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The general quaternionic M–J sets on the mapping $z \leftarrow z^\alpha + c$ $(\alpha \in \mathbf{N})$

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Abstract

Fractal nature exists not only in complex plane but also in higher dimensional space. It is a focus to find new modals to construct new 3-D fractals. In this paper, the general Mandelbrot sets and Julia sets on the mapping $f : z \leftarrow z^\alpha + c$ $(\alpha \in \mathbf{N})$ are discussed. The 3-D projections of M–J sets are constructed using escape time algorithm and ray-tracing method. And their properties are theoretically analysed. The connectness of the general quaternionic M sets is proved and the boundary of the stability region of the fixed point is calculated. It is found that if the parameter c of Julia sets is chosen from the M sets, they share the same cycle number and stable points. It can be concluded that the quaternionic M sets contain sufficient information of quaternionic Julia sets.

Keywords: General Mandelbrot sets; General Julia sets; Quaternion; 3-D fractals

1. Introduction

In 1970s, Mathematician Mandelbrot combined computer graphics and the "Rational Map Iteration Theory in Complex Plane" advanced by Julia and Faton, and studied the simple mapping $z \leftarrow z^2+c$ in complex plane, which can produce very complicated sets called Julia sets. If parameter c is classified according to the corresponding connectivity of Julia sets, then the sets of c constitute Mandelbrot sets [1].

In 1982, Alan Norton found that Julia sets exist not only in complex plane, but also in the 4-dimensional quaternions. Quaternionic Julia sets are nontrivial, and they have many properties that complex Julia sets don't have [2,3]. Since quaternionic fractals belong to 4-dimension, we construct their 3-D projection pictures to observe their structures. There are many 3-D graphics algorithms, such as Norton [2] boundary tracking method, Hart [4,5] ray-tracing algorithm and inverse iterate method, Dang [6] distant estimation method and Cheng [7] volume rendering algorithm. On the other hand, trying to find new modals to construct new 3-D fractals is another focus. Cheng [7], Gintz [8] and Nikiel [9] construct various 3-D fractals exploiting the models of ternary number, customized complexified quaternion and quadratic functions respectively. What's more, the topology structures and interior properties are drawing people's attention. Holbrook [10] examines the symmetries and connectivity of quadratic quaternion M and Julia sets. Although Bedding [11] believes that the quaternionic map $z \leftarrow z^2 + c$ does not play

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any fundamental role analogous to that in the complex plane, people still did do much work on it. Gomatam [12] presents explicit calculations of the stability domain for k-cycle of M set. Bogush [13] discusses the symmetries of the quaternions which give rise to the class of identical Julia sets. Dominic [14] constructs the biocomplex M and Julia sets and discusses their connectivity. Buchanan [15] analyses the cycle sets for the meromorphic complex and quaternionic maps. Shizuo [16] investigates the dynamics of quaternion quadratic maps and proves the connectivity of quaternion M and Julia set. Lakner [17] finds that the quadratic quaternionic Julia sets can be derived from a certain symbolic dynamics giving rise to fractal fibrations.

Although the above research work expands the theory of 3-D fractal fields, little work has been done on the quaternion mapping $f : z \leftarrow z^{\alpha} + c(\alpha \in \mathbf{N})$. Only M and Julia sets of $\alpha = 2$, $\alpha = 3$ and $\alpha = -1$ are explored. Since M sets contain sufficient information of Julia sets, we will construct general M–J sets on the mapping $f : z \leftarrow z^{\alpha} + c(\alpha \in \mathbf{N})$, and discuss their properties and the relationship between them.

2. Quaternions

2.1. Definition of quaternions

The quaternions were discovered by Hamilton in 1843 as a method of performing 3-D multiplication. A quaternion q can be represented as follows:

$$q = [S, V] = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k},$$

where scalar part is $S = q_0$ and vector part is $V = (q_1, q_2, q_3)$. The four-tuple of independent real values (q_0, q_1, q_2, q_3) is assigned to one real axis and three orthonormal imaginary axes: $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

2.2. Quaternion operations

(1) $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \mathbf{ki} = -\mathbf{ik} = \mathbf{j}, \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$;
(2) addition: $q_1 + q_2 = [S_1, V_1] + [S_2, V_2] = [S_1 + S_2, V_1 + V_2]$;
(3) additive identity: $0 = [0, 0]$;
(4) scalar multiplication: $kq = [kS, kV]$;
(5) multiplication: $q_1q_2 = [S_1, V_1][S_2, V_2] = [S_1S_2 - V_1V_2, S_1V_1 + S_2V_2 + V_1 \times V_2]$;
(6) multiplicative identity: $1 = [1, 0]$;
(7) module: $\|q\| = \sqrt{S^2 + V^2}$ (where V^2 denotes $\|V\|^2$);
(8) multiplicative inverse: $q^{-1} = \frac{1}{\|q\|^2}[S, -V]$.

Quaternions are an extension of the complex number to four-space, but quaternion multiplication is not commutative. From the above operations it should be noted that quaternions form a non-abelian division ring [18].

2.3. Quaternion trigonometric representation

Suppose quaternion $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$, whose module is $h = \|q\| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$. Suppose trigonometric function $\cos\alpha = q_0/h$, then $\sin\alpha = \sqrt{1 - \cos^2\alpha} = \sqrt{q_1^2 + q_2^2 + q_3^2}/h$. So q can be represented as follows: $q = h\left[\frac{q_0}{h} + \frac{\sqrt{q_1^2+q_2^2+q_3^2}}{h} \cdot \frac{q_1\mathbf{i}+q_2\mathbf{j}+q_3\mathbf{k}}{\sqrt{q_1^2+q_2^2+q_3^2}}\right] = h(\cos\theta + n\sin\theta)$, where $n = \frac{q_1\mathbf{i}+q_2\mathbf{j}+q_3\mathbf{k}}{\sqrt{q_1^2+q_2^2+q_3^2}}$.

Suppose that $q = [\cos\theta, n\sin\theta]$ is a unit quaternion, where n is unit axis of rotation, then $q = \exp(n\theta)$, $q^u = [\cos u\theta, n\sin u\theta]$, $u \in R$ [19]. Now we give the theorem as follows:

Theorem 2.1. *For any quaternion* $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$, *if* $q_i \in R$, *and* $q_i \neq 0$ $(i = 0, 1, 2, 3)$) *then* $q^u = h^u(\cos u\theta + n\sin u\theta)$, $u \in \mathbf{N}$, *where* $h = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$, $n = \frac{q_1\mathbf{i}+q_2\mathbf{j}+q_3\mathbf{k}}{\sqrt{q_1^2+q_2^2+q_3^2}}$.

Theorem 2.1 expands De-Moivre's Theorem in complex plane to quaternion space, which offers efficient mathematical method to compute general quaternionic M–J sets.

By analogy, we can expand Theorem 2.1 to the N-dimensional space. Suppose $\mathbf{R}^n$ is N-dimensional space on the real field $\mathbf{R}$, whose orthogonal basis is $(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, \ldots, 0, 1)$ written as $\mathbf{e}, \mathbf{i}_1, \ldots \mathbf{i}_{n-1}$. Then the element in $\mathbf{R}^n$ can be written as $q = [S, V] = q_0 + q_1\mathbf{i}_1 + \cdots + q_{n-1}\mathbf{i}_{n-1}$, where scalar part is $S = q_0$ and vector part is $V = (q_1, q_2, \ldots, q_{n-1})$.

Theorem 2.1′. *For any $q \in \mathbf{R}^n$, $q = q_0 + q_1\mathbf{i}_1 + \cdots + q_{n-1}\mathbf{i}_{n-1}$, if $q_i \in R$, and $q_i \neq 0$ $(i = 0, 1, \ldots, n-1)$, then $q^u = h^u\ (\cos u\theta + n \sin u\theta)$, $u \in \mathbf{N}$, where*

$$h = \sqrt{q_0^2 + q_1^2 + \cdots + q_{n-1}^2}, \qquad n = \frac{q_1\mathbf{i}_1 + q_2 i_2 + \cdots + q_{n-1}\mathbf{i}_{n-1}}{\sqrt{q_1^2 + q_2^2 + \cdots + q_{n-1}^2}}.$$

The proof of Theorem 2.1′ *is the same with that of Theorem 2.1, which may be provided if needed.*

3. General quaternionic M sets

3.1. Definition of general quaternionic M *sets*

Since M sets contain sufficient information and have close relationship with iterate function $\{f^k(z)\}$, we give the equivalent definition as follows according to the definition of the complex M sets:

Definition 3.1. Suppose the mapping $f : z \leftarrow z^\alpha + c(\alpha \in \mathbf{N})$ in the quaternionic space $\mathbf{H}$, U is the set of quaternion q whose orbit is bounded, that is:

$$U = \{c \in H : \{f^k(c)\}_{k=1}^{\infty} \text{ remains bounded}\} = \{c \in H : c, c^\alpha + c, (c^\alpha + c)^\alpha + c, \cdots \not\to \infty, k \to \infty\},$$

then U is called the general M sets on the f.

3.2. Properties of general quaternionic M *sets*

According to the above definition, we use escape time algorithm and ray-tracing method to compute and construct general M sets. Since quaternion belongs to 4-dimensional space, we can only sample their 3-D projections. A quaternion value $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ is a quadruplet of independent real values (q_0, q_1, q_2, q_3) assigned to one real axis ($\mathbf{r}$ axis) and three imaginary axes: $\mathbf{i}, \mathbf{j}, \mathbf{k}$. Not losing generality, all the pictures in this paper are the projections on the axes $\mathbf{r}, \mathbf{i}, \mathbf{j}$, where the value of the $\mathbf{k}$ axis is zero (see Figs. 1 and 2).

General quaternionic M sets have the following characters:

Theorem 3.1. *If the quaternion $q(q_0, q_1, q_2, q_3)$ $(q_i \in R, i = 0, 1, 2, 3)$ is in the general qaternionic* M *sets on the mapping $f : z \leftarrow z^\alpha + c(\alpha \in \mathbf{N})$, suppose $q_1^2 + q_2^2 + q_3^2 = \rho^2(\rho > 0)$, then any quaternion $q'(q'_0, q'_1, q'_2, q'_3)$ $(q'_i \in R, i = 0, 1, 2, 3)$ that satisfies $q_0 = q'_0$ and $q_1'^2 + q_2'^2 + q_3'^2 = \rho^2$ is also in the* M *sets.*

Proof. We prove it by mathematical induction.

$$\begin{aligned} z_1 &= z_0^\alpha + c = (q_0 + \rho n)^\alpha + (q_0 + \rho n) = [h(\cos\theta + n\sin\theta)]^\alpha + h(\cos\theta + n\sin\theta) \\ &= h^\alpha(\cos\alpha\theta + n\sin\alpha\theta) + h(\cos\theta + n\sin\theta) = (h^\alpha\cos\alpha\theta + h\cos\theta) + n(h^\alpha\sin\alpha\theta + h\sin\theta). \end{aligned} \tag{3.1}$$

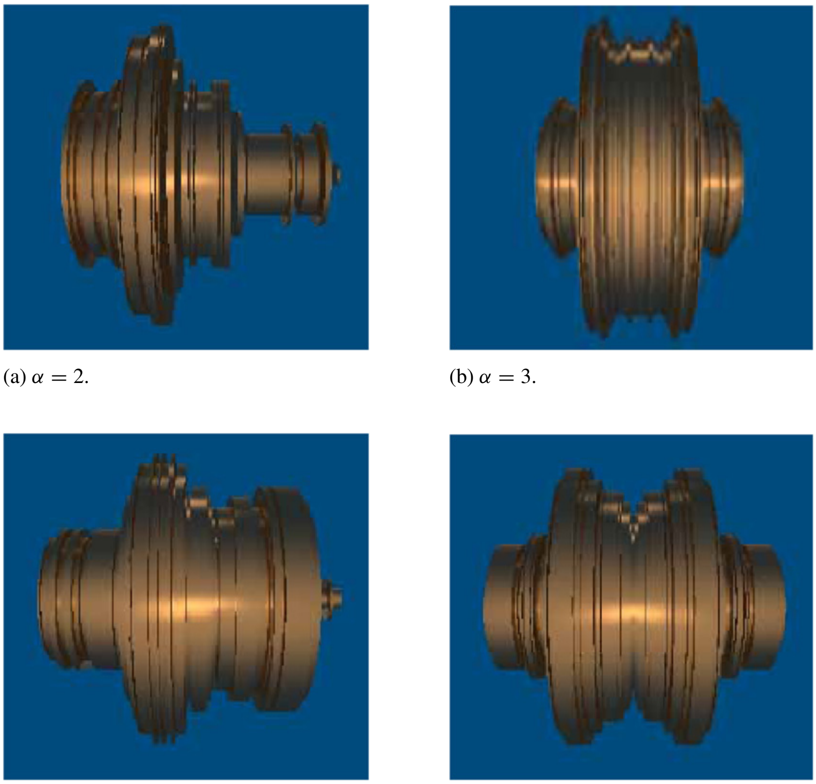
In Eq. (3.1), $h = \sqrt{q_0^2 + \rho^2}$, $\cos\theta = q_0/h$, $\sin\theta = \rho/h$, n is unit axis of rotation (the detailed definition see also Theorem 2.1). By analogy,

$$z'_1 = z_0'^\alpha + c' = (q'_0 + \rho n')^\alpha + (q'_0 + \rho n') = [h'(\cos\theta' + n'\sin\theta')]^\alpha + h'(\cos\theta' + n'\sin\theta').$$

In above equation, $h' = \sqrt{q_0'^2 + \rho^2}$, $\cos\theta' = q'_0/h$, $\sin\theta' = \rho/h$, n' is unit axis of rotation. Since $q_0 = q'_0$, so $h' = h$, $\cos\theta' = \cos\theta$, $\sin\theta' = \sin\theta$, then we have

$$z'_1 = [h(\cos\theta + n'\sin\theta)]^\alpha + h(\cos\theta + n'\sin\theta) = (h^\alpha\cos\alpha\theta + h\cos\theta) + n'(h^\alpha\sin\alpha\theta + h\sin\theta) \tag{3.2}$$

From Eqs. (3.1) and (3.2), we can get three properties as follows:



(a) $\alpha = 2$. (b) $\alpha = 3$.

(c) $\alpha = 8$. (d) $\alpha = 9$.

Fig. 1. General quaternionic M sets ($\alpha > 0$).

(1) $\|z_1\| = \|z_1'\|$;
(2) The scalar of z_1 is equal to that of z_1';
(3) The directions of the vector of z_1 and z_1' are not changed after iterations, which are the same with q and q' respectively.

Suppose z_m and z_m' retain the above properties after m iterations, then after $m + 1$ iterations we have

$$\begin{aligned} z_{m+1} &= z_m^\alpha + c = (q_m + \rho_m n)^\alpha + (q_0 + \rho n) = [h_m(\cos\theta_m + n\sin\theta_m)]^\alpha + h(\cos\theta + n\sin\theta) \\ &= (h_m^\alpha \cos\alpha\theta_m + h\cos\theta) + n(h_m^\alpha \sin\alpha\theta_m + h\sin\theta) \end{aligned} \tag{3.3}$$

$$\begin{aligned} z'_{m+1} &= z_m'^\alpha + c = (q_m' + \rho_m' n')^\alpha + (q_0' + \rho n') = [h_m'(\cos\theta_m' + n'\sin\theta_m')]^\alpha + h(\cos\theta + n'\sin\theta) \\ &= [h_m(\cos\theta_m + n'\sin\theta_m)]^\alpha + h(\cos\theta + n'\sin\theta) \\ &= (h_m^\alpha \cos\alpha\theta_m + h\cos\theta) + n'(h_m^\alpha \sin\alpha\theta_m + h\sin\theta). \end{aligned} \tag{3.4}$$

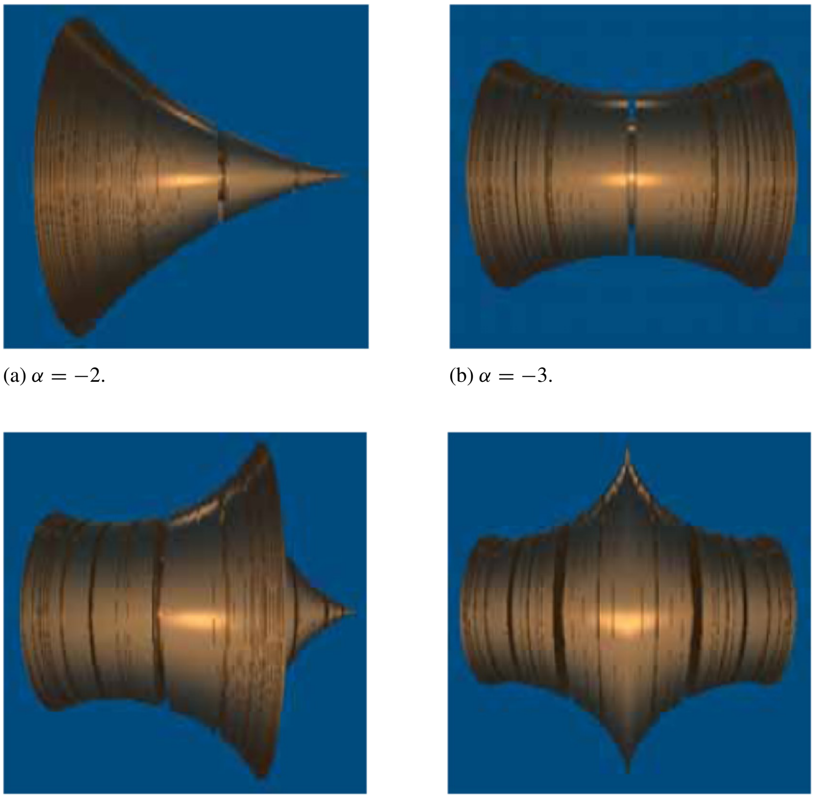
From Eqs. (3.3) and (3.4), we conclude that z_{m+1} and z'_{m+1} still satisfy the above three conditions by induction. Therefore, if $q \in$ M, $q' \in$ M. □

Theorem 3.2. *The 3-D projection of the general quaternionic* M *sets on the three image axes* **i**, **j**, **k** *is a sphere.*

Theorem 3.3. *The general quaternionic* M *sets is symmetrical around the* **r** *real axis.*

It is easy to prove the Theorems 3.2 and 3.3 referenced that of the Theorem 3.1, this paper wouldn't present the detailed. According to the Theorem 3.2, we can conclude that the fractal characteristic of the general quaternionic M sets mainly exists in the projection on the plane of real and imaginary axis.

Theorem 3.4. *The general quaternionic* M *sets on the mapping* $f : z \leftarrow z^\alpha + c(\alpha \in \mathbf{N})$ *can be obtained by rotation of the corresponding complex* M *sets (written as* M_C*) around the real axis.*



(a) $\alpha = -2$.

(b) $\alpha = -3$.

(c) $\alpha = -4$.

(d) $\alpha = -5$.

Fig. 2. General quaternionic M sets ($\alpha < 0$).

Proof. For any quaternion q, there exist a unit quaternion p and a complex number c so that

$$pqp^{-1} = c \text{ [20]}. \tag{3.5}$$

Eq. (3.5) shows the inner automorphism transformation of the division ring of quaternions, in which the scalar and module of q retain unchanged, that is $q_0 = c_0$ and $\|q\| = \|c\|$. Now we prove by mathematical induction that $q \in \mathrm{M}$ if $c \in M_C$.

$$Z_1 = Z_0^\alpha + q = q^\alpha + q,$$

$$z_1 = z_0^\alpha + c = c^\alpha + c = \left(pqp^{-1}\right)^\alpha + pqp^{-1} = pq^\alpha p^{-1} + pqp^{-1} = p(q^\alpha + q)p^{-1} = pZ_1p^{-1}.$$

Suppose $z_m = pZ_mp^{-1}$ after m iterations, then after $m + 1$ iterations,

$$z_{m+1} = z_m^\alpha + c = (pZ_mp^{-1})^\alpha + c = pZ_m^\alpha p^{-1} + pqp^{-1} = p(Z_m^\alpha + q)p^{-1} = pZ_{m+1}^\alpha p^{-1}.$$

The above equation shows that if $c \in M_C$, then $q \in \mathrm{M}$. Since $q_0 = c_0$, $\|q\| = \|c\|$, c and q are in the same sphere. □

According to Theorem 3.4, we can get the following naturally:

Theorem 3.5. *When $\alpha = 2n + 1$, $n \in \mathbf{N}$, the general quaternionic* M *sets are symmetrical around the plane* $\mathbf{r} = 0$.

Theorem 3.5 exhibits the general complex M sets' characters extending in the quaternionic space. The proof of the theorem can be accomplished by combining Theorem 3.4 and the properties of the general complex M sets, this paper wouldn't present the detailed.

Theorem 3.4 explains the relationship between the general quaternionic M sets and the general complex M sets on the mapping $f : z \leftarrow z^\alpha + c(\alpha \in \mathbf{N})$, and proves the followed property of the M sets in another point of view.

Proposition 3.1. *$q(q_0, q_1, q_2, q_3) \in \mathrm{M}$ if and only if $q_0 + \mathbf{i}\rho \in \mathrm{M}$ (The definition of ρ see Theorem 3.1).*

It can be concluded that the dynamics of $f : z \leftarrow z^\alpha + c(\alpha \in \mathbf{N})$ is equivalent to the dynamics of the complex map $p_{c_0+\mathbf{i}\rho} : s + \mathbf{i}t \leftarrow (s + \mathbf{i}t)^\alpha + c_0 + \mathbf{i}\rho(\alpha \in \mathbf{N})$. Now we get the following:

Theorem 3.6. *The general quaternionic M sets on the mapping $f : z \leftarrow z^\alpha + c(\alpha \in \mathbf{N})$ are connected.*

Proof. Suppose the M sets are disconnected, then there must exit two points $q(q_0, q_1, q_2, q_3) \in \mathrm{M}$ and $q'(q'_0, q'_1, q'_2, q'_3) \in \mathrm{M}$, there is no path between whom. Since $c(q_0, \rho, 0, 0) \in \mathrm{M}$ and q are in the same sphere, there is no path between c and q'. And it is the same with $c'(q'_0, \rho', 0, 0) \in \mathrm{M}$ and q. That is, there is no path between c and c'. But this contradicts the fact that the complex Mandelbrot sets are connected. □

3.3. Stability region of general quaternionic M *sets*

The stability regions of the general quaternionic M sets are calculated in Appendices A–C. When $\alpha = 3$, the boundary of the one-cycle stability region satisfies

$$\|Q\| = \left(\frac{1}{3}\right)^{\frac{1}{2}}, \tag{3.6}$$

where $Q = S + \mathbf{i}\sigma \cdot V$. Concerning parameter $C = a + \mathbf{i}\sigma \cdot K$, the boundary is given by

$$a = \frac{\sqrt{3}}{3}\cos t - \frac{\sqrt{3}}{9}\cos 3t, \qquad K = \left(\frac{\sqrt{3}}{3}\sin t - \frac{\sqrt{3}}{9}\sin 3t\right) n \quad (0 \le t < 2\pi). \tag{3.7}$$

When $\alpha = 4$, the boundary of the one-cycle stability region satisfies

$$\|Q\| = \left(\frac{1}{4}\right)^{\frac{1}{3}} \tag{3.8}$$

and the boundary of the fixed points can be given by

$$a = \frac{\sqrt[3]{2}}{2}\cos t - \frac{\sqrt[3]{2}}{8}\cos 4t, \qquad K = \left(\frac{\sqrt[3]{2}}{2}\sin t - \frac{\sqrt[3]{2}}{8}\sin 4t\right) n \quad (0 \le t < 2\pi). \tag{3.9}$$

And when $\alpha = -3$, the boundary of the one-cycle stability region satisfies

$$\|Q\| = \sqrt[4]{3} \tag{3.10}$$

and the boundary of the fixed points is given by

$$a = \sqrt[4]{3}\cos t - \frac{\sqrt[4]{3}}{3}\cos 3t, \qquad K = \left(\sqrt[4]{3}\sin t + \frac{\sqrt[4]{3}}{3}\sin 3t\right) n. \tag{3.11}$$

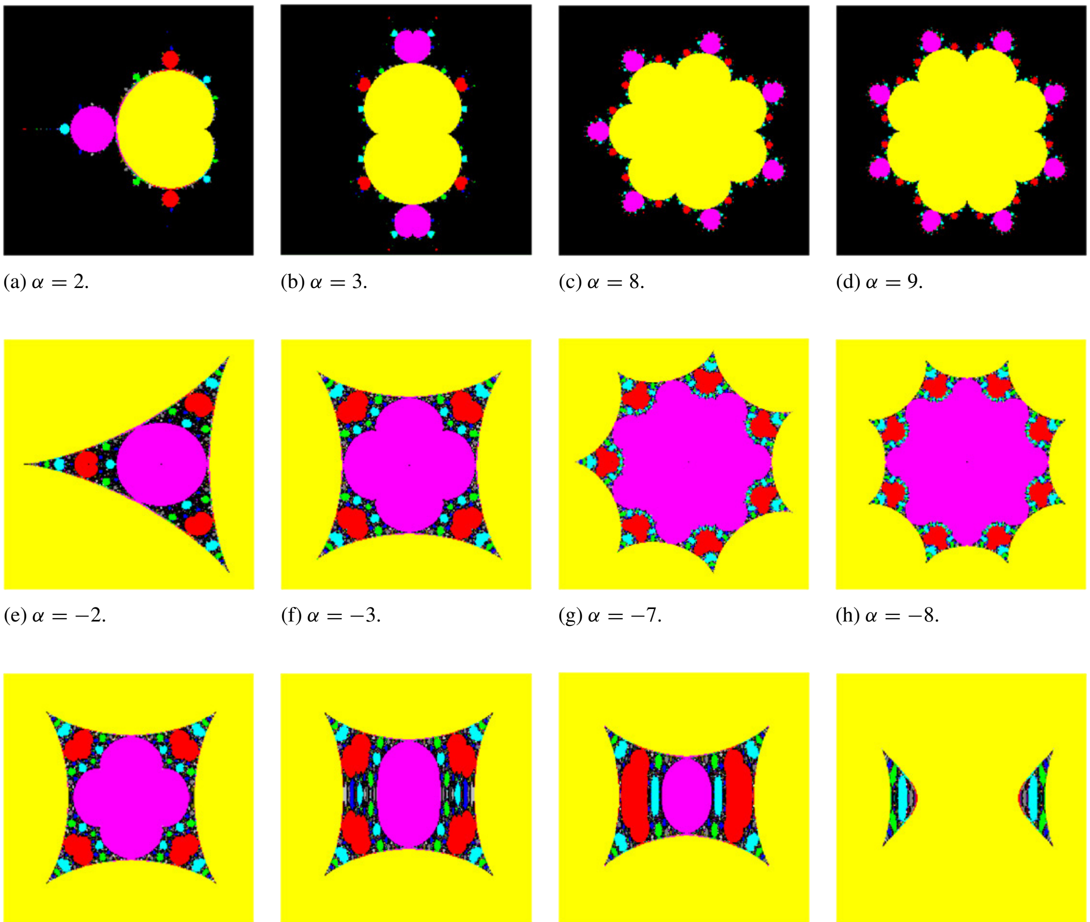
According to the results of Gomatam [12], Walter [15] and this paper, it can be concluded that the boundary of one-cycle stability region of general quaternionic M sets satisfies

$$\|Q\| = |\alpha|^{-1/(\alpha-1)}, \tag{3.12}$$

and it can be given by

$$\begin{cases} a = |\alpha|^{-1/(\alpha-1)}\cos\theta - |\alpha|^{-\alpha/(\alpha-1)}\cos\alpha\theta \\ K = |\alpha|^{-1/(\alpha-1)}\sin\theta - |\alpha|^{-\alpha/(\alpha-1)}\sin\alpha\theta. \end{cases} \tag{3.13}$$

The stability region of various cycle of general quaternionic M sets are presented in Fig. 3. Different colours represent different cycles: yellow is 1-cycle, pink is 2-cycle, red is 3-cycle, cyan is 4-cycle, green is 5-cycle, blue



(a) $\alpha = 2$. (b) $\alpha = 3$. (c) $\alpha = 8$. (d) $\alpha = 9$.

(e) $\alpha = -2$. (f) $\alpha = -3$. (g) $\alpha = -7$. (h) $\alpha = -8$.

(i) $k = 0.2$. (j) $k = 0.5$. (k) $k = 0.7$. (l) $k = 1.0$.

Fig. 3. Stability regions of general quaternionic M sets. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

is 6-cycle, grey is 7-cycle, white is 8-cycle ..., and black is unstable region. In Fig. 3(a)–(h), the third and fourth dimension are zero. It is noted that these quaternionic fractals are the same with the corresponding complex ones under the above condition. In fact, Eq. (3.13) coincides with the stability region of the general complex M sets. In Fig. 3(i) to (l), quaternionic M sets of $\alpha = -3$ with different fourth dimension are proposed (the third dimension is 0), which manifest the diversification of quaternionic M sets in higher dimension.

4. General quaternionic Julia sets

4.1. Definition of general quaternionic Julia sets

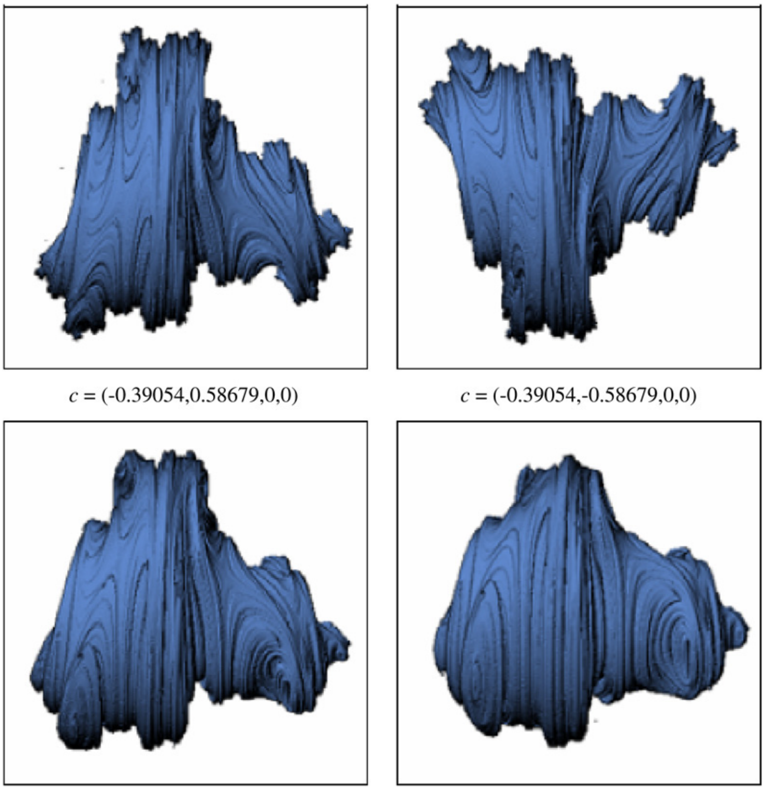
Definition 4.1. Suppose $f : z \leftarrow z^\alpha + c(\alpha \in \mathbf{N})$ is the mapping in the quaternionic space, F_f represents the sets of the points whose orbits don't tend to infinity, that is:

$$F_f = \{z \in H : \{|f^k(z)|\}_{k=1}^{\infty} \text{ remains bouded}\}.$$

Then the sets are the filled-in general J sets on the f, and the boundary of F_f is general quaternionic J sets, written as J_f, that is $J_f = \partial F_f$.

4.2. Properties of general quaternionic Julia sets

The general quaternionic Julia sets have several properties as follows:



c = (-0.39054,0.58679,0,0) c = (-0.39054,-0.58679,0,0)

c = (-0.39054,0.3071196,0.5,0) c = (-0.39054,0.5,0.3071196,0)

Fig. 4. General quaternionic Julia sets ($\alpha = 3$).

Theorem 4.1. *The general quaternionic Julia sets on the mapping $f : z \leftarrow z^\alpha + c(\alpha \in \mathbf{N})$ can be described only by two variables of c, namely the scalar part c_0 and the module $\|c\|$.*

It is proposed in [13] that the quaternionic Julia set on the mapping $f : z \leftarrow z^2 + c$ is completely defined by just two number, c_0 and $\|c\|$. Now we prove that it is also appropriate for the general quaternionic Julia sets on the mapping $f : z \leftarrow z^\alpha + c(\alpha \in \mathbf{N})$.

Set $c = c_0 + \mathbf{i}c_1$, where $c_1 = \sqrt{\|c\|^2 - c_0^2}$. Transformations on the mapping f are as follows:

$$z' = qzq^{-1} \tag{4.1}$$

$$c' = qcq^{-1}. \tag{4.2}$$

In Eq. (4.1), q is a unit quaternion. z'_n and z_n retain the following relationship after n iterations,

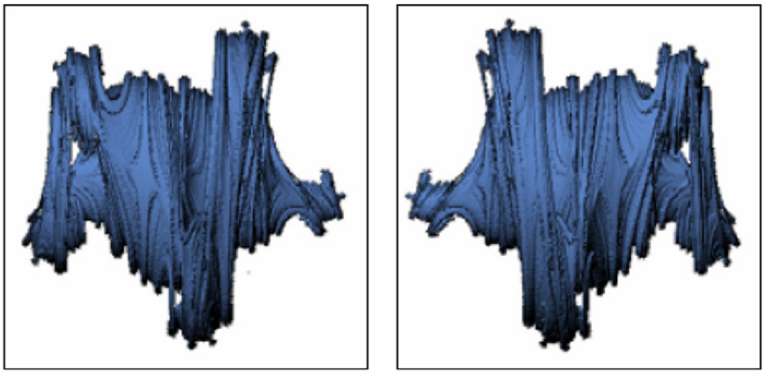
$$z'_n = z'^\alpha_{n-1} + c' = (qz_{n-1}q^{-1})^\alpha + qcq^{-1} = q(z^\alpha_{n-1} + c)q^{-1}$$

that is

$$z'_n = qz_nq^{-1}. \tag{4.3}$$

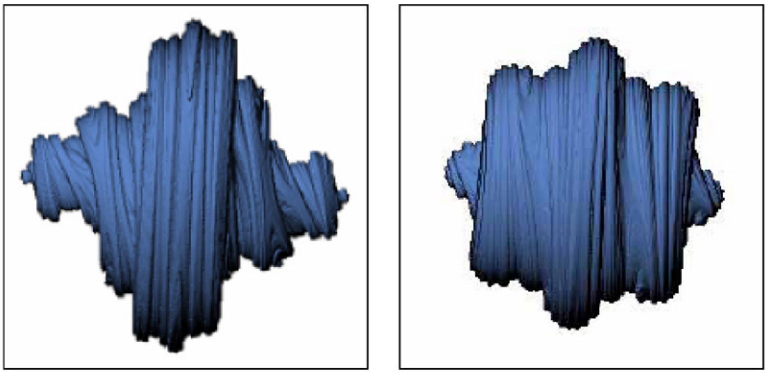
It is easy to check that $\|z\|$ stay invariant in Eqs. (4.1) and (4.3), and it is the same with c_0 and $\|c\|$ in Eq. (4.2). So J_f keeps its structure unchanged under the above transformations, but its 3-D projection rotates by angle θ around the axis that c lies on (see Fig. 4). Then the unit quaternion q is

$$q = \cos\frac{\theta}{2} + n\sin\frac{\theta}{2},$$



c = (-0.61,0.75,0,0) c = (0.61,0.75,0,0)

Fig. 5. General quaternionic Julia sets ($\alpha = 5$).



$\alpha = 4$, c = (-0.61,0.3,0,0) $\alpha = 8$, c = (-0.45,0.51,0,0)

Fig. 6. General quaternionic Julia sets.

where $n = \frac{v}{\sqrt{\|c\|^2 - c_0^2}}$ and the vector direction of n conforms with that of c' [21]. The value of θ isn't mentioned in [13]. According to the pictures in this paper, we hypothesize θ is the angle between the vectors of c' and c.

Theorem 4.2. *When $\alpha = 2n + 1$, $n \in \mathbf{N}$, all the Julia sets which satisfy the condition $c = \pm c_0 + \|c_1\| n$ are of the same structure, only the pictures are rotated by some angle.*

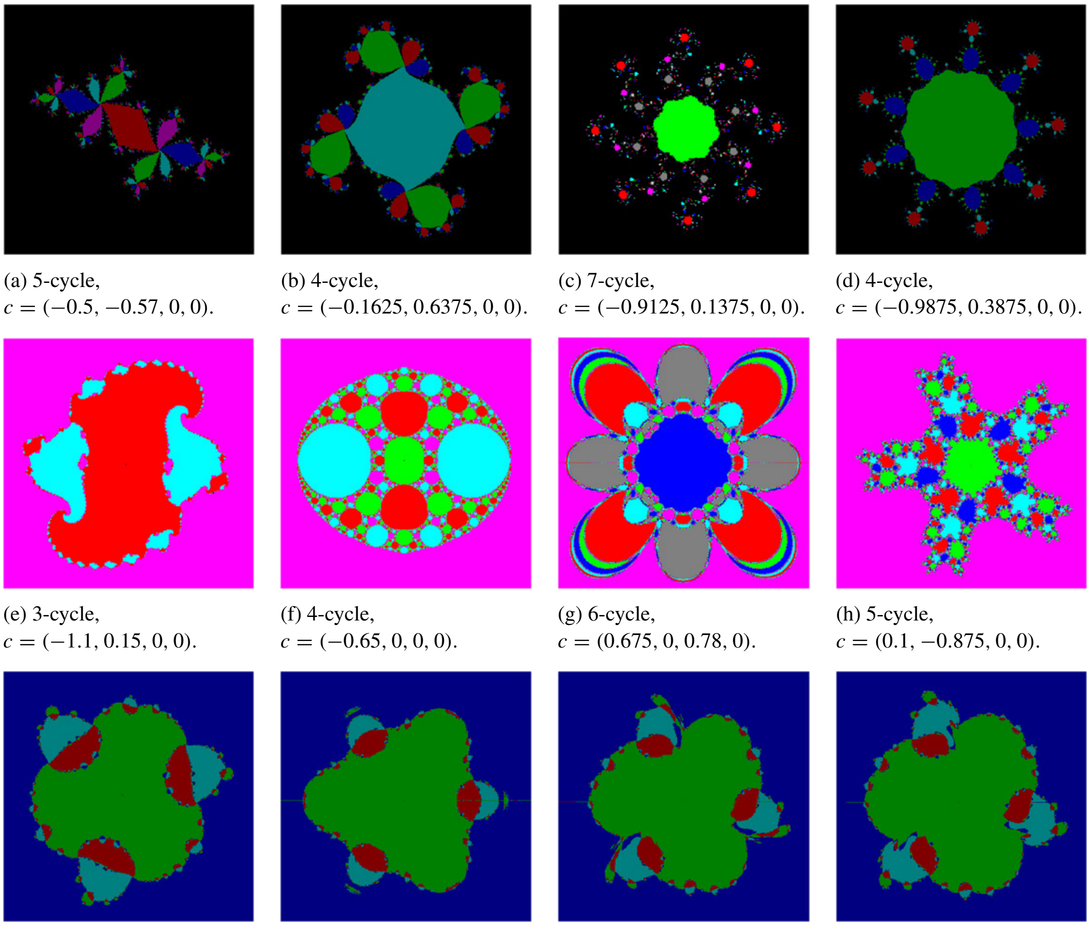
Theorem 4.2 relaxes the condition of Theorem 4.1, which conforms with the Theorem 3.4 of M sets (see Fig. 5). Since M sets are the parameter sets of Julia sets, the symmetries of M sets when $\alpha = 2n + 1$, $n \in \mathbf{N}$ determine the symmetries on the choice of the parameter c.

Theorem 4.3. *When $\alpha = 2n$, $n \in \mathbf{N}$, the general Julia sets are symmetric around the zero point.*

Proof. When $\alpha = 2n$, $n \in \mathbf{N}$, for any quaternions $q(q_0, q_1, q_2, q_3)$ and $q'(-q_0, -q_1, -q_2, -q_3)$, we have $z_1' = z_0'^{\alpha} + c = (-z_0)^{\alpha} + c = z_0^{\alpha} + c = z_1$, so if $q \in J_f$, $q' \in J_f$ (see Fig. 6). □

4.3. Stability region of general quaternionic Julia sets

The two-dimensional projection of quaternionic Julia sets are shown in Fig. 7 (the third and fourth dimension are zero), whose parameters c are selected from the stable cycle points of corresponding M sets. The colourful region is stable and the black region is unstable. Different colour represents convergence to different stable points. According to the results of large experiments, it is found that the cycle number of the quaternionic Julia sets is equivalent to that of the quaternionic M sets and, further, they share the same stable points. What's more, a group of Julia sets with the



(a) 5-cycle, $c = (-0.5, -0.57, 0, 0)$. (b) 4-cycle, $c = (-0.1625, 0.6375, 0, 0)$. (c) 7-cycle, $c = (-0.9125, 0.1375, 0, 0)$. (d) 4-cycle, $c = (-0.9875, 0.3875, 0, 0)$.

(e) 3-cycle, $c = (-1.1, 0.15, 0, 0)$. (f) 4-cycle, $c = (-0.65, 0, 0, 0)$. (g) 6-cycle, $c = (0.675, 0, 0.78, 0)$. (h) 5-cycle, $c = (0.1, -0.875, 0, 0)$.

(i) $c = (-1.025, -0.925, 0, 0)$. (j) $c = (-1.025, 0, 0.925, 0)$. (k) $c = (-1.025, 0.6, 0.704006392, 0)$. (l) $c = (-1.025, 0.704006392, 0.6, 0)$.

Fig. 7. **i**, **j** Slice of general quaternionic Julia sets with different stability regions. (a) $\alpha = 2$, (b) $\alpha = 4$, (c) $\alpha = 8$, (d) $\alpha = 9$, (e) $\alpha = -2$, (f) $\alpha = -2$, (g) $\alpha = -4$, (h) $\alpha = -5$, (i)–(l) $\alpha = -3$.

Table 1
The stable points of the quaternionic Julia sets of $\alpha = -3$

c	(−1.025, −0.925, 0, 0)	(−1.025, 0, 0.925, 0)	(−1.025, 0.6, 0.704006392, 0)	(−1.025, 0.704006392, 0.6, 0)
Stable points	(−1.03845984, −0.92368912, 0, 0)	(−1.03845984, 0.92368912, 0, 0)	(−1.03845984, 0.59914970, 0.70300870, 0)	(−1.03845984, 0.70300870, 0.59914970, 0)
	(−4.19510240, −0.13580870, 0, 0)	(−4.19510240, 0.13580870, 0, 0)	(−4.19510240, 0.08809213, 0.10336237, 0)	(−4.19510240, 0.10336237, 0.08809213, 0)
	(−0.67171534, −0.05475104, 0, 0)	(−0.67171534, 0.05475104, 0, 0)	(−0.67171534, 0.03551419, 0.04167036, 0)	(−0.67171534, 0.041670365, 0.03551419, 0)
	(−0.81157108, −0.61970845, 0, 0)	(−0.81157108, 0.61970845, 0, 0)	(−0.81157108, 0.40197305, 0.47165266, 0)	(−0.81157108, 0.47165266, 0.40197305, 0)

parameter c following the rule of Theorem 4.1 are constructed, as shown in Fig. 7(i)–(l). Their stable points are listed in Table 1. It can be concluded that these stable points are of the same module, and this further validates the result of Theorem 4.1. All these point to the fact that the quaternionic M sets have lot of information of quaternionic Julia sets.

5. Conclusions

This paper discusses the general quaternionic M sets and Julia sets on the mapping $f: z \leftarrow z^\alpha + c(\alpha \in \mathbf{N})$. The 3-D projections and 2-D slices are constructed. The stability regions of the fixed points of M sets are calculated and it is found that the results coincide with that of the complex case. The Julia sets with the parameter c selected from the M sets have some consistent properties with the M sets.

The paper presents a new modal for constructing quaternionic fractals. Since the quaternionic M sets have infinite stability regions, in the future we will further investigate the topology of the quaternionic M sets and the relationship between the M sets and Julia sets, and try to find the quaternionic fractals of the other mapping.

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Appendix A. Stability region of the one-cycle of quaternionic map $f: z \leftarrow z^3 + c$

Let Q define the fixed point of quaternionic map $f: z \leftarrow z^3 + c$, then

$$Q_{n+1} = Q_n^3 + C \tag{A.1}$$

where $Q, C \in \mathbf{H}$, $Q = S + \mathbf{i}\sigma \cdot V$. We may write

$$Q^3 = S(S^2 - 3V^2) + \mathbf{i}\sigma(3S^2 - V^2) \cdot V, \tag{A.2}$$

where V^2 denotes $\|V\|^2$. The iterations of the map $Q_{n+1} = Q_n^3 + C$, (where $C = a + \mathbf{i}\sigma \cdot K$) are therefore given by

$$S_{n+1} = S_n(S_n^2 - 3V_n^2) + a, \qquad V_{n+1} = (3S_n^2 - V_n^2)V_n + K. \tag{A.3}$$

Since Q is the fixed point, the components S and V satisfy

$$S = S(S^2 - 3V^2) + a, \quad V = (3S^2 - V^2)V + K. \tag{A.4}$$

Introducing the perturbation of Q:

$$S = S_n + \delta S_n, \qquad V = V_n + \delta V_n. \tag{A.5}$$

Then using Eqs. (A.3) and (A.4), it can be calculated that

$$\delta S_{n+1} = S - S_{n+1} = S(S^2 - 3V^2) - S_n(S_n^2 - 3V_n^2), \tag{A.6}$$

which can be simplified to

$$\delta S_{n+1} = 3(S^2 - V^2)\delta S_n - 6SV \cdot \delta V_n + O(\delta S_n^2, \delta V_n^2), \tag{A.7}$$

where O denotes "of the order of" (using Eq. (A.5)). Similarly,

$$\delta V_{n+1} = V - V_{n+1} = (3S^2 - V^2)V - (3S_n^2 - V_n^2)V_n, \tag{A.8}$$

which can be simplified to

$$\delta V_{n+1} = 6SV \cdot \delta S_n + 3(S^2 - V^2)\delta V_n + O(\delta S_n^2, \delta V_n^2). \tag{A.9}$$

Now we expand δV_n in terms of the orthogonal triplet (K, I, J) so that $\delta V_n = \delta a_n K + \delta\beta_n I + \delta\chi_n J$, which reduce a 4×4 system to a 2×2 one [12]. Ignoring terms of second order and higher, Eq. (A.7) becomes

$$\delta S_{n+1} = 3(S^2 - V^2)\delta S_n - 6SV \cdot (\delta a_n K), \tag{A.10}$$

where $V = K/(1 - 3S^2 + V^2)$ (from Eq. (A.4)).

Eq. (A.9) becomes

$$\delta V_{n+1} = \frac{6S}{1 - 3S^2 + V^2} K \cdot \delta S_n + 3(S^2 - V^2)(\delta a_n K). \tag{A.11}$$

From (A.10) and (A.11), we can get

$$\begin{pmatrix} \delta S_{n+1} \\ \delta a_{n+1} \end{pmatrix} = J \begin{pmatrix} \delta S_n \\ \delta a_n \end{pmatrix}, \tag{A.12}$$

where $J = \begin{bmatrix} 3(S^2 - V^2) & -6SV \cdot K \\ \frac{6S}{1 - 3S^2 + V^2} & 3(S^2 - V^2) \end{bmatrix}$. The eigenvalues of J are given by

$$\lambda = 3(S^2 - V^2) \pm 6\mathbf{i}\sigma \cdot SV. \tag{A.13}$$

Noting that $|\lambda| < 1$, and $|\lambda| = 3(S^2 + V^2)$, then the boundary of the region of the stability can be determined by the condition as follows:

$$S^2 + V^2 = \frac{1}{3}, \tag{A.14}$$

that is

$$\|Q\| = \left(\frac{1}{3}\right)^{\frac{1}{2}} \tag{A.15}$$

(according to the definition of $\|Q\|$ in Section 2.2). And S and V may be parameterized by

$$S = \frac{\sqrt{3}}{3} \cos t, \qquad V = \frac{\sqrt{3}}{3} \sin t \cdot n \quad (0 \le t < 2\pi) \tag{A.16}$$

where n is an arbitrary unit vector. Thus the boundary of the region of the fixed points is given by

$$a = \frac{\sqrt{3}}{3} \cos t - \frac{\sqrt{3}}{9} \cos 3t, \qquad K = \left(\frac{\sqrt{3}}{3} \sin t - \frac{\sqrt{3}}{9} \sin 3t\right) n \quad (0 \le t < 2\pi). \tag{A.17}$$

Appendix B. Stability region of the one-cycle of quaternionic map $f : z \leftarrow z^4 + c$

Let Q define the fixed point of quaternionic map $f : z \leftarrow z^4 + c$, then

$$Q_{n+1} = Q_n^4 + C \tag{B.1}$$

where $Q, C \in \mathbf{H}$, $Q = S + \mathbf{i}\sigma \cdot V$. We may write

$$Q^4 = (S^4 - 6S^2V^2 + V^4) + 4\mathbf{i}\sigma S(S^2 - V^2) \cdot V. \tag{B.2}$$

The iterations of the map $Q_{n+1} = Q_n^4 + C$ are therefore given by

$$S_{n+1} = (S_n^4 - 6S_n^2V_n^2 + V_n^4) + a, \qquad V_{n+1} = 4S_n(S_n^2 - V_n^2)V_n + K. \tag{B.3}$$

Since Q is the fixed point, the components S and V satisfy

$$S = (S^4 - 6S^2V^2 + V^4) + a, \quad V = 4S(S^2 - V^2)V + K. \tag{B.4}$$

Then it can be calculated that

$$\delta S_{n+1} = S - S_{n+1} = (S^4 - 6S^2V^2 + V^4) - (S_n^4 - 6S_n^2V_n^2 + V_n^4) \tag{B.5}$$

(the definition of δS_n and δV_n see Appendix A), which can be simplified to

$$\delta S_{n+1} = 4S(S^2 - 3V^2)\delta S_n - 4(3S^2 - V^2)V \cdot \delta V_n + O(\delta S_n^2, \delta V_n^2). \tag{B.6}$$

Similarly,

$$\delta V_{n+1} = V - V_{n+1} = 4S(S^2 - V^2)V - 4S_n(S_n^2 - V_n^2)V_n, \tag{B.7}$$

which can be simplified to

$$\delta V_{n+1} = 4(3S^2 - V^2)V \cdot \delta S_n + 4S(S^2 - 3V^2)\delta V_n + O(\delta S_n^2, \delta V_n^2). \tag{B.8}$$

Using the method mentioned in Appendix A, Eq. (B.6) can be written as

$$\delta S_{n+1} = 4S(S^2 - 3V^2)\delta S_n - 4(3S^2 - V^2)V \cdot (\delta a_n K), \tag{B.9}$$

where $V = K/[1 - 4S(S^2 - V^2)]$ (from Eq. (B.4)).

And Eq. (B.8) becomes

$$\delta V_{n+1} = \frac{4(3S^2 - V^2)}{1 - 4S(S^2 - V^2)} K \cdot \delta S_n + 4S(S^2 - 3V^2)(\delta a_n K). \tag{B.10}$$

From (B.9) and (B.10), we can get

$$\begin{pmatrix} \delta S_{n+1} \\ \delta a_{n+1} \end{pmatrix} = J \begin{pmatrix} \delta S_n \\ \delta a_n \end{pmatrix}, \tag{B.11}$$

where $J = \begin{bmatrix} 4S(S^2 - 3V^2) & -4(3S^2 - V^2)V \cdot K \\ \frac{4(3S^2 - V^2)}{1 - 4S(S^2 - V^2)} & 4S(S^2 - 3V^2) \end{bmatrix}$. The eigenvalues of J are given by

$$\lambda = 4[S(S^2 - 3V^2) \pm \mathbf{i}\sigma \cdot (V^2 - 3S^2)V]. \tag{B.12}$$

Noting that $|\lambda| < 1$, and $|\lambda|^2 = 16(S^2 + V^2)^3$, then the boundary of the region of the stability can be determined by the condition as follows:

$$S^2 + V^2 = \left(\frac{1}{4}\right)^{\frac{2}{3}}, \tag{B.13}$$

that is

$$\|Q\| = \left(\frac{1}{4}\right)^{\frac{1}{3}}. \tag{B.14}$$

And S and V may be parameterized by

$$S = \frac{\sqrt[3]{2}}{2} \cos t, \qquad V = \frac{\sqrt[3]{2}}{2} \sin t \cdot n \quad (0 \le t < 2\pi) \tag{B.15}$$

where n is an arbitrary unit vector. Thus the boundary of the region of the fixed points is given by

$$a = \frac{\sqrt[3]{2}}{2} \cos t - \frac{\sqrt[3]{2}}{8} \cos 4t, \qquad K = \left(\frac{\sqrt[3]{2}}{2} \sin t - \frac{\sqrt[3]{2}}{8} \sin 4t\right) n \quad (0 \le t < 2\pi). \tag{B.16}$$

Appendix C. Stability region of the one-cycle of quaternionic map $f : z \leftarrow z^{-3} + c$

Let Q define the fixed point of quaternionic map $f : z \leftarrow z^{-3} + c$, then

$$Q_{n+1} = Q_n^{-3} + C \tag{C.1}$$

where $Q, C \in \mathbf{H}$, $Q = S + \mathbf{i}\sigma \cdot V$. We may write

$$Q^{-3} = \frac{S(S^2 - 3V^2) + \mathbf{i}\sigma(V^2 - 3S^2) \cdot V}{(S^2 + V^2)^3}. \tag{C.2}$$

The iterations of the map $Q_{n+1} = Q_n^{-3} + C$ are therefore given by

$$S_{n+1} = \frac{S_n(S_n^2 - 3V_n^2)}{(S_n^2 + V_n^2)^3} + a, \qquad V_{n+1} = \frac{(V_n^2 - 3S_n^2)V_n}{(S_n^2 + V_n^2)^3} + K. \tag{C.3}$$

Since Q is the fixed point, the components S and V satisfy

$$S = \frac{S(S^2 - 3V^2)}{(S^2 + V^2)^3} + a, \quad V = \frac{(V^2 - 3S^2)V}{(S^2 + V^2)^3} + K. \tag{C.4}$$

Then it can be calculated that

$$\delta S_{n+1} = S - S_{n+1} = \frac{-3(S^4 - 6S^2V^2 + 3V^4)}{MM_n^3}\delta S_n + \frac{12S(V^2 - S^2)}{MM_n^3} V \cdot \delta V_n + O(\delta S_n^2, \delta V_n^2), \tag{C.5}$$

(the definition of δS_n and δV_n see Appendix A), and

$$\delta V_{n+1} = V - V_{n+1} = \frac{12S(S^2 - V^2)}{MM_n^3} V \cdot \delta S_n + \frac{-3(S^4 - 6S^2V^2 + 3V^4)}{MM_n^3}\delta V_n + O(\delta S_n^2, \delta V_n^2), \tag{C.6}$$

where $M = S^2 + V^2$ and $M_n = S_n^2 + V_n^2$. The latter can be further calculated as follows:

$$\frac{1}{M_n^3} = \frac{1}{M^3 + 6SM^2\delta S_n + 6M^2V \cdot \delta V_n + O(\delta S_n^2, \delta V_n^2)} = \frac{1}{M^3\left[1 + \frac{6}{M}(S\delta S_n + V \cdot \delta V_n) + O(\delta S_n^2, \delta V_n^2)\right]}$$
$$= \frac{1}{M^3}\left[1 - \frac{6}{M}(S\delta S_n + V \cdot \delta V_n) + O(\delta S_n^2, \delta V_n^2)\right],$$

so that

$$\frac{1}{M_n^3} = \frac{1}{M^3}\left[1 + O(\delta S_n, \delta V_n)\right]. \tag{C.7}$$

If $\delta S_n, \delta V_n$ are sufficiently small, it may be regarded as $\frac{1}{M_n^3} \approx \frac{1}{M^3}$.

Now using the method mentioned in Appendix A, Eq. (C.5) can be written as

$$\delta S_{n+1} = \frac{-3(S^4 - 6S^2V^2 + 3V^4)}{MM_n^3}\delta S_n + \frac{12S(V^2 - S^2)}{MM_n^3} V \cdot (\delta a_n K), \tag{C.8}$$

where $V = [M^3/(M^3 + 3M - 4V^2)]K$ (from Eq. (C.4)).

And Eq. (C.6) becomes

$$\delta V_{n+1} = \frac{12SM^2(S^2 - V^2)}{M_n^3(M^3 + 3M - 4V^2)} K \cdot \delta S_n + \frac{-3(S^4 - 6S^2V^2 + 3V^4)}{MM_n^3}(\delta a_n K). \tag{C.9}$$

From (C.8) and (C.9), we can get

$$\begin{pmatrix} \delta S_{n+1} \\ \delta a_{n+1} \end{pmatrix} = J \begin{pmatrix} \delta S_n \\ \delta a_n \end{pmatrix}, \tag{C.10}$$

where $J = \begin{bmatrix} \frac{-3(S^4 - 6S^2V^2 + 3V^4)}{MM_n^3} & \frac{12S(V^2 - S^2)}{MM_n^3} V \cdot K \\ = \frac{12SM^2(S^2 - V^2)}{M_n^3(M^3 + 3M - 4V^2)} & \frac{-3(S^4 - 6S^2V^2 + 3V^4)}{MM_n^3} \end{bmatrix}$. The eigenvalues of J are given by

$$\lambda = \frac{3}{M^4}[(8S^2V^2 - M^2) \pm 4\mathbf{i}\sigma \cdot S(S^2 - V^2)V]. \tag{C.11}$$

Noting that $|\lambda| < 1$, and $|\lambda|^2 = 9/M^4$, then the boundary of the region of the stability can be determined by the condition as follows:

$$S^2 + V^2 = \sqrt{3}, \tag{C.12}$$

that is

$$\|Q\| = \sqrt[4]{3}. \tag{C.13}$$

And S and V may be parameterized by

$$S = \sqrt[4]{3}\cos t, \qquad V = \sqrt[4]{3}\sin t \cdot n \quad (0 \le t < 2\pi) \tag{C.14}$$

where n is an arbitrary unit vector. Thus the boundary of the region of the fixed points is given by

$$a = \sqrt[4]{3}\cos t - \frac{\sqrt[4]{3}}{3}\cos 3t, \qquad K = \left(\sqrt[4]{3}\sin t + \frac{\sqrt[4]{3}}{3}\sin 3t\right) n. \tag{C.15}$$

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