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Solution of the knight's Hamiltonian path problem on chessboards

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Abstract

Is it possible for a knight to visit all squares of an $n \times n$ chessboard on an admissible path exactly once? The answer is yes if and only if $n \geqslant 5$. The kth position in such a path can be computed with a constant number of arithmetic operations. A Hamiltonian path from a given source s to a given terminal t exists for $n \geqslant 6$ if and only if some easily testable color criterion is fulfilled. Hamiltonian circuits exist if and only if $n \geqslant 6$ and n is even.

1. Introduction

In many textbooks on algorithmic methods two chess problems are presented as examples for backtracking algorithms:
– is it possible to place n queens on an $n \times n$ chessboard such that no one threatens another?
– is it possible for a knight to visit all squares of an $n \times n$ cheesboard on an admissible path exactly once?

Both problems are special cases of NP-complete graph problems, namely the maximal independent set problem and the Hamiltonian path problem (see [4]).

We consider the second problem. The graph $G_n = (V_n, E_n)$ (always $n \geqslant 2$) for $n \times n$ chessboard C_n consists of the vertex set $V_n := \{(i,j) \mid 1 \leqslant i,j \leqslant n\}$ for the squares of the chessboard and the edge set E_n describing the admissible moves of a knight, i.e.

$$\{(i,j),(i',j')\} \in E_n$$
$$:\Leftrightarrow\ [|i-i'|=1 \text{ and } |j-j'|=2] \text{ or } [|i-i'|=2 \text{ and } |j-j'|=1].$$

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The problem is to decide whether G_n contains a Hamiltonian path, i.e. a path which contains each vertex exactly once, and to construct, in the affirmative case, efficiently a solution.

We solve this problem completely. G_n contains a Hamiltonian path if and only if $n \geqslant 5$. The Hamiltonian path can be computed in optimal sequential time $O(n^2)$ and by n^2 processors in constant time (the model of parallel computers is almost insignificant).

The s–t Hamiltonian path problem is to decide whether a Hamiltonian path from the source s to the terminal t exists. The usual color function for chessboards is defined by $c(i,j)$:= white if $i+j$ is even, and $c(i,j)$:= black if $i+j$ is odd. We define the color criterion for generalized chessboards of size $n \times m$:

$$nm \text{ even:} \quad c(s) \neq c(t),$$

$$nm \text{ odd:} \quad c(s) = c(t) = \text{white}.$$

We prove that an s–t Hamiltonian path exists for $n \times n$ chessboards and $n \geqslant 6$ if and only if the color criterion is fulfilled for s and t. In the affirmative case, we again obtain efficient algorithms for the construction of a Hamiltonian path. As a corollary we show that G_n contains a Hamiltonian circuit if and only if $n \geqslant 6$ and n is even.

Our solution is based on an efficient backtracking algorithm for the solution of a finite number of special cases, among them some problems on nonrectangular chessboards. Afterwards a divide-and-conquer strategy is applied. The $n \times n$ chessboard C_n is divided into small subboards such that the problems on the subboards can be solved by table-look-up and such that the solutions for the small problems can be combined to a solution of the problem for C_n.

Partial solutions of the problem are known since a long time (see the references). The existence of Hamiltonian paths is stated but no complete proof has been published. For the more general and more complex s–t Hamiltonian path problem no solution has been stated before.

In Section 2 we solve the Hamiltonian path problem and in Section 3 we solve the s–t Hamiltonian path problem. Although the results of Section 3 imply the results of Section 2, we think that it is helpful to present a simpler proof for the simpler problem.

2. The Hamiltonian path problem

For special cases we have used a computer program based on the backtracking paradigm. In order to obtain reasonable run times it is always tested whether the graph on all not yet visited vertices is connected. It is also tested whether at least two of these vertices have degree one only. In either case our path cannot be the beginning of a Hamiltonian path and backtracking is necessary. Otherwise we choose as successor on our path among the admissible vertices one with the smallest degree in the graph of all vertices not visited yet. This algorithm is interesting for its own sake. Its idea has been used already by Warnsdorff [8].

We start with the negative results. G_2 and G_3 do not have Hamiltonian paths, since they have isolated vertices. G_4 does not contain a Hamiltonian path. The four corner vertices (1, 1), (1, 4), (4, 1) and (4, 4) have degree 2 only. The opposite corner vertices have the same neighbors. These are (2, 3) and (3, 2) for (1, 1) and (4, 4), and (2, 2) and (3, 3) for (1, 4) and (4, 1). If (1, 1) is neither the source nor the terminal of a Hamiltonian path h, it has to be enclosed by (2, 3) and (3, 2). The same holds for (4, 4). Hence (1, 1), (2, 3), (3, 2) and (4, 4) have to be the first or the last four vertices of h. The same holds for (1, 4), (2, 2), (3, 3) and (4, 1). Hence, the other eight vertices have to be combined by a Hamiltonian path. This is impossible, since the graph on these eight vertices is not connected.

G_n contains, if $n \in \{5, 6, 7, 8, 9\}$, a Hamiltonian path. We present solutions in the appendix and the reader may easily check their correctness. Later we shall also use slightly modified versions of these Hamiltonian paths. Obviously, it is possible to start in any given corner. For $n = 5$ the Hamiltonian path may start in any corner and may arrive in the middle position of any nonadjacent side of the board C_5.

Furthermore, we need Hamiltonian paths for four Γ-shaped chessboards. For $k \in \{6, 7, 8, 9\}$ the pseudo chessboard Γ_k consists of 5 rows of length k and $k - 5$ rows of length 5. Hence, the corresponding graph $G_k^* = (V_k^*, E_k^*)$ has the vertex set

$$V_k^* := \{(i, j) \mid (1 \leqslant i \leqslant 5,\ 1 \leqslant j \leqslant k) \text{ or } (6 \leqslant i \leqslant k,\ 1 \leqslant j \leqslant 5)\}.$$

The edge set E_k^* is defined by the same rules as E_n. Hamiltonian paths for G_k^* starting at $(1, k)$ and arriving at $(k, 3)$ are presented in the appendix. Obviously, we also know Hamiltonian paths from $(k, 1)$ to $(3, k)$. We identify the board Γ_5 with the board C_5.

Theorem 2.1. *The knight problem has a solution if and only if $n \geqslant 5$.*

Proof. We have already considered the cases $n \leqslant 9$. If $n > 9$, we consider the Γ-strip of all positions (i, j) where $i \leqslant 5$ or $j \leqslant 5$. Let $k \equiv n \bmod 5$, where $0 \leqslant k \leqslant 4$. We place the pseudo chessboard Γ_{k+5} into the upper left corner and fill the rest of the Γ-stripe with $2(\lfloor n/5 \rfloor - 1)$ boards of side length 5 (see Fig. 1 for $n = 28$).

The Hamiltonian path starts at $(1, n)$. We run through C_5 on a Hamiltonian path and reach $(3, n - 4)$. Then we jump to $(1, n - 5)$, i.e. to the start position of the next subregion. In this way we run through all boards C_5 in the upper 5 rows of C_n. If $n \equiv 0 \bmod 5$, we use a slightly modified Hamiltonian path for the board C_5 in the upper left corner. The Hamiltonian path goes from (1, 5) to (5, 3) such that we can jump to (6, 1). If $n \not\equiv 0 \bmod 5$, we reach position $(1, k + 5)$. We use our special solution for Γ_{k+5} and reach position $(k + 5, 3)$ from which we jump to $(k + 6, 1)$. In any case we have reached the left upper corner of the first C_5-board not visited yet. We run through the column of C_5-boards always starting at position (1, 1) (for the particular C_5) and arriving at (5, 3). From (5.3) we may jump to (1, 1) of the next C_5-board. Only for the last C_5-board we go from (1, 1) to (3, 5) which is position $(n - 2, 5)$ of C_n. From this position we jump to position $(n, 6)$, the lower left corner of the remaining C_{n-5}-board.

The board C_{n-5} is treated in the same (symmetric) way until we obtain a board C_l where $l \in \{5, \ldots, 9\}$. For this board we know how to construct a Hamiltonian path. □

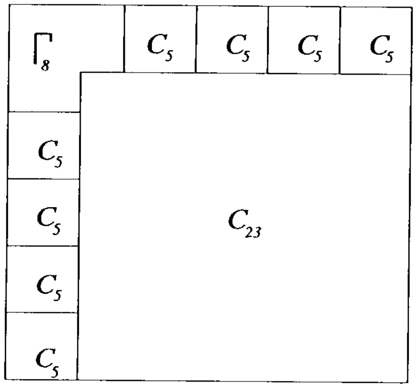

Fig. 1. $n = 28$, $C_{28} - C_{23}$ is a Γ-stripe.

By H_n we denote the Hamiltonian path which has been constructed in the proof of Theorem 2.1. H_n can be computed efficiently by sequential and parallel algorithms.

Definition. For $n \geqslant 5$ let $p_n: \{1, \ldots, n^2\} \to \{1, \ldots, n\} \times \{1, \ldots, n\}$ be the one-to-one mapping describing H_n, i.e. $p_n(t)$ is the tth vertex of H_n on the board C_n.

Proposition 2.2. (i) *The function p_n can be evaluated in a constant number of arithmetic operations.*

(ii) *The Hamiltonian path H_n through C_n can be computed in time $\mathrm{O}(n^2)$ or by n^2 processors in constant time.*

Proof. The first part follows by a simple but lengthy proof which is omitted here. The second part is then a simple corollary. For the sequential algorithm we can compute H_n step by step. For the parallel algorithm, the tth processor, $1 \leqslant t \leqslant n^2$, computes $p_n(t)$ in a constant number of arithmetic operations. The array $p_n(t)$, $1 \leqslant t \leqslant n^2$, represents H_n. Knowing $p_n(t)$, the tth processor may write its number t into position $p_n(t)$ of C_n. Then we obtain that description of H_n which we have used in the appendix. □

3. The *s–t* Hamiltonian path problem

We prove the following theorem.

Theorem 3.1. (i) *Let $n \geqslant 6$. An s–t Hamiltonian path exists if and only if the color criterion is fulfilled for s and t.*

(ii) *Let $n = 5$. An s–t Hamiltonian path exists if and only if the color criterion is fulfilled for s and t and at least one of the vertices s and t is one of the corner vertices* (1, 1), (1, 5), (5, 1) *and* (5, 5).

The necessity of the color criterion for the existence of an s–t Hamiltonian path is easy to see. A knight changes with each legal move the color of the visited square. For a chessboard with an even (odd) number of squares, a Hamiltonian path has odd (even) length. Furthermore, the number of white squares is larger by 1 than the number of black squares if nm is odd.

For $n = 5$ the four corner vertices have degree 2 each, but only four different neighbors. The corner vertices together with their neighbors build a graph on eight vertices consisting of a Hamiltonian circuit only:

$$(1, 1)–(2, 3)–(1, 5)–(3, 4)–(5, 5)–(4, 3)–(5, 1)–(3, 2)–(1, 1).$$

Only the four noncorner vertices are connected with the rest of the graph. If all corner vertices are in the interior of the path, they have to be surrounded by at least five noncorner vertices, which do not exist. Hence, s or t is necessarily a corner vertex. The positive cases are proved with the help of our backtracking algorithm.

In the following let $n \geqslant 6$ and s and t fulfil the color criterion. Our construction of an s–t Hamiltonian path is based on a divide-and-conquer approach. The $n \times n$ chessboard is divided into small rectangular chessboards for which the theorem is proved by case inspection with the backtracking algorithm. Then we concatenate solutions for the small boards to a solution for the $n \times n$ chessboard. If s and t are in the same subboard, we cut the s–t Hamiltonian path for this board into two pieces and concatenate these pieces and solutions for the other boards.

First we discuss how we divide the $n \times n$ chessboard. Obviously, we prefer boards with at least 6 rows and 6 columns. If several subboards have odd size, there are subboards with a majority of black squares. If s (or t) is a white square within such a subboard, we cannot construct a Hamiltonian path through this board starting at s (ending at t). Hence, we consider only partitions of the $n \times n$ chessboard with at most one subboard of odd size. If n is odd and s and t are within the same subboard of even size, there does not exist any s–t Hamiltonian path within this board. Hence, for odd n we like to force s into the only subboard of odd size.

For some $n_1, \ldots, n_r$, where $n_1 + \cdots + n_r = n$, we partition the rows and columns into intervals of length $n_1, \ldots, n_r$ in arbitrary order. By our considerations only one n_i may be odd.

We have proved the theorem by case inspection and the backtracking algorithm for the following types of chess boards: 6×6, 7×7, 8×8, 9×9, 10×10, 11×11, 6×7, 6×8, 7×8, 6×9 and 8×9.

For $n \geqslant 12$ we use the following partition. If $n \notin \{15, 17, 23\}$, there exist $k, l \in \mathbb{N}_0$ and $m \in \{0, 1\}$ such that

$$n = 6k + 7m + 8l$$

and $k \geqslant 1$ if n is odd. For even n we arbitrarily divide the rows and columns into k intervals of length 6 and l intervals of length 8. For odd n let (s_1, s_2) be the position of the source s. Obviously $m = 1$, and we have to force s into the only 7×7 board. We choose the largest $l' \leqslant l$ such that $8l' < s_1$. Then we choose the largest $k' \leqslant k$ such that $8l' + 6k' < s_1$. Since $8l + 6k = n - 7$ and $k \geqslant 1$,

$$8l' + 6k' < s_1 \leqslant 8l' + 6k' + 7.$$

Hence, the s_1th row is in the interval of length 7, if we start with l' intervals of length 8 and k' intervals of length 6, before we choose the interval of length 7. For the columns we use the same strategy with respect to s_2. Finally, $15 = 6 + 9$, $17 = 8 + 9$ and $23 = 6 + 8 + 9$. In each case we can force s into the 9×9 board.

The second problem is to find the right order to visit the subboards. Afterwards, we have to be able to concatenate the Hamiltonian paths for the subboards.

We consider an $n \times n$ chessboard and its partition into r^2 subboards, i.e. r rows with r subboards each. The length of any side of a board is at least 6 and $r \geqslant 2$. From a given subboard, we can directly reach only the eight adjacent boards, if existent. The knight moves with respect to the subboards like a king. More formally, we investigate the following graph $G'_r = (V'_r, E'_r)$ where V'_r:$\{1, \ldots, r\} \times \{1, \ldots, r\}$ and $\{(i, j), (i', j')\} \in E'_r$ if and only if $|i - i'| = 1$ and $|j - j'| \in \{0, 1\}$ or $|i - i'| \in \{0, 1\}$ and $|j - j'| = 1$.

If s and t are within the different subboards s' and t', we look for a king's s'–t' Hamiltonian path with respect to G'_r. We face one additional problem. If n is odd and s is, e.g., the lower right square of the subboard of odd size, it is impossible to start at s, run on a Hamiltonian through this subboard and then jump to that subboard which is the lower right neighbor of the board containing s. Hence, we get the following problem, which we solve later.

Problem 1. Find an s–t Hamiltonian path for a king which does not start with a diagonal move or find two s–t Hamiltonian paths, which start with different moves.

If s and t are within the same subboard, we look for a king's Hamiltonian circuit with respect to G'_r. Again we meet an additional problem. Later we interrupt the s–t Hamiltonian path through the subboard at a corner and run through the other subboards. In order to continue the s–t Hamiltonian path, it is necessary to reach the subboard containing s and t near that position where we have left it. Hence, we get the following problem, which we solve later.

Problem 2. Find a Hamiltonian circuit for a king where the two neighbors of a given square s are connected by a legal move for a king.

In each subboard we exclusively reserve for each of the (at most) eight neighbored subboards some squares for a move between these subboards. For each of the four diagonally connected neighbors we reserve those three squares from which we can reach that neighbor. For each of the other four neighbors we reserve two black and two white squares from which we can reach that neighbor. Since the side length of our subboards is at least 6, this exclusive reservation is possible.

Now we construct the Hamiltonian paths.

Case 1: *s and t are in different subboards.*

First we construct (see Problem 1) a Hamiltonian path h^* for the king on the $r \times r$ board representing the subboards. This path runs from s^*, the board containing s, to

t^*, the board containing t. Without loss of generality $c(s) =$ white. In s^* we choose a Hamiltonian path from s to some reserved square, from which we can move directly to that subboard which is following s^* on h^*. By the special choice of h^* (see Problem 1) this is always possible, since we do not start with a diagonal move or since we can choose a Hamiltonian path, where we do not have to leave s^* via s. If n is odd, s^* has odd size and we reach the next board at a black square. If n is even, we reach the next board at a white square. It is now always possible to choose Hamiltonian paths through the subboards that end at a reserved square from which we can reach the successor board with respect to h^*. Since all subboards with the only possible exception of s^* have even size, all subboards are reached at white squares if n is even, and at black squares if n is odd. Hence, we do not reach the subboard t^* at t and we can choose the last Hamiltonian path such that we finally reach t.

Case 2: s and t are in the same subboard s^*.

Since s^* has odd size if and only if n is odd, we can construct an s–t Hamiltonian path h' within s^*. We also can construct (see Problem 2) a king's Hamiltonian circuit h^* with the special property for the vertex representing s^*. The subboard s^* and its two neighbors on h^* share on the large chessboard one point, w.l.o.g. the lower right corner of s^*. Also, w.l.o.g. we assume that one of the neighbors of s^* on h^* is the subboard adjacent to the lower side of s^*. We start at s and follow h' until we reach the lower right square $v = (v_1, v_2)$ of s^*. Since we can interchange the roles of s and t, we can assume that $t \neq v$ and that $s = v$ or we have reached v via $(v_1 - 2, v_2 - 1)$. Here we use the fact that within s^* the square v has only the neighbors $(v_1 - 2, v_2 - 1)$ and $(v_1 - 1, v_2 - 2)$. Now we jump into that subboard which is a neighbor of s^* on h^* and not the lower neighbor. We know that v is a white square and we reach the next board on a black square.

Similarly to Case 1 we can run through all subboards according to h^* and, finally, we reach the lower neighbor of s^* on a black square. Since this board has even size and $v' = (v_1 + 1, v_2 - 1)$ is a white square, we may choose a Hamiltonian path through this board ending at v'. Then we jump to $(v_1 - 1, v_2 - 2)$ and follow the remaining part of h' until we reach t.

Finally, we have to solve Problems 1 and 2.

A formal and complete solution of Problem 1 is lengthy and tiring. Hence, we are satisfied with an informal description. For $r = 2$ and $r = 3$ the solution is found by case inspection. The situations $r = 3$, $s = (2, 2)$ and $t \in \{(1, 2), (2, 1), (2, 3) (3, 2)\}$ are the only ones where we have to look for two s–t Hamiltonian paths starting with different moves.

In the following let $r \geqslant 4$, $s = (s_1, s_2)$, $t = (t_1, t_2)$ and w.l.o.g. $s_1 < t_1$ and $s_2 \leqslant t_2$. For $s' \geqslant \max\{s_1, 2\}$ we easily find a Hamiltonian path through the first s' rows of the board starting at s and ending at $(s', 1)$ if $(s', 1) \neq s$, and at $(s', 2)$ otherwise. The same considerations hold for t and the last t' rows where $t' \geqslant \max\{r + 1 - t_1, 2\}$. If $t_1 \geqslant s_1 + 2$ or $s_1 \geqslant 2$ and $t_1 \leqslant r - 1$, we can concatenate such paths to an s–t Hamiltonian path.

We still have to consider the situation $s_1 = 1$ and $t_1 = 2$ (or $s_1 = r - 1$ and $t_1 = r$). The solution is obvious if $s_2 = t_2 \in \{1, r\}$. If $s_2 = t_2 \in \{2, \ldots, r - 1\}$, we start at s with a move to the left and run through all squares of the first two rows, which are lying to the left s and t. Then we run through the last $r - 2$ rows ending at $(3, r)$. This path can easily be completed to an s–t Hamiltonian path. Finally, $s_2 < t_2$. We start from s with a move to the right and go to the right, until we reach $(1, t_2)$. Then we go to $(2, t_2 - 1)$, and go to the left to $(2, s_2)$. From here we can follow the strategy for the case $s_2 = t_2 \in \{2, \dots, r-1\}$.

Now it is easy to solve Problem 2. If $s = (s_1, s_2)$ is not in the interior of the board, say $s_1 = 1$, let $t := (2, s_2)$. We use the s–t Hamiltonian path constructed for the solution of Problem 1 and close this path to a circuit. Because of the special choice of t this circuit has the desired properties. Otherwise s is in the interior of the board. For $r = 3$ and, therefore, $s = (2, 2)$, we directly construct a solution. If $r \geqslant 4$, w.l.o.g. we assume that $s_1 > 2$ and $s_2 > 2$. We start from s with a move to the left and run through all squares of the rows $s_1, \dots, r$ ending at (s_1, r). Then we can run through the first $s_1 - 1$ rows and may reach s again from the top neighbor.

Altogether we have proved Theorem 3.1. □

It is simple but tedious to prove the counterpart of the Proposition 2.2.

Corollary 3.2. *G_n contains a Hamiltonian circuit if and only if $n \geqslant 6$ and n is even.*

Proof. The cases $n \leqslant 4$ follow from the fact that a Hamiltonian circuit contains a Hamiltonian path. For odd n Hamiltonian circuits would have odd length and starting and terminal vertex (which are the same for a circuit) would have different colors.

For even $n \geqslant 6$ we take a Hamiltonian path from (1, 1) to (2, 3). This path together with the move from (2, 3) to (1, 1) results in a Hamiltonian circuit. □

Altogether we have presented a complete solution of the knight's Hamiltonian path problem on $n \times n$ chessboards. A list of solutions for special problems is now the basis for a divide-and-conquer algorithm using table-look-up for small problems. The solution of the special problems is efficiently possible by our heuristic backtracking algorithm.

Appendix

Solutions for C_n, $5 \leqslant n \leqslant 9$:

1	18	13	22	7
12	23	8	19	14
17	2	21	6	9
24	11	4	15	20
3	16	25	10	5

1	22	19	8	33	28
18	7	34	29	20	9
23	2	21	32	27	30
6	17	4	35	10	13
3	24	15	12	31	26
16	5	36	25	14	11

1	4	23	40	11	6	9
24	41	2	5	8	45	12
3	22	47	44	39	10	7
42	25	34	21	46	13	38
31	28	43	48	15	20	17
26	35	30	33	18	37	14
29	32	27	36	49	16	19

1	34	3	18	37	32	13	16
4	19	36	33	14	17	38	31
35	2	49	52	47	40	15	12
20	5	64	41	50	53	30	39
57	42	51	48	59	46	11	26
6	21	58	63	54	27	60	29
43	56	23	8	45	62	25	10
22	7	44	55	24	9	28	61

1	36	3	22	49	38	65	20	17
4	23	48	37	78	21	18	39	66
35	2	53	50	47	64	77	16	19
24	5	34	81	52	79	46	67	40
33	58	51	54	69	74	63	76	15
6	25	70	59	80	45	68	41	62
57	32	55	28	73	60	75	14	11
26	7	30	71	44	9	12	61	42
31	56	27	8	29	72	43	10	13

Solutions for Γ_n, $6 \leqslant n \leqslant 9$:

32	3	14	9	26	1
13	24	33	2	15	10
4	31	12	25	8	27
23	20	29	34	11	16
30	5	18	21	28	7
19	22	35	6	17	

9	6	11	34	23	4	1
12	31	8	5	2	29	24
7	10	33	30	35	22	3
32	13	40	21	28	25	36
17	20	15	44	37	42	27
14	39	18	41	26		
19	16	45	38	43		

46	23	6	25	30	35	4	1
7	26	45	36	5	2	31	34
22	47	24	29	52	33	42	3
27	8	49	44	37	40	51	32
48	21	28	53	50	43	38	41
9	12	19	16	39			
20	17	14	11	54			
13	10	55	18	15			

27	42	37	30	21	14	19	6	1
36	31	28	43	38	7	2	13	18
41	26	33	22	29	20	15	10	5
32	35	24	39	44	3	8	17	12
25	40	45	34	23	16	11	4	9
46	63	50	57	54				
51	58	55	62	49				
64	47	60	53	56				
59	52	65	48	61				

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