + 1}(\log n)^{\frac{1}{2}}. \]

(2.13)

By means of Lemma 2, we obtain

\[ \left\{ \frac{zp'(z)}{p(z)} \right\}_n \leq \frac{r^{-n}}{2\pi} \int_{0}^{2\pi} \left| \frac{zp'(z)}{p(z)} \right| d\theta \leq Ar^{-n+1} \log \frac{1}{1 - r}. \]

(2.14)

Let \( r = 1 - \frac{1}{n} \); we obtain from (2.14) that, for \( n = 2, 3, \ldots \),

\[ \left\{ \frac{zp'(z)}{p(z)} \right\}_n \leq A \log n. \]

(2.15)

We obtain from (2.12) and (2.15) that

\[ \left\{ \frac{zp'(z)}{p(z)} h_1(z) \right\}_n = \left\{ \sum_{m=0}^{n} |[h_1(z)]_m| \left\{ \frac{zp'(z)}{p(z)} \right\}_{n-m} \right\}_n \leq A(\log n) \sum_{m=0}^{n} |[h_1(z)]_m| \]

\[ \leq A(\log n)n^{\frac{1}{2}} \left( \sum_{m=0}^{n} |[h_1(z)]_m|^2 \right)^{\frac{1}{2}} \leq An^{2\lambda + 1}(\log n)^{\frac{3}{2}}. \]

(2.16)

By means of Lemmas 1 and 3 we obtain that

\[ \left\{ \frac{zf'(z)}{f(z)} h_1(z) \right\}_n \leq \frac{r^{-n}}{2\pi} \int_{0}^{2\pi} \left| \frac{zf'(z)}{f(z)} \right| |h_1(z)| d\theta \]

\[ = \frac{r^{-n}}{2\pi} \int_{0}^{2\pi} |z|^{2-\lambda} |\frac{f'(z)}{f(z)}|^2 |f(z)|^2 d\theta \leq Ar^{-n+2}(1 - r)^{-2\lambda - 1} \log \frac{1}{1 - r}. \]

(2.17)

Let \( r = 1 - \frac{1}{n} \); we obtain from (2.17) that, for \( n = 2, 3, \ldots \),

\[ \left\{ \frac{zf'(z)}{f(z)} h_1(z) \right\}_n \leq An^{2\lambda + 1} \log n. \]

(2.18)

Because \( g(z) \in S^* \subset S \), we get from (2.18) that

\[ \left\{ \frac{zg'(z)}{g(z)} h_1(z) \right\}_n \leq An^{2\lambda + 1} \log n. \]

(2.19)

Combining (2.13), (2.16), (2.18) and (2.19), we obtain from (2.10) that, for \( n = 2, 3, \ldots \),

\[ n|[h_1(z)]_n| = |[zh'_1(z)]_n| \leq An^{2\lambda + 1}(\log n)^{\frac{3}{2}}. \]

(2.20)
Then
\[ |h_n(z)| \leq A n^{2\lambda} (\log n)^{\frac{3}{2}}. \tag{2.21} \]
Combining (2.7) and (2.21), we obtain from (2.5) that, for \( n = 2, 3, \ldots, \)
\[ n|D_n| \leq A n^{2\lambda} (\log n)^{\frac{3}{2}}, \tag{2.22} \]
which implies that, for \( n = 2, 3, \ldots, \)
\[ |D_n| \leq A n^{2\lambda-1} (\log n)^{\frac{3}{2}}. \tag{2.23} \]
The Koebe function \( k(z) = z(1 - z)^{-2} \) implies that the exponent \( 2\lambda - 1 \) is the best possible.

Since \( S_c \subset B_{\alpha} \), we obtain the following corollary:

**Corollary 1.** Let \( f(z) \in S_c, \psi(z) = \left[ \frac{f(z)}{z} \right]^\lambda = 1 + \sum_{n=1}^{\infty} D_n(\lambda)z^n, \) \( 0 < \lambda < 1. \) Then
\[ |D_n| \leq A n^{2\lambda-1} (\log n)^{\frac{3}{2}}. \tag{2.24} \]
The exponent \( 2\lambda - 1 \) is the best possible.

### 3. On the coefficients of circularly symmetric functions

In this section, by using the Milin–Lebejev method, we shall study the coefficients \( |D_n| \) when \( f(z) \in Y. \) In order to obtain **Theorem 2**, we need the following lemma.

**Lemma 4** ([9]). Let \( f(z) \in Y. \) Then, for \( n \geq 2, \)
\[ |\gamma_n| \leq A n^{-1} \log n, \tag{3.1} \]
where the exponent \(-1\) is the best possible.

**Theorem 2.** Let \( f(z) \in Y, \psi(z) = \left[ \frac{f(z)}{z} \right]^\lambda = 1 + \sum_{n=1}^{\infty} D_n(\lambda)z^n, \) \( 0 < \lambda < 1. \) Then
\[ |D_n| \leq A n^{2\lambda-1} \log n. \tag{3.2} \]
The exponent \( 2\lambda - 1 \) is the best possible.

**Proof.** We define \( d_k(2\lambda) \) by the expansion
\[ \frac{1}{(1 - z)^{2\lambda}} = \sum_{k=0}^{\infty} d_k(2\lambda)z^k, \quad |z| < 1. \tag{3.3} \]
In the paper [7], it is shown that
\[ \sum_{k=0}^{n-1} d_k(2\lambda) = d_n(2\lambda + 1) \leq A n^{2\lambda}. \tag{3.4} \]
It is clear that
\[ z\psi'(z) = z\frac{\psi'(z)}{\psi(z)} \psi(z). \tag{3.5} \]
So
\[ \sum_{k=1}^{\infty} kD_kz^k = \sum_{k=1}^{\infty} 2\lambda k\gamma_k z^k \sum_{k=0}^{\infty} D_kz^k, \quad D_0 = 1. \tag{3.6} \]
Comparing the coefficients of the same powers of \( z \) in (3.6) and applying Lemma 4, we obtain
\[ |nD_n| = \left| 2\lambda \sum_{k=1}^{n} k\gamma_k D_{n-k} \right| \leq A \log n \sum_{k=0}^{n-1} |D_k|. \tag{3.7} \]
Applying Schwarz’s inequality, we obtain from (3.7) that
\[ |D_n|^2 \leq \frac{A}{n^2} \left( \log n \right)^2 \left[ \sum_{k=0}^{n-1} \frac{|D_k|^2}{d_k(2\lambda)} \cdot \sum_{k=0}^{n-1} d_k(2\lambda) \right]. \] (3.8)

It is well known that (see [7])
\[ n^2 - k = \sum_{l=1}^{k} |\gamma_l|^2 - \sum_{l=1}^{k} \frac{1}{l} < 0.312. \] (3.10)

Combining (3.4) and (3.10), we obtain from (3.9) that
\[ \sum_{k=0}^{n-1} \frac{|D_k|^2}{d_k(2\lambda)} \leq Ad_n(2\lambda + 1) \leq An^{2\lambda}. \] (3.11)

Combining (3.4) and (3.11), we obtain from (3.8) that
\[ |D_n|^2 \leq An^{4\lambda - 2}(\log n)^2. \] (3.12)

which implies that
\[ |D_n| \leq An^{2\lambda - 1} \log n. \] (3.13)

The Koebefunction \( k(z) = z(1 - z)^{-2} \) implies that the exponent \( 2\lambda - 1 \) is the best possible.

Since \( \mathcal{K} \subset S^* \), we have:

**Corollary 2.** Let \( f(z) \in \mathcal{S}^* \), \( \psi(z) = \left[ \frac{f^{(2)}(z)}{2} \right]^\lambda = 1 + \sum_{n=1}^{\infty} D_n(\lambda)z^n, 0 < \lambda < 1. \) Then
\[ |D_n| \leq An^{2\lambda - 1}. \] (3.14)

The exponent \( 2\lambda - 1 \) is the best possible.

Similarly, by using the Milin–Lebejev method we obtain:

**Corollary 3.** Let \( f(z) \in \mathcal{K}, \psi(z) = \left[ \frac{f^{(2)}(z)}{2} \right]^\lambda = 1 + \sum_{n=1}^{\infty} D_n(\lambda)z^n, 0 < \lambda < 1. \) Then
\[ |D_n| \leq An^{2\lambda - 1}. \] (3.15)

The exponent \( 2\lambda - 1 \) is the best possible.

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