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Note

Embedding two edge-disjoint Hamiltonian cycles into locally twisted cubes

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ABSTRACT

The n-dimensional hypercube network Q_n is one of the most popular interconnection networks since it has simple structure and is easy to implement. The n-dimensional locally twisted cube LTQ_n, an important variation of the hypercube, has the same number of nodes and the same number of connections per node as Q_n. One advantage of LTQ_n is that the diameter is only about half of the diameter of Q_n. Recently, some interesting properties of LTQ_n have been investigated in the literature. The presence of edge-disjoint Hamiltonian cycles provides an advantage when implementing algorithms that require a ring structure by allowing message traffic to be spread evenly across the interconnection network. The existence of two edge-disjoint Hamiltonian cycles in locally twisted cubes has remained unknown. In this paper, we prove that the locally twisted cube LTQ_n with $n \geqslant 4$ contains two edge-disjoint Hamiltonian cycles. Based on the proof of existence, we further provide an $O(n2^n)$-linear time algorithm to construct two edge-disjoint Hamiltonian cycles in an n-dimensional locally twisted cube LTQ_n with $n \geqslant 4$, where LTQ_n contains 2^n nodes and $n2^{n-1}$ edges.

1. Introduction

Parallel computing is important for speeding up computation. The design of an interconnection network is the first thing to be considered. Many topologies have been proposed in the literature [3,5,6,9], and the desirable properties of an interconnection network include symmetry, relatively small degree, small diameter, embedding capabilities, scalability, robustness, and efficient routing. Among the proposed interconnection networks, the hypercube is a popular interconnection network with many attractive properties such as regularity, symmetry, small diameter, strong connectivity, recursive construction, partition ability, and relatively low link complexity [24]. The architecture of an interconnection network is usually modelled by a graph in which the nodes represent the processing elements and the edges represent the communication links. In this paper, we will use graph and network interchangeably.

The n-dimensional locally twisted cube, denoted by LTQ_n, was first proposed by Yang et al. [27,28] and is a better hypercube variant which is conceptually closer to the comparable hypercube Q_n than existing variants. The n-dimensional locally twisted cube LTQ_n is similar to the n-dimensional hypercube Q_n in the sense that the nodes can be one-to-one labelled with 0–1 binary strings of length n, so that the labels of any two adjacent nodes differ in at most two successive bits. One advantage is that the diameter of an n-dimensional locally twisted cube is only about half the diameter of an n-dimensional hypercube [28]. Some interesting properties of the locally twisted cube LTQ_n have been investigated. In the following, we give a brief survey on the properties of locally twisted cubes. Yang et al. [28] proved that LTQ_n has a connectivity of n. They also showed that locally twisted cubes are 4-pancyclic, i.e. they contain a cycle of length from 4 to 2^n for $n \geqslant 3$, and that

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a locally twisted cube is superior to a hypercube in terms of ring embedding capability [27]. Ma and Xu [21] improved the result in [28] by showing that for any two different nodes u and v in LTQ_n, with $n \geqslant 3$, there exists a uv-path of length ℓ with $d(u, v) + 2 \leqslant \ell \leqslant 2^n - 1$ except for the shortest uv-path, where $d(u, v)$ is the length of the shortest path between u and v. Ma and Xu [22] also proved that the n-dimensional locally twisted cube LTQ_n is edge-pancyclic, i.e. for any edge (u, v) in LTQ_n and integer ℓ, $4 \leqslant \ell \leqslant 2^n$, there exists a cycle C of length ℓ in LTQ_n such that (u, v) is in C. Yang and Yang [26] addressed the fault diagnosis of locally twisted cubes under the MM^* comparison model. Hsieh et al. [12] constructed n edge-disjoint spanning trees in an n-dimensional locally twisted cube, where two spanning trees in a graph are said to be edge-disjoint if they are rooted at the same vertex without sharing any common vertex. Recently, Lin et al. [20] proved that all spanning trees constructed in [12] are independent, i.e. any two spanning trees are rooted at the same node, say r, and for any other node $v \neq r$, the two different paths from v to r, one path in each tree, are internally node-disjoint. On the other hand, Hsieh et al. [14] showed that for any LTQ_n, $n \geqslant 3$, with at most $2n - 5$ faulty edges in which each node is incident to at least two fault-free edges, there exists a fault-free Hamiltonian cycle.

A *Hamiltonian cycle* in a graph is a simple cycle that passes through every node of the graph exactly once. Two Hamiltonian cycles in a graph are said to be *edge-disjoint* if they do not share any common edge. The edge-disjoint Hamiltonian cycles can provide advantages for algorithms that make use of a ring structure [25]. The following application for edge-disjoint Hamiltonian cycles can be found in [25]. Consider the problem of all-to-all broadcasting in which each node sends an identical message to all other nodes in the network. There is a simple solution for the problem using an n-node ring that requires $n - 1$ steps, i.e., at each step, every node receives a new message from its ring predecessor and passes the previous message to its ring successor. If the network admits edge-disjoint rings, then messages can be divided and the parts broadcast along different rings without any edge contention. If the network can be decomposed into edge-disjoint Hamiltonian cycles, then the message traffic will be evenly distributed across all communication links. Edge-disjoint Hamiltonian cycles also form the basis of an efficient all-to-all broadcasting algorithm for networks that employ wormhole or cut-through routing [18]. Further, edge-disjoint Hamiltonian cycles also provide the edge-fault tolerant hamiltonicity of an interconnected network; that is, when a Hamiltonian cycle of an interconnected network contains one faulty edge, then the other edge-disjoint Hamiltonian cycle can be used to replace it for transmission. In this paper, we will construct two edge-disjoint Hamiltonian cycles of locally twisted cubes.

Previous related works are summarized below. The edge-disjoint Hamiltonian cycles in k-ary n-cubes and hypercubes have been constructed in [1]. Barden et al. [2] constructed the maximum number of edge-disjoint spanning trees in a hypercube. Petrovic and Thomassen [23] characterized the number of edge-disjoint Hamiltonian cycles in hyper-tournaments. Hsieh et al. [12] constructed n edge-disjoint spanning trees in an n-dimensional locally twisted cube. In [14], Hsieh et al. investigated the edge-fault tolerant hamiltonicity of an n-dimensional locally twisted cube. Hsieh et al. [10] showed that the arrangement graph contains a Hamiltonian cycle even if it is faulty, i.e. edge faults and vertex faults. Hsieh and Chang [11] showed that Möbius cubes with faulty nodes and faulty edges are 4-pancyclic. Hsieh and Lee [13] determined the conditional edge-fault hamiltonicity of hypercube-like networks, including crossed cubes, twisted cubes, locally twisted cubes, and generalized twisted cubes. They also showed that these hypercube-like networks are all conditional $(2n-5)$-edge-fault pancyclic, where n is the number of dimensions of these networks [15]. Recently, Hsieh and Cian [16] determined the conditional edge-fault hamiltonicity of augmented cubes. The n-dimensional twisted cube is derived from the n-dimensional hypercube by twisting some edges similarly to locally twisted cubes. An n-dimensional twisted cube is $(n-3)$-Hamiltonian connected [17] and $(n-2)$-pancyclic [19], whereas the hypercube is not. In [8], Fu showed that an n-dimensional twisted cube can tolerate up to $2n - 5$ edge faults, while retaining a fault-free Hamiltonian cycle. Fan et al. [7] showed that the twisted cube is edge-pancyclic and provided an $O(l \log l + n^2 + nl)$ time algorithm to find a cycle of length l containing a given edge of the twisted cube. In [7], Fan et al. also asked if an n-dimensional twisted cube is edge-pancyclic with $n - 3$ faults for $n \geqslant 3$. Yang [29] answered the question and showed that an n-dimensional twisted cube is not edge-pancyclic with only one faulty edge for $n \geqslant 3$, and that it is node-pancyclic with $(\lfloor n/2 \rfloor - 1)$-faulty edges for $n \geqslant 3$.

The existence of a Hamiltonian cycle in locally twisted cubes has been verified in [27]. However, there has been no work reported so far on edge-disjoint hamiltonicity properties in locally twisted cubes. In this paper, we show that there exist two edge-disjoint Hamiltonian cycles in an n-dimensional locally twisted cube LTQ_n, for any integer $n \geqslant 4$. Based on the proof of existence, we present an $O(n2^n)$ time algorithm to construct two edge-disjoint Hamiltonian cycles in LTQ_n, where LTQ_n contains 2^n nodes and $n2^{n-1}$ edges.

The rest of this paper is organized as follows. In Section 2, the structure of locally twisted cubes is introduced and some notations are given. Section 3 shows the existence of two edge-disjoint Hamiltonian cycles in locally twisted cubes. In Section 4, we provide a recursive algorithm to construct two edge-disjoint Hamiltonian cycles in an n-dimensional locally twisted cube. Finally, some concluding remarks and future work are given in Section 5.

2. Preliminaries

We usually use a graph to represent the topology of an interconnection network. A graph $G = (V, E)$ is a pair of the node set V and the edge set E, where V is a finite set and E is a subset of $\{(u, v)|(u, v)$ is an unordered pair of $V\}$. We will use $V(G)$ and $E(G)$ to denote the node set and the edge set of G, respectively. If (u, v) is an edge in a graph G, we say that *u is adjacent to v* and *u, v are incident to* edge (u, v). A *neighbor* of a node v in a graph G is any node that is adjacent to v. We write $N_G(v)$ for the set of neighbors of v in G. The subscript 'G' of $N_G(v)$ can be removed from the notation if it has no ambiguity.

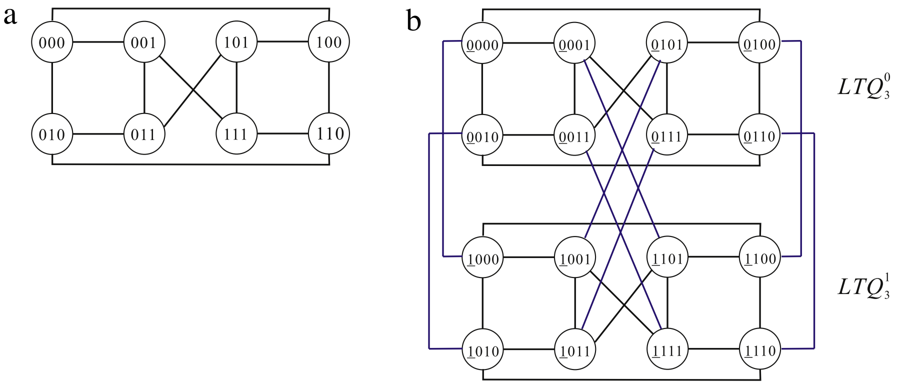

Fig. 1. (a) The 3-dimensional locally twisted cube LTQ_3, and (b) the 4-dimensional locally twisted cube LTQ_4 containing sub-locally twisted cubes LTQ_3^0 and LTQ_3^1, where the leading bits of nodes are underlined.

Let $G = (V, E)$ be a graph with node set V and edge set E. A path P of length ℓ in G, denoted by $v_0 \to v_1 \to \cdots \to v_{\ell-1} \to v_\ell$, is a sequence $(v_0, v_1, \ldots, v_{\ell-1}, v_\ell)$ of nodes such that $(v_i, v_{i+1}) \in E$ for $0 \leqslant i \leqslant \ell - 1$. The first node v_0 and the last node v_ℓ visited by P are called the *path-start* and *path-end* of P, denoted by $start(P)$ and $end(P)$, respectively, and they are called the *end nodes* of P. Path $v_\ell \to v_{\ell-1} \to \cdots \to v_1 \to v_0$ is called the *reversed path*, denoted by P_{rev}, of path P. That is, path P_{rev} visits the nodes of path P from $end(P)$ to $start(P)$ sequentially. In addition, P is a cycle if $|V(P)| \geqslant 3$ and $end(P)$ is adjacent to $start(P)$. A path $P = v_0 \to v_1 \to \cdots \to v_{\ell-1} \to v_\ell$ may contain another subpath Q, denoted as $v_0 \to v_1 \to \cdots \to v_i \to Q \to v_j \to \cdots \to v_{\ell-1} \to v_\ell$, where $Q = v_i \to v_{i+1} \to \cdots \to v_j$, $start(Q) = v_i$, and $end(Q) = v_j$ for $0 \leqslant i \leqslant j \leqslant \ell$. A path (or cycle) in G is called a *Hamiltonian path* (or *Hamiltonian cycle*) if it contains every node of G exactly once. Two paths (or cycles) P_1 and P_2 connecting a node u to a node v are said to be *edge-disjoint* if and only if $E(P_1) \cap E(P_2) = \emptyset$. Two paths (or cycles) Q_1 and Q_2 of graph G are called *node-disjoint* if and only if $V(Q_1) \cap V(Q_2) = \emptyset$. Two node-disjoint paths Q_1 and Q_2 can be *concatenated* into a path, denoted by $Q_1 \Rightarrow Q_2$, if $end(Q_1)$ is adjacent to $start(Q_2)$.

Now, we introduce locally twisted cubes. A node of the n-dimensional locally twisted cube LTQ_n is represented by a 0–1 binary string of length n. A binary string b of length n is denoted by $b_{n-1}b_{n-2}\cdots b_1b_0$, where b_{n-1} is the most significant bit. We then give the recursive definition of the n-dimensional locally twisted cube LTQ_n, for any integer $n \geqslant 2$, as follows.

Definition 2.1 ([27,28]). Let $n \geqslant 2$. The n-dimensional locally twisted cube, denoted by LTQ_n, is defined recursively as follows.
(1) LTQ_2 is a graph consisting of four nodes labelled with 00, 01, 10, and 11, respectively, connected by four edges (00, 01), (00, 10), (01, 11), and (10, 11).
(2) For $n \geqslant 3$, LTQ_n is built from two disjoint copies LTQ_{n-1} according to the following steps. Let LTQ_{n-1}^0 denote the graph obtained by prefixing the label of each node of one copy of LTQ_{n-1} with 0, let LTQ_{n-1}^1 denote the graph obtained by prefixing the label of each node of the other copy of LTQ_{n-1} with 1, and connect each node $b = 0b_{n-2}b_{n-3}\cdots b_1b_0$ of LTQ_{n-1}^0 with the node $1(b_{n-2} \oplus b_0)b_{n-3}\cdots b_1b_0$ of LTQ_{n-1}^1 by an edge, where '$\oplus$' represents the modulo 2 addition.

By the above definition, LTQ_n is an n-regular graph with 2^n nodes and $n \cdot 2^{n-1}$ edges. The n-dimensional locally twisted cube LTQ_n is closed to an n-dimensional hypercube Q_n except that the labels of any two adjacent nodes in LTQ_n differ in at most two successive bits. In addition, LTQ_n can be decomposed into two sub-locally twisted cubes LTQ_{n-1}^0 and LTQ_{n-1}^1, where for each $i \in \{0, 1\}$, LTQ_{n-1}^i consists of those nodes $b = b_{n-1}b_{n-2}\cdots b_1b_0$ with leading bit $b_{n-1} = i$. For each $i \in \{0, 1\}$, LTQ_{n-1}^i is isomorphic to LTQ_{n-1}. For example, Fig. 1(a) shows LTQ_3 and Fig. 1(b) depicts LTQ_4 containing two sub-locally twisted cubes LTQ_3^0 and LTQ_3^1.

Let b be a binary string $b_{n-1}b_{n-2}\cdots b_1b_0$ of length n. We write b^k for the new binary string obtained by repeating the b string k times. Note that if $k = 0$ we say that b^k is an empty string. For instance, $(10)^2 = 1010$ and $0^3 = 000$.

3. The existence of two edge-disjoint Hamiltonian cycles

In this section, we will show that there exist two edge-disjoint Hamiltonian cycles in the n-dimensional locally twisted cube LTQ_n with $n \geqslant 4$. Obviously, LTQ_3 contains no two edge-disjoint Hamiltonian cycles since each node in it is only incident to three edges. We prove the existence of two edge-disjoint Hamiltonian cycles in LTQ_n, $n \geqslant 4$, by induction on n, the dimension of the locally twisted cube. For $n \geqslant 4$, we will show by induction that there are two edge-disjoint Hamiltonian paths P and Q in LTQ_n such that $start(P) = 0(0)^{n-4}010$, $end(P) = 1(0)^{n-4}010$, $start(Q) = 0(0)^{n-4}110$, and $end(Q) = 1(0)^{n-4}110$. By Definition 2.1, $start(P) \in N(end(P))$, $start(Q) \in N(end(Q))$, and the edge $(start(P), end(P))$ is different from the edge $(start(Q), end(Q))$. Thus, P and Q form two edge-disjoint Hamiltonian cycles of LTQ_n for $n \geqslant 4$. In the following, we will show how to construct two such edge-disjoint Hamiltonian cycles. We first show that LTQ_4 contains two edge-disjoint Hamiltonian paths as follows.

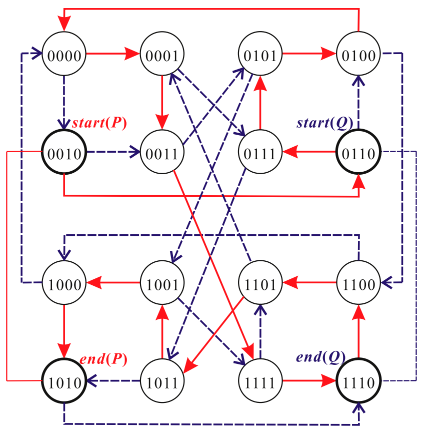

Fig. 2. Two edge-disjoint Hamiltonian paths in LTQ_4, where solid arrow lines indicate a Hamiltonian path P and dotted arrow lines indicate the other edge-disjoint Hamiltonian path Q.

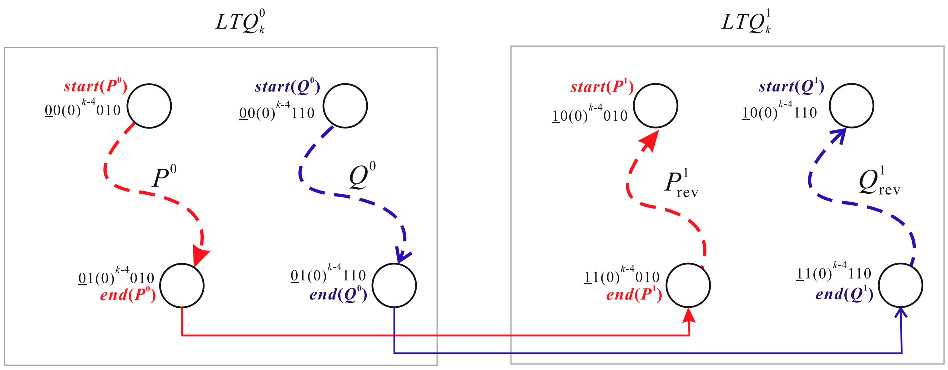

Fig. 3. The construction of two edge-disjoint Hamiltonian paths in LTQ_{k+1} with $k \geqslant 4$, where dotted arrow lines indicate the paths, solid arrow lines indicate concatenated edges, and the leading bits of nodes are underlined.

Lemma 3.1. *There are two edge-disjoint Hamiltonian paths P and Q in LTQ_4 such that $start(P) = 0010$, $end(P) = 1010$, $start(Q) = 0110$, and $end(Q) = 1110$.*

Proof. We prove this lemma by constructing two such paths P and Q. Let
$P = 0010 \to 0110 \to 0111 \to 0101 \to 0100 \to 0000 \to 0001 \to 0011 \to 1111 \to 1110 \to 1100 \to 1101 \to 1011 \to 1001 \to 1000 \to 1010$, and let
$Q = 0110 \to 0100 \to 1100 \to 1000 \to 0000 \to 0010 \to 0011 \to 0101 \to 1001 \to 1111 \to 1101 \to 0001 \to 0111 \to 1011 \to 1010 \to 1110$.
Fig. 2 depicts the construction of P and Q. Clearly, P and Q form two edge-disjoint Hamiltonian paths in LTQ_4. $\square$

Using Lemma 3.1, we prove the following lemma by induction.

Lemma 3.2. *For any integer $n \geqslant 4$, there are two edge-disjoint Hamiltonian paths P and Q in LTQ_n such that $start(P) = 0(0)^{n-4}010$, $end(P) = 1(0)^{n-4}010$, $start(Q) = 0(0)^{n-4}110$, and $end(Q) = 1(0)^{n-4}110$.*

Proof. We prove this lemma by induction on n, the dimension of the locally twisted cube. It follows from Lemma 3.1 that the lemma holds when $n = 4$. Assume that the lemma is true for the case of $n = k \geqslant 4$. Consider LTQ_{k+1}. We first partition LTQ_{k+1} into two sub-locally twisted cubes LTQ_k^0 and LTQ_k^1. By the induction hypothesis, there are two edge-disjoint Hamiltonian paths P^i and Q^i in LTQ_k^i, for $i \in \{0, 1\}$, such that $start(P^i) = i0(0)^{k-4}010$, $end(P^i) = i1(0)^{k-4}010$, $start(Q^i) = i0(0)^{k-4}110$, and $end(Q^i) = i1(0)^{k-4}110$. By Definition 2.1, we have that $end(P^0) \in N(end(P^1))$ and $end(Q^0) \in N(end(Q^1))$.
Let $P = P^0 \Rightarrow P^1_{\text{rev}}$ and let $Q = Q^0 \Rightarrow Q^1_{\text{rev}}$, where P^1_{rev} and Q^1_{rev} are the reversed paths of P^1 and Q^1, respectively. Then, P and Q are two edge-disjoint Hamiltonian paths in LTQ_{k+1} such that $start(P) = 0(0)^{k-3}010$, $end(P) = 1(0)^{k-3}010$, $start(Q) = 0(0)^{k-3}110$, and $end(Q) = 1(0)^{k-3}110$. Fig. 3 depicts the construction of two such edge-disjoint Hamiltonian paths in LTQ_{k+1}. Thus, the lemma holds true when $n = k + 1$. By induction, the lemma holds true. $\square$

By Definition 2.1, the nodes $start(P) = 0(0)^{n-4}010$ and $end(P) = 1(0)^{n-4}010$ are adjacent, the nodes $start(Q) = 0(0)^{n-4}110$ and $end(Q) = 1(0)^{n-4}110$ are adjacent, and the two edges $(start(P), end(P))$ and $(start(Q), end(Q))$ are distinct. Thus the following two theorems hold true.

Theorem 3.3. *There exist two edge-disjoint Hamiltonian paths in LTQ_n for any integer $n \geqslant 4$.*

Theorem 3.4. *There exist two edge-disjoint Hamiltonian cycles in LTQ_n for any integer $n \geqslant 4$.*

4. The algorithm

Based on the proofs of Lemmas 3.1 and 3.2, we design a recursive algorithm to construct two edge-disjoint Hamiltonian paths of an n-dimensional locally twisted cube. The algorithm typically follows a divide-and-conquer approach [4] and is sketched as follows. It is given by an n-dimensional locally twisted cube LTQ_n with $n \geqslant 4$. If $n = 4$, then the algorithm constructs two edge-disjoint Hamiltonian paths according to the proof of Lemma 3.1. Suppose that $n > 4$. It first decomposes LTQ_n into two sub-locally twisted cubes LTQ^0_{n-1} and LTQ^1_{n-1}, where for each $i \in \{0, 1\}$, LTQ^i_{n-1} consists of those nodes $b = b_{n-1}b_{n-2}\cdots b_1b_0$ with leading bit $b_{n-1} = i$. Then, the algorithm computes two edge-disjoint Hamiltonian paths of LTQ^0_{n-1} and LTQ^1_{n-1} recursively. Finally, it concatenates these four cycles into two edge-disjoint Hamiltonian paths of LTQ_n according to the the proof of Lemma 3.2, and outputs these two concatenated paths. The algorithm is formally presented as follows.

Algorithm Constructing-2EDHP
Input: LTQ_n, an n-dimensional locally twisted cube with $n \geqslant 4$.
Output: Two edge-disjoint Hamiltonian paths P and Q in LTQ_n such that $start(P) = 0(0)^{n-4}010$, $end(P) = 1(0)^{n-4}010$, $start(Q) = 0(0)^{n-4}110$, and $end(Q) = 1(0)^{n-4}110$.
Method:

1. **if** $n = 4$, **then**
   begin
2. let $P = 0010 \to 0110 \to 0111 \to 0101 \to 0100 \to 0000 \to 0001 \to 0011 \to 1111 \to 1110 \to 1100 \to 1101 \to 1011 \to 1001 \to 1000 \to 1010$;
3. let $Q = 0110 \to 0100 \to 1100 \to 1000 \to 0000 \to 0010 \to 0011 \to 0101 \to 1001 \to 1111 \to 1101 \to 0001 \to 0111 \to 1011 \to 1010 \to 1110$;
4. **output** "P and Q" as two edge-disjoint Hamiltonian paths of LTQ_n;
   end
5. divide LTQ_n into two sub-locally twisted cubes LTQ^0_{n-1} and LTQ^1_{n-1} such that LTQ^i_{n-1}, $i \in \{0, 1\}$, consists of those nodes $b = b_{n-1}b_{n-2}\cdots b_1b_0$ with leading bit $b_{n-1} = i$;
6. call Algorithm Constructing-2EDHP given LTQ^0_{n-1} to compute two edge-disjoint Hamiltonian paths P^0 and Q^0 of LTQ^0_{n-1}, where $start(P^0) = \underline{0}0(0)^{n-5}010$, $end(P^0) = \underline{0}1(0)^{n-5}010$, $start(Q^0) = \underline{0}0(0)^{n-5}110$, $end(Q^0) = \underline{0}1(0)^{n-5}110$;
7. call Algorithm Constructing-2EDHP given LTQ^1_{n-1} to compute two edge-disjoint Hamiltonian paths P^1 and Q^1 of LTQ^1_{n-1}, where $start(P^1) = \underline{1}0(0)^{n-5}010$, $end(P^1) = \underline{1}1(0)^{n-5}010$, $start(Q^1) = \underline{1}0(0)^{n-5}110$, $end(Q^1) = \underline{1}1(0)^{n-5}110$;
8. compute $P = P^0 \Rightarrow P^1_{\text{rev}}$ and $Q = Q^0 \Rightarrow Q^1_{\text{rev}}$, where P^1_{rev} and Q^1_{rev} are the reversed paths of P^1 and Q^1, respectively;
9. **output** "P and Q" as two edge-disjoint Hamiltonian paths of LTQ_n.

For example, Fig. 4 shows two edge-disjoint Hamiltonian paths of LTQ_5, consisting of two sub-locally twisted cubes LTQ^0_4 and LTQ^1_4, constructed by Algorithm Constructing-2EDHP. The correctness of Algorithm Constructing-2EDHP follows from Lemmas 3.1 and 3.2. Now, we analyze its time complexity. Let m be the number of the nodes in LTQ_n. Then, $m = 2^n$. Let $T(m)$ be the running time of Algorithm Constructing-2EDHP given LTQ_n. It is easy to verify from lines 2 and 3 that $T(m) = O(1)$ if $n = 4$. Suppose that $n > 4$. By visiting every node of LTQ_n once, dividing LTQ_n into LTQ^0_{n-1} and LTQ^1_{n-1} can be done in $O(m)$ time, where each node in LTQ^i_{n-1}, $i \in \{0, 1\}$, is labelled with leading bit i. Thus, line 5 of the algorithm runs in $O(m)$ time. Then, our division of the problem yields two subproblems, each of which is 1/2 the size of the original. It takes time $T(m/2)$ to solve one subproblem, and so it takes time $2 \cdot T(m/2)$ to solve the two subproblems. In addition, concatenating four paths into two paths (line 8) can be easily done in $O(m)$ time. Thus, we get the following recurrence equation:

$$T(m) = \begin{cases} O(1), & \text{if } n = 4; \\ 2 \cdot T(m/2) + O(m), & \text{if } n > 4. \end{cases}$$

The solution of the above recurrence is $T(m) = O(m \log m) = O(n2^n)$. Thus, the running time of Algorithm Constructing-2EDHP given LTQ_n is $O(n2^n)$. Since an n-dimensional locally twisted cube LTQ_n contains 2^n nodes and $n2^{n-1}$ edges, the algorithm is a linear time algorithm.

Let P and Q be the edge-disjoint Hamiltonian paths output by Algorithm Constructing-2EDHP given LTQ_n. By Definition 2.1, $start(P) \in N(end(P))$ and $start(Q) \in N(end(Q))$. In addition, the edge connecting $start(P)$ with $end(P)$ is different from the edge connecting $start(Q)$ with $end(Q)$. Thus, P and Q are two edge-disjoint Hamiltonian cycles of LTQ_n. We finally conclude the following theorem.

Theorem 4.1. *Algorithm* Constructing-2EDHP *can correctly construct two edge-disjoint Hamiltonian cycles (paths) of an n-dimensional locally twisted cube LTQ_n, with $n \geqslant 4$, in $O(n2^n)$-linear time.*

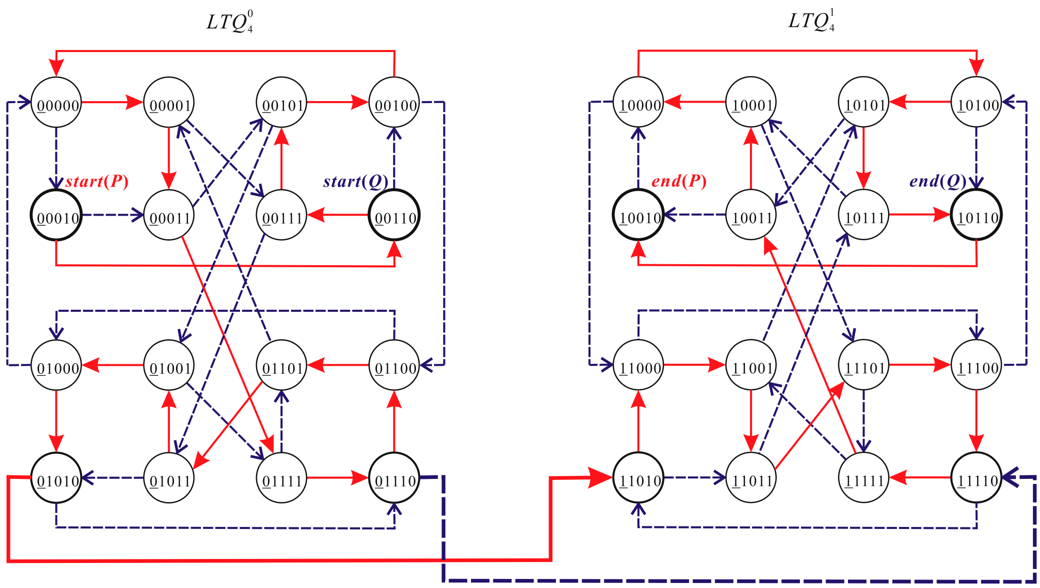

Fig. 4. Two edge-disjoint Hamiltonian paths of LTQ_5 constructed by Algorithm Constructing-2EDHP, where solid arrow lines indicate a Hamiltonian path P, dotted arrow lines indicate the other edge-disjoint Hamiltonian path Q, and the leading bits of nodes are underlined.

5. Concluding remarks

The existence of two edge-disjoint Hamiltonian cycles on locally twisted cubes was unknown prior to our work. Obviously, there exist no two edge-disjoint Hamiltonian cycles in a 3-dimensional locally twisted cube since it is a 3-regular graph. In this paper, we first show that an n-dimensional locally twisted cube LTQ_n, with $n \geqslant 4$, contains two edge-disjoint Hamiltonian cycles (paths). Then, we provide an $O(n2^n)$-linear time algorithm to construct two edge-disjoint Hamiltonian cycles (paths) of LTQ_n. However, many edges are not used in our construction of two edge-disjoint Hamiltonian cycles (paths) in LTQ_n for $n > 4$. Thus, our result may not be optimal about the number of edge-disjoint Hamiltonian cycles in LTQ_n when $n \geqslant 6$. The maximum number of edge-disjoint Hamiltonian cycles in LTQ_n, with $n \geqslant 4$, is bounded by $\lfloor n/2 \rfloor$. It would be interesting to see whether there is maximum number of edge-disjoint Hamiltonian cycles in LTQ_n for $n \geqslant 6$. We would like to post this as an open problem for interested readers.

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