Available online at www.sciencedirect.com
ScienceDirect
European Journal of Combinatorics 29 (2008) 159–170
European Journal of Combinatorics
www.elsevier.com/locate/ejc

A modular triple characterization of circuit signatures

Lori Koban [1]

Department of Mathematics and Computer Science, University of Maine at Farmington, United States

Received 29 September 2005; received in revised form 9 February 2006; accepted 27 November 2006
Available online 30 December 2006

Abstract

We study the modular triples of circuits of a matroid and use them to characterize four types of circuit signatures, three of which are known (weak orientations, orientations, and ternary signatures) and one of which is new (lifting signatures). Lifting signatures allow us to specify a linear class of circuits in a matroid, and thereby the lift of the matroid, by labeling the elements from a group in the manner of Dowling and Kelly.

1. Introduction

We use the notion of *modular triples* of circuits (a generalization of theta graphs) to characterize four types of circuit signatures of matroids, and we apply this result to prove that it is possible to construct elementary lifts of ternary or binary matroids by labeling their elements with members of Abelian groups.

Our work was inspired by an attempt to generalize Zaslavsky's theory of lifts of graphic matroids [14, Section 3] to all matroids. In this theory, one assigns *gains* (elements of a group) to the edges of the graph in order to pick out a linear class of *balanced* circles (simple closed paths). One then applies the lift construction of Dowling and Kelly [5, Section 6] to produce a lift of the graphic matroid that is determined by the balanced circles.

We assign gains to the elements of a matroid. To pick out a linear class of balanced circuits, we rely on *lifting signatures* instead of graphs. We characterize this special class of circuit signatures in terms of modular triples of circuits. We find that binary and ternary matroids have

E-mail address: lori.koban@maine.edu.

[1] Also at: UMF, Math Department, 117 South Street, Farmington, ME 04938, United States. Tel.: +1 207 778 7117; fax: +1 207 778 7125.

0195-6698/$ - see front matter
doi:10.1016/j.ejc.2006.11.003

lifting signatures, which allows the application of Dowling and Kelly's lift construction. We also find that modular triples characterize weak orientations [1], orientations [2], and ternary signatures [10].

2. Background

2.1. Linear classes and matroid lifts

Let M be a matroid with ground set E. In [5], Dowling and Kelly use a linear class of circuits to construct an elementary lift of M. We use some terminology from [9] to discuss their construction.

We say that (C_1, C_2, C_3) is a *modular triple of circuits* of M if the three circuits are distinct and, for distinct i, j, and k, $C_k \subseteq C_i \cup C_j$ and (C_i, C_j) is a modular pair. We say that (H_1, H_2, H_3) is a *modular triple of copoints* of M if the three copoints are distinct and intersect in a coline.

Since (C_1, C_2, C_3) is a modular triple of circuits of M if and only if $(E \setminus C_1, E \setminus C_2, E \setminus C_3)$ is a modular triple of copoints of M^*, we write H_i^* to mean $E \setminus C_i$. If L^* is the coline in M^* at which H_1^*, H_2^*, and H_3^* meet, then $L^* = E \setminus (C_1 \cup C_2 \cup C_3)$.

Lemma 2.1. *A matroid M is binary if and only if, for each modular triple (C_1, C_2, C_3) of circuits, $C_1 \cap C_2 \cap C_3 = \emptyset$.*

Lemma 2.2. *Let (C_1, C_2) be a modular pair of circuits of a matroid M, and let $e \in C_1 \cap C_2$. Then there exists a unique circuit C_3 such that $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$. Moreover, (C_1, C_2, C_3) is a modular triple of circuits of M.*

Let $\mathcal{B}$ be a subclass of circuits of M. If, for each modular triple of circuits, either 0, 1, or 3 of these circuits are in $\mathcal{B}$, we say that $\mathcal{B}$ is a *linear subclass of circuits* of M [14, Section 3]. Let $\mathcal{B}$ be such a subclass, and define $\mathcal{B}^* = \{E \setminus C : C \in \mathcal{B}\}$. Then the set

$$\mathcal{M}_0 = \{F \in \mathcal{F}(M^*) : \text{ every copoint containing } F \text{ is in } \mathcal{B}^*\} \tag{2.1}$$

is a modular cut of M^* (see [4, Section 6]). As long as $\mathcal{B}$ does not contain all circuits of M, $((M^* +_{\mathcal{M}_0} e)/\{e\})^*$ is an elementary lift of M. We denote this elementary lift by $L(M, \mathcal{B})$.

2.2. Using gains to lift graphic matroids

A *gain graph* $\Phi = (\Gamma, \phi)$ [13, Section 5] consists of a graph Γ and a *gain mapping* ϕ from the edges of Γ into a group $\mathfrak{G}$, the *gain group*. We require that $\phi(e^{-1}) = \phi(e)^{-1}$, where e^{-1} means e with its orientation reversed. Associated with Φ is a class $\mathcal{B}(\Phi)$ of *balanced circles*. Let B be a circle of Γ. To decide whether or not B is balanced, choose an edge e_1 of B and a direction (clockwise or counterclockwise) to traverse B. Let $e_1, e_2, \ldots, e_k$ be the edges of B in the order in which they are traversed, and let them be oriented in this direction. The gain of B is $\phi(B) = \phi(e_1)\phi(e_2)\cdots\phi(e_k)$. Then $B \in \mathcal{B}(\Phi)$ if $\phi(B) = 1$.

The set of balanced circles of a gain graph is a linear subclass of circuits of the graphic matroid $G(\Gamma)$ [13, Proposition 5.1]. Accordingly, $L(G(\Gamma), \mathcal{B}(\Phi))$ is an elementary lift of $G(\Gamma)$ (unless all circles are balanced). We usually denote this lift by $L(\Phi)$ [14, Section 3].

2.3. Circuit signatures

A *signed set* is a set X together with an ordered bipartition (X^+, X^-) of X. Sometimes we write $a_1 \cdots a_p \overline{b}_1 \cdots \overline{b}_n$ to mean $(\{a_1, \ldots, a_p\}, \{b_1, \ldots, b_n\})$. The signed set has *support* X. We denote both the signed set and its support by X. We write $-X = (X^-, X^+)$, and $X \setminus T = (X^+ \setminus T, X^- \setminus T)$.

If X is a signed subset of E, then for each $e \in E$, define

$$X(e) = \begin{cases} +1 & \text{if } e \in X^+, \\ -1 & \text{if } e \in X^-, \\ 0 & \text{if } e \notin X^+ \cup X^-. \end{cases}$$

Let M be a matroid on E, and let $\mathcal{C}$ be a collection of signed subsets of E. We say that $\mathcal{C}$ is a *circuit signature* of M if:

(1) every signed set in $\mathcal{C}$ has a circuit of M as support, and
(2) for every circuit C of M, there are precisely two members of $\mathcal{C}$ with support C, and these two signed sets are negatives of each other.

Let $\mathcal{C}$ be a circuit signature of M, and let $e \in E$. The *reorientation of* $\mathcal{C}$ *on* A is the circuit signature ${}_{\overline{A}}\mathcal{C} = \{{}_{\overline{A}}X : X \in \mathcal{C}\}$, where ${}_{\overline{A}}X$ is the signed set derived from X by reversing the signs of the elements of A. We define $\mathcal{C} \setminus e = \{X \in \mathcal{C} : e \notin X\}$, the *deletion of* $\mathcal{C}$ *by* e. Also, we define the *contraction of* $\mathcal{C}$ *by* e, denoted by $\mathcal{C}/e$, to be the set of elements of $\{X \setminus \{e\} : X \in \mathcal{C} \text{ and } X \setminus \{e\} \neq \emptyset\}$ that have setwise minimal support. Every collection obtained from $\mathcal{C}$ by a succession of deletions and contractions is called a *minor* of $\mathcal{C}$. We call (C_1, C_2, C_3) a *modular triple of signed circuits* of $\mathcal{C}$ if, as supports, (C_1, C_2, C_3) is a modular triple of circuits of M. A *modular pair of signed circuits* is defined similarly.

2.4. Orientations of matroids

Bland and Las Vergnas introduced oriented matroids in [2]. Let $\mathcal{C}$ be a circuit signature of M. Then $\mathcal{C}$ determines (i.e. is the set of signed circuits of) an oriented matroid if the following *circuit elimination* property holds: for all $X, Y \in \mathcal{C}$ such that $X \neq -Y$, and all $e \in X^+ \cap Y^-$, there is a $Z \in \mathcal{C}$ such that $Z^+ \subseteq (X^+ \cup Y^+) \setminus \{e\}$ and $Z^- \subseteq (X^- \cup Y^-) \setminus \{e\}$. We say that $\mathcal{C}$ is an *orientation* of M and that M is *orientable*.

2.5. Weak orientations of matroids

Weakly oriented matroids were introduced by Bland and Jensen in [1]. They are matroids together with a special type of circuit signature, called a *weak orientation*.

Theorem 2.3. *Let $\mathcal{C}$ be a circuit signature of a matroid M. Then $\mathcal{C}$ is a weak orientation of M if and only if for every $X_1, X_2 \in \mathcal{C}$ with $e \in X_1^+ \cap X_2^-$ and $X_1 \neq -X_2$,*

(i) *if $f \in (X_1^+ \setminus X_2^-) \cup (X_1^- \setminus X_2^+)$, then there exists $X_3 \in \mathcal{C}$ with $f \in X_3 \subseteq (X_1 \cup X_2) \setminus \{e\}$; and*
(ii) *there are $e_1 \in X_1 \setminus X_2$, $e_2 \in X_2 \setminus X_1$, and $X_4 \in \mathcal{C}$ satisfying $X_4 \subseteq (X_1 \cup X_2) \setminus \{e\}$ so that $X_4(e_1)X_4(e_2) = X_1(e_1)X_2(e_2)$.*

Theorem 2.4 (*[7, Theorem 1, p. 173]*). *A circuit signature $\mathcal{C}$ of a matroid is a weak orientation of M if and only if $\mathcal{C}$ has no minor isomorphic to a reorientation of the signature* $\{12, 13, 23\}$ *of* $U_{1,3}$.

2.6. Ternary signatures of matroids

The construction of ternary signatures stems from Tutte's theory of chain groups (see [11, Chapter 8] and [12, Section 9.4]). Let M be a ternary matroid, and let N be the rowspace of a GF(3)-representation matrix of M^*. Then $M \cong M(N)$ where $M(N)$ is the chain-group matroid of N. The elementary vectors of N can be used to obtain a circuit signature of M: an element s in a signed circuit is positive or negative, depending on whether the value in the s-coordinate of the elementary vector is $+1$ or -1, respectively. This circuit signature is called *ternary signature* of M [10].

Example 2.5. Consider the representation

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

of $U_{2,4}$ over GF(3). The elementary vectors of the row space of this representation are

$$\begin{pmatrix} 1 & 1 & -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 1 & -1 \end{pmatrix},$$

and their negatives. This gives the ternary signature $\{12\overline{3}, 1\overline{24}, 134, 23\overline{4}\}$ of $U_{2,4}$.

Fortunately, ternary signatures do not depend on the representation matrix of M^*.

Theorem 2.6. *A ternary matroid has a unique ternary signature, up to reorientation.*

Proof. This result follows from the fact that ternary matroids are uniquely GF(3)-representable [6, Theorem 3.2]. □

Ternary signatures can be characterized by a signed circuit elimination axiom [10, Theorem 3.1].

Theorem 2.7. *Let M be a matroid and $\mathcal{C}$ a signature of its circuits. Then the following properties are equivalent:*

(1) *$\mathcal{C}$ is a ternary signature.*
(2) *For any $X_1, X_2 \in \mathcal{C}$ with $(X_1^+ \cap X_2^-) \cup (X_1^- \cap X_2^+) \neq \emptyset$ and for any $f \in (X_1^+ \setminus X_2^-) \cup (X_1^- \setminus X_2^+)$, there exist $X_3 \in \mathcal{C}$ such that $f \in X_3 \subseteq (X_1 \cup X_2) \setminus ((X_1^+ \cap X_2^-) \cup (X_1^- \cap X_2^+))$, and there exist $e_1 \in X_1 \cap X_3$ and $e_2 \in X_2 \cap X_3$ such that $X_1(e_1)X_2(e_2) = X_3(e_1)X_3(e_2)$.*
(3) *$\mathcal{C}$ has no minor isomorphic to a reorientation of the circuit signature* $\{12, 13, 23\}$ *of $U_{1,3}$, or to a reorientation of the circuit signature* $\{123, 1\overline{2}4, 134, 23\overline{4}\}$ *of $U_{2,4}$.*

3. How to characterize weak orientations, orientations, and ternary signatures by modular triples

Weak orientations, orientations, and ternary signatures are characterized in the literature in a variety of ways. We provide a new characterization of these circuit signatures, as well as of lifting signatures (see Section 4.1), in terms of modular triples of circuits.

Let $\mathcal{C}$ be a circuit signature of a matroid M. We now define the *Well-Distribution Property* (WDP): for each modular triple of signed circuits, (C_1, C_2, C_3), there exist sets I_1, I_2, I_3, and I_4

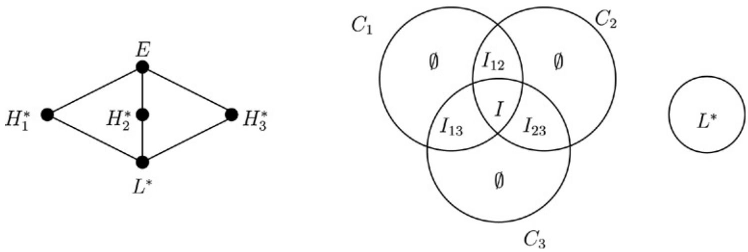

Fig. 2.1. On the left is a portion of the lattice of flats of M^* in which (H_1^*, H_2^*, and H_3^*) constitute a modular triple of copoints. On the right, we see (C_1, C_2, C_3), the corresponding modular triple of circuits of M.

with $I_1 \mathbin{\dot\cup} I_2 = I_3 \mathbin{\dot\cup} I_4 = I$ so that, up to reorientation,

$$C_1 = (I \cup I_{13}, I_{12}),$$
$$C_2 = \pm(I_1 \cup I_{12}, I_2 \cup I_{23}), \quad \text{and}$$
$$C_3 = \pm(I_3 \cup I_{23}, I_4 \cup I_{13}).$$

(The sets I, I_{13}, I_{12}, and I_{23} are defined in Fig. 2.1.)

Theorem 3.1. *Let $\mathcal{C}$ be a circuit signature of a matroid M.*

(1) *$\mathcal{C}$ is a weak orientation of M if and only if the Well-Distribution Property (WDP) holds.*
(2) *$\mathcal{C}$ is an orientation of M if and only if the Well-Distribution Property holds with $I_3 \subseteq I_2$.*
(3) *$\mathcal{C}$ is a ternary signature of M if and only if the Well-Distribution Property holds with $I_1 = I_3 = I$.*
(4) *Let $\mathfrak{A}$ be an Abelian group with* $\exp(\mathfrak{A}) > 2$.
   (a) *Assume M is binary. Then $\mathcal{C}$ is a lifting signature for gains in $\mathfrak{A}$ if and only if the Well-Distribution Property holds with $I_1 = I_3 = I$.*
   (b) *Assume M is not binary. Then $\mathcal{C}$ is a lifting signature for gains in $\mathfrak{A}$ if and only if* $\exp(\mathfrak{A}) = 3$ *and the Well-Distribution Property holds with $I_1 = I_3 = I$.*

We use two different techniques to prove Theorem 3.1(1)–(3). To prove the sufficiency of part (2), we use Las Vergnas' result that signed circuit elimination for orientations is equivalent to modular signed circuit elimination [8, Theorem 2.1]. To prove the sufficiency of parts (1) and (3), we use forbidden minor arguments, which require Lemma 3.2.

Let Property P be the WDP together with one (or none) of the additional requirements involving the sets I_1, I_2, and I_3 that appear in Theorem 3.1(2)–(4).

Lemma 3.2. *Let $\mathcal{C}$ be a circuit signature of a matroid M that satisfies Property P. Then any minor of $\mathcal{C}$ also satisfies Property P.*

Proof. We show that if $\mathcal{C}$ satisfies Property P, then $\mathcal{C} \setminus s$ and $\mathcal{C}/s$ also satisfy Property P. We prove the contrapositive of this statement by induction on $|E(M)|$. Assume that $|E(M)| > 1$.

If $\mathcal{C} \setminus s$ does not satisfy Property P, then neither does $\mathcal{C}$, because the signature of a signed set in $\mathcal{C} \setminus s$ and in $\mathcal{C}$ are the same. Let (C_1, C_2, C_3) be a modular triple of signed circuits in $\mathcal{C}/s$ that does not satisfy Property P. Ignoring signatures momentarily, we claim that M has a modular triple of circuits, (D_1, D_2, D_3), such that $C_i \subseteq D_i$. The signature of D_i is an extension of the signature of C_i, so (D_1, D_2, D_3) cannot satisfy Property P, because this would imply that (C_1, C_2, C_3) satisfies Property P.

To prove our claim, we observe that M has circuits D_1, D_2, and D_3 such that $C_i = D_i \setminus \{s\}$, and (D_1, D_2), (D_1, D_3), and (D_2, D_3) are modular pairs of circuits [8, Lemma 2.3]. We know that $D_i = C_i$ or $D_i = C_i \cup \{s\}$. Also, D_1, D_2, and D_3 are distinct circuits because C_1, C_2, and C_3 are distinct. To prove that (D_1, D_2, D_3) is a modular triple of circuits of M, we need only prove that $D_i \subseteq (D_j \cup D_k)$ for distinct i, j, and k. The only way that $D_i \subseteq (D_j \cup D_k)$ can fail is when $D_i = C_i \cup \{s\}$, $D_j = C_j$, and $D_k = C_k$. Suppose this happens. Let E be the ground set of M. Define $H_i^* = (E \setminus \{s\}) \setminus C_i$. Then (H_1^*, H_2^*, H_3^*) is a modular triple of copoints of $M^* \setminus s$. Also, $(H_i^*, H_j^* \cup \{s\}, H_k^* \cup \{s\})$ is a modular triple of copoints of M^*. Since M^* is a single-element extension of $M^* \setminus s$, H_j^* and H_k^* are in the associated modular cut of $M^* \setminus s$, but H_i^* is not in the modular cut. According to the definition of a modular cut, this is impossible because (H_j^*, H_k^*) is a modular pair and so H_i^* must be in the modular cut as well. This contradiction proves that $D_i \subseteq (D_j \cup D_k)$. $\square$

Lemma 3.3. *Let $\mathcal{C}$ be a circuit signature of M, and let $A \subseteq E$.*

(1) [1, *Proposition 1.7*] *If $\mathcal{C}$ is a weak orientation of M, then ${}_{\overline{A}}\mathcal{C}$ is also a weak orientation of M.*

(2) [2, *Section* 2] *If $\mathcal{C}$ is an orientation of M, then ${}_{\overline{A}}\mathcal{C}$ is also an orientation of M.*

(3) [10, *Section* 2] *If $\mathcal{C}$ is a ternary signature of M, then ${}_{\overline{A}}\mathcal{C}$ is also a ternary signature of M.*

Lemma 3.4. *Let $\mathcal{C}$ be a weak orientation of M, and let (C_1, C_2, C_3) be a modular triple of signed circuits. Let i, j, and k be distinct elements of $\{1, 2, 3\}$. If $\{x_1, x_2\} \subseteq (C_i \cap C_j) \setminus C_k$ and $x_1 \in (C_i^+ \cap C_j^+) \cup (C_i^- \cap C_j^-)$, then $x_2 \in (C_i^+ \cap C_j^+) \cup (C^- \cap C_j^-)$.*

Proof. Assume $x_1 \in C_i^+ \cap C_j^+$ and $x_2 \in C_i^+ \cap C_j^-$. If the conclusion is false, then Theorem 2.3(i) guarantees the existence of $X_3 \in \mathcal{C}$ where $x_1 \in X_3 \subseteq (C_i \cup C_j) \setminus \{x_2\}$. By Lemma 2.2, C_k is the unique circuit in $(C_i \cup C_j) \setminus \{x_2\}$. Thus $X_3 = \pm C_k$, which contradicts $x_1 \notin C_k$. $\square$

Proof of Theorem 3.1(1). Assume $\mathcal{C}$ is a weak orientation of M, and let (C_1, C_2, C_3) be a modular triple of signed circuits. By Lemma 3.3(1), we may assume that $C_1 = (I \cup I_{13}, I_{12})$.

We show that either $I_{12} \subseteq C_2^+$ or $I_{12} \subseteq C_2^-$. If not, there exist y_1 and y_2 in I_{12} such that $y_1 \in C_2^+ \cap C_1^-$ and $y_2 \in C_2^- \cap C_1^-$. This contradicts Lemma 3.4. We may assume that $I_{12} \subseteq C_2^+$. By reorientation in I_{23}, we may also assume that $I_{23} \subseteq C_2^-$. We have found that $C_2 = (I_1 \cup I_{12}, I_2 \cup I_{23})$, where $I_1 \uplus I_2 = I$.

Similarly, $I_{13} \subseteq C_3^+$ or $I_{13} \subseteq C_3^-$, and $I_{23} \subseteq C_3^+$ or $I_{23} \subseteq C_3^-$. Suppose that the elements of $I_{13} \cup I_{23}$ all have the same sign in C_3. We may assume that $I_{13} \cup I_{23} \subseteq C_3^-$. Choose $x \in I_{13}$. When we apply Theorem 2.3(ii) to x, C_1, and C_3, we find that $e_1 \in I_{12}$ and $e_2 \in I_{23}$, so $C_1(e_1)C_3(e_2) = +1$. However, by Lemma 2.2, $X_4 = \pm C_2$, and in both cases $X_4(e_1)X_4(e_2) = -1$, a contradiction. Thus, $C_3 = (I_3 \cup I_{23}, I_4 \cup I_{13})$, where $I_3 \uplus I_4 = I$.

Now assume that $\mathcal{C}$ satisfies the WDP. By Lemma 3.2, any minor of $\mathcal{C}$ must also satisfy the WDP. Thus an induced $U_{1,3}$ circuit signature must be isomorphic to a reorientation of $\{12, 13, 2\overline{3}\}$. Using Theorem 2.4, we conclude that $\mathcal{C}$ is a weak orientation of M. $\square$

Proof of Theorem 3.1(2). Assume that $\mathcal{C}$ is an orientation of M, and let (C_1, C_2, C_3) be a modular triple of signed circuits. By Theorem 3.1(1) and Lemma 3.3(2), we may assume that $C_1 = (I \cup I_{13}, I_{12})$, $C_2 = (I_1 \cup I_{12}, I_2 \cup I_{23})$, and $C_3 = (I_3 \cup I_{23}, I_4 \cup I_{13})$, where $I_1 \uplus I_2 = I_3 \uplus I_4 = I$.

Choose $y \in I_{12}$. By applying signed circuit elimination to y, C_1, and C_2, we find that $\mathcal{C}$ has a signed circuit $C \subseteq (C_1 \cup C_2) \setminus \{y\}$ with $C^+ \subseteq I \cup I_{13} \cup I_{12} \setminus \{y\}$ and $C^- \subseteq I_2 \cup I_{12} \setminus \{y\} \cup I_{23}$. By Lemma 2.2, $C = \pm C_3$. Thus $(I_3 \cup I_{23}) \subseteq (I_2 \cup I_{23})$, which implies that $I_3 \subseteq I_2$.

Now assume that $\mathcal{C}$ satisfies the WDP with $I_3 \subseteq I_2$. Let (C_1, C_2) be a modular pair of signed circuits such that $C_1 \neq \pm C_2$, and let $e \in C_1^+ \cap C_2^-$. By Lemma 2.2, there exists a unique circuit C_3 such that $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$. Moreover, (C_1, C_2, C_3) is a modular triple, and $e \in I_{12}$.

We must prove that for $\tau = +$ or $\tau = -$, $(\tau C_3)^+ \subseteq (C_1^+ \cup C_2^+) \setminus \{e\}$ and $(\tau C_3)^- \subseteq (C_1^- \cup C_2^-) \setminus \{e\}$. Up to reorientation, we know that $C_1 = (I \cup I_{13}, I_{12})$, $C_2 = \pm(I_1 \cup I_{12}, I_2 \cup I_{23})$, and $C_3 = \pm(I_3 \cup I_{23}, I_4 \cup I_{13})$, where $I_1 \uplus I_2 = I_3 \uplus I_4 = I$ and $I_3 \subseteq I_2$. By construction, e has opposite signs in C_1 and C_2. Thus $C_2 = (I_1 \cup I_{12}, I_2 \cup I_{23})$. Since we assumed that $I_3 \subseteq I_2$, the result follows. $\square$

Proof of Theorem 3.1(3). Assume that $\mathcal{C}$ is a ternary signature, and let (C_1, C_2, C_3) be a modular triple of signed circuits. By Theorem 3.1(1) and Lemma 3.3(3), we may assume that $C_1 = (I \cup I_{13}, I_{12})$, $C_2 = (I_1 \cup I_{12}, I_2 \cup I_{23})$, and $C_3 = (I_3 \cup I_{23}, I_4 \cup I_{13})$, where $I_1 \uplus I_2 = I_3 \uplus I_4 = I$.

Choose $x \in I_{13}$, and let C_1, C_2, and x play the respective roles of X_1, X_2, and f in Theorem 2.7(2). Thus there exists $X_3 \in \mathcal{C}$ such that $X_3 \subseteq (C_1 \cup C_2) \setminus (I_2 \cup I_{12})$. But C_3 and $-C_3$ are the only signed circuits contained in $(C_1 \cup C_2) \setminus I_{12}$, so $X_3 = \pm C_3$. However, $I \subseteq C_3$, so $I_2 = \emptyset$. An identical argument shows that $I_4 = \emptyset$.

To prove sufficiency, assume that $\mathcal{C}$ satisfies the WDP with $I_1 = I_3 = I$. By Theorems 3.1(1) and 2.4, $\mathcal{C}$ has no minor isomorphic to a reorientation of $\{12, 13, 23\}$.

According to Lemma 3.2, any minor that is a signature of $U_{2,4}$ must also satisfy the WDP with $I_1 = I_3 = I$. We claim that such a minor is a reorientation of $\{12\overline{3}, 1\overline{2}4, 13\overline{4}, 234\}$. There is no way to reorient this signature so that exactly two circuits have positive signatures; thus $\mathcal{C}$ has no minor isomorphic to a reorientation of the signatures in Theorem 2.7(3). It follows that $\mathcal{C}$ is a ternary signature of M.

We conclude with a proof of our claim. Since any three circuits of $U_{2,4}$ form a modular triple, we know that the signatures of $\{1, 2, 3\}$, $\{1, 2, 4\}$, and $\{1, 3, 4\}$ are some reorientation of $12\overline{3}$, $1\overline{2}4$, and $13\overline{4}$. We also know that the signatures of $\{1, 2, 3\}$, $\{1, 2, 4\}$, and $\{2, 3, 4\}$ are some reorientation of $12\overline{3}$, $\overline{1}24$, and $23\overline{4}$. Putting these two facts together, we see that the circuit signature of $U_{2,4}$ must be some reorientation of $\{12\overline{3}, 1\overline{2}4, 13\overline{4}, 234\}$. $\square$

4. Lifting signatures

4.1. Definitions

Now we generalize the results of Section 2.2, where we saw that gains enabled the construction of graphic-matroid lifts. The main idea is to replace information obtained from graphs with information obtained from matroid circuit signatures.

Let $\Phi = (\Gamma, \phi)$ be a gain graph with gain group $\mathfrak{G}$. We can think of Γ as being a directed graph because ϕ oriented the edges in order to assign gains. There is a standard way of associating this directed graph with an orientation $\mathcal{C}$ of the graphic matroid $G(\Gamma)$ (see [3, Section 1.1]). Arbitrarily, we assign an orientation to each circle of Γ; an element of a signed circuit is positive if its direction agrees with the orientation assigned that circle, and it is negative otherwise.

Suppose the circle B in Fig. 4.1 is in Γ. According to Section 2.2, B is balanced if and only if $\phi(e_1)\phi(e_2)\phi(e_3)^{-1}\phi(e_4)\phi(e_5)^{-1} = 1$. Balance can also be defined using the circuit signature

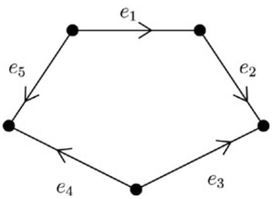

Fig. 4.1. This is a circle of Φ. The arrows on the edges indicate the orientations prescribed by ϕ.

$\mathcal{C}$ which we described above. In our example, $(\{e_1, e_2, e_4\}, \{e_3, e_5\}) \in \mathcal{C}$. Assuming that the gain group is Abelian, B is balanced if and only if

$$\prod_{e \in B^+} \phi(e) \prod_{e \in B^-} \phi(e)^{-1} = 1.$$

We require that the gain group be Abelian; otherwise, this product may not be well defined.

Our example illustrates how circuit signatures determine whether or not a matroid circuit is balanced. Let M be a matroid on E, let $\mathcal{C}$ be a circuit signature of M, and let $\mathfrak{A}$ be an Abelian group. A *gain mapping* ϕ is a function from E into $\mathfrak{A}$. We call $\mathfrak{A}$ the *gain group*. Let C be a circuit of M, so C is the support of two signed circuits in $\mathcal{C}$. Suppose one of these signed circuits is $(\{a_1, \ldots, a_p\}, \{b_1, \ldots, b_n\})$. We define the *gain* of C to be

$$\phi(C) = \prod_{a \in C^+} \phi(a) \prod_{b \in C^-} \phi(b)^{-1}.$$

We say that C is *balanced* if $\phi(C) = 1$. Let $\mathcal{B}(\phi, \mathcal{C})$ denote the class of balanced circuits. If $\mathcal{C}$ is clear from the context, we write $\mathcal{B}(\phi)$. If $\mathcal{B}(\phi)$ is a linear class of circuits, we can apply Dowling and Kelly's lift construction to obtain $L(M, \mathcal{B}(\phi))$.

It is certainly not the case that $\mathcal{B}(\phi, \mathcal{C})$ is linear for all choices of ϕ and $\mathcal{C}$. To generalize the graphic case, where $\mathcal{B}(\phi)$ is always linear for all gain mappings, we must be selective when choosing a circuit signature. A matroid M can be *lifted by gains in* $\mathfrak{A}$ if M has a circuit signature $\mathcal{C}$ such that $\mathcal{B}(\phi, \mathcal{C})$ is linear for all $\phi : E \to \mathfrak{A}$. In this case, we call $\mathcal{C}$ a *lifting signature for gains in* $\mathfrak{A}$. Which matroids have lifting signatures? Since linear classes of circuits are central to the definition of a lifting signature, and since they are defined in terms of modular triples, it is natural that modular triples be used to characterize lifting signatures.

4.2. Using gains to lift binary and ternary matroids

Our goal is to classify the matroids that can be lifted by gains. Given a gain mapping ϕ, define a new gain mapping ϕ_A by

$$\phi_A(e) = \begin{cases} \phi(e) & \text{if } e \notin A \\ \phi(e)^{-1} & \text{if } e \in A. \end{cases}$$

Lemma 4.1. *Let $\mathcal{C}$ be a circuit signature of M, let $A \subseteq E$, and let $\mathfrak{A}$ be an Abelian group.*

(1) *For each gain mapping ϕ, $\mathcal{B}(\phi_A, {}_{\overline{A}}\mathcal{C}) = \mathcal{B}(\phi, \mathcal{C})$.*

(2) *$\mathcal{C}$ is a lifting signature for gains in $\mathfrak{A}$ if and only if ${}_{\overline{A}}\mathcal{C}$ is a lifting signature for gains in $\mathfrak{A}$.*

Proof. Let ϕ be a gain mapping, and let C be a circuit of M. Throughout this proof, $\phi(C)$ is calculated using $\mathcal{C}$ and $\phi_A(C)$ is calculated using ${}_{\overline{A}}\mathcal{C}$.

Assume that C is the support of the signed circuit $(\{p_1, \ldots, p_r, a_1, \ldots, a_s\}, \{n_1, \ldots, n_t, b_1, \ldots, b_q\})$ of $\mathcal{C}$, where $A \cap C = \{a_1, \ldots, a_s, b_1, \ldots, b_q\}$. Then C is the support of the signed circuit $(\{p_1, \ldots, p_r, b_1, \ldots, b_q\}, \{n_1, \ldots, n_t, a_1, \ldots, a_s\})$ of ${}_{\overline{A}}\mathcal{C}$. Accordingly,

$$\begin{aligned}\phi_A(C) &= \prod_{i=1}^{r} \phi_A(p_i) \prod_{i=1}^{q} \phi_A(b_i) \prod_{i=1}^{t} \phi_A(n_i)^{-1} \prod_{i=1}^{s} \phi_A(a_i)^{-1} \\ &= \prod_{i=1}^{r} \phi(p_i) \prod_{i=1}^{s} \phi(a_i) \prod_{i=1}^{t} \phi(n_i)^{-1} \prod_{i=1}^{q} \phi(b_i)^{-1} = \phi(C).\end{aligned}$$

It follows immediately that $\mathcal{B}(\phi_A, {}_{\overline{A}}\mathcal{C}) = \mathcal{B}(\phi, \mathcal{C})$.

Assume that $\mathcal{C}$ is not a lifting signature for gains in $\mathfrak{A}$. So there exist a gain mapping ϕ and a modular triple of circuits, (C_1, C_2, C_3), such that $\phi(C_1) = \phi(C_2) = 1$ and $\phi(C_3) \neq 1$. From part (1), it follows that $\phi_A(C_1) = \phi_A(C_2) = 1$ and $\phi_A(C_3) \neq 1$. Thus ${}_{\overline{A}}\mathcal{C}$ is not a lifting signature for gains in $\mathfrak{A}$. If ${}_{\overline{A}}\mathcal{C}$ is not a lifting signature for gains in $\mathfrak{A}$, then ${}_{\overline{A}}({}_{\overline{A}}\mathcal{C}) = \mathcal{C}$ is not a lifting signature for gains in $\mathfrak{A}$. $\square$

Lemma 4.2. *Let $\mathfrak{A}$ be an Abelian group with $\exp(\mathfrak{A}) > 2$, let $\mathcal{C}$ be a lifting signature of M for gains in $\mathfrak{A}$, and let (C_1, C_2, C_3) be a modular triple of signed circuits. Let i, j, and k be distinct elements of $\{1, 2, 3\}$.*

(1) *Assume $\{x_1, x_2\} \subseteq (C_i \cap C_j) \setminus C_k$. If $x_1 \in (C_i^+ \cap C_j^+) \cup (C_i^- \cap C_j^-)$, then $x_2 \in (C_i^+ \cap C_j^+) \cup (C_i^- \cap C_j^-)$.*
(2) *Assume x, y, and z are each in exactly two of C_i, C_j, and C_k. If $x \in C_i^+ \cap C_k^-$, $y \in C_j^+ \cap C_i^-$, $z \in C_j^-$, and $z \in C_k$, then $z \in C_k^+$.*
(3) *Assume $y \in (C_i \cap C_j) \setminus C_k$ and $w \in C_i \cap C_j \cap C_k$. If $y \in C_i^- \cap C_j^+$ and $w \in C_i^+$, then $w \in C_j^+$.*

Proof. Throughout this proof, let $g \in \mathfrak{A}$ have order greater than 2.

For part (1), we may assume that $x_1 \in (C_i^+ \cap C_j^+)$. Otherwise, we could proceed with the proof using the modular triple $(-C_i, -C_j, C_k)$. Suppose the conclusion is false. By relabeling, if necessary, we can assume that $x_2 \in (C_i^+ \cap C_j^-)$. Define a gain mapping ϕ by

$$\phi(e) = \begin{cases} g & \text{if } e \in \{x_1, x_2\}, \\ 1 & \text{otherwise.} \end{cases}$$

Then $\phi(C_j) = g \cdot g^{-1} = 1$ and $\phi(C_k) = 1$, but $\phi(C_i) = g \cdot g = g^2 \neq 1$, which contradicts the assumption that $\mathcal{C}$ is a lifting signature for gains in $\mathfrak{A}$.

For part (2), suppose $z \in C_k^-$. Define a gain mapping ϕ by

$$\phi(e) = \begin{cases} g & \text{if } e \in \{x, y, z\}, \\ 1 & \text{otherwise.} \end{cases}$$

Then $\phi(C_i) = \phi(C_j) = g \cdot g^{-1} = 1$, but $\phi(C_k) = g^{-1} \cdot g^{-1} = (g^{-1})^2 \neq 1$, a contradiction.

For part (3), suppose $w \in C_j^-$. Define a gain mapping ϕ by

$$\phi(e) = \begin{cases} g & \text{if } e \in \{w, y\}, \\ 1 & \text{otherwise.} \end{cases}$$

Then $\phi(C_i) = \phi(C_j) = g \cdot g^{-1} = 1$. It is not known whether w is positive or negative in C_k, so $\phi(C_k) = g$ or $\phi(C_k) = g^{-1}$. In either case, $\phi(C_k) \neq 1$, a contradiction. □

Proof of Theorem 3.1(4). Assume $\mathcal{C}$ is a lifting signature for gains in $\mathfrak{A}$, and let (C_1, C_2, C_3) be a modular triple of signed circuits. By Lemma 4.1(2), we may assume that $C_1 = (I \cup I_{13}, I_{12})$.

We will show that either $I_{12} \subseteq C_2^+$ or $I_{12} \subseteq C_2^-$. If not, there exist y_1 and y_2, both elements of I_{12}, such that $y_1 \in C_1^- \cap C_2^-$ and $y_2 \in C_1^- \cap C_2^+$. This contradicts Lemma 4.2(1). We may assume that $I_{12} \subseteq C_2^+$ and that $I_{23} \subseteq C_2^-$. Applying Lemma 4.2(3), the elements of I have the same sign in C_2 as the elements of I_{12}. Thus $C_2 = (I \cup I_{12}, I_{23})$. Similarly, $I_{13} \subseteq C_3^+$ or $I_{13} \subseteq C_3^-$. Furthermore, by Lemma 4.2(2,3), we find that the elements of $I \cup I_{23}$ and those of I_{13} have opposite signs in C_3. Thus $C_3 = (I \cup I_{23}, I_{13})$. We have proved the necessity of part (4a).

To prove the necessity of part (4b), we must show that $\exp(\mathfrak{A}) = 3$. Suppose $\exp(\mathfrak{A}) \neq 3$, so that there exists $g \in \mathfrak{A}$ such that $g^3 \neq 1$. Since M is not binary, we apply Lemma 2.1 to find a modular triple of signed circuits, (C_1, C_2, C_3), with nonempty intersection. We must show that $\mathcal{C}$ is not a lifting signature. By the above argument, we may assume that $C_1 = (I \cup I_{13}, I_{12})$, $C_2 = \pm(I \cup I_{12}, I_{23})$, and $C_3 = \pm(I \cup I_{23}, I_{13})$. Choose $w \in I$, $x \in I_{13}$, and $z \in I_{23}$, and define a gain mapping ϕ by

$$\phi(e) = \begin{cases} g & \text{if } e = x, \\ g^{-1} & \text{if } e = w \text{ or } z, \\ 1 & \text{otherwise.} \end{cases}$$

Then $\phi(C_1) = \phi(C_2) = 1$, but $\phi(C_3)$ is $(g^{-1})^3$ or g^3, neither of which is 1. Thus $\mathcal{C}$ is not a lifting signature for gains in $\mathfrak{A}$. This contradicts our hypothesis, so $\exp(\mathfrak{A}) = 3$.

Let (C_1, C_2, C_3) be a modular triple of signed circuits. We must prove that $\mathcal{C}$ is a lifting signature. We may assume that $C_1 = (I \cup I_{13}, I_{12})$, $C_2 = (I \cup I_{12}, I_{23})$, and $C_3 = (I \cup I_{23}, I_{13})$.

Let ϕ be a gain mapping. We must show that if $\phi(C_1) = \phi(C_2) = 1$, then $\phi(C_3) = 1$. If $\phi(C_1) = \phi(C_2) = 1$, then

$$\prod_{w \in I} \phi(w) \prod_{x \in I_{13}} \phi(x) \prod_{y \in I_{12}} \phi(y)^{-1} = \prod_{w \in I} \phi(w) \prod_{y \in I_{12}} \phi(y) \prod_{z \in I_{23}} \phi(z)^{-1} = 1.$$

Thus

$$\begin{aligned} \phi(C_3) &= \prod_{w \in I} \phi(w) \prod_{z \in I_{23}} \phi(z) \prod_{x \in I_{13}} \phi(x)^{-1} \\ &= \left(\prod_{w \in I} \phi(w)\right) \left(\prod_{w \in I} \phi(w) \prod_{y \in I_{12}} \phi(y)\right) \left(\prod_{w \in I} \phi(w) \prod_{y \in I_{12}} \phi(y)^{-1}\right) \\ &= \prod_{w \in I} (\phi(w))^3. \end{aligned}$$

If M is binary, then $I = \emptyset$ (see Lemma 2.1). If M is not binary, then $\exp(\mathfrak{A}) = 3$. In both cases, $\phi(C_3) = 1$. □

Theorem 4.3. *Let $\mathcal{C}$ be a circuit signature of a matroid M, and let $\mathfrak{A}$ be an Abelian group with* $\exp(\mathfrak{A}) > 2$. *Then $\mathcal{C}$ is a lifting signature for gains in $\mathfrak{A}$ if and only if $\mathcal{C}$ is a ternary signature and, when M is not binary,* $\exp(\mathfrak{A}) = 3$.

Theorem 4.4 classifies the matroids that can be lifted by gains from a group of exponents greater than 2. Theorem 4.5 is a classification for gain groups of exponent 2.

Theorem 4.4. *Let M be a matroid, and let $\mathfrak{A}$ be an Abelian group such that* $\exp(\mathfrak{A}) > 2$*. Then M can be lifted by gains in $\mathfrak{A}$ if and only if M is ternary and, when M is not binary,* $\exp(\mathfrak{A}) = 3$*. Moreover, the lifting signature is the ternary signature associated with M, which is unique up to reorientation.*

Proof. M can be lifted by gains in $\mathfrak{A}$ if and only if M has a lifting signature for gains in $\mathfrak{A}$; call it $\mathcal{C}$. From Theorem 4.3, we see that $\mathcal{C}$ is also a ternary signature. But M has a ternary signature if and only if M is ternary. Moreover, Theorem 2.6 guarantees that a ternary matroid has precisely one ternary signature, up to reorientation. $\square$

Theorem 4.5. *Let $\mathfrak{A}$ be an Abelian group with* $\exp(\mathfrak{A}) = 2$*, and let $\mathcal{C}$ be a circuit signature of a matroid M. Then $\mathcal{C}$ is a lifting signature for gains in $\mathfrak{A}$ if and only if M is binary.*

Proof. If M is not binary, then there exists a modular triple of signed circuits, (C_1, C_2, C_3), with nonempty intersection. Let $w \in I$ and $y \in I_{12}$, and let $g \in \mathfrak{A}$ be any element other than 1. We define a gain mapping ϕ by

$$\phi(e) = \begin{cases} g & \text{if } e \in \{w, y\}, \\ 1 & \text{otherwise.} \end{cases}$$

Then $\phi(C_1) = \phi(C_2) = 1$, but $\phi(C_3) = g \neq 1$. Thus $\mathcal{C}$ is not a lifting signature for gains in $\mathfrak{A}$.

Now assume that M is binary. Let (C_1, C_2, C_3) be a modular triple of signed circuits (so $I = \emptyset$), and let ϕ be a gain mapping. Since $\exp(\mathfrak{A}) = 2$, $\phi(e) = \phi(e)^{-1}$ for all $e \in E$, we may assume that $\mathcal{C}$ is the all-positive signature. For distinct i, j, and k, we must show that if $\phi(C_i) = \phi(C_j) = 1$, then $\phi(C_k) = 1$ as well. Assume $\phi(C_1) = \phi(C_2) = 1$. Thus

$$\prod_{x \in I_{13}} \phi(x) \prod_{y \in I_{12}} \phi(y) = \prod_{y \in I_{12}} \phi(y) \prod_{z \in I_{23}} \phi(z) = 1.$$

Then

$$\phi(C_3) = \prod_{x \in I_{13}} \phi(x) \prod_{x \in I_{23}} \phi(z) = \prod_{y \in I_{12}} (\phi(y))^2 = 1. \quad \square$$

5. Applications

Here we provide quick proofs of four known facts about circuit signatures. The reference to lifting signatures in part (1) is new.

Corollary 5.1. *Let $\mathcal{C}$ be a circuit signature of a matroid M, and let $\mathfrak{A}$ be an Abelian group with* $\exp(\mathfrak{A}) > 2$*, and with* $\exp(\mathfrak{A}) = 3$ *if M is not binary.*

(1) *Assume M is binary. The following are equivalent: $\mathcal{C}$ is a lifting signature for gains in $\mathfrak{A}$, $\mathcal{C}$ is an orientation, $\mathcal{C}$ is a weak orientation, and $\mathcal{C}$ is a ternary signature.*
(2) *A binary matroid is orientable if and only if it is regular.*
(3) *Assume M is regular and $\mathcal{C}$ is an orientation. Then, up to reorientation, $\mathcal{C}$ is unique.*
(4) *If M is not binary and $\mathcal{C}$ is a ternary signature, then $\mathcal{C}$ is a weak orientation but is not an orientation.*

Proof. (1) By Lemma 2.1, if (C_1, C_2, C_3) is a modular triple of circuits of M, then $C_1 \cap C_2 \cap C_3 = \emptyset$. The proof follows immediately from Theorem 3.1.

(2) Let $\mathcal{C}$ be an orientation of a binary matroid M. By part (1), this is equivalent to $\mathcal{C}$ being a ternary signature. Hence M is ternary and binary, and therefore M is regular.

(3) M is both binary and ternary. Since it is binary, part (1) says that $\mathcal{C}$ is also a ternary signature. But by Theorem 2.6, $\mathcal{C}$ is unique up to reorientation.

(4) Since M is not binary, there is a modular triple of circuits, (C_1, C_2, C_3), such that $I \neq \emptyset$. Since $\mathcal{C}$ is ternary, Theorem 3.1 indicates that, up to reorientation and negation, $C_1 = (I \cup I_{13}, I_{12})$, $C_2 = (I \cup I_{12}, I_{23})$, and $C_3 = (I \cup I_{23}, I_{13})$. If $\mathcal{C}$ is also an orientation, then, up to reorientation and negation, $C_1 = (I \cup I_{13}, I_{12})$, $C_2 = (I_1 \cup I_{12}, I_2 \cup I_{23})$, and $C_3 = (I_3 \cup I_{23}, I_4 \cup I_{13})$ for some $I_1 \uplus I_2 = I_3 \uplus I_4 = I$ with $I_3 \subseteq I_2$. Thus $I_1 = I_3 = I$ and $I_2 = I_4 = \emptyset$, which contradicts $I_3 \subseteq I_2$. The result follows because ternary signatures are weak orientations. $\square$

References

[1] Robert G. Bland, David L. Jensen, Weakly oriented matroids, Manuscript, 1987.

[2] Robert G. Bland, Michel Las Vergnas, Orientability of matroids, J. Combin. Theory Ser. B 24 (1978) 94–123.

[3] Anders Björner, Michel Las Vergnas, Bernd Sturmfels, Neil White, Gunter Zeigler, Oriented Matroids, second ed., in: Encycl. of Math. and its Appl., vol. 46, Cambridge Univ. Press, Cambridge, 1999.

[4] Henry H. Crapo, Single-element extensions of matroids, J. Res. Nat. Bur. Standards (USA) 69B (1965) 55–65.

[5] Thomas A. Dowling, Douglas G. Kelly, Elementary strong maps between combinatorial geometries, in: Colloquio Internationale sulle Teorie Combinatorie (Proc., Rome, 1973), vol. II, Accad. Naz. Lincei, Rome, 1976, pp. 121–152.

[6] Thomas H. Brylawski, Dean Lucas, Uniquely representable combinatorial geometries, in: Colloquio Internationale sulle Teorie Combinatorie (Proc., Rome, 1973), vol. I, Accad. Naz. Lincei, Rome, 1976, pp. 83–104.

[7] David L. Jensen, Coloring and duality: Combinatorial augmentation methods, Ph.D. Thesis, Cornell University, 1985.

[8] Michel Las Vergnas, Oriented matroids as signed geometries real in corank 2, in: Finite and Infinite Sets (Proc. 6th Hungarian Comb. Conf., Eger, 1981), North-Holland, Amsterdam, 1984, pp. 555–565.

[9] James G. Oxley, Matroid Theory, Oxford Univ. Press, New York, 1992.

[10] Jean-Pierre Roudneff, Marc Wagowski, Characterizations of ternary matroids in terms of circuit signatures, J. Combin. Theory Ser. B 47 (1989) 93–106.

[11] William Tutte, Graph Theory, in: Encycl. Math. Appl., vol. 21, Cambridge Univ. Press, Cambridge, 2001.

[12] Dominic Welsh, Matroid Theory, Academic Press, London, 1976.

[13] Thomas Zaslavsky, Biased graphs. I. Bias, balance, and gains, J. Combin. Theory Ser. B 47 (1989) 32–52.

[14] Thomas Zaslavsky, Biased graphs. II. The three matroids, J. Combin. Theory Ser. B 51 (1991) 46–72.