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An L^2-Index Theorem for Dirac Operators on $S^1 \times \mathbb{R}^3$

Tom M. W. Nye and Michael A. Singer

Department of Mathematics and Statistics, University of Edinburgh, James Clerk Maxwell Building, The King's Buildings, Edinburgh EH9 3JZ, Scotland
E-mail: tmwn@maths.ed.ac.uk, michael@maths.ed.ac.uk

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An expression is found for the L^2-index of a Dirac operator coupled to a connection on a U_n vector bundle over $S^1 \times \mathbb{R}^3$. Boundary conditions for the connection are given which ensure the coupled Dirac operator Fredholm. Callias' index theorem

theorem to this special case. © 2000 Academic Press

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1. INTRODUCTION

Let X be a compact, oriented smooth manifold with boundary ∂X, and let $X^o = X \backslash \partial X$ be the corresponding open manifold. Let g be a complete Riemannian metric on X^o and let $E \to X^o$ be a complex vector bundle with a hermitian structure and unitary connection A. If X is a spin-manifold, we can introduce the coupled Dirac operator

$$D_A\colon C^\infty(X^o, S \otimes E) \to C^\infty(X^o, S \otimes E)$$

and it is natural to attempt to obtain first, conditions on A and g near ∂X which ensure that D_A is a Fredholm operator on L^2, and second, to obtain a formula for the L^2-index. Since with standard conventions D_A is self-adjoint, we must explain how one obtains interesting L^2-index problems from it.

If dim X is *even*, then the total spin-bundle decomposes as $S = S^+ \oplus S^-$ and so D_A gives rise to the "chiral" Dirac operators

$$D_A^\pm\colon C^\infty(X^o, S^\pm \otimes E) \to C^\infty(X^o, S^\mp \otimes E)$$

which have equal and opposite L^2-indices. If the geometry near ∂X is restricted so that g is a *b-metric*, then the celebrated index theorem of

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Atiyah, Patodi and Singer [3] gives a formula for the L^2-index of D_A^+ when this operator is Fredholm. (The notion of b-metric was not used explicitly in [3]; they worked with the equivalent idea of X^o having cylindrical ends, a simple condition on g near ∂X. The APS index theorem is discussed from the point of view of b-metrics in [11].) The APS theorem expresses the L^2-index of D_A^+ as a sum of two terms, one an integral of characteristic classes over X and a boundary contribution involving the famous η-invariant of ∂X.

By contrast, when dim X is *odd*, an interesting index-problem arises for operators of the form

$$D_{A,\Phi}\colon D_A + 1\otimes\Phi : C^\infty(X^o, S\otimes E) \to C^\infty(X^o, S\otimes E), \tag{1}$$

where Φ is a suitable skew-adjoint endomorphism of E. According to work of Callias, Anghel and Råde [2, 4, 13], $D_{A,\Phi}$ is Fredholm in L^2 under mild conditions on Φ, the most important being that it be invertible on ∂X. Then, with no further restrictions on the geometry of g near ∂X, the index is given by integrating over ∂X a certain characteristic class constructed from $\Phi \mid \partial X$ and $E\otimes S \mid \partial X$. We shall refer to this as the CAR index theorem in this paper, even though the result is closely related to pre-existing index theorems (cf. [13] for a discussion of this point).

The purpose of this note is to state and prove an index formula for D_A^+ when X is even-dimensional, but the geometry near the boundary is not that of a b-metric. We shall restrict ourselves to a very simple special case: we take (X^o, g) to be isometric to $S^1\times\mathbb{R}^3$, with a standard flat product metric. Then X is diffeomorphic to $S^1\times\bar{B}^3$ where $\bar{B}^3$ is the closed unit ball in $\mathbb{R}^3$. Despite its simplicity, this example already leads to an interesting index theorem, thereby answering a question posed by Mazzeo and Melrose in their study of Φ-pseudodifferential operators [10], at least in this very special case. (This Φ stands for "fibred-cusp" and has nothing to do with the Φ in (1). This notational clash is unfortunate but seems unlikely to lead to serious confusion.)

This index theorem is also important in the study of self-dual calorons (periodic instantons) of which more will be said in Section 1.2.

1.1. *Statement of Results*

The following notation will be used throughout this paper: let $X = S^1\times\bar{B}^3$, $S^2_\infty = \partial\bar{B}^3$, so that $\partial X = S^1\times S^2_\infty$; let $p\colon X\to\bar{B}^3$ be the projection. Let the metric g on X^o be the standard flat product metric on $S^1\times\mathbb{R}^3$ giving the circle a length $2\pi/\mu_0$, where $\mu_0>0$. Let $z\in\mathbb{R}/(2\pi/\mu_0)\mathbb{Z}$ be a standard coordinate on the circle, and x_1, x_2, x_3 standard coordinates on $\mathbb{R}^3$. Finally, fix an orientation on X by decreeing that the ordered basis (dz, dx_1, dx_2, dx_3) be positive.

Let $\mathbb{E} \to X$ be a smooth U_n-bundle and let $\mathbb{A}$ be a smooth unitary connection in $\mathbb{E}$. $\mathbb{A}$ will be identified with the corresponding covariant derivative operator ∇, which has components ∇_z, ∇_1, ∇_2, ∇_3 in the frame $(\partial_z, \partial_1, \partial_2, \partial_3)$. Each of the two spin-bundles $S^{\pm}$ over X^o can be identified with $p^*S_{(3)}$, where $S_{(3)}$ is the spin-bundle of $\mathbb{R}^3$. This is a complex vector bundle of rank 2 and comes equipped with skew-adjoint Clifford multiplication operators e_1, e_2, e_3 associated with ∂_1, ∂_2, ∂_3. The two coupled Dirac operators over X^o can now be written

$$D^{\pm}_{\mathbb{A}} = \pm\nabla_z + D_A : C^{\infty}(X^o, p^*S_{(3)} \otimes \mathbb{E}) \to C^{\infty}(X^o, p^*S_{(3)} \otimes \mathbb{E}), \qquad (2)$$

where $D_A = \sum_1^3 e_j \nabla_j$. The first term in (2) operates on sections of the tensor product by $\nabla_z(p^*(s) \otimes u) = p^*(s) \otimes \nabla_z u$ for any $s \in C^{\infty}(\mathbb{R}^3, S_{(3)})$ and $u \in C^{\infty}(X^o, \mathbb{E})$.

Note that the "3 + 1" decomposition of $D^{\pm}_{\mathbb{A}}$ in (2) depends only upon the product structure of X; we introduced bases only for ease of presentation. We now focus on $D^{+}_{\mathbb{A}}$; by abuse of notation, denote by the same symbol the extension of $D^{+}_{\mathbb{A}}$ to Sobolev spaces

$$D^{+}_{\mathbb{A}} : W^1(X, S^+ \otimes \mathbb{E}) \to W^0(X, S^- \otimes \mathbb{E}), \qquad (3)$$

where W^k is the space of sections with k derivatives in L^2, the latter space being defined in terms of the metric g.

In order to fix boundary conditions that make (3) a Fredholm operator it is convenient to fix a trivialization of $\mathbb{E} \,|\, \partial X$:

Definition 1.1. A *framing* of $\mathbb{E}$ is a choice of trivialization f of $\mathbb{E} \,|\, \partial X$. The pair $(\mathbb{E}, f)$ is called a *framed* bundle.

As we shall see in Section 2.1, framed bundles of rank $\geqslant 2$ are classified by an integral topological invariant analogous to the second Chern class, which we shall denote by $c_2(\mathbb{E}, f)[X]$.

For later convenience, we identify the trivial bundle that is implicit in Definition 1.1 with p^*E_{∞}, where E_{∞} denotes the trivial bundle over S^2_{∞}. Now we can write down boundary conditions for $\mathbb{A}$:

Definition 1.2. Let $\mathbb{A}$ be a connection on a framed bundle $(\mathbb{E}, f)$, smooth up to the boundary of X, let A_{∞} be a U_n-connection on E_{∞} and let Φ_{∞} be a skew-adjoint endomorphism of E_{∞}.

(i) $\mathbb{A}$ is called a *caloron configuration framed by* $(A_{\infty}, \Phi_{\infty})$ if

$$\mathbb{A} = p^*A_{\infty} + p^*\Phi_{\infty}\, dz$$

on ∂X, where f has been used to identify $\mathbb{E} \,|\, \partial X$ with p^*E_{∞}.

(ii) The pair (A_∞, Φ_∞) is called *admissible* if $\nabla_\infty \Phi_\infty = 0$, where ∇_∞ is the covariant derivative operator induced by A_∞ on $\mathrm{End}(E_\infty)$.

Our first main result asserts that the Fredholm properties of (3) are entirely determined by Φ_∞:

THEOREM 1.1. *Let $\mathbb{A}$ be a caloron configuration framed by an admissible pair (A_∞, Φ_∞). Then the operator in* (3) *is Fredholm if and only if* $1 - \exp(2\pi\Phi_\infty/\mu_0)$ *is invertible.*

If (A_∞, Φ_∞) is admissible, the eigenvalues $i\mu_1, ..., i\mu_n$ of Φ_∞ are constant. An equivalent formulation of this theorem is thus that (3) is *not* Fredholm if and only if there exist $j \neq 0$ and an integer N such that $\mu_j = N\mu_0$.

We now come to a statement of our L^2-index theorem:

THEOREM 1.2. *Suppose that* (3) *is Fredholm. Then*

$$ind(D^+_{\mathbb{A}}) = -c_2(\mathbb{E}, f)[X] - \sum_k c_1(E^+_{(k)})[S^2_\infty], \tag{4}$$

where for each integer k, $E^+_{(k)}$ is the sub-bundle of E_∞ on which $k\mu_0 - i\Phi_\infty$ is positive-definite.

It is clear that $E^+_{(k)}$ is either 0 or E_∞ for all but a finite number of integers k. Since E_∞ is trivial, it follows that the sum on the RHS of (4) is finite.

1.2. *Remarks*

It is instructive to compare Theorem 1.2 with the APS theorem on the one hand and the CAR theorem on the other. Our formula (4) is a sum of two terms, $-c_2(\mathbb{E}, f)[X]$ being an integral over the interior of X, while the sum over k is a contribution from the boundary. The first term is analogous to the 4-dimensional contribution to the index of the APS theorem, while the boundary contribution is reminiscent of the CAR index formula. Our proof of Theorem 1.2 "explains" why (4) has this shape. Unfortunately, our proof does not give many clues about possible generalizations of this result, in particular whether a contribution analogous to the η-invariant is to be expected in general. We should note also that an L^2-index theorem for the coupled Dirac operator over $S^1 \times S^1 \times \mathbb{R}^2$ has been obtained by Jardim [8]. This is another natural example of a Φ-geometry, but with fibre dimension 2 rather than 1.

Connections $\mathbb{A}$ of the type we have considered here are of interest in gauge theory, especially when they are required to satisfy the self-duality equations [6]; the term *caloron* was introduced in this context by Nahm [12]. Some other terminology used in this paper has also been borrowed from the mathematical physics literature, but no detailed knowledge of this area is assumed. The next paragraph is devoted to a quick discussion of the relevance of Theorem 1.2 to a gauge-theoretic study of self-dual calorons. The reader who is unfamiliar with gauge theory can skip it, for it is intended only as motivation.

It is natural to think of calorons as hybrids between monopoles and ordinary instantons, these corresponding respectively to the limit $\mu_0 \to \infty$ and $\mu_0 \to 0$. (These limits make the length of S^1 go to 0 or to ∞, respectively.) From this point of view it might be expected that calorons in general have two types of "topological charge": these are the "instanton charge" $c_2(\mathbb{E}, f)[X]$ and the "monopole" or "magnetic charges" which are essentially the degrees of the eigenbundles of Φ_∞. A physical interpretation of Theorem 1.2 is thus that the number of zero-modes of the Dirac-operator in the background $\mathbb{A}$ is completely determined by these topological charges. The determination of these zero-modes is crucial in proving the completeness of the *Nahm description* of calorons [12] and this is the original reason for our interest in this index problem. At the time of writing, the Nahm description has only been proved in detail for calorons with unit instanton charge and zero magnetic charge [9]. Details of the full transform will be the subject of a future publication.

1.3. *A Sketch of the Proof*

For the proof of Theorem 1.1, we apply general results of earlier authors. One approach is to check the conditions written down by Anghel, who has given necessary and sufficient conditions for the Dirac operator over a complete Riemannian manifold to be Fredholm in L^2. The alternative is to use the characterisation of Fredholm operators in the calculus of Φ-pseudo-differential operators [10]. It is worth remarking that this latter approach gives necessary and sufficient conditions for any "natural" operator on $S^1 \times \mathbb{R}^3$ to be Fredholm, not only operators of Dirac type. The two proofs appear in Section 4.

The proof of Theorem 1.2 involves two main steps. The first is a calculation of the index in the case that there is a trivialization of $\mathbb{E}$ such that $\mathbb{A}$ is independent of z. By Fourier analysis in the S^1-variable, the index can be identified in this case with a sum of indices of CAR-operators of the form (1). By topological arguments which we begin in Section 2, this calculation gives the index for any $\mathbb{A}$ over a framed bundle with $c_2(\mathbb{E}, f)[X] = 0$.

The second step invokes an excision theorem for operators of Dirac type due to Anghel [1] and Gromov–Lawson [7]. In our case, this result gives $\mathrm{ind}(D^+_{\mathbb{A}}) - \mathrm{ind}(D^+_{\mathbb{B}}) = -c_2(\mathbb{E}, f)[X]$ if $\mathbb{B}$ is any caloron configuration agreeing with $\mathbb{A}$ near ∂X but living on a new framed bundle $(\mathbb{F}, f)$, with $c_2(\mathbb{F}, f)[X] = 0$. Since we calculated $\mathrm{ind}(D^+_{\mathbb{B}})$ in the first step, that completes the proof of Theorem 1.2. The details appear in Section 5.

2. ON THE TOPOLOGY OF CALORONS

In this section we define the invariant $c_2(\mathbb{E}, f)[X]$ of a framed bundle over X from two points of view, the first homotopy-theoretic, the second a version of Chern–Weil theory. Proposition 2.1 and formula (10) will be used in the proof of Theorem 1.2.

2.1. *Topological Classification of Framed Bundles*

We start with a useful way of thinking of framed bundles and calorons in terms of the "rectangle" $R = [0, 2\pi/\mu_0] \times \bar{B}^3$. There is a natural map $R \to X$ which will be used to identify objects defined over X with corresponding objects over R. By abuse of notation we denote the second projection of R by p and we shall denote by the same symbol $(\mathbb{E}, f)$ the pull-back to R of a framed bundle over X; similarly for caloron configurations $\mathbb{A}$. In particular, when a framed bundle $(\mathbb{E}, f)$ is transferred to R, we obtain a bundle over R, framed over $[0, 2\pi/\mu_0] \times S^2_\infty$, and with a "clutching map"

$$\phi\colon \mathbb{E} \mid \{0\} \times \bar{B}^3 \simeq \mathbb{E} \mid \{2\pi/\mu_0\} \times \bar{B}^3.$$

Since E_∞ is trivial, we can regard it as the restriction to S^2_∞ of a trivial bundle $E \to \bar{B}^3$, say, and we can extend the framing of f of $\mathbb{E}$ to a bundle isomorphism $F\colon \mathbb{E} \to p^*E$ over R. In this way ϕ becomes a unitary endomorphism c of E which shall refer to as a clutching function for $\mathbb{E}$. Because of the periodicity, c then lies in the group

$$\mathscr{C} = \{\text{unitary automorphisms } c \text{ of } E : c \mid S^2_\infty = 1\}.$$

Now $\pi_0(\mathscr{C}) = \mathbb{Z}$, for any element c extends to a continuous map from the one-point compactification S^3 of $\bar{B}^3$ into U_n and $\pi_3(U_n) = \mathbb{Z}$. We *define* $c_2(\mathbb{E}, f)[X] = -\deg(c)$. $c_2(\mathbb{E}, f)[X]$ is the obstruction to extending the framing f to the interior of $\mathbb{E}$—an extension exists iff $c_2(\mathbb{E}, f)[X] = 0$.

We now reintroduce calorons, continuing to work over R, with $\mathbb{E} = p^*E$. Because of this identification, we can take the "3 + 1" decomposition

$$\nabla_{\mathbb{A}} = \nabla_{A_{(z)}} + dz(\partial_z + \Phi_{(z)}) \tag{5}$$

along $\{z\} \times \bar{B}^3$, where $A_{(z)}$ is a unitary connection on E and $\Phi_{(z)}$ is a skew-adjoint endomorphism of E. Thus we have obtained from $\mathbb{A}$ a *path* $(A_{(z)}, \Phi_{(z)})$ in

$$\mathscr{A} = \{(A, \Phi)\colon A \text{ is a } U_n \text{ connection on } E, \Phi \text{ is a skew-adjoint endomorphism of } E, (A, \Phi) \mid S^2_\infty = (A_\infty, \Phi_\infty)\}.$$

For periodicity, the end-points of this path must be related by the clutching function c:

$$A_{(2\pi/\mu_0)} = c^*(A_{(0)}) = cA_{(0)}c^{-1} - dcc^{-1} \tag{6}$$

and

$$\Phi_{(2\pi/\mu_0)} = c^*(\Phi_{(0)}) = c\Phi_{(0)}c^{-1}. \tag{7}$$

In other words, $\mathbb{A}$ can be identified with a loop in the quotient space $\mathscr{A}/\mathscr{C}$. Conversely, any such loop gives rise to a caloron configuration $\mathbb{A}$ framed by (A_∞, Φ_∞), subject to further matching conditions needed to ensure the smoothness of $\mathbb{A}$ on X when the two edges $\{z=0\}$ and $\{z=2\pi/\mu_0\}$ are glued together.

The simplest example of this correspondence is of course the case that the path in $\mathscr{A}$ is constant so that c is identically 1 and $c_2(\mathbb{E}, f)[X]=0$. Then we say that $\mathbb{A}$ is the *pull-back of a monopole*. Here is a sort of converse:

PROPOSITION 2.1. *Let $\mathbb{A}$ be a framed caloron in a framed bundle with $c_2(\mathbb{E}, f)[X]=0$. Then there is a deformation $\mathbb{B}$ of $\mathbb{A}$ (through framed caloron configurations), such that $\mathbb{B}$ is the pull-back of a monopole.*

Proof. Since $c_2(\mathbb{E}, f)[X]=0$, we can find a unitary automorphism C of $\mathbb{E}$ over R which is equal to 1 on $\{0\} \times \bar{B}^3$ and $[0, 2\pi/\mu_0] \times S^2_\infty$, and equal to c on $\{2\pi/\mu_0\} \times \bar{B}^3$. Pulling $\mathbb{A}$ back by C, we reduce to the case $c=1$, so that the caloron configuration is now a *loop* in $\mathscr{A}$. But this space is contractible, so the result follows. ∎

2.2. *Chern–Weil Theory for Frame Bundles*

Another way to think about the invariant $c_2(\mathbb{E}, f)[X]$ is in terms of the integral

$$\int_X \mathrm{ch}(\mathbb{E}) = -\frac{1}{8\pi^2}\int_X \mathrm{tr}\, F_{\mathbb{A}} \wedge F_{\mathbb{A}}, \tag{8}$$

where $\mathbb{A}$ is some framed caloron configuration. If X has no boundary, this integral would give minus the second Chern class, but here there are additional contributions from ∂X. This integral has been calculated by different means in [5] and [6]. As in the previous section, we work over R, so that $\mathbb{E}$ is identified with p^*E, together with a clutching function c. Using the familiar trick of writing

$$\mathrm{tr}\, F_{\mathbb{A}} \wedge F_{\mathbb{A}} = d\,\mathrm{tr}\{d\mathbb{A} \wedge \mathbb{A} + \tfrac{2}{3}\mathbb{A} \wedge \mathbb{A} \wedge \mathbb{A}\}$$

the integral (8) becomes an integral over the boundary of the rectangle $[0, 2\pi/\mu_0] \times \bar{B}^3$:

$$-\frac{1}{8\pi^2}\int_X \operatorname{tr} F_{\mathbb{A}} \wedge F_{\mathbb{A}}$$

$$= -\frac{1}{8\pi^2}\int_{\partial([0,\, 2\pi/\mu_0] \times \bar{B}^3)} \operatorname{tr}\left\{d\mathbb{A} \wedge \mathbb{A} + \frac{2}{3}\mathbb{A} \wedge \mathbb{A} \wedge \mathbb{A}\right\}. \tag{9}$$

Now regard $\mathbb{A}$ as a path $(A_{(z)}, \Phi_{(z)})$ satisfying (6) and (7). Evaluating (9) on $(\partial[0, 2\pi/\mu_0]) \times \bar{B}^3$ and using the clutching formulas gives

$$-\frac{1}{8\pi^2}\int_{(\partial[0,\, 2\pi/\mu_0]) \times \bar{B}^2} \operatorname{tr}\left\{d\mathbb{A} \wedge \mathbb{A} + \frac{2}{3}\mathbb{A} \wedge \mathbb{A} \wedge \mathbb{A}\right\}$$

$$= -\frac{1}{24\pi^2}\int_{\bar{B}^3} \operatorname{tr}(dcc^{-1})^3 + \frac{1}{8\pi^2}\int_{\bar{B}^3} d\operatorname{tr}\{A(0)\, c^{-1}\, dc\}.$$

The first term is $\deg c = -c_2(\mathbb{E}, f)[X]$, and the second vanishes because $c = 1$ on S^2_∞. On the other piece of the boundary we obtain

$$-\frac{1}{8\pi^2}\int_{[0,\, 2\pi/\mu_0] \times S^2_\infty} \operatorname{tr}\left\{d\mathbb{A} \wedge \mathbb{A} + \frac{2}{3}\mathbb{A} \wedge \mathbb{A} \wedge \mathbb{A}\right\}$$

$$= -\frac{1}{8\pi^2}\int_{[0,\, 2\pi/\mu_0] \times S^2_\infty} \operatorname{tr}\{2F_A \wedge \Phi\, dz - dA \wedge \Phi\, dz$$

$$+ A \wedge d\Phi \wedge dz + \partial_z A \wedge A \wedge dz\}.$$

The final term vanishes because the restriction of A to $[0, 2\pi/\mu_0] \times S^2_\infty$ is pulled back from S^2_∞ so that $\partial_z A = 0$ there (condition (i) of Definition 1.2). On the other hand, the sum of the middle two terms is exact, so does not contribute to the integral. The condition that A_∞ is compatible with Φ_∞ implies that A_∞ decomposes as a direct sum of connections, one on each eigenbundle of E_∞. Suppose E_μ is the eigenbundle of Φ_∞ with eigenvalue $i\mu$. Then $A_\infty = \bigoplus a_\mu$ where a_μ is a connection on E_μ, and $F_A \mid S^2_\infty = \bigoplus f_\mu$ where f_μ is the curvature of a_μ. Since the first Chern class of E_μ is given by

$$c_1(E_\mu)[S^2_\infty] = \frac{i}{2\pi}\int_{S^2_\infty} \operatorname{tr} f_\mu,$$

we have

$$-\frac{1}{8\pi^2}\int_{[0,\, 2\pi/\mu_0] \times S^2_\infty} \operatorname{tr} 2F_A \wedge \Phi\, dz = -\frac{1}{\mu_0}\sum_\mu \mu c_1(E_\mu)[S^2_\infty].$$

Putting the terms together, we arrive at the expression

$$\int_X \mathrm{ch}(\mathbb{E}) = -c_2(\mathbb{E}, f)[X] - \frac{1}{\mu_0} \sum_\mu \mu c_1(E_\mu)[S^2_\infty]. \tag{10}$$

3. BOUNDARY CONDITIONS VERSUS ASYMPTOTICS

In previous work, calorons have been studied exclusively as connections over $S^1 \times \mathbb{R}^3$, with decay conditions imposed near ∞; the compact manifold X was not used. The purpose of this section is to show how the boundary conditions that are implicit in Definition 1.2 translate into the "BPS" decay conditions for calorons that were written down in [6].

In order to compare the asymptotic region of $S^1 \times \mathbb{R}^3$ with a neighbourhood of the boundary of X, choose polar coordinates r, y_1, y_2 in $\mathbb{R}^3$, and continue to use z as a coordinate in S^1. Thus r is the distance from the origin in $\mathbb{R}^3$ and y_1, y_2 are some local angular coordinates on S^2_∞. We suppose y_1 and y_2 are chosen so that g takes the form

$$g = dr^2 + r^2(h_1\, dy_1^2 + h_2\, dy_2^2) + dz^2,$$

for some positive locally-defined functions h_1, h_2. Local coordinates near the boundary of X will be $x = r^{-1}$, y_1, y_2 and z, so that x becomes a boundary defining function: $x \geqslant 0$ on X, with equality only at ∂X. Writing g in terms of x,

$$g = \frac{dx^2}{x^4} + h_1 \frac{dy_1^2}{x^2} + h_2 \frac{dy_2^2}{x^2} + dz^2.$$

Now denote the components of $\mathbb{A}$, in some gauge (trivialisation) that is smooth up to the boundary, by

$$\nabla_x = \partial_x + A_x, \qquad \nabla_{y_j} = \partial_{y_j} + A_{y_j}, \qquad \nabla_z = \partial_z + \Phi.$$

Performing a $3+1$-decomposition of $\mathbb{A} = (A_{(z)}, \Phi_{(z)})$ as before, we have, near the boundary,

$$\begin{aligned} \|A\|^2 &= |A_x|^2\, |dx|_g^2 + |A_{y_1}|^2\, |dy_1|_g^2 + |A_{y_2}|^2\, |dy_2|_g^2 \\ &= x^2(x^2\, |A_x|^2 + h_1^{-1}\, |A_{y_1}|^2 + h_2^{-1}\, |A_{y_2}|^2). \end{aligned}$$

Since $x = r^{-1}$, we see that $\|A\| = O(r^{-1})$ as $r \to \infty$, uniformly in the angular variables (y_1, y_2). This statement is not gauge invariant: a better formulation is that there exist preferred gauges near ∞ in X^o (namely those that extend smoothly on ∂X), such that the connection 1-form satisfies $\|A\| = O(r^{-1})$ in such a gauge. In such gauges we also have $\|\Phi\| = O(1)$.

Similarly, the assumptions in Definition 1.2 imply that $\nabla_{y_j}\Phi$ and $\partial_z A_{y_j}$ are $O(x)$ as $x \to 0$, while $\nabla_x \Phi$ and $\partial_z A_x$ are $O(1)$. It follows that $\|\nabla_A \Phi - \partial_z A\| = O(r^{-2})$ as $r \to \infty$, a fact that will be used in the next section.

4. PROOF OF THEOREM 1.1

In this section we give two proofs of Theorem 1.1 about the Fredholm properties of $D^+_{\mathbb{A}}$. The first proof rests on a result of Anghel in [1], the second on the general theory of Φ-pseudodifferential operators. Of course both methods give the same answer, and indeed the key point is the same in each case.

4.1. *First Proof*

Theorem (2.1) of [1] gives conditions for $D_{\mathbb{A}} = D^+_{\mathbb{A}} \oplus D^-_{\mathbb{A}}$ to be Fredholm: $D_{\mathbb{A}}$ is Fredholm if and only if there is a compact set $K \subset X^o$ and a constant $C > 0$ such that

$$\|D_{\mathbb{A}}\psi\|_{L^2} \geqslant C\,\|\psi\|_{L^2}, \qquad \text{when} \quad \psi \in W^1(S \otimes \mathbb{E}) \quad \text{and} \quad \operatorname{Supp}(\psi) \subset X^o \backslash K.$$

If $D_{\mathbb{A}}$ is Fredholm then $D^+_{\mathbb{A}}$ must be Fredholm too. Now

$$D^*_{\mathbb{A}} D_{\mathbb{A}} = (D^-_{\mathbb{A}} D^+_{\mathbb{A}}) \oplus (D^-_{\mathbb{A}} D^+_{\mathbb{A}})^*$$

so to obtain estimates on $\|D_{\mathbb{A}}\|_{L^2}$ we consider the operator $D^-_{\mathbb{A}} D^+_{\mathbb{A}}$. Using the notation and conventions of Section 1.1 we have from (2),

$$D^-_{\mathbb{A}} D^+_{\mathbb{A}} = -\nabla_z^2 + [D_A, \nabla_z] + D_A^2.$$

The third term here is clearly positive, and the boundary conditions allow us to estimate the other two as follows.

The first term. Extend the framing f to a neighbourhood of ∂X; this gives a gauge near ∞ in which the "3 + 1" decomposition (5) can be performed. As the boundary ∂X is approached the eigenvalues of Φ converge to the eigenvalues of Φ_∞. Using spherical polar coordinates on $\mathbb{R}^3$, let $i\lambda_j(r, y_1, y_2, z)$ be the eigenvalues of Φ, and $i\mu_j$ be the eigenvalues of Φ_∞ ($j = 1, ..., n$) such that $\lambda_j \to \mu_j$ as $r \to \infty$. Let $\lambda(r, y_1, y_2, z)$ be the smallest element in $\{|\lambda_j + k\mu_0| : j = 1, ..., n \in \mathbb{Z}\}$ and μ be the smallest element of the set $\{|\mu_j + k\mu_0| : j = 1, ..., n, k \in \mathbb{Z}\}$. The invertibility condition on Φ_∞ in the statement of the theorem implies that $\mu > 0$, so there exists a compact set $K_1 \subset X^o$ such that $\lambda > \mu/2$ on $X^o \backslash K_1$.

Suppose $\psi \in W^2(S^+ \otimes \mathbb{E})$ and $\operatorname{Supp} \psi \subset X^o \backslash K_1$. Using the isomorphism $S^+ \cong p^* S_{(3)}$, ψ can be written as a Fourier series

$$\psi = \sum_k \exp(ik\mu_0 z)\, \phi_k,$$

where ϕ_k is a section of $S_{(3)} \otimes E$. Let

$$\psi^{(k)} = \exp(ik\mu_0 z)\, \phi_k.$$

Then

$$\nabla_z \psi^{(k)} = (ik\mu_0 + \Phi)\, \psi^{(k)}$$

so

$$(-\nabla_z \nabla_z \psi^{(k)}, \psi^{(k)}) \geqslant \tfrac{1}{4}\, \mu^2\, \|\psi^{(k)}\|^2, \qquad \text{on} \quad X^o \backslash K_1$$

as a pointwise estimate. (Since $\psi^{(k)} \in W^2(S^+ \otimes \mathbb{E})$, $\psi^{(k)}$ is actually continuous so both sides of the inequality exist.) Since the inequality is independent of k it holds for general ψ and we obtain

$$\operatorname{Supp}(\psi) \subset X^o \backslash K_1 \Rightarrow (-\nabla_z \nabla_z \psi, \psi)_{L^2} \geqslant \tfrac{1}{4}\, \mu^2\, \|\psi\|^2_{L^2}. \tag{11}$$

The second term. We have

$$[D_A, \nabla_z] = \sum_j e_j [\nabla_j, \nabla_z] = \sum_j \iota(\partial_j)(\nabla_A \Phi - \partial_z A),$$

where $\iota(\xi)$ denotes interior product with ξ. But $\|\nabla_A \Phi - \partial_z A\| \to 0$ as $r \to \infty$, so there exists a compact set $K_2 \subset X^o$ such that

$$\operatorname{Supp}(\psi) \subset X^o \backslash K_2 \Rightarrow |([D_A, \nabla_z]\, \psi, \psi)_{L^2}| \leqslant \tfrac{1}{8}\, \mu^2\, \|\psi\|^2_{L^2}. \tag{12}$$

Now let K be a compact set containing K_1 and K_2. Combining (11) and (12) we obtain

$$\operatorname{Supp}(\psi) \subset X^o \backslash K \Rightarrow (D^-_{\mathbb{A}} D^+_{\mathbb{A}} \psi, \psi)_{L^2} \geqslant \tfrac{1}{8}\, \mu^2\, \|\psi\|^2_{L^2}.$$

A similar bound is obtained for $D^+_{\mathbb{A}} D^-_{\mathbb{A}} = (D^-_{\mathbb{A}} D^+_{\mathbb{A}})^*$, and so we obtain the following bound for $D_{\mathbb{A}}$:

$$\psi \in W^2(S \otimes \mathbb{E}), \qquad \operatorname{Supp}(\psi) \subset X^o \backslash K \Rightarrow \|D_{\mathbb{A}} \psi\|_{L^2} \geqslant \frac{1}{\sqrt{8}}\, \mu\, \|\psi\|_{L^2}.$$

By density, the inequality in fact holds for $\psi \in W^1(S \otimes \mathbb{E})$. This completes the verification of Anghel's criterion and gives a proof of the "if" part of

Theorem 1.1. The "only if" part can also be proved in this framework but this is omitted. This converse statement also follows at once from the discussion of the next section.

4.2. *Second Proof*

Recall the boundary-adapted coordinate system x, y_1, y_2, z introduced in Section 3, and let the components of $\nabla_{\mathbb{A}}$ in these coordinates be

$$\nabla_x = \partial_x + A_x, \qquad \nabla_{y_j} = \partial_{y_j} + A_{y_j}, \qquad \nabla_z = \partial_z + A_z.$$

Relative to a suitable choice of basis for the spin-bundles, we have then

$$D_{\mathbb{A}}^+ = \nabla_z + e_1 x \nabla_{y_1} + e_2 x \nabla_{y_2} + e_3 x^2 \nabla_x.$$

Strictly speaking, we are making a choice of normal coordinates here; otherwise there will be additional zero-order terms coming from connection coefficients. This is an example of a Φ-differential operator in the sense of [10]; more generally the algebra of Φ-differential operators on X consists of all differential operators which take the form

$$P(x, y, z; x^2\partial_x, x\partial_y, \partial_z), \tag{13}$$

near ∂X, where P is smooth in the first three variables and polynomial in the last three variables. In [10] it is shown that such an operator is Fredholm in L^2 if and only if it is *fully elliptic* in the following sense. First, (13) must be elliptic in the usual sense over X^o. Secondly, the associated *indicial family* must be invertible on every fibre $p^{-1}(y) \subset \partial X$. Given such a fibre, the indicial family on $p^{-1}(y)$ is defined by picking a real number ξ and a real cotangent vector $\eta \in T_y^* S_\infty^2$, and defining

$$\hat{P}_{(y, \eta, \xi)} = P(0, y, z; i\xi, i\eta, \partial_z)$$

as a differential operator on $p^{-1}(y)$. To say that the indicial family is invertible is to say that $\hat{P}_{(y, \eta, \xi)}$ is invertible (in any reasonable space of sections over $p^{-1}(y)$), for each choice of (y, η, ξ) as above.

Following this recipe for $D_{\mathbb{A}}^+$, we obtain

$$\hat{P}_{(y, \eta, \xi)} = \nabla_z + i(\eta_1 e_1 + \eta_2 e_2 + \xi e_3).$$

This operator in $C^\infty(S^1, p^* S_{(3)} \otimes E_\infty)$ is a sum of two terms $B + A$, where $A = i(\eta_1 e_1 + \eta_2 e_2 + \xi e_3)$ is self-adjoint, $B = \nabla_z$ is skew-adjoint and $[A, B] = 0$. It follows by considering $(A+B)^* (A+B)$ that $(A+B) u = 0$ if and only if $Au = 0$ and $Bu = 0$. Now B has a non-trivial null-space only if one of the μ_j is an integral multiple of μ_0. Hence under the assumption of Theorem 1, $A + B$ is injective. Similarly the adjoint $(A+B)^* = A - B$ is injective, so that the hypothesis of Theorem 1.1 implies that the indicial family is invertible, and so $D_{\mathbb{A}}^+$ is Fredholm in L^2. Conversely, if the condition fails, then B is not invertible, and nor is $B + A$ when $\eta_j = 0 = \xi$. So in this case $D_{\mathbb{A}}^+$ is not fully elliptic and hence cannot be Fredholm in L^2. The proof of Theorem 1.1 is now complete.

4.3. *Remarks about the L^2-condition*

According to [10, Proposition 9], elements of the null-space of a fully elliptic Φ-differential operator decay very rapidly at the boundary. More precisely, if $D_{\mathbb{A}}^+$ is fully elliptic, if $D_{\mathbb{A}}^+ \psi = 0$, and if for some real m, $x^m \psi \in L^2(X)$, then $\psi \in C^\infty(X)$ and ψ vanishes to all orders in x at ∂X. There is a similar statement for the cokernel. Now in terms of the boundary-adapted coordinates (x, y_j, z), the volume element determined by the metric g has the form $x^{-4}\, d\mu$ where $d\mu = h_1 h_2\, dx\, dy_1\, dy_2\, dz$. It follows from the above that the index of (3) is the same as the index of

$$D_{\mathbb{A}}^+ : W^1(X, \mathbb{E} \otimes S^+, d\mu) \to W^0(X, \mathbb{E} \otimes S^-, d\mu). \tag{14}$$

This fact makes the next result almost obvious:

Proposition 4.1. *Let $\mathbb{A}$, $\mathbb{B}$ be two caloron configurations on $(\mathbb{E}, f)$, both framed by (A_∞, Φ_∞). Then $D_{\mathbb{A}}^+$ is Fredholm if and only if $D_{\mathbb{B}}^+$ is so, and their L^2-indices coincide.*

Proof. The space of calorons on a given framed bundle, with given boundary data (A_∞, Φ_∞) is contractible. It is easy to see that any continuous path joining $\mathbb{A}$ to $\mathbb{B}$ gives rise to a norm-continuous path of Dirac operators between the Sobolev spaces in (14). Since each of these is Fredholm by Theorem 1.1, it follows that the index is constant on this path. ∎

5. PROOF OF THE INDEX THEOREM

5.1. *Proof When $c_2(\mathbb{E}, f)[X] = 0$*

In this case, by Proposition 2.1 and 4.1 it is enough to compute the index when $\mathbb{E} = p^*(E)$ and $\mathbb{A} = p^*A + p^*\Phi\, dz$ is the pull-back of a monopole (cf. Section 2.1). Then the coefficients of $D_{\mathbb{A}}^+$ are independent of z and we can use Fourier analysis in the S^1-variable to reduce the calculation of the index to that of a collection of operators of the form (1) on $\mathbb{R}^3$. These operators are precisely the subject of the CAR theorem in is simplest form [4].

Let

$$Y_k = \{\psi = \exp(ik\mu_0 z)\, \phi : \phi \in W^0(\mathbb{R}^3, S_{(3)} \otimes E)\}$$

so that

$$W^0(S^+ \otimes \mathbb{E}) = \left\{ \sum \psi^{(k)} : \psi^{(k)} \in Y_k \text{ and } \sum \|\psi^{(k)}\|^2 < \infty \right\}.$$

Since by assumption the coefficients are independent of z, $D^+_{\mathbb{A}}$ maps $Y_k \cap W^1$ into Y_k and its restriction to this subspace is equal to

$$D_k \colon W^1(S_{(3)} \otimes E) \to W^0(S_{(3)} \otimes E)$$

$$D_k = D_A + ik\mu_0 + 1 \otimes \Phi.$$

According to the theory developed by Callias–Anghel–Råde, D_k is Fredholm for every $k \in \mathbb{Z}$ iff $D^+_{\mathbb{A}}$ is Fredholm, and [13] shows that

$$\begin{aligned} \operatorname{ind} D_k &= -\int_{S^2_\infty} \hat{A}(S^2_\infty) \wedge \operatorname{ch}(E^+_{(k)}) \\ &= -c_1(D^+_{(k)})[S^2_\infty], \end{aligned}$$

where $E^+_{(k)}$ is the subbundle of E_∞ on which $(k\mu_0 - i\Phi_\infty)$ has positive eigenvalues. (We have already noted that this sum is finite.) Since $Y_j \cap Y_k = 0$ if $j \neq k$, the index of $D^+_{\mathbb{A}}$ is the sum of the indices of the D_k, i.e.

$$\operatorname{ind} D^+_{\mathbb{A}} = \sum_k \operatorname{ind} D_k = -\sum_k c_1(E^+_{(k)})[S^2_\infty].$$

That completes the proof of Theorem 1.2 when $c_2(\mathbb{E}, f)[X] = 0$.

5.2. *Proof of the Index Theorem when* $c_2(\mathbb{E}, f)[X] \neq 0$

Anghel [1], generalizing work of Gromov and Lawson [7], has given an excision theorem which compares the L^2-indices of a pair of Dirac operators over a complete manifold that agree near infinity. In our case this result yields the following statement. Let $\mathbb{E}$ and $\mathbb{F}$ be a pair of bundles over X^o and let $\mathbb{A}$ and $\mathbb{B}$ be unitary connections on $\mathbb{E}$ and $\mathbb{F}$ (respectively). Suppose that there is a bundle isometry $\theta\colon \mathbb{E} \mid X^o \backslash K \to \mathbb{F} \mid X^o \backslash K$ which carries $\mathbb{A}$ to $\mathbb{B}$ outside some compact set $K \subset X^o$. Then

$$\operatorname{ind} D^+_{\mathbb{A}} - \operatorname{ind} D^+_{\mathbb{B}} = \int_{X^o} \operatorname{ch}(\mathbb{E}) - \int_{X^o} \operatorname{ch}(\mathbb{F}). \tag{15}$$

We are going to deduce Theorem 1.2 by taking for $\mathbb{B}$ a connection which agrees with $\mathbb{A}$ near ∞, but which lives on a framed bundle $(\mathbb{F}, f)$ with

$c_2(\mathbb{F}, f)=0$. This will complete the proof in view of the results of the previous section.

Let then $(\mathbb{E}, f)$ be a framed bundle and $\mathbb{A}$ a framed caloron configuration on $\mathbb{E}$. As in Section 2.1, identify $\mathbb{E}$ with p^*E, together with a clutching function $c \in \mathscr{C}$. Extend the framing smoothly from the boundary to a region $[0, 2\pi/\mu_0] \times U$ where $K \subset \mathbb{R}^3$ is compact and $U = X^o \backslash K$. By a deformation of $\mathbb{A}$ over $\{2\pi/\mu_0\} \times U$ which vanishes at ∞, we can assume that $c=1$ on U. Now define $\mathbb{F} = p^*E$ and $\mathbb{B}$ to agree with $\mathbb{A}$ over $S^1 \times U$, but extended over $S^1 \times K$ to define a smooth connection on $\mathbb{F}$. (This can be achieved by a suitable use of cut-off functions.)

Applying (15),

$$\operatorname{ind} D^+_{\mathbb{A}} - \operatorname{ind} D^+_{\mathbb{B}} = \int_{X^o} \operatorname{ch}(\mathbb{E}) - \int_{X^o} \operatorname{ch}(\mathbb{F}).$$

But

$$\int_{X^o} \operatorname{ch}(\mathbb{E}) = -c_2(\mathbb{E}, f)[X] - \frac{1}{\mu_0} \sum_{\mu} \mu c_1(E_\mu)[S^2_\infty]$$

from (10), and

$$\int_{X^o} \operatorname{ch}(\mathbb{F}) = -\frac{1}{\mu_0} \sum_{\mu} \mu_c(E_\mu)[S^2_\infty].$$

So

$$\operatorname{ind} D^+_{\mathbb{A}} = \operatorname{ind} D^+_{\mathbb{B}} - c_2(\mathbb{E}, f)[X].$$

From Section 5.1 we know that $\operatorname{ind} D^+_{\mathbb{B}} = -\sum_k c_1(E^+_{(k)})[S^2_\infty]$ so we have proved that

$$\operatorname{ind} D^+_{\mathbb{A}} = -c_2(\mathbb{E}, f)[X] - \sum_k c_1(E^+_{(k)})[S^2_\infty].$$

This completes the proof of Theorem 1.2.

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