Similarity Preserving Linear Maps on Matrices

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ABSTRACT

We determine all similarity preserving linear maps on the space of $n \times n$ complex matrices and all unitary equivalence preserving linear maps on the space of $n \times n$ Hermitian matrices. (Sub)majorization preserving linear maps on Hermitian matrices are also determined.

INTRODUCTION

Concerning linear maps on matrices, much work has been done on the problem described generally as follows. Let $\mathcal{A}$ be a space of matrices, and $Q(A)$ a quantity or property of matrices $A \in \mathcal{A}$. Then determine the structure of linear maps $\Phi: \mathcal{A} \rightarrow \mathcal{A}$ which preserve $Q$: $Q(\Phi(A)) = Q(A)$, $A \in \mathcal{A}$. Some aspects of this problem were surveyed in [10], and in particular the characterizations of linear maps which preserve a fixed rank were established in [11, 3, 9]. Moreover, when $R(A, B)$ is a relation between matrices $A, B \in \mathcal{A}$, it is interesting to determine all linear maps $\Phi: \mathcal{A} \rightarrow \mathcal{A}$ which preserve $R$ [i.e. $R(\Phi(A), \Phi(B))$ holds whenever $R(A, B)$ does]. For example, several authors have discussed commutativity preserving linear maps on some spaces of matrices (see [6, 14, 7] and references therein). This is the case when $R(A, B)$ is "$AB = BA$." The purpose of this paper is to discuss the problems when $R(A, B)$ is "$A$ is similar to $B$" or "$A$ is unitarily equivalent to $B$.”
In Section 1 of this paper, we determine all similarity preserving linear maps on the space of $n \times n$ complex matrices. To do this, we relate the rank of a matrix with the dimension of tangent space of its similarity orbit, and then apply the structure theorem [11] for rank 1 preserving linear maps. The geometric method using tangent spaces of similarity orbits was employed in [4] for studying the spectral variation of matrices. It may be pointed out that our result for similarity preserving linear maps has a strong resemblance to those in [6, 14, 7] for commutativity preserving ones. In Section 2, unitary equivalence preserving linear maps on the space of $n \times n$ Hermitian matrices are characterized in a similar manner. Furthermore the notions of majorization and submajorization (= weak majorization) between Hermitian matrices are very important in matrix theory (see [1, 12]). In the final Section 3, we consider (sub)majorization preserving linear maps on the space of Hermitian matrices. We notice however that majorization preserving linear maps are nothing but unitary equivalence preserving ones. Linear maps on $\mathbb{R}^n$ which preserve (sub)majorization were discussed in [8] and [1, §2].

1. SIMILARITY PRESERVING LINEAR MAPS

Let $M_n(\mathbb{C})$ denote the space of all $n \times n$ complex matrices. For $A \in M_n(\mathbb{C})$, let $\text{tr}(A)$ be the trace of $A$, $\text{rank}(A)$ the rank of $A$, and $A^t$ the transpose of $A$. The identity matrix is denoted by $I$. Let $E_{ij}$ be the matrix with 1 in position $(i, j)$ and 0 elsewhere. For $A, B \in M_n(\mathbb{C})$, we write $A \sim B$ if $A$ is similar to $B$. For a map $\Phi$ from $M_n(\mathbb{C})$ into itself, we say that $\Phi$ preserves similarity or $\Phi$ is a similarity preserving map if $A \sim B$ implies $\Phi(A) \sim \Phi(B)$ for every $A, B \in M_n(\mathbb{C})$. The next theorem characterizes similarity preserving linear maps on $M_n(\mathbb{C})$.

**Theorem 1.1.** Let $\Phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ be a linear map. Then $\Phi$ preserves similarity if and only if either there exists an $A_0 \in M_n(\mathbb{C})$ such that

(i) \[ \Phi(X) = \text{tr}(X) A_0 \quad \text{for all} \quad X \in M_n(\mathbb{C}), \]

or there exist a nonsingular matrix $S$ and $\alpha, \beta \in \mathbb{C}$ such that $\Phi$ has one of the following forms:

(ii) \[ \Phi(X) = \alpha S X S^{-1} + \beta \text{tr}(X) I \quad \text{for all} \quad X \in M_n(\mathbb{C}), \]

(iii) \[ \Phi(X) = \alpha S X^t S^{-1} + \beta \text{tr}(X) I \quad \text{for all} \quad X \in M_n(\mathbb{C}). \]
The "if" part of the theorem is obvious. To prove the "only if" part, we first determine the kernels of similarity preserving linear maps.

**Lemma 1.2.** Let $\Phi$ be a similarity preserving linear map on $M_n(\mathbb{C})$, and $\ker(\Phi)$ its kernel. Then either $\ker(\Phi) \subseteq C I$ or $\ker(\Phi) \supseteq \ker(\text{tr})$ holds.

**Proof.** Since $\Phi$ preserves similarity, it follows that $\ker(\Phi)$ is invariant under any similarity transformation. Suppose an $A \in \ker(\Phi)$ exists outside $C I$. By taking the Jordan canonical form of $A$, we may consider the following two cases:

1. $A = \begin{bmatrix} \alpha & 1 & 0 & \cdots & 0 \\ 0 & \ddots & & & \\ \vdots & & \ddots & & \\ 0 & & & \ddots & \\ 0 & & & & \end{bmatrix}$

(a) $A = \text{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n)$, where $\alpha_1 \neq \alpha_2$.

In case (a), since

$$\text{diag}(-1, 1, \ldots, 1)A\text{diag}(-1, 1, \ldots, 1) = \begin{bmatrix} \alpha & -1 & 0 & \cdots & 0 \\ 0 & \ddots & & & \\ \vdots & & \ddots & & \\ 0 & & & \ddots & \\ 0 & & & & \end{bmatrix},$$

we get $E_{12} \in \ker(\Phi)$ and hence $E_{21} \in \ker(\Phi)$. In view of

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

we get $E_{11} - E_{22} \in \ker(\Phi)$. In case (b),

$$E_{11} - E_{22} = (\alpha_1 - \alpha_2)^{-1}\{\text{diag}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) - \text{diag}(\alpha_2, \alpha_1, \alpha_3, \ldots, \alpha_n)\}$$

is in $\ker(\Phi)$. Therefore it follows in either case that $E_{11} - E_{1i} \in \ker(\Phi)$ for $2 \leq i \leq n$. If $X = \text{diag}(\lambda_1, \ldots, \lambda_n)$ and $\text{tr}(X) = \sum_{i=1}^{n} \lambda_i = 0$, then

$$X = - \sum_{i=2}^{n} \lambda_i (E_{11} - E_{1i}) \in \ker(\Phi).$$
For every $X \in M_n(\mathbb{C})$ with $\text{tr}(X) = 0$, since $(X + X^*)/2$ and $(X - X^*)/2i$ are unitarily diagonalizable, we have $(X + X^*)/2, (X - X^*)/2i \in \ker(\Phi)$, so that $X \in \ker(\Phi)$.

By Lemma 1.2, it suffices to prove Theorem 1.1 when $\Phi$ is a similarity preserving linear bijection. Indeed, if $\ker(\Phi) \supseteq \ker(\text{tr})$, then $\Phi$ is of the form (i) in Theorem 1.1 with $A_0 = n^{-1}\Phi(I)$. If $\ker(\Phi) = \mathbb{C}I$, then $\Phi(\cdot) + \text{tr}(\cdot)I$ is a similarity preserving linear bijection.

For each $A \in M_n(\mathbb{C})$, let $\mathcal{O}_A$ be the similarity orbit of $A$, that is,

$$\mathcal{O}_A = \left\{ B \in M_n(\mathbb{C}) : B \sim A \right\}.$$

As explained in [2] (also [4]), $\mathcal{O}_A$ is a smooth submanifold in $M_n(\mathbb{C}) = \mathbb{C}^{n^2}$, and the tangent space $T_A\mathcal{O}_A$ to $\mathcal{O}_A$ at $A$ is given by

$$T_A\mathcal{O}_A = \left\{ [A, X] : X \in M_n(\mathbb{C}) \right\},$$


**Lemma 1.3.** Let $A \in M_n(\mathbb{C})$, and $\dim(T_A\mathcal{O}_A)$ be the dimension of $T_A\mathcal{O}_A$ over $\mathbb{C}$. Assume $A \notin \mathbb{C}I$. Then $\dim(T_A\mathcal{O}_A) = 2n - 2$ if $\text{rank}(A - \alpha I) = 1$ for some $\alpha \in \mathbb{C}$, and $\dim(T_A\mathcal{O}_A) > 2n$ otherwise.

**Proof.** The linear map $X \mapsto [A, X]$ on $M_n(\mathbb{C})$ is represented by the matrix $\tilde{A} = A \otimes I - I \otimes A^t$ in $M_n^2(\mathbb{C}) = M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$. Hence $\dim(T_A\mathcal{O}_A) = \text{rank}({\tilde{A}})$ from the fact noted above. To show the lemma, we may assume that $A$ is a Jordan matrix:

$$A = \begin{bmatrix}
\alpha_1 & \delta_1 & & 0 \\
& \alpha_2 & \delta_2 & \\
& & \ddots & \ddots \\
& & & 0 & \delta_n \\
& & & & \alpha_n
\end{bmatrix},$$

where $\delta_i = 0$ or 1.
Then

$$\tilde{A} = \begin{bmatrix}
    A - \alpha_1 I \\
    -\delta_1 I & A - \alpha_2 I & & 0 \\
    -\delta_2 I & & & 0 \\
    \vdots & & & \ddots \\
    0 & & & 0 & -\delta_{n-1} I & A - \alpha_n I
\end{bmatrix}.$$  

When $\delta_i = 1$ for at least two $i$, rank($\tilde{A}$) $\geq 2n$ and rank($A - \alpha I$) $\geq 2$ for all $\alpha \in \mathbb{C}$. When $\delta_i = 1$ for exactly one $i$, we can suppose $\delta_1 = 1$ (so $\alpha_1 = \alpha_2$) and $\delta_2 = \cdots = \delta_{n-1} = 0$. Then rank($\tilde{A}$) $= 2n - 2$ if all $\alpha_i = \alpha_1$ [equivalently rank($A - \alpha_1 I$) $= 1$], and rank ($\tilde{A}$) $\geq 2n$ otherwise. Finally when all $\delta_i = 0$, rank($\tilde{A}$) is equal to the number of $(i, j)$ with $\alpha_i \neq \alpha_j$. Since $A \notin \mathbb{C} I$, not all $\alpha_i$ are equal. Hence rank($\tilde{A}$) $= 2n - 2$ occurs if and only if all $\alpha_i$ except one are equal.

**Lemma 1.4.** Let $\Phi$ be a similarity preserving linear bijection on $M_n(\mathbb{C})$. Then:

1. $\Phi(I) \in \mathbb{C} I$.
2. If $A \in M_n(\mathbb{C})$ and rank($\Phi(A)$) $= 1$, then rank($A - \alpha I$) $= 1$ for some $\alpha \in \mathbb{C}$.

**Proof.**

(1): Let $\Phi(B) = I$. Since $\Phi$ preserves similarity, $\Phi$ sends $\mathcal{O}_B$ to $\mathcal{O}_I = \{I\}$. Hence $\mathcal{O}_B = \{B\}$ from injectivity of $\Phi$. This implies $B \in \mathbb{C} I$, so that $\Phi(I) \in \mathbb{C} I$.

(2): Because the similarity preserving property of $\Phi$ implies $\Phi(T_A \mathcal{O}_A) \subseteq T_{\Phi(A)} \mathcal{O}_{\Phi(A)}$, it follows from injectivity of $\Phi$ and Lemma 1.3 that

$$\dim(T_A \mathcal{O}_A) \leq \dim(T_{\Phi(A)} \mathcal{O}_{\Phi(A)}) = 2n - 2.$$  

Since $A \notin \mathbb{C} I$ by (1) (except the trivial case $n = 1$), the desired conclusion follows from Lemma 1.3.

**Lemma 1.5.** Let $n \geq 3$, and $\Phi$ be a similarity preserving linear bijection on $M_n(\mathbb{C})$. Let $A, B \in M_n(\mathbb{C})$ and $\alpha, \beta \in \mathbb{C}$. Let $X = \Phi(A + \alpha I)$ and $Y = \Phi(B + \beta I)$. If rank($A$) $= \text{rank}(B) = 1$ and rank($X$) $= \text{rank}(Y) = \text{rank}(X + Y) = 1$, then rank($A + B$) $\leq 1$. 


**Proof.** We write \( A = ac^*, \) \( B = bd^*, \) \( X = xu^*, \) and \( Y = yv^* \) where \( a, c, \ldots \) are nonzero vectors in \( \mathbb{C}^n. \) If \( \{c, d\} \) is linearly dependent, then \( \text{rank}(A + B) \leq 1. \) So assume that \( \{c, d\} \) is linearly independent. Because \( \Phi \) can be replaced by \( \Phi(S \cdot S^{-1}) \) for any nonsingular matrix \( S, \) we may assume that \( c = e_1 \) and \( d = e_2, \) where \( \{e_i\} \) is the canonical basis for \( \mathbb{C}^n. \) Thus \( A \) and \( B \) are written as follows:

\[
A = \begin{bmatrix}
\alpha_1 \\
\vdots \\
\alpha_n
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & \beta_1 \\
\vdots & \vdots \\
0 & \beta_n
\end{bmatrix}.
\]

Since \( \Phi(A + B + (\alpha + \beta)I) = X + Y \) and \( \text{rank}(X + Y) = 1, \) it follows from Lemma 1.4(2) that \( \text{rank}(A + B - \xi I) = 1 \) for some \( \xi \in \mathbb{C}. \) Now suppose \( \xi \neq 0, \) which implies \( \text{rank}(A + B) > 2 \) because \( n > 3. \) Since

\[
A + B - \xi I = \begin{bmatrix}
\alpha_1 - \xi & \beta_1 \\
\alpha_2 & \beta_2 - \xi \\
\alpha_3 & \beta_3 - \xi \\
\vdots & \vdots \\
\alpha_n & \beta_n
\end{bmatrix}
\]

has rank 1, we get \( \alpha_1 = \beta_2 = \xi. \) Because \( \text{rank}(X + Y) = 1 \) implies that either \( \{x, y\} \) or \( \{u, v\} \) is linearly dependent, there exists a \( \rho \in \mathbb{C} \) such that \( \rho \neq 0, 1 \) and \( \text{rank}(X + \rho Y) = 1. \) Then, from Lemma 1.4(2), \( \text{rank}(A + \rho B - \eta I) = 1 \) for some \( \eta \in \mathbb{C}. \) When \( \eta = 0, \) \( \text{rank}(A + \rho B) = 1 \) and hence \( \text{rank}(A + B) \leq 1, \) a contradiction. When \( \eta \neq 0, \) we get \( \alpha_1 = \rho \beta_2 = \eta \) as above, and hence \( \xi = \rho \xi \neq \xi, \) a contradiction. Therefore \( \xi = 0, \) showing \( \text{rank}(A + B) = 1 \) as desired. \( \blacksquare \)

We now prove Theorem 1.1 in the cases when \( n \geq 3 \) and when \( n = 2 \) separately.

**Proof of Theorem 1.1 (when \( n \geq 3 \)).** As remarked before, we may assume that \( \Phi \) is a similarity preserving linear bijection on \( M_n(\mathbb{C}). \) Furthermore \( \Phi(I) = I \) may be assumed by Lemma 1.4(1). Let \( \mathcal{R}_1 \) be the set of all rank 1 matrices in \( M_n(\mathbb{C}). \) For each \( X \in \mathcal{R}_1, \) by Lemma 1.4(2) there exist (unique) \( F(X) \in \mathcal{R}_1 \) and \( f(X) \in \mathbb{C} \) such that \( \Phi^{-1}(X) = F(X) + f(X)I. \) It follows from Lemma 1.5 that \( F(X + Y) = F(X) + F(Y) \) if \( X, Y \) and \( X + Y \) are in \( \mathcal{R}_1. \) Thus

\[
f(X + Y) = f(X) + f(Y) \quad \text{if} \quad X, Y, X + Y \in \mathcal{R}_1.
\]

\((*)\)
Moreover it is obvious that $f(\xi X) = \xi f(X)$ for all $X \in \mathcal{R}_1$ and $\xi \in \mathbb{C}$ with convention $f(0) = 0$. For every $X = xu^* \in \mathcal{R}_1$ where $x = \sum_{i=1}^n \xi_i e_i$ and $u = \sum_{i=1}^n \eta_i e_i$, we have $X = \sum_{i,j=1}^n \xi_i \bar{\eta}_j E_{ij}$ and, by using the above (*) repeatedly,

$$f(X) = f(\xi_1 e_1 u^*) + f\left(\sum_{i=2}^n \xi_i e_i u^*\right)$$

$$\vdots$$

$$= \sum_{i=1}^n f(\xi_i e_i u^*)$$

$$= \sum_{i=1}^n \left(f(\xi_i e_i (\eta_1 e_1)^*) + f\left(\xi_i e_i \left(\sum_{j=2}^n \eta_j e_j\right)^*\right)\right)$$

$$\vdots$$

$$= \sum_{i,j=1}^n f(\xi_i e_i (\eta_j e_j)^*) = \sum_{i,j=1}^n \xi_i \bar{\eta}_j f(E_{ij}).$$

Now define a linear functional $\psi$ on $M_n(\mathbb{C})$ by $\psi(\sum_{i,j=1}^n \alpha_{ij} E_{ij}) = \sum_{i,j=1}^n \alpha_{ij} f(E_{ij})$ and a linear map $\Phi$ on $M_n(\mathbb{C})$ by

$$\Phi(A) = \Phi^{-1}(A) - \psi(A)I \quad \text{for} \quad A \in M_n(\mathbb{C}).$$

For every $X \in \mathcal{R}_1$, we then have $\psi(X) = f(X)$ and $\Phi(X) = F(X)$. This shows that $\Phi$ preserves rank 1 matrices. Hence, by virtue of [11], there exist nonsingular matrices $Q$ and $S$ such that either $\Phi(A) = QAS$ for all $A \in M_n(\mathbb{C})$ or $\Phi(A) = QA^tS$ for all $A \in M_n(\mathbb{C})$. Suppose $\Phi$ is of the first form. Since $\Phi(I) = I$, we have

$$\Phi(A) = Q^{-1}AS^{-1} + \varphi(A)I \quad \text{for all} \quad A \in M_n(\mathbb{C}),$$

where $\varphi(\cdot) = -\psi(Q^{-1} \cdot S^{-1})$. Letting $A = I$, we get $Q^{-1} = \alpha S$, where $\alpha = 1 - \varphi(I)$, so that $\Phi(A) = \alpha SAS^{-1} + \varphi(A)I$ and hence

$$\varphi(A) = n^{-1}\{\text{tr}(\Phi(A)) - \alpha \text{tr}(A)\} \quad \text{for all} \quad A \in M_n(\mathbb{C}).$$

This shows that $\varphi$ is invariant under any similarity transformation. Thus $\varphi = \beta \text{tr}$ for some $\beta \in \mathbb{C}$, so that $\Phi$ has the form (ii) in Theorem 1.1.
Analogously, when $\Psi$ is of the second form, $\Phi$ has the form (iii) in Theorem 1.1.

Proof of Theorem 1.1 (when $n = 2$). Let $\Phi$ be a similarity preserving linear bijection on $M_2(\mathbb{C})$ with $\Phi(I) = I$. As the Jordan canonical form of $\Phi(E_{11})$, the following two cases are considered:

(a) $\Phi(E_{11}) \sim \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$,   (b) $\Phi(E_{11}) \sim \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$.

We first show that the case (a) cannot occur. Suppose there exists a nonsingular matrix $Q$ such that

$$Q\Phi(E_{11})Q^{-1} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}.$$ 

Since

$$Q\Phi(E_{22})Q^{-1} = Q\Phi(I - E_{11})Q^{-1} = \begin{bmatrix} 1 - \lambda & -1 \\ 0 & 1 - \lambda \end{bmatrix},$$

we have

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \sim \begin{bmatrix} 1 - \lambda & -1 \\ 0 & 1 - \lambda \end{bmatrix}$$

and hence $\lambda = \frac{1}{2}$. Define a similarity preserving linear map $\Psi$ on $M_2(\mathbb{C})$ by

$$\Psi(A) = Q\Phi(A)Q^{-1} - \frac{1}{2}\text{tr}(A)I \quad \text{for} \quad A \in M_2(\mathbb{C}).$$

Then

$$\Psi(\alpha_1E_{11} + \alpha_2E_{22}) = \begin{bmatrix} \frac{\alpha_1 + \alpha_2}{2} & \frac{\alpha_1 - \alpha_2}{2} \\ 0 & \frac{\alpha_1 + \alpha_2}{2} \end{bmatrix} - \frac{\alpha_1 + \alpha_2}{2}I$$

$$= (\alpha_1 - \alpha_2)F_{12} \quad \text{for} \quad \alpha_1, \alpha_2 \in \mathbb{C},$$

so that $\text{tr}(\Psi(A)) = 0$ and $\Psi(A)^2 = 0$ for every Hermitian $A \in M_2(\mathbb{C})$. Hence $\text{ran}(\Psi)$, the range of $\Psi$, is included in $\ker(\text{tr})$. But, since $\ker(\Psi) = \mathbb{C}I$,
ran(Ψ) is of codimension 1, so that ran(Ψ) = ker(tr). On the other hand, for each Hermitian A, B ∈ M₂(ℂ), we get Ψ(A)² = Ψ(B)² = Ψ(A + B)² = 0 and hence Ψ(A)Ψ(B) + Ψ(B)Ψ(A) = 0, implying Ψ(A + iB)² = 0. Therefore

\[ \ker(tr) = \text{ran}(Ψ) \subseteq \{ X \in M₂(ℂ) : X² = 0 \}, \]

a contradiction.

We next consider the case (b). There exists a nonsingular matrix Q such that

Here X ≠ ZJ from \( \Phi(E_{11}) \notin 𝕊 \). Since

we have

and hence \( μ = 1 - λ, \; λ \neq \frac{1}{2} \). Define a similarity preserving linear map Ψ on M₂(ℂ) by

Then

so that Ψ preserves trace and diagonalizability. Furthermore Ψ is bijective from \( \lambda \neq \frac{1}{2} \). Now let X ∈ M₂(ℂ) be such that Ψ(X) = E₁₂. Since X is not diagonalizable and tr(X) = 0, we observe X ∼ E₁₂. Therefore \( Ψ(E₁₂) \sim E₁₂ \). Because each rank 1 matrix in M₂(ℂ) is similar to either E₁₁ or E₁₂, Ψ preserves rank 1 matrices. By [11], there exist nonsingular matrices S₁ and S₂ such that either Ψ(A) = S₁AS₂ for all A ∈ M₂(ℂ) or Ψ(A) = S₁A'S₂ for all A ∈ M₂(ℂ). Since Ψ(I) ∈ 𝕊, Φ has one of the forms (ii) and (iii) in Theorem 1.1. □
2. UNITARY EQUIVALENCE PRESERVING LINEAR MAPS

Let $H_n(C)$ denote the real space of all $n \times n$ complex Hermitian matrices. For $A, B \in H_n(C)$, we write $A \sim B$ if $A$ is unitarily equivalent to $B$. We say that a map $\Phi: H_n(C) \to H_n(C)$ preserves unitary equivalence or is a unitary equivalence preserving map if $A \sim B$ implies $\Phi(A) \sim \Phi(B)$ for every $A, B \in H_n(C)$. The next theorem characterizes unitary equivalence preserving linear maps on $H_n(C)$. The proof can be done in a manner similar to that of Theorem 1.1 and is indeed considerably easier.

**Theorem 2.1.** Let $\Phi: H_n(C) \to H_n(C)$ be a linear map. Then $\Phi$ preserves unitary equivalence if and only if either there exists an $A_0 \in H_n(C)$ such that

\[
\Phi(X) = \text{tr}(X)A_0 \quad \text{for all} \quad X \in H_n(C),
\]

or there exist a unitary matrix $U$ and $\alpha, \beta \in \mathbb{R}$ such that $\Phi$ has one of the following forms:

\[
\begin{align*}
(ii) & \quad \Phi(X) = \alpha UXU^* + \beta \text{tr}(X)I \quad \text{for all} \quad X \in H_n(C), \\
(iii) & \quad \Phi(X) = \alpha UXU^* + \beta \text{tr}(X)I \quad \text{for all} \quad X \in H_n(C).
\end{align*}
\]

**Proof.** Let $\Phi$ be a unitary equivalence preserving linear map on $H_n(C)$. It follows from the proof of Lemma 1.2 that either $\ker(\Phi) \subseteq \mathbb{R}I$ or $\ker(\Phi) \supseteq \ker(\text{tr}) \cap H_n(C)$. So we may assume that $\Phi$ is bijective. Since $\Phi(I) \in \mathbb{R}I$ as in Lemma 1.4(1), we furthermore assume $\Phi(I) = I$. For $A \in H_n(C)$, let $\mathcal{U}_A = \{B \in H_n(C) : B \sim A\}$, the unitary orbit of $A$. Then $\mathcal{U}_A$ is a smooth submanifold in $H_n(C) = \mathbb{R}^{n^2}$, and the tangent space $T_A\mathcal{U}_A$ to $\mathcal{U}_A$ at $A$ is given by

\[
T_A\mathcal{U}_A = \{[A, X] : X \in M_n(C), X^* = -X\}.
\]

We hence observe that if $A \sim \text{diag}(\alpha_1, \ldots, \alpha_n)$, then $\dim(T_A\mathcal{U}_A)$ over $\mathbb{R}$ is equal to the number of $(i, j)$ with $\alpha_i \neq \alpha_j$. Now let $\Phi(A) = E_{11}$. Then, since $\Phi(T_A\mathcal{U}_A) \subseteq T_{E_{11}}\mathcal{U}_{E_{11}}$, we get

\[
\dim(T_A\mathcal{U}_A) \leq \dim(T_{E_{11}}\mathcal{U}_{E_{11}}) = 2n - 2.
\]
Since $A \notin \mathbb{R}I$ (except the trivial case $n = 1$), there exist a rank 1 Hermitian projection $P$ and $\alpha, \beta \in \mathbb{R}$ with $\alpha \neq 0$ such that $A = \alpha P + \beta I$. Define a unitary equivalence preserving map $\Psi$ on $H_n(\mathbb{C})$ by

$$\Psi(X) = \alpha \Phi(X) + \beta \text{tr}(X) I \quad \text{for} \quad X \in H_n(\mathbb{C}).$$

It follows from $\Psi(P) = E_{11}$ that $\Psi$ preserves rank 1 projections in $H_n(\mathbb{C})$. From the argument of spectral decompositions, we observe that $\Psi$ is a Jordan automorphism [i.e. $\Psi(X^2) = \Psi(X)^2$ for all $X \in H_n(\mathbb{C})$]. Thus, as is well known (see [5, Example 3.2.14] for instance), there exists a unitary matrix $U$ such that either $\Psi(X) = UXU^*$ for all $X \in H_n(\mathbb{C})$ or $\Psi(X) = UXU^*$ for all $X \in H_n(\mathbb{C})$. Therefore $\Phi$ has the desired form.

Let $H_n(\mathbb{R})$ be the space of all $n \times n$ real symmetric matrices. For $A, B \in H_n(\mathbb{R})$, $A \preceq B$ means that $A$ is orthogonally equivalent to $B$. Orthogonal equivalence preserving linear maps on $H_n(\mathbb{R})$ are determined in the next theorem. We omit the proof, because it is almost the same as that of Theorem 2.1 (but, in the final step, use [13, Lemma 1] for instance).

**Theorem 2.2.** Let $\Phi: H_n(\mathbb{R}) \to H_n(\mathbb{R})$ be a linear map. Then $\Phi$ preserves orthogonal equivalence if and only if either there exists an $A_0 \in H_n(\mathbb{R})$ such that

$$\Phi(X) = \text{tr}(X) A_0 \quad \text{for all} \quad X \in H_n(\mathbb{R}),$$

or there exist a real orthogonal matrix $T$ and $\alpha, \beta \in \mathbb{R}$ such that

$$\Phi(X) = \alpha TXT^{-1} + \beta \text{tr}(X) I \quad \text{for all} \quad X \in H_n(\mathbb{R}).$$

### 3. MAJORIZATION PRESERVING LINEAR MAPS

For each $A \in H_n(\mathbb{C})$, let $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$ be the eigenvalues of $A$ arranged in decreasing order. For $A, B \in H_n(\mathbb{C})$, $A$ is said to be submajorized by $B$ (we write $A \prec B$) if $\sum_{i=1}^k \lambda_i(A) \leq \sum_{i=1}^k \lambda_i(B)$ for all $1 \leq k \leq n$, and $A$ is said to be majorized by $B$ (we write $A \prec B$) if $A \prec B$ and $\sum_{i=1}^n \lambda_i(A) = \sum_{i=1}^n \lambda_i(B)$ [i.e. $\text{tr}(A) = \text{tr}(B)$]. We say that a map $\Phi: H_n(\mathbb{C}) \to H_n(\mathbb{C})$ preserves majorization [submajorization] or is a majorization [submajorization] preserving map if $A \prec B$ [i.e. $A \prec B$] implies $\Phi(A) \prec \Phi(B)$ [i.e. $\Phi(A) \prec \Phi(B)$] for every $A, B \in H_n(\mathbb{C})$. A majorization [submajorization] preserving map is also called strictly [strongly] isotone in [1]. For instance, if $f$ is a
monotone increasing and convex function on \( \mathbb{R} \), then the map \( A \rightarrow f(A) \) preserves submajorization (see [1]). But the next theorem shows that, for linear maps on \( H_n(\mathbb{C}) \), the notion of majorization preserving coincides with that of unitary equivalence preserving. Consequently Theorem 2.1 is just the characterization of majorization preserving linear maps on \( H_n(\mathbb{C}) \).

**Theorem 3.1.** For each linear map \( \Phi \) on \( H_n(\mathbb{C}) \), the following conditions are equivalent:

(i) \( \Phi \) preserves majorization;

(ii) \( A < B \) implies \( \Phi(A) < \Phi(B) \) for every \( A, B \in H_n(\mathbb{C}) \) (i.e. \( \Phi \) is isotone [1]);

(iii) \( \Phi \) preserves unitary equivalence.

**Proof.** It is obvious that (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii). To show (iii) \( \Rightarrow \) (i), let \( A, B \in H_n(\mathbb{C}) \) and \( A < B \). Then there exist unitary matrices \( U_k, s_k > 0, 1 \leq k \leq N \), with \( \sum_{k=1}^N s_k = 1 \) such that \( A = \sum_{k=1}^N s_k U_k B U_k^* \) (see [1, Theorem 7.1]). By (iii), there exist unitary matrices \( V_k \) such that \( \Phi(U_k B U_k^*) = V_k \Phi(B) V_k^*, 1 \leq k \leq N \). Hence \( \Phi(A) = \sum_{k=1}^N s_k V_k \Phi(B) V_k^* \), implying \( \Phi(A) < \Phi(B) \).

Finally, submajorization preserving linear maps on \( H_n(\mathbb{C}) \) are determined in the next theorem.

**Theorem 3.2.** Let \( \Phi : H_n(\mathbb{C}) \rightarrow H_n(\mathbb{C}) \) be a linear map. Then \( \Phi \) preserves submajorization if and only if \( \Phi \) preserves unitary equivalence and is positive (i.e. \( \Phi(A) \geq 0 \) whenever \( A \geq 0 \)). Consequently \( \Phi \) preserves submajorization if and only if either \( \Phi \) has the form (i) in Theorem 2.1 with \( \lambda_0 \geq 0 \) or \( \Phi \) has one of the forms (ii) and (iii) in Theorem 2.1 with \( \alpha + \beta > 0 \) and \( \beta \geq 0 \).

**Proof.** Suppose \( \Phi \) preserves submajorization. By Theorem 3.1, \( \Phi \) preserves unitary equivalence. Let \( A \in H_n(\mathbb{C}) \) and \( A \geq 0 \). Then, since \( - A \leq 0 \) implies \( - A \leq 0 \), we get \( - \Phi(A) \leq 0 \), so that \( - \lambda_n(\Phi(A)) = \lambda_1(-\Phi(A)) \leq 0 \). Hence \( \lambda_n(\Phi(A)) \geq 0 \), showing \( \Phi(A) \geq 0 \). Thus \( \Phi \) is positive. The converse implication follows immediately from Theorem 3.1 and the fact that \( A < B \) implies \( A \leq A' < B \) for some \( A' \in H_n(\mathbb{C}) \). Moreover the second assertion is easily seen from Theorem 2.1 and the first assertion. Indeed suppose \( \Phi \) is of the form (ii) or (iii) in Theorem 2.1. We then have \( \Phi(P) \cong \text{diag}(\alpha + \beta, \beta, \ldots, \beta) \) for every rank 1 projection \( P \in H_n(\mathbb{C}) \). Hence, except the trivial case \( n = 1 \), \( \Phi \) is positive if and only if \( \alpha + \beta \geq 0 \) and \( \beta \geq 0 \).
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