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Running effects on lepton mixing angles in flavour models with type I seesaw

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Abstract

We study renormalization group running effects on neutrino mixing patterns when a (type I) seesaw model is implemented by suitable flavour symmetries. We are particularly interested in mass-independent mixing patterns to which the widely studied tribimaximal mixing pattern belongs. In this class of flavour models, the running contribution from neutrino Yukawa coupling, which is generally dominant at energies above the seesaw threshold, can be absorbed by a small shift on neutrino mass eigenvalues leaving mixing angles unchanged. Consequently, in the whole running energy range, the change in mixing angles is due to the contribution coming from charged lepton sector. Subsequently, we analyze in detail these effects in an explicit flavour model for tribimaximal neutrino mixing based on an A_4 discrete symmetry group. We find that for normally ordered light neutrinos, the tribimaximal prediction is essentially stable under renormalization group evolution. On the other hand, in the case of inverted hierarchy, the deviation of the solar angle from its TB value can be large depending on mass degeneracy, putting strong constraints on the flavour model.

Keywords: Running group equations; Neutrino masses and mixings; Tribimaximal pattern; Flavour symmetries

1. Introduction

The seesaw mechanism [1] explains the lightness of left-handed (LH) neutrinos in a simple and elegant way. However, the seesaw models usually contain many more parameters than low energy observables. In order to economically describe the neutrino oscillations, new ingredients

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are needed: the introduction of a horizontal symmetry could improve the situation and indeed many examples of this kind are present in literature. A common feature of these models is to provide a description of neutrino masses and mixings only at a very high energy scale. On the other hand, for a comparison of experimental results with the high energy predictions from flavour symmetries, it is important to evolve the observables to low energies through renormalization group (RG) running. In general, the deviations from high energy values due to the RG running consist in minor corrections, but the future improvements of neutrino experiments could hopefully bring the precision down to these small quantities.

In the MSSM context, the RG effect on the light neutrino mass operator m_ν in the leading Log approximation can approximately be parametrized as

$$m_{\nu\,(\text{lower energy})} = I_U J_e^T J_\nu^T m_{\nu\,(\text{higher energy})} J_\nu J_e$$

where I_U is a universal contribution, J_e, that is proportional to $Y_e^\dagger Y_e$ with Y_e the charged Yukawa coupling, is always flavour-dependent and, J_ν may depend on flavour or not. The running contributions to m_ν from J_e are similar to those below the lightest right-handed (RH) neutrino mass and it is as known well under control. In fact, large RG effects can be expected only in the case of a degenerate light neutrino spectrum or for large values of $\tan\beta$ [2]. The running contribution from J_ν, however, is generally more complicated depending non-trivially on the neutrino Yukawa coupling Y_ν. Moreover, the last contribution can even be the dominant one since Y_ν is expected to be of order one [3]. Similar conclusions can be driven in the SM context, when assuming a flavour origin of the neutrino mass term m_ν. In seesaw models implemented by flavour symmetries, Y_ν is usually subject to constraints in order to efficiently describe the observed neutrino mixing structure. We should expect that these constraints have also some important impacts on running effects. In this paper, we will show that, under quite general assumptions, flavour symmetries imply a J_ν contribution which has not effects on mixing angles. As a consequence even including energy ranges above and between the RH neutrino scales, the contribution from the charged lepton Yukawa coupling J_e on mixing patterns dominates.

In the first part of the paper, we will focus on a general class of flavour models in which the mixing textures are independent from the mass eigenstates. These mass-independent patterns usually exhibit a underlying discrete symmetry nature. A very well known example is the tribimaximal (TB) mixing scheme [4], defined by

$$U_{\text{TB}} = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & +1/\sqrt{2} \end{pmatrix}, \tag{1}$$

which provides a very simple first order description of the existing oscillation data [5–7]. It has been widely studied in the last years, naturally arising in the context of non-Abelian discrete flavour symmetries, such as A_4 [8–11], T' [12] and S_4 [13,14]. Another important historical example of mass-independent mixing scheme is the bimaximal (BM) mixing (see [15] for a recent revival of BM in the context of the discrete symmetry S_4 and for an up-to-date list of references). The running effects on BM [16] and TB [17] mixing patterns have already been studied in literature without, however, consider an explicit realization based on flavour symmetry.

As an explicit example, in the second part of the paper, we describe in detail the RG effects on the TB mixing texture in the lepton flavour model proposed by Altarelli and Feruglio (AF) [8]. At leading order, this model contains less parameters than the case of a general TB pattern considered in [17]. Then our result may not a priori reproduce the same results obtain in [17]. In our analysis, the RG corrections to mixing angles and the phases are discussed as functions of the lightest neutrino mass and the type of spectrum. The analysis has been performed both in the Standard Model (SM) framework and the its minimal SUSY extension (MSSM).

The outline of the paper is organized as follows. In Section 2, we analytically discuss the RG effects on the neutrino mass operator m_ν in a type I seesaw model. In Section 3, we characterize the general class of flavour models in which J_ν has no effects on mixing angles. Then we turn to RG effects from J_e on TB mixing pattern. In Section 4 we introduce the AF model and show its main features including also the next-to-leading (NLO) contributions coming from higher-dimensional operators. In Section 5, a more detailed numerical analysis on the impact of RG running in the AF model is given. The result has been compared also with the NLO contributions. In the end in Section 6 we conclude summarizing our main results.

2. RG effects on neutrino mass operator m_ν

In this section we begin to analyze, in a general context, the RG equations for neutrino masses below and above the seesaw threshold, both in the SM and in MSSM extended with three right-handed neutrinos. For definiteness, we consider the following Lagrangian

$$\mathcal{L} = e^{cT} Y_e H^\dagger \ell + \nu^{cT} Y_\nu \tilde{H}^\dagger \ell + \nu^{cT} M_R \nu^c + \text{h.c.} \tag{2}$$

where ℓ are the LH lepton doublets, e^c the RH charged lepton singlets and H ($\tilde{H} \equiv i\sigma_2 H^*$) is the Higgs doublet. In the supersymmetric case, this Lagrangian should be identified to a superpotential where $H(\tilde{H})$ is replaced by $h_d(h_u)$ and all the fields are instead supermultiplets. In what follows we concentrate only on the SM particles, postponing the study of their supersymmetric partners elsewhere: for this reason in our notation a chiral superfield and its R-parity even component are denoted by the same letter.

Given the heavy Majorana and the Dirac neutrino mass matrices, M_R and $m_D = Y_\nu v/\sqrt{2}$ respectively, the light neutrino one is obtained from block-diagonalizing the complete 6×6 neutrino mass matrix:

$$m_\nu = -\frac{v^2}{2} Y_\nu^T M_R^{-1} Y_\nu, \tag{3}$$

where v refers to the VEV of the Higgs field, $\langle H \rangle \equiv v/\sqrt{2}$ ($v \approx 246$ GeV). The equivalent relation in the supersymmetric case is achieved by replacing v with v_u. In our notation the VEV of the Higgs fields h_u and h_d are given by $\langle h_{u,d} \rangle \equiv v_{u,d}/\sqrt{2}$, with $\sqrt{v_u^2 + v_d^2} = v$. The matrix m_ν is modified by quantum corrections according to the renormalization group equations (RGEs) widely studied in literature [3]. For completeness, in Appendix A, we report the full RGEs for all the relevant quantities. From these RGEs, one can obtain the RG evolution for the running composite operator $m_\nu(\mu)$ defined in Eq. (3). In order to analytically study the change of $m_\nu(\mu)$ from high to low energy, it is useful to work in the basis in which the Majorana neutrino mass is diagonal and real, $\hat{M}_R = \text{diag}(M_S, M_M, M_L)$. The mass eigenvalues can be ordered as $M_S < M_M < M_L$. Furthermore, we can divide the RG effects in three distinct energy ranges: from the cutoff Λ of the theory down to M_L, the mass of the heaviest RH neutrino; from M_L down to M_S, the mass of the lightest RH neutrino; below M_S down to λ, which can be either m_Z, considered as the electroweak scale, or m_{SUSY}, the average energy scale for the supersymmetric particles.

$\Lambda_f \to M_L$. Above the highest seesaw scale the three RH neutrinos are all active and the dependence of the effective light neutrino mass matrix from the renormalization scale μ is given by mean of the μ-dependence of Y_ν and M_R:

$$m_\nu(\mu) = -\frac{v^2}{2} Y_\nu^T(\mu) M_R^{-1}(\mu) Y_\nu(\mu). \tag{4}$$

Then from the RGEs in Eqs. (64), (65), it is not difficult to see that the evolution of the composite operator m_ν is given by:

$$16\pi^2 \frac{\mathrm{d}m_\nu}{\mathrm{d}t} = \left(C_e Y_e^\dagger Y_e + C_\nu Y_\nu^\dagger Y_\nu\right)^T m_\nu + m_\nu\left(C_e Y_e^\dagger Y_e + C_\nu Y_\nu^\dagger Y_\nu\right) + \bar{\alpha} m_\nu \tag{5}$$

with

$$C_e = -\frac{3}{2}, \qquad C_\nu = \frac{1}{2} \quad \text{in the SM,}$$
$$C_e = C_\nu = 1 \quad \text{in the MSSM} \tag{6}$$

and

$$\bar{\alpha}_{\mathrm{SM}} = 2\,\mathrm{Tr}\left[3Y_u^\dagger Y_u + 3Y_d^\dagger Y_d + Y_\nu^\dagger Y_\nu + Y_e^\dagger Y_e\right] - \frac{9}{10} g_1^2 - \frac{9}{2} g_2^2,$$
$$\bar{\alpha}_{\mathrm{MSSM}} = 2\,\mathrm{Tr}\left[3Y_u^\dagger Y_u + Y_\nu^\dagger Y_\nu\right] - \frac{6}{5} g_1^2 - 6 g_2^2. \tag{7}$$

$M_L \to M_S$. The effective neutrino mass matrix m_ν below the highest seesaw scale can be obtained by sequentially integrating out ν_n^c with $n = L, M, S$:

$$m_\nu = -\frac{v^2}{4}\left(\overset{(n)}{\kappa} + 2\overset{(n)}{Y_\nu^T}\,\overset{(n)}{M_R^{-1}}\,\overset{(n)}{Y_\nu}\right) \tag{8}$$

where $\overset{(n)}{\kappa}$ is the coefficient of the effective neutrino mass operator $\widetilde{H}^\dagger \ell \widetilde{H}^\dagger \ell$. From the (tree-level) matching condition, it is given by

$$\overset{(n)}{\kappa}_{ij} = 2\left(Y_\nu^T\right)_{in} M_n^{-1} (Y_\nu)_{nj}, \tag{9}$$

which is imposed at $\mu = M_n$. At M_L, the 2×3 Yukawa matrix $\overset{(L)}{Y_\nu}$ is obtained by simply removing the Lth row of Y_ν and the 2×2 mass matrix $\overset{(L)}{M_R}$ is found from M_R by removing the Lth row and Lth column. Further decreasing the energy scale down to M_M, $\overset{(M)}{Y_\nu}$ is a single-row matrix, obtained by removing the Mth row from $\overset{(L)}{Y_\nu}$, and $\overset{(M)}{M_R}$ consists of a single parameter, found by removing the Mth row and Mth column from $\overset{(L)}{M_R}$. Finally at M_S, $\overset{(S)}{Y_\nu}$ and $\overset{(S)}{M_R}$ are vanishing.

In the SM, the two parts which define m_ν in Eq. (8) evolve in different ways. We can summarize the corresponding RGEs as follows:

$$16\pi^2 \frac{\mathrm{d}\overset{(n)}{X}}{\mathrm{d}t} = \left(\frac{1}{2}\overset{(n)}{Y_\nu^\dagger}\overset{(n)}{Y_\nu} - \frac{3}{2} Y_e^\dagger Y_e\right)^T \overset{(n)}{X} + \overset{(n)}{X}\left(\frac{1}{2}\overset{(n)}{Y_\nu^\dagger}\overset{(n)}{Y_\nu} - \frac{3}{2} Y_e^\dagger Y_e\right) + \overset{(n)}{\bar{\alpha}}_X \overset{(n)}{X} \tag{10}$$

where

$$\overset{(n)}{\bar{\alpha}}_\kappa = 2\,\mathrm{Tr}\left[3Y_u^\dagger Y_u + 3Y_d^\dagger Y_d + \overset{(n)}{Y_\nu^\dagger}\overset{(n)}{Y_\nu} + Y_e^\dagger Y_e\right] - 3 g_2^2 + \lambda_H,$$
$$\overset{(n)}{\bar{\alpha}}_{Y_\nu^T M_R^{-1} Y_\nu} = 2\,\mathrm{Tr}\left[3Y_u^\dagger Y_u + 3Y_d^\dagger Y_d + \overset{(n)}{Y_\nu^\dagger}\overset{(n)}{Y_\nu} + Y_e^\dagger Y_e\right] - \frac{9}{10} g_1^2 - \frac{9}{2} g_2^2, \tag{11}$$

with λ_H the Higgs self-coupling.[1]

In MSSM the running of $\overset{(n)}{\kappa}$ and of $\overset{(n)}{Y}{}_\nu^T \overset{(n)}{M}{}_R^{-1} \overset{(n)}{Y}_\nu$ is the same and therefore we can write

$$16\pi^2 \frac{\mathrm{d}m_\nu}{\mathrm{d}t} = \big(Y_e^\dagger Y_e + \overset{(n)}{Y}{}_\nu^\dagger \overset{(n)}{Y}_\nu\big)^T m_\nu + m_\nu\big(Y_e^\dagger Y_e + \overset{(n)}{Y}{}_\nu^\dagger \overset{(n)}{Y}_\nu\big) + \overset{(n)}{\bar{\alpha}} m_\nu, \tag{12}$$

where

$$\overset{(n)}{\bar{\alpha}} = 2\,\mathrm{Tr}\big[3Y_u^\dagger Y_u + \overset{(n)}{Y}{}_\nu^\dagger \overset{(n)}{Y}_\nu\big] - \frac{6}{5}g_1^2 - 6g_2^2. \tag{13}$$

$M_S \to \lambda$. For energy range below the mass scale of the lightest RH neutrino, all the ν_n^c are integrated out and $\overset{(S)}{Y}_\nu$ and $\overset{(S)}{M}_R$ vanish. In the right-hand side of Eq. (8) only the term $\overset{(S)}{\kappa}$ is not vanishing and in this case the composite operator m_ν evolves as:

$$16\pi^2 \frac{\mathrm{d}m_\nu}{\mathrm{d}t} = \big(C_e Y_e^\dagger Y_e\big)^T m_\nu + m_\nu\big(C_e Y_e^\dagger Y_e\big) + \overset{(S)}{\bar{\alpha}} m_\nu \tag{14}$$

with

$$\begin{aligned}\overset{(S)}{\bar{\alpha}}_{\mathrm{SM}} &= 2\,\mathrm{Tr}\big[3Y_u^\dagger Y_u + 3Y_d^\dagger Y_d + Y_e^\dagger Y_e\big] - 3g_2^2 + \lambda_H,\\ \overset{(S)}{\bar{\alpha}}_{\mathrm{MSSM}} &= 6\,\mathrm{Tr}\big[Y_u^\dagger Y_u\big] - \frac{6}{5}g_1^2 - 6g_2^2.\end{aligned} \tag{15}$$

2.1. Analytical approximation to RG evolution of m_ν

Now we analytically solve the RG equations for m_ν in the leading Log approximation. All the Yukawa couplings $Y_i^\dagger Y_i$ for $i = \nu, e, u, d$ are valuated at their initial value at the cutoff Λ_f. Furthermore, we will keep only the leading contributions from each $Y_i^\dagger Y_i$ term, for $i = e, u, d$, i.e. $|y_\tau|^2$, $|y_t|^2$ and $|y_b|^2$ respectively. The RG corrections to these quantities would contribute to the final result as subleading effects and we can safely neglect them in the analytical estimate.

In the MSSM context, the general solution to Eqs. (5), (12) and (14) have all the same structure, which is approximately given by

$$m_{\nu\,(\text{lower energy})} \approx I_U J_e^T J_\nu^T m_{\nu\,(\text{higher energy})} J_\nu J_e \tag{16}$$

where I_U, J_e and J_ν are all exponentials of integrals containing loop suppressing factors and as a result they are close to $\mathbb{1}$. Note that I_U is a universal contribution defined as

$$I_U = \exp\left[-\frac{1}{16\pi^2}\int \overset{(n)}{\bar{\alpha}}\,\mathrm{d}t\right] \tag{17}$$

where the integral runs between two subsequent energy scales and we have extended the definition of $\overset{(n)}{\bar{\alpha}}$ by identifying $\overset{(\Lambda)}{\bar{\alpha}} \equiv \bar{\alpha}$ in order to include the range from Λ down to M_L. J_e is the contribution from charged lepton Yukawa couplings which is always flavour-dependent and is given by[2]

[1] We use the convention that the Higgs self-interaction term in the Lagrangian is $-\lambda_H(H^\dagger H)^2/4$.

[2] In Eq. (18), the combination $Y_e^\dagger Y_e$ should enter with $\overset{(n)}{Y}_e$ instead of Y_e, as one can see from the RGEs in Appendix A. In our approximation, however, they coincide.

$$J_e = \exp\left[-\frac{1}{16\pi^2}\int Y_e^\dagger Y_e \, dt\right]. \tag{18}$$

Finally, J_ν is the contribution from the neutrino Yukawa coupling

$$J_\nu = \exp\left[-\frac{1}{16\pi^2}\int \overset{(n)}{Y}{}_\nu^\dagger \overset{(n)}{Y}_\nu \, dt\right], \tag{19}$$

where also here we have extended the definition of $\overset{(n)}{Y}_\nu$ by identifying $\overset{(\Lambda)}{Y}_\nu$ with Y_ν in order to include the range between Λ and M_L. Differently from J_e, J_ν can be flavour-dependent or not.

In the SM context, the RG effect does not factorize, due to the different RG evolution of $\overset{(n)}{\kappa}$ and $\overset{(n)}{Y}{}_\nu^T \overset{(n)}{M}{}_R^{-1} \overset{(n)}{Y}_\nu$ between the seesaw mass thresholds. However, Eq. (16) applies also to the SM context when m_ν is a result of a flavour symmetry: in this case, by a suitable redefinition of the mass eigenvalues, the sum $\overset{(n)}{\kappa} + \overset{(n)}{Y}{}_\nu^T \overset{(n)}{M}{}_R^{-1} \overset{(n)}{Y}_\nu$ after the RG evolution has exactly the same flavour structure of $m_{\nu\,(\text{higher energy})}$. For the purposes of the present discussion we simply assume that Eq. (16) is valid also in the SM context and an explicit example will be proposed in Section 3.3.

Expanded J_e and J_ν in Taylor series and summing up (16) on several energy ranges one can approximately calculate the neutrino mass at low energy as

$$m_{\nu(\lambda)} \simeq I_U\left(m_{\nu(\Lambda)} + \Delta m_\nu^{(J_e)} + \Delta m_\nu^{(J_\nu)}\right), \tag{20}$$

where the low energy scale λ is m_Z in the case of SM and m_{SUSY} for MSSM. The explicit form of the universal part I_U is not useful for the following analysis and we simply omit them. $\Delta m_\nu^{(J_e)}$ is the contribution from J_e and can easily be calculated as:

$$\Delta m_\nu^{(J_e)} = m_{\nu(\Lambda)}\,\text{diag}(0,0,\Delta_\tau) + \text{diag}(0,0,\Delta_\tau)m_{\nu(\Lambda)} \tag{21}$$

where the small parameter Δ_τ is given by

$$\Delta_\tau \equiv -\frac{3m_\tau^2}{16\pi^2 v^2}\ln\frac{\Lambda}{m_Z} \quad \text{in the SM,}$$
$$\Delta_\tau \equiv \frac{m_\tau^2}{8\pi^2 v^2}\left(1+\tan^2\beta\right)\ln\frac{\Lambda}{m_{\text{SUSY}}} \quad \text{in the MSSM} \tag{22}$$

where $\tan\beta$ is the ratio between the VEVs of the neutral spin zero components of h_u and h_d, the two doublets responsible for electroweak symmetry breaking in the MSSM. On the other hand, the contribution from J_ν, $\Delta m_\nu^{(J_\nu)}$, non-trivially depends on the neutrino Yukawa coupling Y_ν which cannot be determined by low energy observables without additional ingredients. In Section 3, we will analyze strong impacts of the flavour symmetries on J_ν.

3. Flavour symmetries and RGE effects

In the present section, we will apply the general results of the RG evolution of the neutrino mass operator m_ν to models beyond the Standard Model, where a flavour symmetry is added to the gauge group of the SM. The main aim is to track some interesting connections between the running effects and how the flavour symmetry is realized in nature.

In a given basis, $Y_e^\dagger Y_e$ and m_ν can be diagonalized by unitary matrices, U_e and U_ν, respectively. The lepton mixing matrix is given by $U_{\text{PMNS}} = U_e^\dagger U_\nu$. The analysis of how U_{PMNS} changes with the RG running has already extensively performed [2] in the context of SM and MSSM. On the other hand, only few studies [3] are present in literature considering the presence of additional RH neutrinos, which originate the type I seesaw mechanism. Here we develop a general RG analysis for seesaw models in which the lepton mixing matrix $U_{\rm PMNS}$ is dictated by a flavour symmetry G_f. It is a common feature in flavour model building that G_f must be spontaneously broken in order to naturally describe fermion masses and mixings, as we will see in Section 4. Here, we simply assume that G_f is spontaneously broken by a set of flavon fields Φ at a very high scale. The symmetry group can be discrete or continuous, global or local (or even a combination of them). Suppose that, at leading order, the neutrino mixing matrix is given by U_0 which differs from $U_{\rm PMNS}$ by subleading contributions $\sim \langle\Phi\rangle/\Lambda_f$ where Λ_f is the cutoff scale of the flavour symmetry G_f. We will begin with some general assumptions on U_0 without, however, specifying its form. Then we will move to specialize in a concrete case in which U_0 is given by the TB mixing pattern.

3.1. Running effects on neutrino mixing patterns

As described in Section 2 the relevant running effects on m_ν are encoded in the combinations $Y_e^\dagger Y_e$ and $Y_\nu^\dagger Y_\nu$. Furthermore, we observe that a relevant contribution to the running of $Y_e^\dagger Y_e$ is encoded by $Y_\nu^\dagger Y_\nu$. We perform the analysis in the basis in which the charged leptons are diagonal, then at high energy we have

$$Y_e^\dagger Y_e = \mathrm{diag}\big(m_e^2, m_\mu^2, m_\tau^2\big)\frac{2}{v^2}. \tag{23}$$

From now on, we will use v in the notation of the SM and in order to convert similar expressions to the MSSM, it is sufficient to substitute v with $v_{u,d}$, when dealing with neutrinos or charged leptons, respectively. Naturally, this simple form should change when evolving down to low energies. The running effect of $Y_e^\dagger Y_e$ on m_ν is of second order and we can safely forget it. However, it can generate a non-trivial U_e and consequently introduces additional corrections to $U_{\rm PMNS}$. We will return to this effect in Section 3.2.

Since flavour symmetries impose constraints on Y_ν, they should have some impacts also on running effects. It is interesting to first comment on Y_ν proportional to a unitary matrix. In this case the study of RG evolutions becomes trivial, since m_ν does not receive any flavour violating contribution from J_ν and all the new effects are encoded in J_e. This case is relevant in flavour physics since $Y_\nu^\dagger Y_\nu \sim \mathbb{1}$ or $Y_\nu Y_\nu^\dagger \sim \mathbb{1}$ is quite frequent in the presence of a flavour symmetry. It is, for example, a consequence of the first Schur's lemma when ℓ or ν^c transforms in a irreducible representation of the group G_f [18].

A second relevant case is when m_ν can be exactly diagonalized by U_0 according to

$$\hat{m}_\nu = U_0^T m_\nu U_0 \tag{24}$$

where $\hat{m}_\nu = \mathrm{diag}(m_1, m_2, m_3)$ with m_i positive and U_0 is a mass-independent mixing pattern enforced by the flavour symmetry G_f. The TB mixing pattern, independently from what is the underlying flavour symmetry, is one of the examples of this class of models. Other examples are given by flavour symmetries which give rise, at leading order, to the bimaximal mixing pattern [15], to the golden ratio mixing [19] and some (but not all) cases of the trimaximal mixing [20].

Independently from the way G_f is broken, the neutrino Yukawa coupling in the basis of diagonal RH Majorana neutrinos, which we indicate as $\hat{Y}_\nu$, has the following simple form

$$\hat{Y}_\nu = i D U_0^\dagger \tag{25}$$

where $D = \text{diag}(\pm\sqrt{2m_1 M_1}, \pm\sqrt{2m_2 M_2}, \pm\sqrt{2m_3 M_3})/v$ [21–23]. Observe that $\hat{Y}_\nu$ becomes unitary if $D = \mathbb{1}$. However, the present case is not strictly a generalization of the previous one since a unitary Y_ν does not necessarily imply a mass-independent mixing pattern.

For energy larger than M_L the neutrino mass matrix is fully given by seesaw formula (8). The initial condition for m_ν is given by

$$m_{\nu(\Lambda)} = U_0^* \hat{m}_\nu U_0^\dagger. \tag{26}$$

In the hatted basis J_ν is proportional to

$$\hat{Y}_\nu^\dagger \hat{Y}_\nu = U_0 D^2 U_0^\dagger. \tag{27}$$

Since U_0 is a mass-independent mixing matrix, we should expect that the effect of J_ν is only to change slightly the mass eigenvalues m_i but not the mixing angles. In fact, the running effect from J_ν is then proportional to

$$\left(\hat{Y}_\nu^\dagger \hat{Y}_\nu\right)^T m_{\nu(\Lambda)} + \text{symmetrization} = \frac{2}{v^2} U_0^* \,\text{diag}\left(m_1^2 M_1, m_2^2 M_2, m_3^2 M_3\right) U_0^\dagger \tag{28}$$

which has exactly the same flavour structure of $m_{\nu(\Lambda)}$.

Now we can move to the energy range between M_L and M_S in which the seesaw formula is only partial as given in Eq. (8). We can exactly proceed in the same way as the previous case considering first the running effect from J_ν on $\overset{(n)}{\kappa}$ in the hatted basis:

$$\left(\overset{(n)}{Y_\nu^\dagger}\overset{(n)}{Y_\nu}\right)^T \overset{(n)}{\kappa} + \text{symmetrization}.$$

Explicitly we have

$$2\sum_{m\neq l}\sum_k \left(U_0^*\right)_{im} D_m^2 \left(U_0^T\right)_{mk} \left(U_0^*\right)_{kl} m_l \left(U_0^\dagger\right)_{lj} = 0$$

where we have used the unitary condition for U_0. As a result, this contribution is only global.

Now we move to analyze the second term in (8). Observing that in the hatted basis we can write

$$\overset{(n)}{Y_\nu^T}\overset{(n)}{M_R^{-1}}\overset{(n)}{Y_\nu} = \sum_{m\neq n} \left(U_0^*\right)_{im} (\hat{m})_m \left(U^\dagger\right)_{mj},$$

using the unitary condition for U_0, we obtain

$$\sum_{m,p\neq n}\sum_k \left(U_0^*\right)_{ip} D_p^2 \left(U_0^T\right)_{pk} \left(U_0^*\right)_{km} (\hat{m})_m \left(U^\dagger\right)_{mj} = \sum_{m\neq n} \left(U_0^*\right)_{km} D_m^2 (\hat{m})_m \left(U^\dagger\right)_{mj},$$

and the same for the symmetrization part. As we can see, this term also is a global contribution. Similarly as in the energy range higher than M_L, also here, the form of m_ν remains invariant and only some of m_i are slightly shifted. These shifts can be resorbed by redefinition of m_i and do not change anyway the mixing angles which are contained in U_0 and independent from mass eigenvalues.

Then we arrive to a very general conclusion, in any flavour symmetries with a mass-independent mixing pattern, the running effects from J_ν correct only the neutrino mass eigenvalues but not the mixing angles. As in the previous class of models the only flavour-dependent RG contribution to m_ν is encoded in J_e.

3.1.1. A special case $U_0 = iU_{TB}P^*$ and $D \propto \mathrm{diag}(1,1,-1)$

In this section we consider a special case of $\hat{Y}_\nu = iDU_0^\dagger$ in which the expression of U_0 is enforced by a flavour symmetry based on A_4 group. The main feature of this model will be presented in the next section together with a more detailed analysis of the running effects. Here we need only the constraint on the mixing matrix $U_0 = iU_{TB}P^*$ and the neutrino Yukawa coupling in the hatted basis:

$$\hat{Y}_\nu \equiv yPU_{TB}^T O_{23} = yP\begin{pmatrix} \sqrt{2/3} & -1/\sqrt{6} & -1/\sqrt{6} \\ 1/\sqrt{3} & +1/\sqrt{3} & +1/\sqrt{3} \\ 0 & +1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \tag{29}$$

where y is a positive parameter of order $\mathcal{O}(1)$, P is a diagonal matrix of phases which corresponds to the Majorana phases and can be written as

$$P = \mathrm{diag}\left(e^{i\alpha_1/2}, e^{i\alpha_2/2}, e^{i\alpha_3/2}\right) \tag{30}$$

and O_{23} is defined as

$$O_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

In order to confront (29) with the general expression $\hat{Y}_\nu = iDU_0^\dagger$ we observe that

$$\hat{Y}_\nu = yPU_{TB}^T O_{23} U_{TB} U_{TB}^T = \mathrm{diag}(y, y, -y) PU_{TB}^T.$$

Then we conclude that (29) corresponds to the special case in which $D = \mathrm{diag}(y, y, -y)$. Furthermore, in the A_4 model considered in this paper, there is a very simple relation between m_i and M_i given by $m_i = v^2 y^2/2M_i$.

Now we explicitly calculate the RG running from Λ_f down to λ for this special case using the approximate analytical expressions given in Section 2.1. In the physical basis, the light neutrino mass matrix from Eq. (3) at the initial energy scale Λ_f can be recovered by imposing the condition in Eq. (26):

$$\begin{aligned} m_\nu^{TB} &= -U_{TB}P\hat{m}_\nu PU_{TB}^T \\ &= -\left[\frac{\tilde{m}_3}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} + \frac{\tilde{m}_2}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \frac{\tilde{m}_1}{6}\begin{pmatrix} 4 & -2 & -2 \\ -2 & 1 & 1 \\ -2 & 1 & 1 \end{pmatrix}\right], \end{aligned} \tag{31}$$

where $\tilde{m}_i = m_i e^{i\alpha_i}$. It is obvious now the meaning of the matrix P as the matrix of the Majorana phases of the light neutrinos. It is necessary to specify the kind of neutrino mass spectrum: in the Normal Hierarchy (NH) case the light neutrinos are ordered as $m_1 < m_2 < m_3$ and the heavy ones as $M_3 < M_2 < M_1$; while in the Inverse Hierarchy (IH) case they are arranged as $m_3 < m_1 \lesssim m_2$ and $M_2 \lesssim M_1 < M_3$.

The general result of the running effects on m_ν is given by Eq. (20) which in our case becomes

$$m_{\nu(\lambda)} = I_U\left(m_\nu^{TB} + \Delta m_\nu^{(J_e)} + \Delta m_\nu^{(J_\nu)}\right). \tag{32}$$

The analytical result for both I_U and $\Delta m_\nu^{(J_e)}$ (see Section 2.1) does not depend on the type of the neutrino spectrum, it is sufficient to identify M_S, M_M, M_L with the correct hierarchy between M_1, M_2, M_3. In particular, for the TB mixing pattern, the contribution from J_e is given by

$$\Delta m_\nu^{(J_e)} = m_\nu^{\rm TB}\,{\rm diag}(0,0,\Delta_\tau) + {\rm diag}(0,0,\Delta_\tau) m_\nu^{\rm TB}$$
$$= -\begin{pmatrix} 0 & 0 & \frac{\tilde m_1}{3}-\frac{\tilde m_2}{3} \\ 0 & 0 & -\frac{\tilde m_1}{6}-\frac{\tilde m_2}{3}+\frac{\tilde m_3}{2} \\ \frac{\tilde m_1}{3}-\frac{\tilde m_2}{3} & -\frac{\tilde m_1}{6}-\frac{\tilde m_2}{3}+\frac{\tilde m_3}{2} & -\frac{\tilde m_1}{3}-\frac{2\tilde m_2}{3}-\tilde m_3 \end{pmatrix}\Delta_\tau. \tag{33}$$

Naturally, the contribution from J_ν depends on the type of the neutrino spectrum, however, it can be written in the same form for both the spectra:

$$\Delta m_\nu^{(J_\nu)} = -\left[\frac{\tilde m_1'}{6}\begin{pmatrix} 4 & -2 & -2 \\ -2 & 1 & 1 \\ -2 & 1 & 1 \end{pmatrix} + \frac{2\tilde m_2'}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \tilde m_3'\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}\right] \tag{34}$$

where $\tilde m_i'$ are redefinitions of the light neutrino masses:

NH case:

$$\tilde m_1' = \tilde m_1(p+q), \qquad \tilde m_2' = \tilde m_2(x+q), \qquad \tilde m_3' = \tilde m_3(x+z) \quad \text{in the SM,}$$
$$\tilde m_1' = 0, \qquad \tilde m_2' = 2\tilde m_2 x, \qquad \tilde m_3' = 2\tilde m_3(x+z) \quad \text{in the MSSM} \tag{35}$$

with

$$p = -\frac{1}{16\pi^2}\left(-3g_2^2 + \lambda + \frac{9}{10}g_1^2 + \frac{9}{2}g_2^2\right)\ln\frac{M_1}{M_2},$$
$$q = -\frac{1}{16\pi^2}\left(-3g_2^2 + \lambda + \frac{9}{10}g_1^2 + \frac{9}{2}g_2^2\right)\ln\frac{M_2}{M_3},$$
$$x = -\frac{y^2}{32\pi^2}\ln\frac{M_1}{M_2},$$
$$z = -\frac{y^2}{32\pi^2}\ln\frac{M_2}{M_3}; \tag{36}$$

IH case:

$$\tilde m_1' = \tilde m_1(x+q), \qquad \tilde m_2' = \tilde m_2(x+z), \qquad \tilde m_3' = \tilde m_3(p+q) \quad \text{in the SM,}$$
$$\tilde m_1' = 2\tilde m_1 x, \qquad \tilde m_2' = 2\tilde m_2(x+z), \qquad \tilde m_3' = 0 \quad \text{in the MSSM} \tag{37}$$

with

$$p = -\frac{1}{16\pi^2}\left(-3g_2^2 + \lambda + \frac{9}{10}g_1^2 + \frac{9}{2}g_2^2\right)\ln\frac{M_3}{M_1},$$
$$q = -\frac{1}{16\pi^2}\left(-3g_2^2 + \lambda + \frac{9}{10}g_1^2 + \frac{9}{2}g_2^2\right)\ln\frac{M_1}{M_2},$$
$$x = -\frac{y^2}{32\pi^2}\ln\frac{M_3}{M_1},$$
$$z = -\frac{y^2}{32\pi^2}\ln\frac{M_1}{M_2}. \tag{38}$$

Comparing $m_\nu^{\rm TB}$ of Eq. (31) with the perturbations Δm_ν of Eqs. (34), we note the presence of the same flavour structure for several matrices and in particular, by redefining $\tilde{m}_i$ to absorb the terms $\tilde{m}'_i$ it is possible to account for the seesaw contributions from the RG running into $m_\nu^{\rm TB}$. As a consequence the LO predictions for the TB angles receive corrections only from the terms proportional to Δ_τ. This result explicitly confirms what we outlined in the previous section.

3.2. RGE effects in the charged lepton sector

The presence of a term proportional to $\hat{Y}_\nu^\dagger \hat{Y}_\nu$ in the RG equation for Y_e can switch on off-diagonal entries in the charged lepton Yukawa matrix Y_e. When rotated away, this additional contribution introduces a non-trivial U_e and consequently corrects the lepton mixing matrix $U_{\rm PMNS}$. For a unitary $\hat{Y}_\nu$, this correction appears only between the seesaw mass scales while, in the general case discussed in Section 3.1, it appears already from the cutoff Λ_f.

In close analogy with the running effects on neutrino mass matrix (32), the full result of the running for charged lepton mass matrix can conventionally be written as

$$\big(Y_e^\dagger Y_e\big)_{(\lambda)} = I_e\big[\big(Y_e^\dagger Y_e\big)_{(\Lambda_f)} + \Delta\big(Y_e^\dagger Y_e\big)\big], \tag{39}$$

where I_e is an irrelevant global coefficient which can be absorbed by, for example, y_τ. Now we move to the case of TB mixing pattern. In this case, the flavour-dependent corrections can be explicitly calculated:

NH case:

$$\Delta\big(Y_e^\dagger Y_e\big) \simeq y_\tau^2 \left[a_e \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -\frac{1}{2} \\ 1 & -\frac{1}{2} & 5 \end{pmatrix} + b_e \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 2 \end{pmatrix} + c_e \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \right], \tag{40}$$

IH case:

$$\Delta\big(Y_e^\dagger Y_e\big) \simeq y_\tau^2 \left[a'_e \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} + b'_e \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} + c'_e \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \right], \tag{41}$$

where the coefficients are

$$\begin{aligned} a_e = b'_e &= -\frac{C'_\nu}{16\pi^2}\frac{y^2}{3}\ln\frac{M_1}{M_2}, \qquad & b_e &= -\frac{C'_\nu}{16\pi^2}\frac{y^2}{2}\ln\frac{M_2}{M_3}, \\ c_e = c'_e &= -\frac{3C'_e y_\tau^2}{16\pi^2}\ln\frac{\Lambda_f}{m_{\rm SUSY}(m_Z)}, \qquad & a'_e &= -\frac{C'_\nu}{16\pi^2}\frac{y^2}{2}\ln\frac{M_3}{M_1}, \end{aligned} \tag{42}$$

and $C'_\nu = -3/2$ (1), $C'_e = 3/2$ (3) in the SM (MSSM). Here we observe that the off-diagonal contributions to $Y_e^\dagger Y_e$ are encoded in a_e, b_e, a'_e and b'_e which depend only on the seesaw scales M_i. As a result, as we will show in the next section, c_e and c'_e do not affect the lepton mixing angles.

3.3. Full RGE effects on the TB mixing pattern

In this section, we combine various contributions discussed in previous sections into the observable matrix $U_{\rm PMNS}$ from which we extract angles and phases at low energy. Since we are interested in physical quantities, we eliminate one of the phases of P defined in (30) and in particular we express each result in function of $\alpha_{ij} \equiv (\alpha_i - \alpha_j)/2$, removing α_3. The corrected mixing angles can be written as

$$\theta_{ij(m_\lambda)} = \theta_{ij}^{\mathrm{TB}} + k_{ij} + \dots \tag{43}$$

where $\theta_{13}^{\mathrm{TB}} = 0$, $\theta_{12}^{\mathrm{TB}} = \arcsin\sqrt{1/3}$, $\theta_{23}^{\mathrm{TB}} = -\pi/4$ and k_{ij} are defined by

$$\begin{aligned}
k_{12} &= \frac{1}{3\sqrt{2}}\left(\frac{|\tilde{m}_1 + \tilde{m}_2|^2}{m_2^2 - m_1^2}\Delta_\tau - 3a_e\right),\\
k_{23} &= \frac{1}{6}\left[\left(\frac{|\tilde{m}_1 + \tilde{m}_3|^2}{m_3^2 - m_1^2} + 2\frac{|\tilde{m}_2 + \tilde{m}_3|^2}{m_3^2 - m_2^2}\right)\Delta_\tau - 3a_e - 6b_e\right] \quad \text{for NH}\\
&= \frac{1}{6}\left[\left(\frac{|\tilde{m}_1 + \tilde{m}_3|^2}{m_3^2 - m_1^2} + 2\frac{|\tilde{m}_2 + \tilde{m}_3|^2}{m_3^2 - m_2^2}\right)\Delta_\tau + 3a_e + 3a'_e\right] \quad \text{for IH},\\
k_{13} &= \frac{1}{3\sqrt{2}}\left(4m_3^2\Delta_\tau^2\left(\frac{m_1\sin\alpha_{13}}{m_1^2 - m_3^2} - \frac{m_2\sin\alpha_{23}}{m_2^2 - m_3^2}\right)^2\right.\\
&\quad \left. + \left[\left(\frac{|\tilde{m}_1 + \tilde{m}_3|^2}{m_1^2 - m_3^2} - \frac{|\tilde{m}_2 + \tilde{m}_3|^2}{\tilde{m}_2^2 - \tilde{m}_3^2}\right)\Delta_\tau - 3a_e\right]^2\right)^{1/2}
\end{aligned}$$

and the dots stand for subleading corrections. In the previous expressions we can clearly distinguish the contributions coming from the diagonalization of the corrected TB neutrino mass matrix (32) and those from the diagonalization of (39). As it is clear from (42), the corrections to the TB mixing from the charged lepton sector is important only for hierarchical RH neutrinos and will approach to zero as soon as the spectrum becomes degenerate. On the other hand, the corrections from the neutrino sector should be enhanced if the light neutrinos are quasi-degenerate and if the $\tan\beta$ is large, in the MSSM case.

The physical Majorana phases are also corrected due to the RG running and we found the following results:

$$\alpha_{ij(m_\lambda)} \simeq \alpha_{ij} + \delta\alpha_{ij}\Delta_\tau + \cdots \tag{44}$$

where α_{ij} are the starting values at Λ_f and

$$\delta\alpha_{13} = \frac{2}{3}\frac{m_1 m_2 \sin(\alpha_{13} - \alpha_{23})}{m_2^2 - m_1^2}, \tag{45}$$

$$\delta\alpha_{23} = \frac{4}{3}\frac{m_1 m_2 \sin(\alpha_{13} - \alpha_{23})}{m_2^2 - m_1^2}. \tag{46}$$

At Λ_f, $\sin\theta_{13}^{\mathrm{TB}}$ is vanishing and as a result the Dirac CP violating phase is undetermined. An alternative is to study the Jarlskog invariants [24] which are well-defined at each energy scale:

$$J_{\mathrm{CP}} = \frac{1}{2}\left|\Im\left\{(U_{\mathrm{PMNS}})^*_{ii}(U_{\mathrm{PMNS}})_{ij}(U_{\mathrm{PMNS}})_{ji}(U_{\mathrm{PMNS}})^*_{jj}\right\}\right|, \tag{47}$$

where $i, j \in \{1, 2, 3\}$ and $i \neq j$. At Λ_f, J_{CP} is vanishing, while after the RG running it is given by

$$J_{\mathrm{CP}} = \frac{1}{18}\left|m_3\left(\frac{m_1\sin\alpha_{13}}{m_1^2 - m_3^2} - \frac{m_2\sin\alpha_{23}}{m_2^2 - m_3^2}\right)\right|\Delta_\tau. \tag{48}$$

Two comments are worth. First of all, in the expression for k_{13}, it is easy to recover the resulting expression for J_{CP} as the first term under the square root, apart global coefficients. This means that the RG procedure introduce a mixing between the expression of the reactor angle and of the Dirac CP-phase. Moreover, we can recover the value of the Dirac CP-phase directly from Eq. (48) and we get the following expression:

$$\cot\delta_{\mathrm{CP}} = -\frac{m_1(m_2^2-m_3^2)\cos\alpha_{13} - m_2(m_1^2-m_3^2)\cos\alpha_{23} - m_3(m_1^2-m_2^2)}{m_1(m_2^2-m_3^2)\sin\alpha_{13} - m_2(m_1^2-m_3^2)\sin\alpha_{23}} - \frac{3a_e(m_2^2-m_3^2)(m_1^2-m_3^2)}{2m_3[m_1(m_2^2-m_3^2)\sin\alpha_{13} - m_2(m_1^2-m_3^2)\sin\alpha_{23}]\Delta_\tau}. \tag{49}$$

In the neutrino sector, the RG contributions from the seesaw terms are present only in the resulting mass eigenvalues:

$$m_{i(\lambda)} \simeq m_i(1+\delta m_i) + \cdots \tag{50}$$

where m_i are the starting values at Λ_f and δm_i, in both the SM and the MSSM and in both the NH and IH spectra, are given by

$$\delta m_1 = \frac{m'_1}{m_1} - \frac{\Delta_\tau}{3}, \qquad \delta m_2 = 2\frac{m'_2}{m_2} - \frac{2\Delta_\tau}{3}, \qquad \delta m_3 = 2\frac{m'_3}{m_3} - \Delta_\tau, \tag{51}$$

with $m'_i \equiv |\tilde{m}'_i|$, given as in Eqs. (35), (37).

4. The Altarelli–Feruglio (AF) model

We recall here the main features of the AF model [8], which is based on the flavour group $G_f = A_4 \times Z_3 \times U(1)_{\mathrm{FN}}$: the spontaneous breaking of A_4, the group of the even permutations of 4 objects, is responsible for the TB mixing; the cyclic symmetry Z_3 prevents the appearance of dangerous couplings and helps keeping separated the charged lepton sector and the neutrino one; the $U(1)_{\mathrm{FN}}$ [25] provides a natural hierarchy among the charged lepton masses. The TB mixing is achieved through a well-defined symmetry breaking mechanism: A_4 is spontaneously broken down to $G_\nu = Z_2$ in the neutrino sector and to a different subgroup $G_\ell = Z_3$ in the charged lepton one. This breaking chain is fundamental in the model, because G_ν and G_ℓ represent the low-energy flavour structures of neutrinos and charged leptons, respectively. This mechanism is produced by a set of very heavy scalar fields, the flavons, which transform only under the flavour group G_f and develop a specific vacuum expectation value alignment. The ratios of these VEVs and the cutoff Λ_f of the theory are small numbers, in particular less than one, which can parametrize by the parameter u, which can vary only inside this range:

$$\begin{aligned} &0.003 \lesssim u \lesssim 0.05 \quad \text{in the SM,} \\ &0.007 \lesssim u \lesssim 0.05 \quad \text{in the MSSM.} \end{aligned} \tag{52}$$

Once defined the transformations of all the fields under G_f, it is possible to write down the Yukawa interactions as an expansion in terms of u. Stopping at the first non-trivial order, the LO results consist in diagonal charged leptons and a neutrino mass matrix which is diagonalized by the TB pattern. The light neutrino masses are directly linked to the heavy neutrino masses through the following relations, which can be found in [18]: the RH neutrino eigenvalues are given by

$$M_1 = |a + b|, \qquad M_2 = |a|, \qquad M_3 = |-a + b| \tag{53}$$

and the corresponding light ones by

$$m_i = \frac{v^2}{2}\frac{y^2}{M_i}. \tag{54}$$

They can be expressed in terms of only three independent parameters. It is possible to choose these parameters in order to simply the analysis: $|a| = |M_2| = v^2|y|^2/(2|m_2|)$ (in the MSSM we replace v with v_u), ρ and Δ, where ρ and Δ are defined as

$$\frac{b}{a} = \rho e^{i\Delta}, \tag{55}$$

where Δ is defined in the range $[0, 2\pi]$. From the experimental side only the squared mass differences have been measured: for the NH (IH) they are [5]

$$\Delta m^2_{\rm sol} \equiv m_2^2 - m_1^2, \qquad \Delta m^2_{\rm atm} \equiv \left|m_3^2 - m_1^2\left(m_2^2\right)\right| \tag{56}$$

As a result the spectrum is not fully determined and indeed Δ is still a free parameter. We can bound this parameter, requiring $|\cos\Delta| \leqslant 1$. In order to get analytical relations for ρ and $\cos\Delta$, we calculate the following mass ratios:

$$\frac{m_2^2}{m_{1(3)}^2} = 1 \pm 2\rho\cos\Delta + \rho^2. \tag{57}$$

It is then easy to express ρ and $\cos\Delta$ as a function of the neutrino masses:

$$\rho = \sqrt{\frac{1}{2}\left(\frac{m_2^2}{m_1^2} + \frac{m_2^2}{m_3^2}\right) - 1}, \qquad \cos\Delta = \frac{\frac{m_2^2}{m_1^2} - \frac{m_2^2}{m_1^2}}{4\sqrt{\frac{1}{2}\left(\frac{m_2^2}{m_1^2} + \frac{m_2^2}{m_3^2}\right) - 1}}. \tag{58}$$

It is interesting to note that this expression holds for both the types of spectra. Using Eqs. (56), it is possible to express $\cos\Delta$ in function of only the lightest neutrino mass and, imposing the constraint $|\cos\Delta| \leqslant 1$, the following ranges can be derived: taking the central values of $\Delta m^2_{\rm sol}$ and $\Delta m^2_{\rm atm}$

$$\begin{aligned} &\text{NH:} \quad 4.46\ \text{meV} < m_1 < 5.91\ \text{meV}, \\ &\text{IH:} \quad 17.1\ \text{meV} < m_3. \end{aligned} \tag{59}$$

For the NH, m_1 spans in a narrow range of values, which corresponds to values of Δ close to zero. On the other hand, for the IH, m_3 is bounded only from below and the minimum is achieved when Δ is close to $\pm\pi$. Furthermore, $\cos\Delta$ is restricted only to negative values (see [14,18] for further details).

From Eq. (54) it is possible to describe the LO spectrum of the RH neutrinos in function of a unique single parameter, which is the lightest LH neutrino mass. In all the allowed range for $m_{1(3)}$, the order of magnitude of the RH neutrino masses is 10^{14-15} GeV. In Fig. 1 we show explicitly the ratios of the RH neutrino masses for NH and the IH cases, on the left and on the right respectively. The ratios are well defined for the NH, thanks to the narrow allowed range for m_1: $M_1/M_3 \sim 11$ and $M_2/M_3 \sim 5$. In the case of the IH, the ratio M_1/M_2 is fixed at 1 while M_3/M_2 varies from about 3 to 1, going from the lower bound of m_3 up to the KATRIN sensitivity.

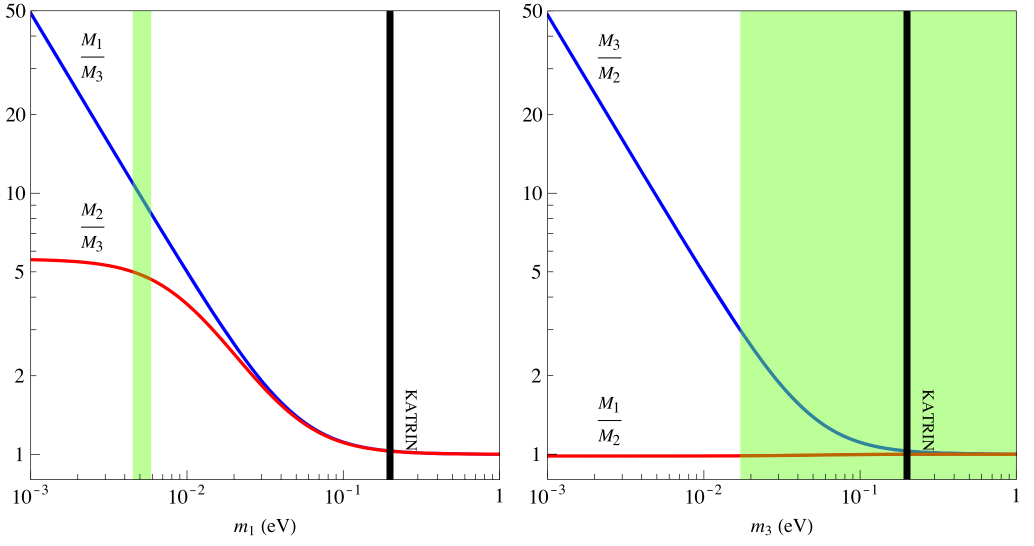

Fig. 1. Plots of the ratios of the heavy RH neutrino masses as a function of the lightest LH neutrino masses. On the left the NH case and on the right the IH one. The green opaque areas refer to the allowed range for $m_{1(3)}$ as in Eq. (59). The vertical black lines correspond to the future sensitivity of 0.2 eV of KATRIN experiment. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

These results are valid only at LO and some deviations are expected with the introduction of the higher order terms. The result of a direct computation shows that for the NH spectrum the corrections leave approximatively unaffected Eqs. (59); this is true for the IH case too, apart when the neutrino masses reach values at about 0.1 eV for which the deviations become significant.

We only commented on the neutrino masses, both light and heavy, and we saw that it is possible to express all of them in function of the lightest LH neutrino mass. The same is true for the phases too: since in the TB mixing the reactor angle is vanishing, the Dirac CP phase is undetermined at LO; on the contrary the Majorana phases are well defined and they can be expressed through ρ and Δ. Since we are interested in physical observables, we report only phases differences, $\alpha_{ij} \equiv (\alpha_i - \alpha_j)/2$:

$$\sin(2\alpha_{23}) = \frac{\rho \sin \Delta}{\sqrt{1 - 2\rho \cos \Delta + \rho^2}},$$
$$\sin(2\alpha_{13}) = \frac{2\rho \sin \Delta}{\sqrt{(\rho^2 - 1)^2 + 4\rho^2 \sin^2 \Delta}}. \tag{60}$$

It will be useful for the following discussion to show also $\sin(2\alpha_{12})$, which enters in the RG evolution of the physical Majorana phases:

$$\sin(2\alpha_{12}) = -\frac{\rho \sin \Delta}{\sqrt{1 + 2\rho \cos \Delta + \rho^2}}. \tag{61}$$

The NLO terms affect also these results for the Dirac and the Majorana phases. All the new parameters which perturb the LO results are complex and therefore they introduce corrections to the phases of the PMNS matrix. Due to the large amount of such a parameters, we expect large deviations from the LO values.

Before concluding, it is interesting to comment on the deviations on the mixing angles coming from the NLO terms: it has already been found [8,18] that these corrections can be safely parametrized as

$$\sin^2\theta_{23} = \frac{1}{2} + \mathcal{O}(u), \qquad \sin^2\theta_{12} = \frac{1}{3} + \mathcal{O}(u), \qquad \sin\theta_{13} = \mathcal{O}(u). \tag{62}$$

5. RG effects in the Altarelli–Feruglio model

In this section we will apply the analysis of RG running effects on the lepton mixing angles to the AF model. In order to perform such a study, it is important to verify the initial assumptions made in Section 3.3, in particular, we see that Eq. (29) exactly corresponds to the one implied by the AF model, when moving to the physical basis. On the other side, the presence of flavon fields has a relevant impact on the results of the analysis. In the unbroken phase, flavons are active fields and should modify the RG equations. Although the study of the relevant Feynman diagrams goes beyond the aim of this work, what follows can be easily proved. Since the only source of the A_4 breaking is the VEVs of the flavons, any flavour structure is preserved above the corresponding energy scale, whatever interactions are present. We can deduce that $\langle\varphi\rangle \sim M_i$ and as a result in the AF model Δ_τ must be proportional to $\ln(\langle\varphi\rangle/\lambda)$ and not to $\ln(\Lambda_f/\lambda)$.

We will separately discuss the evolution of angles and phases for both type of hierarchy. In the following, the results will be shown for the SM and for the MSSM with $\tan\beta = 15$ apart where explicitly indicated otherwise. Without loss of generality, we choose $y = 1$ for our numerical analysis. We also set $\langle\varphi\rangle = 10^{15}$. The spectrum spans the range obtained in (59).

5.1. Running of the angles

Since we are interested in deviations of the corrected mixing angles from the TB predictions and in confronting them with experimental values, it is convenient to relate the coefficients k_{ij} defined in Section 3.3 with physical observables. Keeping in mind that $|k_{ij}| \ll 1$ and that we start from a TB mixing matrix, it follows that

$$\sin\theta_{13} \simeq k_{13}, \qquad \cos 2\theta_{23} \simeq 2k_{23}, \qquad \sin^2\theta_{12} - \frac{1}{3} \simeq \frac{2\sqrt{2}}{3}k_{12}. \tag{63}$$

The corrections to the TB mixing angles as functions of $m_1(m_3)$ in the NH (IH) case are shown in Fig. 2.

We begin with the case of NH. Since the dependence of the corrected mixing angles from Δ_τ is the same, SM corrections are generally expected to be smaller than those in MSSM. However, from Fig. 2 we see that, in NH, there is not a large split between the two curves for SM and MSSM respectively. This fact suggests a dominant contribution coming from the charged lepton sector. For the atmospheric and reactor angles, the deviation from the TB prediction lays roughly one order of magnitude below the 1σ limit. In particular, RG effects on $\sin\theta_{13}$ are even smaller than the NLO contributions which are of $\mathcal{O}(u)$, without cancellations. On the other hand, since the experimental value of the solar angle is better measured than the other two, the running effects become more important in this case. Indeed, RG correction to the TB solar angle evades the 1σ limit as it can be clearly seen in Fig. 2. Anyway, we observe that for both the atmospheric and solar angles, the running contribution is of the same order as the contribution from NLO operators.

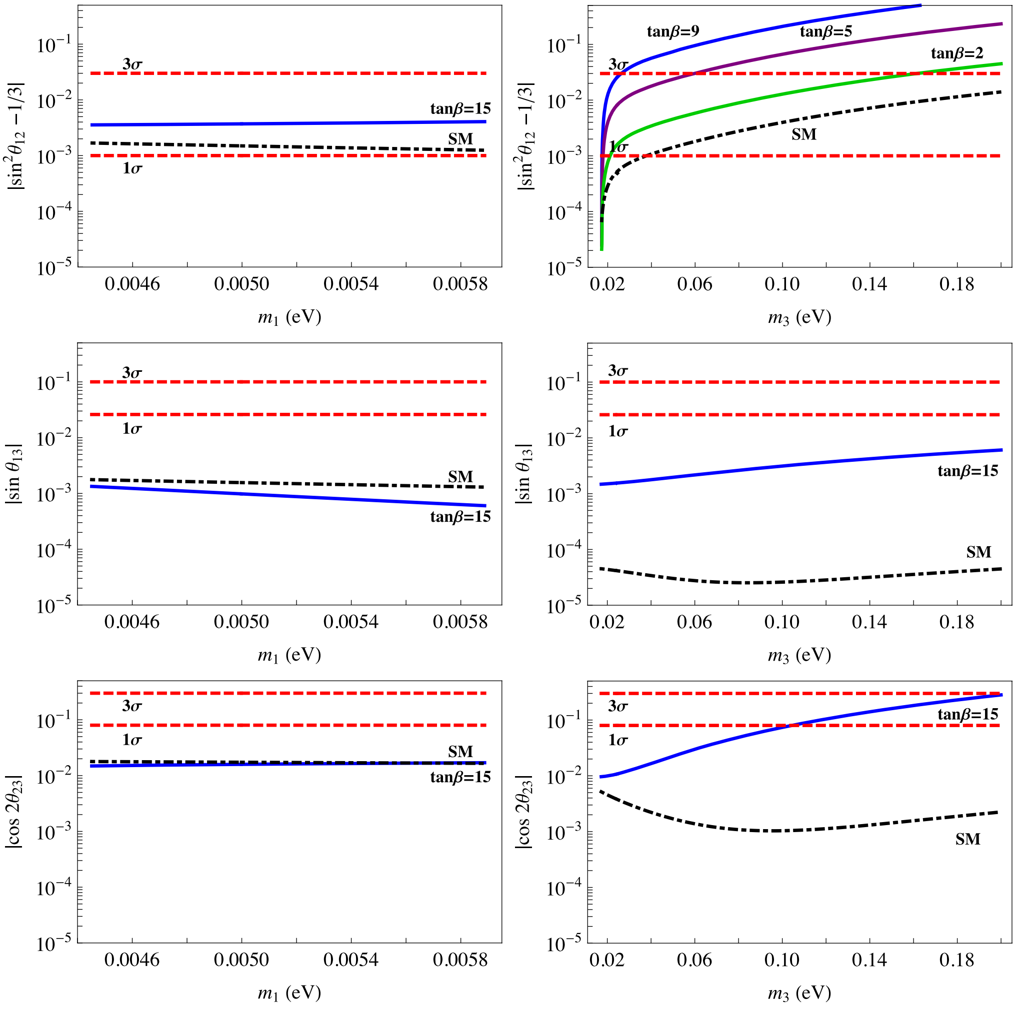

Fig. 2. Corrections to the TB mixing angles as functions of m_1 (m_3) for the NH (IH) are shown. On the left column the plots refer to the NH spectrum, while on the right column the IH case is reported. The plots show the MSSM case with $\tan\beta = 15$ (solid blue) and the SM case (black dashed), compared to the current 1σ and 3σ limits (dashed red). For the specific case of $|\sin^2\theta_{12} - 1/3|$ for the IH, two values for $\tan\beta = 2, 5, 9$ are considered (solid blue, purple and green). All the plots span in a range for m_1 (m_3) which is given by Eq. (59) or by the KATRIN bound. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Now we move to analyze the case of IH. In this case, since the neutrino spectrum predicted by the AF model is almost degenerate, the contribution from the charged lepton sector in Eqs. (41) is subdominant. As a consequence the information which distinguishes the SM case from the MSSM one is mainly dictated by Δ_τ defined in Eq. (22). As a result the running effects in the MSSM are always larger than in the SM and for large $\tan\beta$ they are potentially dangerous. The curves corresponding to the atmospheric and reactor angles do not go above the 3σ and 1σ windows respectively. However, the deviation from θ_{12}^{TB} presents a more interesting situation. For example, for $\tan\beta \gtrsim 10$, the RG effects push the value of the solar angle beyond the 3σ limit for entire spectrum. For lower values of $\tan\beta$, the model is within the 3σ limit only for apart of

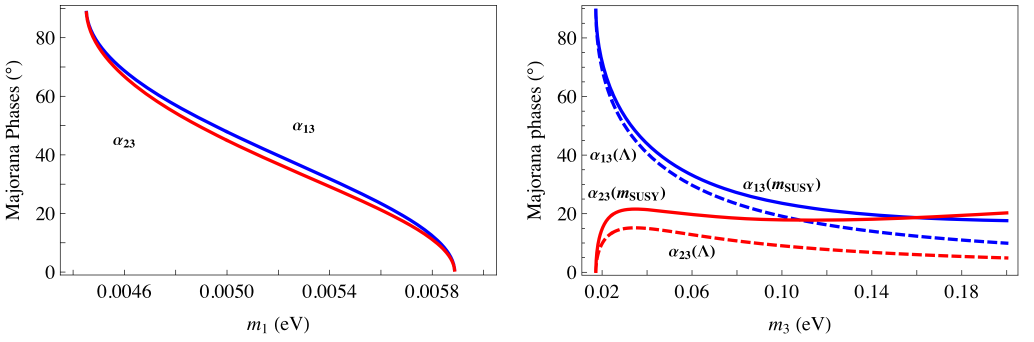

Fig. 3. Majorana phases α_{13} and α_{23} as functions of the lightest LH neutrino masses. For the NH (left panel) the corresponding curves at low and high energies are undistinguishable. For the IH (right panel) the curves refers to low energy values in MSSM with $\tan\beta = 15$ (solid blue or red) and the AF prediction at Λ_f (dashed blue or red). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

the spectrum where the neutrinos are less degenerate. Confronting with the running effects, in the IH case, the contribution from NLO operators in the AF model is under control.

5.2. Running of the phases

Majorana phases are affected by RG running effects too. Since there is not experimental information on Majorana phases available in this moment we will simply show their values at low energy, eventually comparing them with the prediction in the AF model. We stress again that they are completely determined by only one parameter, the mass of the lightest neutrino, m_1 for NH and m_3 for IH.

In the case of NH, Majorana phases are essentially not corrected by RG effects. This feature is due to the fact that $\delta\alpha_{13}$ and $\delta\alpha_{23}$ of Eqs. (45), (46) are proportional to $\sin(\alpha_{13} - \alpha_{23})$ which is close to zero, as we can see looking at the left panel of Fig. 3. In the case of IH, MSSM RG effects always increase the values of phases when moving from high energy to low energy and they are maximized for $\tan\beta = 15$, especially when the neutrino spectrum becomes degenerate. On the contrary, in the SM context, the low energy curves cannot be distinguished from the high energy ones.

As described in Section 3.3, a definite Dirac CP violating phase δ_{CP} arises from running effects even if, in the presence of a TB mixing pattern, it is undetermined in the beginning. Although the final Dirac phase can be large, Jarlskog invariant, which measures an observable CP violation, remains small because of the smallness of θ_{13} (see Fig. 4). We remember that these results are valid both for the SM and for MSSM.

6. Conclusions and outlook

The flavour sector is poorly known within the SM or the MSSM and the masses and mixings are considered as free parameters that can be adjusted to agree with experiments. This lack can be improved by adding to the SM or the MSSM gauge group appropriate flavour symmetries such that Yukawa couplings can be understood in a more fundamental way. Even though the mechanism that generates the fermion masses is not yet completely understood, a great effort has been made in the last years, especially after the discovery of neutrino masses which are sig-

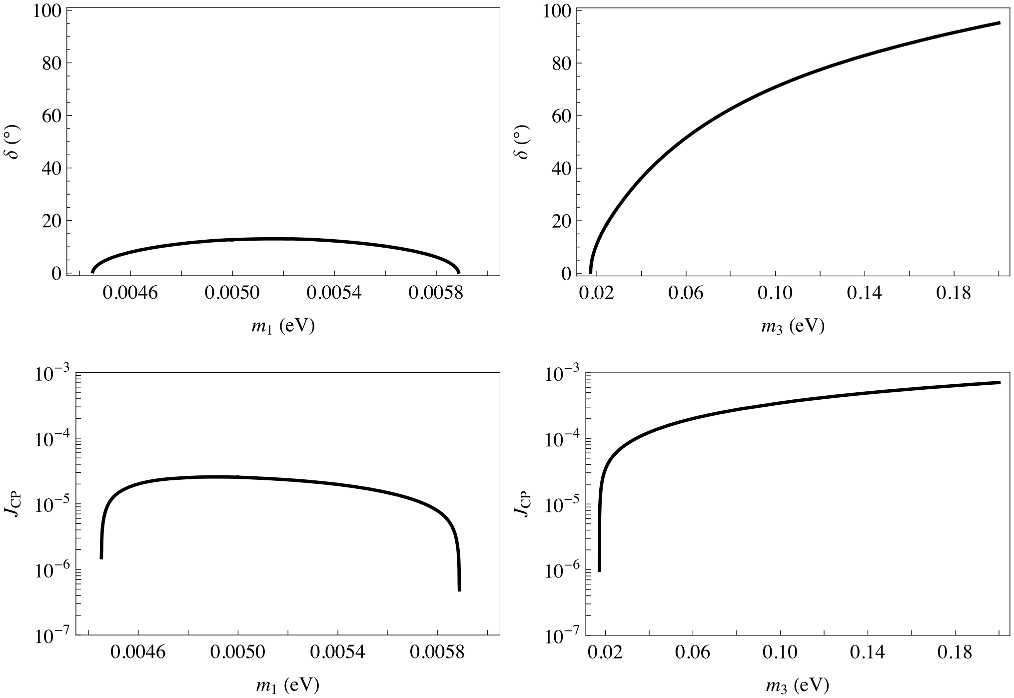

Fig. 4. Dirac CP phase δ_{CP} and Jarlskog invariant J_{CP} as functions of the lightest LH neutrino masses for the NH (left column) and the IH (right column), in the MSSM with $\tan\beta = 15$.

nificantly smaller than those of charged fermions. Furthermore, the leptonic mixing pattern is also very different from V_{CKM} because it contains a nearly maximal atmospheric angle θ_{23} and a solar angle very approximate to the TB prediction. The flavour structure of the neutrino mass matrix can be nicely explained by (type I) seesaw mechanism implemented with flavour symmetries. In order to naturally describe lepton masses and mixings, however, the flavour group must be (spontaneously) broken. Then we can expect that the LO prediction of a flavour symmetry is always subject by subleading corrections characterized by small symmetry breaking parameters. Then, in a consistent flavour model building, these subleading corrections must be safely under control.

In any flavour model, there is also an other independent correction to the LO predictions which is due to the RG evolution of parameters. In this paper we have studied these running effects on neutrino mixing patterns. In seesaw models, the running contribution from the neutrino Yukawa coupling Y_ν, encoded in J_ν, is generally dominant at energy above the seesaw threshold. However, this effect, which in general introduces appreciable deviations from the LO mixing patterns, does not affect the mixing angles, under specific conditions. This happens when Y_ν is proportional to a unitary matrix, corresponding to the case in which the RH singlet neutrinos or the charged leptons are in an irreducible representation of the flavour group. It also happens when considering the so-called mass-independent mixing pattern, in which the effect of J_ν can be absorbed by a small shift on neutrino mass eigenvalues leaving mixing angles unchanged. The widely studied TB mixing pattern belongs, for example, to this class of models.

In the second part of the paper, we focused on a special realization of the general class of flavour models studied in the first part. We were interested in the AF model for TB mixing where the flavour symmetry group is given by $A_4 \times Z_3 \times U(1)_{\mathrm{FN}}$. The aim is to analyze the RG effects on the TB mixing pattern in addition to the NLO corrections already present in this model and to confront them with experimental values. The analysis has been performed both in the SM and MSSM and for both neutrino spectra. We found that for NH light neutrinos, the dominant running contribution comes from the charged lepton sector which weakly depends on both $\tan\beta$ and mass degeneracy. As a result, for this type of spectrum, the tribimaximal prediction is stable under RG evolution. Moreover, the running contribution is of the same order or smaller with respect to the contribution from NLO operators. On the other hand, in the case of IH, the deviation of the solar angle from its TB value can be larger than the NLO contribution and, in particular, for $\tan\beta \gtrsim 10$ an IH spectrum is strongly disfavored. In the end, observe that for both spectra, the reactor angle θ_{13} does not receive appreciable deviations from zero (at a level $\lesssim u$).

The effects of RG running can be manifested also in other phenomena which are not directly related to the neutrino properties, such as Lepton Flavour Violating (LFV) transitions, leptogenesis, etc. For example, in [21,22,18,26], it has been pointed out that, in the limit of an exact A_4 symmetry, all CP violating asymmetries vanish. Then when the flavour symmetry is spontaneously broken, NLO corrections become important in generating the desired leptogenesis. On the other hand, also the RG evolution can introduce symmetry breaking effects which can be in principle dominant over NLO contributions. As a result, the estimate of generated leptogenesis can be quite different from those obtained in [21,22,18,26] since they do not take into account the RG effects. Another important consequence of RG running is the generation of off diagonal terms in the soft SUSY breaking mass matrices, contributing to LFV rare processes. In a series of papers [27–29] it has been studied the impact of using flavour symmetries in order to explain the measured bounds on rare decays. In [27] it has already been shown that, below the seesaw scales, the running effect is negligible with respect to that originated in the corresponding flavour theory. However, between the seesaw scales the threshold effects are important and the two contributions to LFVs can be comparable. All these issues are very interesting and are subject for a further investigation.

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Appendix A. Renormalization group equations

In order to calculate the RG evolution of the light neutrino mass matrix, the RGEs for all the parameters of the theory have to be solved simultaneously. We use the notation defined in the text, where a superscript (n) denotes a quantity between the nth and the $(n+1)$th mass threshold. When all the RH neutrinos are integrated out, the RGEs can be recovered by setting the neutrino Yukawa coupling Y_ν to zero, while in the full theory above the highest seesaw scale, the superscript (n) has to be omitted.

In the SM extended by singlet neutrinos, the RGEs for Y_e, $\overset{(n)}{Y}_\nu$, $\overset{(n)}{M}$, $\overset{(n)}{\kappa}$, Y_d, and Y_u are given by

$$16\pi^2 \frac{\mathrm{d}}{\mathrm{d}t} \overset{(n)}{Y}_e = Y_e \left\{ \frac{3}{2} Y_e^\dagger Y_e - \frac{3}{2} \overset{(n)}{Y}{}_\nu^\dagger \overset{(n)}{Y}_\nu + \mathrm{Tr}\big[3Y_u^\dagger Y_u + 3Y_d^\dagger Y_d + \overset{(n)}{Y}{}_\nu^\dagger \overset{(n)}{Y}_\nu + Y_e^\dagger Y_e\big]\right.$$

$$
\begin{aligned}
&\quad -\frac{9}{4}g_1^2-\frac{9}{4}g_2^2\Bigg\},\\
16\pi^2\frac{\mathrm{d}}{\mathrm{d}t}\overset{(n)}{Y}_\nu &= \overset{(n)}{Y}_\nu\Bigg\{\frac{3}{2}\overset{(n)}{Y}{}_\nu^\dagger\overset{(n)}{Y}_\nu-\frac{3}{2}Y_e^\dagger Y_e+\mathrm{Tr}\big[3Y_u^\dagger Y_u+3Y_d^\dagger Y_d+\overset{(n)}{Y}{}_\nu^\dagger\overset{(n)}{Y}_\nu+Y_e^\dagger Y_e\big]\\
&\quad -\frac{9}{20}g_1^2-\frac{9}{4}g_2^2\Bigg\},\\
16\pi^2\frac{\mathrm{d}}{\mathrm{d}t}\overset{(n)}{M}_R &= \big(\overset{(n)}{Y}_\nu\overset{(n)}{Y}{}_\nu^\dagger\big)\overset{(n)}{M}_R+\overset{(n)}{M}_R\big(\overset{(n)}{Y}_\nu\overset{(n)}{Y}{}_\nu^\dagger\big)^T,\\
16\pi^2\frac{\mathrm{d}}{\mathrm{d}t}\overset{(n)}{\kappa} &= \frac{1}{2}\big[\overset{(n)}{Y}{}_\nu^\dagger\overset{(n)}{Y}_\nu-3Y_e^\dagger Y_e\big]^T\overset{(n)}{\kappa}+\frac{1}{2}\overset{(n)}{\kappa}\big[\overset{(n)}{Y}{}_\nu^\dagger\overset{(n)}{Y}_\nu-3Y_e^\dagger Y_e\big]\\
&\quad +2\,\mathrm{Tr}\big[3Y_u^\dagger Y_u+3Y_d^\dagger Y_d+\overset{(n)}{Y}{}_\nu^\dagger\overset{(n)}{Y}_\nu+Y_e^\dagger Y_e\big]-3g_2^2\overset{(n)}{\kappa}+\lambda_H\overset{(n)}{\kappa},\\
16\pi^2\frac{\mathrm{d}}{\mathrm{d}t}\overset{(n)}{Y}_d &= Y_d\Bigg\{\frac{3}{2}Y_d^\dagger Y_d-\frac{3}{2}Y_u^\dagger Y_u+\mathrm{Tr}\big[3Y_u^\dagger Y_u+3Y_d^\dagger Y_d+\overset{(n)}{Y}{}_\nu^\dagger\overset{(n)}{Y}_\nu+Y_e^\dagger Y_e\big]\\
&\quad -\frac{1}{4}g_1^2-\frac{9}{4}g_2^2-8g_3^2\Bigg\},\\
16\pi^2\frac{\mathrm{d}}{\mathrm{d}t}\overset{(n)}{Y}_u &= Y_u\Bigg\{\frac{3}{2}Y_u^\dagger Y_u-\frac{3}{2}Y_d^\dagger Y_d+\mathrm{Tr}\big[3Y_u^\dagger Y_u+3Y_d^\dagger Y_d+\overset{(n)}{Y}{}_\nu^\dagger\overset{(n)}{Y}_\nu+Y_e^\dagger Y_e\big]\\
&\quad -\frac{17}{20}g_1^2-\frac{9}{4}g_2^2-8g_3^2\Bigg\},\\
16\pi^2\frac{\mathrm{d}}{\mathrm{d}t}\overset{(n)}{\lambda}_H &= 6\lambda_H^2-3\lambda_H\left(3g_2^2+\frac{3}{5}g_1^2\right)+3g_2^4+\frac{3}{2}\left(\frac{3}{5}g_1^2+g_2^2\right)^2\\
&\quad +4\lambda_H\,\mathrm{Tr}\big[3Y_u^\dagger Y_u+3Y_d^\dagger Y_d+\overset{(n)}{Y}{}_\nu^\dagger\overset{(n)}{Y}_\nu+Y_e^\dagger Y_e\big]\\
&\quad -8\,\mathrm{Tr}\big[3Y_u^\dagger Y_u Y_u^\dagger Y_u+3Y_d^\dagger Y_d Y_d^\dagger Y_d+\overset{(n)}{Y}{}_\nu^\dagger\overset{(n)}{Y}_\nu\overset{(n)}{Y}{}_\nu^\dagger\overset{(n)}{Y}_\nu+Y_e^\dagger Y_e Y_e^\dagger Y_e\big],
\end{aligned}
\tag{64}
$$

where $t := \ln(\mu/\mu_0)$. We use the convention that the Higgs self-interaction term in the Lagrangian is $-\lambda_H(H^\dagger H)^2/4$ and the GUT normalization, such that $g_2 = g$ and $g_1 = \sqrt{5/3}g'$.

In the MSSM context the 1-loop RGEs for the same quantities are given by

$$
\begin{aligned}
16\pi^2\frac{\mathrm{d}}{\mathrm{d}t}\overset{(n)}{Y}_e &= Y_e\Bigg\{3Y_e^\dagger Y_e+\overset{(n)}{Y}{}_\nu^\dagger\overset{(n)}{Y}_\nu+\mathrm{Tr}\big[3Y_d^\dagger Y_d+Y_e^\dagger Y_e\big]-\frac{9}{5}g_1^2-3g_2^2\Bigg\},\\
16\pi^2\frac{\mathrm{d}}{\mathrm{d}t}\overset{(n)}{Y}_\nu &= \overset{(n)}{Y}_\nu\Bigg\{3\overset{(n)}{Y}{}_\nu^\dagger\overset{(n)}{Y}_\nu+Y_e^\dagger Y_e+\mathrm{Tr}\big[3Y_u^\dagger Y_u+\overset{(n)}{Y}{}_\nu^\dagger\overset{(n)}{Y}_\nu\big]-\frac{3}{5}g_1^2-3g_2^2\Bigg\},\\
16\pi^2\frac{\mathrm{d}}{\mathrm{d}t}\overset{(n)}{M}_R &= 2\big(\overset{(n)}{Y}_\nu\overset{(n)}{Y}{}_\nu^\dagger\big)\overset{(n)}{M}_R+2\overset{(n)}{M}_R\big(\overset{(n)}{Y}_\nu\overset{(n)}{Y}{}_\nu^\dagger\big)^T,\\
16\pi^2\frac{\mathrm{d}}{\mathrm{d}t}\overset{(n)}{\kappa} &= \big[\overset{(n)}{Y}{}_\nu^\dagger\overset{(n)}{Y}_\nu+Y_e^\dagger Y_e\big]^T\overset{(n)}{\kappa}+\overset{(n)}{\kappa}\big[\overset{(n)}{Y}{}_\nu^\dagger\overset{(n)}{Y}_\nu+Y_e^\dagger Y_e\big]+2\,\mathrm{Tr}\big[3Y_u^\dagger Y_u+\overset{(n)}{Y}{}_\nu^\dagger\overset{(n)}{Y}_\nu\big]\overset{(n)}{\kappa}\\
&\quad -\frac{6}{5}g_1^2\overset{(n)}{\kappa}-6g_2^2\overset{(n)}{\kappa},\\
16\pi^2\frac{\mathrm{d}}{\mathrm{d}t}\overset{(n)}{Y}_d &= Y_d\Bigg\{3Y_d^\dagger Y_d+Y_u^\dagger Y_u+\mathrm{Tr}\big[3Y_d^\dagger Y_d+Y_e^\dagger Y_e\big]-\frac{7}{15}g_1^2-3g_2^2-\frac{16}{3}g_3^2\Bigg\},\\
16\pi^2\frac{\mathrm{d}}{\mathrm{d}t}\overset{(n)}{Y}_u &= Y_u\Bigg\{Y_d^\dagger Y_d+3Y_u^\dagger Y_u+\mathrm{Tr}\big[3Y_u^\dagger Y_u+\overset{(n)}{Y}{}_\nu^\dagger\overset{(n)}{Y}_\nu\big]-\frac{13}{15}g_1^2-3g_2^2-\frac{16}{3}g_3^2\Bigg\}.
\end{aligned}
\tag{65}
$$

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