Geometrically exact beam theory without Euler angles

P. Frank Pai

Department of Mechanical and Aerospace Engineering, University of Missouri, Columbia, MO 65211, United States

A R T I C L E  I N F O

Article history:
Received 15 February 2011
Received in revised form 30 May 2011
Available online 19 July 2011

Keywords:
Geometrically exact beam theory
Total-Lagrangian singularity-free formulation
Large rotation variables
Nonlinear structural mechanics

A B S T R A C T

In modeling highly flexible beams undergoing arbitrary rigid–elastic deformations, difficulties exist in describing large rotations using rotational variables, including three Euler angles, two Euler angles, one principal rotation angle plus three direction cosines of the principal rotation axis, four Euler parameters, three Rodrigues parameters, and three modified Rodrigues parameters. The main problem is that such rotational variables are either sequence-dependent and/or spatially discontinuous because they are not mechanics-based variables. Hence, they are not appropriate for use as nodal degrees of freedom in total-Lagrangian finite-element modeling. Moreover, it is difficult to apply boundary conditions on such discontinuous and/or sequence-dependent rotational variables. This paper presents a new geometrically exact beam theory that uses no rotation variables and has no singular points in the spatial domain. The theory fully accounts for geometric nonlinearities and initial curvatures by using Jaumann strains, exact coordinate transformations, and orthogonal virtual rotations. The derivations are presented in detail, fully nonlinear governing equations and boundary conditions are presented, a finite element formulation is included, and the corresponding governing equations for numerically exact analysis using a multiple shooting method is also derived. Numerical examples are used to illustrate the problems of using rotational variables and to demonstrate the accuracy of the proposed geometrically exact displacement-based beam theory.

© 2011 Elsevier Ltd. All rights reserved.

1. Introduction

A flexible multibody system consists of interconnected rigid and deformable components and each component may undergo large translations and rotations (Shabana, 2005; Bauchau, 2010; Kane et al., 1983). Modeling and analysis of a flexible multibody system that undergoes large rotations is very challenging because geometric nonlinearities exist in flexible components and equations of motion of rigid components are nonlinear ordinary differential equations (Shabana, 2005; Bauchau, 2010). Hence, nonlinear finite-element modeling with iteration techniques is often used in the modeling and analysis of flexible multibody systems. Even with the use of finite elements, however, many challenging problems still exist, and the most challenging task is how to accurately describe large rotations of flexible and rigid components without singularity problems in the space and time domains. One way to reduce the coupling-induced complexity of governing equations is to derive and use total-Lagrangian structural theories referred directly to an inertial reference frame without using any floating reference frames (Kane et al., 1987). Because the strain–displacement relations of a total-Lagrangian structural theory fully account for both rigid and elastic deformations, there is no need of complicated, problem-dependent nonlinear terms to describe the coupling of rigid and flexible components. Moreover, total-Lagrangian nonlinear rotary inertial terms of a differential flexible component have the same form as those of a rigid body (Pai, 2007).

However, challenging issues exist in the derivation and analysis of geometrically exact total-Lagrangian displacement-based structural theories.

An initially curved beam undergoing large rigid–elastic deformation requires three coordinate systems to describe its motion, as shown in Fig. 1(a). The abc is a fixed rectangular coordinate system used for reference, the xyz is a fixed orthogonal curvilinear coordinate system used to describe the undeformed beam geometry, and the ζηξ is a moving orthogonal curvilinear coordinate system used to describe the deformed beam geometry. Let \( \mathbf{i}_x, \mathbf{i}_y, \) and \( \mathbf{i}_z \) be the unit vectors of the abc system; \( \mathbf{i}_x, \mathbf{i}_y, \) and \( \mathbf{i}_z \) be the unit vectors of the xyz system; and \( \mathbf{i}_\zeta, \mathbf{i}_\eta, \) and \( \mathbf{i}_\xi \) be the unit vectors of the \( \zeta\eta\xi \) system. Moreover, \( u, v, \) and \( w \) represent the absolute displacements of the observed reference point \( O \) with respect to \( \text{(w.r.t.)} \) the \( x, y, \) and \( z \) axes, respectively, and \( s \) denotes the undeformed arc length along the reference line starting from the beam root. Because \( u, v, \) and \( w \) are continuous functions of the spatial coordinate \( s \) (and the time \( t \) if a dynamic problem), \( v' = \frac{\partial v}{\partial s} \), \( w' \), and \( u' \) exist and they can exactly describe the reference line’s bending rotations of any magnitude (Pai, 2007). However, a torsional angle \( \phi \) (see, e.g., Fig. 1(a)) is still needed in order to describe the twisting of
the translation vector w.r.t. a unit vector coordinate systems for modeling, and (b) displacement variables.

**Fig. 1.** A flexible beam undergoing large rigid-elastic deformation: (a) three coordinate systems for modeling, and (b) displacement variables.

According to Euler’s principal rotation theorem, the rigid-body movement of the cross section in Fig. 1(a) can be described by the translation vector $\mathbf{D}_0$ and a unique principal rotation angle $\Phi$ w.r.t. a unit vector $\mathbf{n} = n_1\mathbf{i}_1 + n_2\mathbf{i}_2 + n_3\mathbf{i}_3$. The coordinate transformation matrix $[T]$ describing the relation between the $\mathbf{abc}$ and $\mathbf{xyz}$ systems can be presented in terms of four Euler parameters (quaternion representation) derived from $\Phi$ and $n_i$ ($i = 1, 2, 3$) with no singular points, in terms of three Rodrigues parameters with a singular point at $\Phi = 180^\circ$, or in terms of three modified Rodrigues parameters with a singular point at $\Phi = 360^\circ$ (Shuster, 1993). The best approach is often problem dependent. Unfortunately, $\Phi$ cannot be explicitly expressed in terms of $u$, $v$, $w$ and $\Phi$, as shown later in Section 2.4. Because Euler and Rodrigues parameters are defined using $n_1$, $n_2$, $n_3$ and $\Phi$, they also cannot be expressed in terms of $u$, $v$, $w$ and $\Phi$. If they are used as nodal degrees of freedom (DOFs) in total-Lagrangian finite-element modeling, shape functions for these DOFs can only be derived using only these DOFs without the involvement of displacement DOFs (i.e., $u$, $v$ and $w$). Then, the order of polynomial shape functions for these DOFs is inconsistent with that of shape functions for displacement DOFs. Moreover, although $\Phi$ and $n_i$ are continuous on the $s$ domain, their spatial derivatives can be discontinuous and have large local gradients, as shown later in Section 5.1. Hence, Euler and Rodrigues parameters are not really appropriate for use as nodal DOFs in finite element modeling. Furthermore, deformations described by these parameters are often difficult to recognize two close orientations by direct inspection of these parameters because they are not directly related to the deformed structural geometry (Shabana, 2005; Bauchau, 2010). Hence, for finite element modeling of 1D (one-dimensional) and 2D structures, it is inconvenient or even inappropriate to use Euler parameters, Rodrigues parameters, modified Rodrigues parameters, or Cayley–Klein parameters as nodal DOFs. Furthermore, the use of different rotational variables to model a beam or a multibody results in different models with different mathematical characteristics and singular points. Equations linearized w.r.t. a deformed state in terms of such variables can have different sets of eigenvalues and different stability predictions when different sets of rotational variables are used, and non-zero oscillation frequencies may be obtained for rigid-body motions (Shabana, 2010).

Here we present a geometrically exact beam theory that uses only mechanics-based variables without Euler angles. Moreover, we illustrate the problems about using rotation variables and Euler and Rodrigues parameters in modeling and analysis of geometrically nonlinear beams.

### 2. Geometrically exact beam theories

Fig. 1(b) shows that the displacement vector $\mathbf{D}$ of an arbitrary point on the observed cross section consists of a rigid-body motion that moves a rectangle of side lengths $y$ and $z$ on the $yz$ plane to that on the $\eta_z$ plane and a small local relative displacement vector $\mathbf{u}$ with respect to the $\eta_z$ plane. The $\mathbf{u}$ accounts for out-of-plane shear and torsional warping and in-plane warping due to Poisson’s effect. For Euler–Bernoulli beams, out-of-plane warping are neglected. Here we consider the Euler–Bernoulli theory extended for nonlinear elastic beams in order to clearly illustrate the derivation and reveal the main characteristics of geometrically exact beam theories without complex mathematics and notations. For fully nonlinear strain–displacement relations and inertial terms that include all influences of $\mathbf{u}$, the derivation steps are essentially the same and the reader is referred to Hodges (2006), Nayfeh and Pai (2004), and Pai (2007) for details.

The direction cosines of $\mathbf{i}_x$, $\mathbf{i}_y$, and $\mathbf{i}_z$ with respect to the $a$, $b$, and $c$ axes result in a transformation matrix $[T^0]$ that represents the relative orientation between the two coordinate systems $abc$ and $xyz$ as

$$\begin{align*}
\{\mathbf{k}_{xyz}\} &= [T^0]\{\mathbf{k}_{abc}\}, \\
\{\mathbf{i}_{xyz}\} &= \begin{pmatrix} \mathbf{i}_x \\ \mathbf{i}_y \\ \mathbf{i}_z \end{pmatrix}, \\
\{\mathbf{i}_{abc}\} &= \begin{pmatrix} \mathbf{i}_a \\ \mathbf{i}_b \\ \mathbf{i}_c \end{pmatrix}
\end{align*} \quad (1)
$$

One can use Eq. (1) and the orthonormality of $\mathbf{i}_x$, $\mathbf{i}_y$, and $\mathbf{i}_z$ to obtain

$$\{\mathbf{k}_{xyz}\} = [T][\{\mathbf{k}_{xyz}\}],$$

$$[k] = \begin{bmatrix}
0 & k_2 & -k_3 \\
-k_2 & 0 & k_1 \\
k_3 & -k_1 & 0
\end{bmatrix} = [T^0][T^0]^T \quad (2)
$$

where $(\gamma = d\theta)/ds$, and $[T^0]^{-1} = [T^0]^T$ because $[T^0]$ is a unitary matrix. Then, $k_1$, $k_2$, and $k_3$ are initial curvatures with respect to axes $x$, $y$, and $z$, respectively. $[T^0]$ and $k_i$ can be calculated from the known undeformed beam geometry defined w.r.t. the rectangular coordinate system $abc$.

The deformed reference axis $\xi$ can be exactly and explicitly described by displacements $u$, $v$, and $w$, as shown next. In Fig. 1(b), if $\mathbf{OP} = ds$, the corresponding deformed vector $\mathbf{OP}$ can be used to define $\mathbf{i}_1$ as
\[ i_{11} \equiv T_{11} k_1 + T_{12} k_2 + T_{13} k_3 = -v l_1 + v l_2 + w l_1 + w l_2 + \frac{w}{1 + e} d s \]
\[ T_{11} = 1 + \frac{u' - v k_3 + w k_2}{1 + e}, \quad T_{12} = \frac{v u k_1 - w k_3}{1 + e}, \quad T_{13} = \frac{w u k_2 + v k_1}{1 + e} \]
\[ e = \sqrt{1 + \left( u' - v k_3 + w k_2 \right)^2 + \left( v u k_1 - w k_3 \right)^2 + \left( w u k_2 + v k_1 \right)^2} - 1 \]  
(3a)

where Eq. (2) is used, \( e \) is the axial strain on the \( z \) axis, and the expression of \( e \) is derived from the identity \( \sqrt{T_{11}^2 + T_{12}^2 + T_{13}^2} = 1 \). The deformed system \( \eta^* \) can be related to the undeformed system \( xyz \) as

\[ \{i_{123}\} = [T]\{i_{123}\}, \quad [T] = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}, \quad \{i_{123}\} = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} \]

\[ \{i_{123}\} \equiv \{i_1, i_2, i_3\} \]

(3b)

\[ T_{11} = T_{12} T_{21} - T_{13} T_{22}, \quad T_{12} = T_{13} T_{21} - T_{11} T_{23}, \quad T_{13} = T_{11} T_{22} - T_{12} T_{21} \]

(3c)

The direction cosines of \( T_{ij} \) of the deformed reference axis \( z \) are fully described by \( u, v, w, u', v', w' \), as shown in Eq. (3a). However, one, two or three rotation angles are needed in order to describe the rotation of the deformed cross section, and they result in different forms of \( T_{ij} (i = 1, 2, 3) \). Because \( k_1 = l_1 \times l_2 \), when \( T_{11} \) and \( T_{22} \) (\( i = 1, 2, 3 \)) are known, \( T_{12} \) can be obtained as shown in Eq. (3c). Because \( |k_3| = 1 \) and \( l_1 \cdot k_2 = 0 \), we have

\[ T_{22} = \pm \sqrt{1 - T_{11}^2 - T_{12}^2}, \quad T_{22} = -T_{11} T_{21} - T_{12} T_{23} \]

(3d)

In Eq. (3d), the first equation determines the value of \( T_{22} \) and the second one determines the sign and the value of \( T_{22} \) for double checking. If \( T_{12} = 0 \), the condition of smooth continuity of the \( T_{22} \) of two neighboring material points and/or \( T_{12} \) of the previous and current loading steps (if static problems) or time steps (if dynamic problems) to determine the sign of \( T_{22} \). Hence, only \( T_{21} \) and \( T_{23} \) need to be determined by rotation angles. Different choices for the number and sequence of rotations result in different \( [T] \) matrices with different singularity problems, as shown earlier in Sections 2.2–2.5. The expression of \( e \) in Eq. (3a) guarantees \( T_{11}^2 + T_{12}^2 + T_{13}^2 = 1 \). Eq. (3d) guarantees \( T_{21}^2 + T_{22}^2 + T_{23}^2 = 1 \) and \( l_1 \cdot l_2 = 0 \), and Eq. (3c) warrants \( i_1 = 1 \) and \( l_1 \cdot k_2 = 0 \) and \( T_{11}^2 + T_{12}^2 + T_{13}^2 = 1 \). After \( [T] \) is derived, it follows from Eqs. (3b) and (2) and the identity \( [T]^{-1} = [T]^T \) that

\[ \{i_{123}\} = [K]\{i_{123}\}, \quad [K] = \begin{bmatrix} 0 & \rho_3 & -\rho_2 \\ -\rho_3 & 0 & \rho_1 \\ \rho_2 & -\rho_1 & 0 \end{bmatrix} = [T][T]^T + [T][k][T]^T \]

(4a)

where

\[ \rho_1 \equiv l_1 \cdot k_3 = \sum_{i=1}^{3} (T_{1i} T_{3i} + T_{3i} k_i) \]

\[ \rho_2 \equiv -l_1 \cdot l_2 = \sum_{i=1}^{3} (-T_{1i} T_{3i} + T_{1i} k_i) \]

\[ \rho_3 \equiv l_1 \cdot k_2 = \sum_{i=1}^{3} (T_{1i} T_{2i} + T_{2i} k_i) \]

(4b)

Here, \( \rho_1 \) is the deformed twisting curvature w.r.t. the \( z \) axis, and \( \rho_2 \) and \( \rho_3 \) are the deformed bending curvatures with respect to the \( \eta \) and \( \zeta \) axes, respectively. Eqs. (3a)–(4b) completely describe the \( \eta^* \) system w.r.t. the \( xyz \) system, the deformed beam geometry, and deformed curvatures. However, the explicit forms of \( T_{21}, T_{23} \) and \( \rho_1 \) need to be derived later.

After the undeformed and deformed beam geometries are fully described, a geometrically exact beam theory can be derived using the extended Hamilton principle, i.e.

\[ 0 = \int_0^L \left( \frac{\partial K_e}{\partial \dot{\theta}} - \dot{\theta} \frac{\partial H}{\partial \theta} + \frac{\partial W_{nc}}{\partial \theta} \right) dt \]

where \( t \) is the time, \( K_e \) the kinetic energy, \( H \) the elastic energy, and \( W_{nc} \) the non-conservative work due to external loads. Virtual rotations \( \dot{\theta}_i \) (\( i = 1, 2, 3 \)) w.r.t. the axes \( \xi, \eta, \) and \( \zeta \) are needed in order to derive the variations \( \delta K_e, \delta H, \) and \( \delta W_{nc} \). Virtual rotations \( \delta \theta_i \) result in variations of \( \dot{\theta}_i \) as

\[ \left\{ \begin{array}{c} \delta \dot{\theta}_1 \\ \delta \dot{\theta}_2 \\ \delta \dot{\theta}_3 \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ \delta \theta_3 \\ \delta \theta_2 \end{array} \right\} = \left\{ \begin{array}{c} \delta \theta_3 \\ \delta \theta_2 \\ \delta \theta_1 \end{array} \right\} \]

(6a)

Hence, it follows from Eqs. (3a), (6a), and (3b) that

\[ \delta \dot{e} = T_{11} \delta (1 + u' - v k_3 + w k_2) + T_{12} \delta (v u k_1 - w k_3) + T_{13} \delta (w u k_2 + v k_1) \]

(6b)

Since the function form of \( T_{ij} \) are not known yet, the explicit form of \( \delta \dot{e} \) needs to be derived later. In linear beam theories without initial curvatures, \( \delta \dot{e} = \delta \dot{u}' + \delta \dot{v}' + \delta \dot{w}' \), and \( \delta \dot{e} = \delta \dot{u}' \). In geometrically exact initially-curved beam theories, Eq. (6b) show that \( \delta \dot{e}, \delta \dot{u}', \) and \( \delta \dot{v}' \) are explicitly related to \( \delta \dot{u}', \delta \dot{v}' \), and \( \delta \dot{w}' \). In the form

\[ \left\{ \begin{array}{c} (1 + e) \delta \dot{u}' \\ (1 + e) \delta \dot{v}' \\ (1 + e) \delta \dot{w}' \end{array} \right\} = \left\{ \begin{array}{c} \delta \dot{u}' \\ \delta \dot{v}' \\ \delta \dot{w}' \end{array} \right\} - \{k\} \left\{ \begin{array}{c} \delta u \\ \delta v \\ \delta w \end{array} \right\} \]

(6c)

One can use Eq. (4b) and the Kirchhoff kinetic analogy between curvatures \( \rho_i \) and angular velocities \( \dot{\theta}_i \) to derive the angular velocity vector \( \omega \) of the \( \eta^* \) system as (Pai, 2007)

\[ \omega = \dot{\theta}_1 l_1 + \dot{\theta}_2 l_2 + \dot{\theta}_3 l_3, \quad \omega = \dot{\theta}_1 l_1 + \dot{\theta}_2 l_2 + \dot{\theta}_3 l_3 \]

(6d)

where \( \dot{\theta}_i = \delta \dot{\theta}_i / \delta t, \) and \( \dot{i}_i = \omega \times i_i \) and \( \omega \times \omega = 0 \) are used. Because the local displacement vector \( u \) in Fig. 1(b) is due to in-plane and out-of-plane warping, which are negligibly small for the kinetic energy of a flexible beam and will be neglected here to simplify the demonstration of derivations. Without \( u \), the displacement vector \( D \) in Fig. 1(b) and its time derivatives and variation are given by
\[ D = u \mathbf{i}_x + v \mathbf{i}_y + w \mathbf{i}_z + y \mathbf{j}_x + z \mathbf{j}_y - y \mathbf{j}_z \]

\[ \bar{D} = \bar{u} \mathbf{i}_x + \bar{v} \mathbf{i}_y + \bar{w} \mathbf{i}_z + \omega \times (y \mathbf{j}_x + z \mathbf{j}_y) \]

\[ \delta \bar{D} = \delta \bar{u} \mathbf{i}_x + \delta \bar{v} \mathbf{i}_y + \delta \bar{w} \mathbf{i}_z + \gamma (\delta \bar{\theta}_1 \mathbf{j}_1 - \delta \bar{\theta}_2 \mathbf{j}_2) + z (\delta \bar{\theta}_1 \mathbf{j}_1 - \delta \bar{\theta}_2 \mathbf{j}_2) \]  

where \( E \) is Young’s modulus and \( G \) is the shear modulus. Again, simple isotropic materials are assumed here for illustration purpose. For anisotropic materials, the reduced material property matrix \( \{Q\} \) is a full matrix (Pai, 2007). The \( \{Q\} \) can be determined by experiments using small engineering strain and stress measures because Jaumann strains are co-rotated engineering strains (Pai, 2007). On the other hand, if Green–Lagrange strains are used, the \( \{Q\} \) needs to be determined by experiments using second Piola–Kirchhoff stresses and Green–Lagrange strains, which are nonlinear and computationally awkward and are not usually done in experiments.

Using Eqs. (8a) and (8b) we obtain

\[ \delta \Pi = \int_0^L \int_A \{ \delta \bar{\psi} \}^T [\{ \sigma \}] dA ds = \int_0^L \{ \delta \bar{\psi} \}^T [\{ \psi \}] ds \]

where

\[ [\{ \psi \}] = \left[ \begin{array}{ccc}
EA & 0 & 0 \\
0 & G & 0 \\
0 & 0 & EI \end{array} \right] 
\]

\[ \{ \psi \} = \int_A \rho \bar{D} = \int_A \rho \bar{D} dA = \int_A \rho dA \]

and

\[ \{ \delta \bar{\psi} \} = \int_A \rho \delta \bar{D} = \int_A \rho \delta \bar{D} dA \]

(8c)

Here the axes \( x \) and \( y \) are assumed to be the principal axes of the cross section and hence \( \int_A \rho (x, y, z) dA = \{ 0, 0, 0 \} \). The inertial terms \( A_{ij} \) in Eq. (7c) have exactly the same form of a rigid body, except that the rotary inertias here are for a unit length. Note that the rotary inertias \( j_k \) are often small, especially for flexible beams. If rotary inertias \( j_k \) are neglected, there are no nonlinear inertial terms.

Eqs. (6d) and (7c) show that the kinetic energy has a simple form when expressed in terms of angular velocities \( w \) r.t. the body-fixed orthogonal system \( \xi \zeta \gamma \). Moreover, for feedback control of a rigid-body system, it is more convenient to work with angular velocities \( w \) r.t. the body-fixed axes because sensors measure angular motions and actuators apply torques \( \omega \) r.t. the body-fixed axes. However, although the angular velocities are defined with respect to three orthogonal axes and hence are convenient for use, they cannot be integrated to obtain three sequence-independent large rotation angles because large rotations are essentially sequence-dependent. In other words, \( \theta_1 \) have no exact function forms and are called quasi-coordinates (i.e., not well defined coordinates). However, \( \theta_2 \) and \( \theta_3 \) have exact function forms, as shown in Eqs. (6b) and (6d).

Because Jaumann strains are co-rotated engineering strains without influences of rigid-body movement, fully nonlinear Jaumann strains can be derived using the concept of local displacements relative to the deformed coordinate system \( \xi \zeta \gamma \) (Pai, 2007). If \( u \) is neglected, Jaumann strains can be derived to be (Pai, 2007)

\[ \{ \varepsilon \} = \{ \bar{S} \} \{ \psi \}, \quad \{ \varepsilon \} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \varepsilon_{13} \end{bmatrix} \]

\[ \bar{S} = \begin{bmatrix} 1 & 0 & z \\ 0 & -z & 0 \\ 0 & y & 0 \end{bmatrix} \]

(8a)

The Jaumann strains are related to their work-conjugate stresses, i.e., Jaumann stresses \( \sigma_{ij} \), as

\[ \{ \sigma \} = [\{ Q \}] \{ \varepsilon \} = \begin{bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{13} \end{bmatrix} \]

\[ [\{ Q \}] = \begin{bmatrix} E & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix} \]

(8b)

where \( E \) is Young’s modulus and \( G \) is the shear modulus. Again, simple isotropic materials are assumed here for illustration purpose. For anisotropic materials, the reduced material property matrix \( \{Q\} \) is a full matrix (Pai, 2007). The \( \{Q\} \) can be determined by experiments using small engineering strain and stress measures because Jaumann strains are co-rotated engineering strains (Pai, 2007). On the other hand, if Green–Lagrange strains are used, the \( \{Q\} \) needs to be determined by experiments using second Piola–Kirchhoff stresses and Green–Lagrange strains, which are nonlinear and computationally awkward and are not usually done in experiments.

Using Eqs. (8a) and (8b) we obtain

\[ \delta \Pi = \int_0^L \int_A \{ \delta \bar{\psi} \}^T [\{ \sigma \}] dA ds = \int_0^L \{ \delta \bar{\psi} \}^T [\{ \psi \}] ds \]

(8c)

where

\[ [\{ \psi \}] = \left[ \begin{array}{ccc}
EA & 0 & 0 \\
0 & G & 0 \\
0 & 0 & EI \end{array} \right] 
\]

\[ \{ \psi \} = \int_A \rho \bar{D} = \int_A \rho \bar{D} dA = \int_A \rho dA \]

\[ \{ \delta \bar{\psi} \} = \int_A \rho \delta \bar{D} = \int_A \rho \delta \bar{D} dA \]

\[ (I_1, I_2, I_3) = \int_A \begin{bmatrix} (y^2 + z^2) \mathbf{c}_1 \end{bmatrix} dA \]

Here, \( F_1 \) and \( M_i \) are stress resultants, and \( c_1 \) is a correction factor accounting for the decrease of torsional rigidity due to out-of-plane torsional warping and can be calculated using the theory of elasticity (Timoshenko and Goodier, 1970). For example, \( c_1 = 1 \) for a circular cross section, and \( c_1 < 1 \) for non-circular cross sections.

It follows from Eqs. (4a) and (6a) and \( \bar{x}_2 \cdot \bar{k}_2 = 0 \) that \( \delta \bar{\psi}_t \) are related to \( \delta \bar{\theta}_t \) as (Pai, 2007)

\[ \begin{bmatrix} \delta \bar{\psi}_t \\ \delta \bar{\theta}_t \end{bmatrix} = \begin{bmatrix} \delta \bar{\psi}_t \end{bmatrix} - \begin{bmatrix} \delta \bar{\theta}_t \end{bmatrix} = \begin{bmatrix} \delta \bar{\psi}_t - \delta \bar{\theta}_t \end{bmatrix} \]

(9)

If \( q_1, q_2, \) and \( q_3 \) are distributed forces along the \( x, y, \) and \( z \) axes and \( q_a, q_b, \) and \( q_d \) are distributed torsional and bending loads along the \( \zeta, \eta, \) and \( \zeta \) axes, we have

\[ \delta W_{NC} = \int_0^L \left[ q_1 \delta u + q_2 \delta v + q_3 \delta w + q_4 \delta \theta_1 + q_5 \delta \theta_2 + q_6 \delta \theta_3 \right] ds \]

(10)

Substituting Eqs. (7b), (8c), (9) and (10) into Eq. (5) and integrating by parts yields

\[ 0 = \int_0^L \begin{bmatrix} F_1 \delta \bar{\psi} + \{ \bar{m} - q_1, \bar{m} \bar{v} - q_2, \bar{m} \bar{w} - q_3 \} \{ \delta \bar{\psi}_t, \delta \bar{\theta}_t \}^T \\
-M_1 \delta \bar{\theta}_t + M_2 \delta \bar{\theta}_t - A_{01} + q_4 \delta \theta_1 \\
-M_2 \delta \bar{\theta}_t + M_1 \delta \bar{\theta}_t - A_{02} + q_5 \delta \theta_2 \\
+M_3 \delta \bar{\theta}_t + M_1 \delta \bar{\theta}_t - A_{03} + q_6 \delta \theta_3 \end{bmatrix} \delta \bar{\theta}_t \]

(11a)

The inertia-induced internal transverse shear forces \( F_2 \) and \( F_3 \) are defined as (Pai, 2007)

\[ F_2 = \frac{1}{1 + \varepsilon} \left[ -M_3 \delta \bar{\theta}_t + M_1 \delta \bar{\theta}_t - A_{03} + q_6 \delta \theta_3 \right] \]

(11b)
If the transverse shear strains $\gamma_5$ and $\gamma_6$ are included, the total transverse shear forces are different from $F_2$ and $F_3$ (Pai, 2007). Replacing the $\delta \phi$, $\delta \theta_2$, and $\delta \theta_3$ in Eq. (11a) with Eq. (6c), using Eq. (11b), and taking integration by parts yields (Pai, 2007)

$$0 = \int_{\xi} \left[ -((F_1, F_2, F_3)[T])^T + (F_1, F_2, F_3)[T][k]\{\delta u, \delta v, \delta w\}^T + (m\bar{u} - q_1, m\bar{v} - q_2, m\bar{w} - q_3)\{\delta u, \delta v, \delta w\}^T - (M_1 + M_2\rho_2 - M_2\rho_3 - A_{\phi} + q_4)\delta \theta_1 \right] \, ds$$

(11c)

Setting the coefficients of $\delta u$, $\delta v$, $\delta w$, and $\delta \theta_1$ to zero yields the following governing equations:

$$\frac{\partial}{\partial s} \left( [T]^T \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \right) - [k][T]^T \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} + \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} m\bar{u} \\ m\bar{v} \\ m\bar{w} \end{pmatrix}$$

(12a)

$$M'_1 + M_3\rho_2 - M_2\rho_3 + q_4 = A_{\phi}$$

(12b)

The corresponding boundary conditions are:

$$\delta u = 0 \text{ or } F_3(= F_1T_{11} + F_2T_{21} + F_3T_{31}) \text{ known}$$
$$\delta v = 0 \text{ or } F_3(= F_1T_{12} + F_2T_{22} + F_3T_{32}) \text{ known}$$
$$\delta w = 0 \text{ or } F_3(= F_1T_{13} + F_2T_{23} + F_3T_{33}) \text{ known}$$
$$\delta \theta_1 = 0 \text{ or } M_1 \text{ known}$$
$$\delta \theta_2 = 0 \text{ or } M_2 \text{ known}$$
$$\delta \theta_3 = 0 \text{ or } M_3 \text{ known}$$

(12b)

The actual implications of boundary conditions $\delta \theta_1 = 0$ will be explained later in Section 2.5.

The governing Eqs. (12a) and (11b) can also be derived using a vector approach based on Newton’s second law and the free-body diagram of a differential beam element (Pai, 2007). This shows that the energy formulation starting from the extended Hamilton principle (i.e., Eq. (5)) is fully correlated with the vector formulation, and governing equations obtained from these two different approaches are essentially the same. On the other hand, if Green–Lagrange strains and second Piola–Kirchhoff stresses are used in the extended Hamilton principle and the material stiffness matrix shown in Eq. (8b) is used, one can never show that the so-obtained governing equations are the same as those from the vector formulation (Pai, 2007).

Eqs. (12a), (11b), (12b), (8d), (4b) and (3a)–(3d) show that this geometrically exact beam theory is completely and explicitly described by $u$, $v$, $w$, $T_{12}$, and $T_{23}$. If the influences of transverse shear deformations $\gamma_5$ and $\gamma_6$ are to be included, one just needs to keep the local displacement vector $\mathbf{u}$ in the displacement vector $\mathbf{D}$ in Eq. (7a) and then follow the same derivation process (Pai, 2007).

However, because shear rotations are independent of bending rotations, two equations governing $\gamma_5$ and $\gamma_6$ will be added to Eq. (12a) (Pai, 2007). Next we derive the explicit function forms of $T_{21}$ and $T_{23}$ when different rotation angles are used to rotate the undeformed cross section to its deformed position.

### 2.1. Three rotations $(\alpha, \beta, \phi)$: beam theory $\Theta_{321}$

As shown in Fig. 2(a), the $\alpha$ rotates the axes $x$ and $y$ to $\hat{x}$ and $\hat{y}$, the $-\beta$ $(|\beta| \leq \pi/2)$ rotates the axes $x$ and $z$ to $\hat{z}$ and $\hat{x}$, and the $\phi$ rotates the axes $\hat{y}$ and $z$ to $\eta$ and $\zeta$. It follows from Fig. 2(a) and Eq. (3b) that (Wu et al., 2011)

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi \end{bmatrix} \left[ \begin{array}{c} \hat{x} \\ \hat{y} \\ \hat{z} \end{array} \right]$$

(13a)

$$T_{21} = \frac{T_{11}}{1 - T_{13}^2}, \quad T_{12} = \frac{T_{11}}{\sqrt{1 - T_{13}^2}}, \quad T_{13} = \frac{T_{11}}{\sqrt{1 - T_{13}^2}}$$

(13b)

**Fig. 2.** The cross-section rotation: (a) three angles, (b) three angles, and (c) two angles.
When $T_{11}$, $T_{12}$, $T_{13}$, $T_{21}$ and $T_{23}$ are known, one can obtain a unique value for $0 \leq \psi \leq 2\pi$ from Eq. (13b). The $[T]$ matrix reveals that the singular points of this beam theory happen at $T_{13} = \pm 1$. This beam theory is most appropriate for modeling and analysis of rotor blades and other beam-like structures that mainly undergo large rigid-elastic deformations on the $xy$ plane (Wu et al., 2011). However, this beam theory is valid only if $|\varphi| < \pi$ because $\cos \varphi = \sqrt{1 - \varphi^2}$ is used in Eq. (13a). To extend the theory for $|\varphi| > \pi$ (i.e., the full range for any possible large rotations), one needs to track the possibility of $\cos \varphi = -\sqrt{1 - \varphi^2}$, which requires tracking of the complicated actual rotation sequence of the three rotations and using the continuity of $\varphi$ between two adjacent points (Pai, 2007).

If the three consecutive rotations are $-\varphi$, $\varphi$, and $\varphi$, one can similarly show that

$$
[T] = \begin{bmatrix}
0 & 0 & \cos \varphi \\
0 & \cos \varphi & -\sin \varphi \\
-\sin \varphi & \cos \varphi & 0
\end{bmatrix}
$$

$$
[\tilde{T}] = \begin{bmatrix}
T_{11} & T_{12} & T_{13} \\
-T_{12}/\sqrt{1 - T_{12}^2} & T_{11}/\sqrt{1 - T_{12}^2} & -T_{13}/\sqrt{1 - T_{12}^2} \\
T_{13}/\sqrt{1 - T_{12}^2} & 0 & T_{11}/\sqrt{1 - T_{12}^2}
\end{bmatrix}
$$

When $T_{11}$, $T_{12}$, $T_{13}$, $T_{21}$ and $T_{23}$ are known, one can obtain a unique value for $0 \leq \psi \leq 2\pi$ from Eq. (14b). The singular points of this beam theory happen at $T_{12} = \pm 1$. However, this beam theory is valid only if $|\varphi| < \pi$ because $\cos \varphi = \sqrt{1 - \varphi^2}$ is used in Eq. (14a).

Eqs. (13a) and (14a) show that the different rotation sequences cause $[T]$ and $\varphi$ to be different from $[\tilde{T}]$ and $\varphi$. The zeros of $[\tilde{T}]$ and $[\tilde{T}]$ also reveal that $\varphi$ needs to be different from $\varphi$ because $T_{22} = 0$ but $\tilde{T}_{22} = 0$. However, there is no need to treat Eq. (14a) as a different beam theory because using Eq. (14a) for a beam is the same as using Eq. (13a) for the same beam with the coordinate system $xyz$ being rotated w.r.t. the $x$ axis by 90°.

2.2. Three rotations ($\varphi$, $\varphi$, $\varphi$); beam theory $\Theta_{012}$

As shown in Fig. 2(b), the $\varphi$ rotates axes $x$ and $z$ to axes $y$ and $z$, the $\varphi$ rotates axes $x$ and $y$ to axes $x$ and $z$, and then the $\varphi$ rotates axes $y$ and $z$ to the axes $x$ and $z$. It follows from Fig. 2(b) and Eq. (3b) that

$$
T_{22} = \frac{T_{11}}{\sqrt{1 - T_{12}^2}} \cos \varphi, \quad T_{23} = \frac{T_{11}}{\sqrt{1 - T_{12}^2}} \sin \varphi
$$

(15b)

When $T_{11}$, $T_{12}$, $T_{13}$, $T_{21}$ and $T_{23}$ are known, one can obtain a unique value for $0 \leq \varphi \leq 2\pi$ from Eq. (15b). The singular points of this beam theory happen at $T_{13} = \pm 1$. This beam theory is most appropriate for modeling and analysis of spinning shafts and other beam-like structures that undergo a large or even continuously increasing rotation about the $x$ axis. However, this beam theory is valid only if $|\varphi| < \pi$ because $\cos \varphi = \sqrt{1 - \varphi^2}$ is used in Eq. (15a).

2.3. Two rotations ($\varphi$, $\varphi$) or ($\varphi$, $\varphi$); beam theory $\Theta_{012}$

As shown in Fig. 2(c), the first rotation $\varphi(0 \leq \varphi \leq \pi)$ is w.r.t. the $n$ axis that is perpendicular to axes $x$ and $z$. $\varphi$ rotates the axes $x$, $y$, and $z$ to the axes $z$, $y$, and $z$, and then the $\varphi$ rotates the axes $y$ and $z$ to the axes $n$ and $z$. It follows from Fig. 2(c) that (Pai, 2007)

$$
[T] = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \varphi & -\sin \varphi \\
0 & \sin \varphi & \cos \varphi
\end{bmatrix}
$$

$$
[\tilde{T}] = \begin{bmatrix}
T_{11} & T_{12} & T_{13} \\
-T_{12}/(1 + T_{12}) & T_{11}/(1 + T_{12}) & -T_{13}/(1 + T_{12}) \\
T_{13}/(1 + T_{12}) & 0 & T_{11}/(1 + T_{12})
\end{bmatrix}
$$

When $T_{11}$, $T_{12}$, $T_{13}$, $T_{21}$ and $T_{23}$ are known, one can obtain a unique value for $0 \leq \varphi \leq 2\pi$ from Eq. (16b). The singular points of this beam theory happen at $T_{13} = \pm 1$. This beam theory is most appropriate for modeling and analysis of beam-like structures that undergo weakly nonlinear elastic deformations (Nayfeh and Pai, 2004). If two consecutive rotations $\varphi$ and $\varphi$ are used, $[T]$ is the same as that shown in Eq. (16a). In other words, the model is independent of rotation sequence because the two rotation axes are perpendicular to each other.

2.4. One rotation ($\varphi$, $n_1$, $n_2$, $n_3$)

It follows from Euler’s principal rotation theorem that the rotation of the cross section shown in Fig. 1(b) can be described by a rotation angle $\varphi$ with respect to a unit vector $n$ as (Shabana, 2005; Bauchau, 2010; Shuster, 1993; Marandi and Modi, 1987)

$$
[T] = \begin{bmatrix}
T_{11} & T_{12} & T_{13} \\
-T_{12}/\sqrt{1 - T_{12}^2} & T_{11}/\sqrt{1 - T_{12}^2} & -T_{13}/\sqrt{1 - T_{12}^2} \\
T_{13}/\sqrt{1 - T_{12}^2} & 0 & T_{11}/\sqrt{1 - T_{12}^2}
\end{bmatrix}
$$

$$
T_{12} = T_{12} \cos \varphi + T_{13} \sin \varphi, \quad T_{13} = -T_{12} \sin \varphi + T_{13} \cos \varphi
$$

(15a)

$$
\mathbf{n} = n_1 \mathbf{i}_1 + n_2 \mathbf{i}_2 + n_3 \mathbf{i}_3 = n_1 \mathbf{i}_1 + n_2 \mathbf{i}_2 + n_3 \mathbf{i}_3, \quad n_1^2 + n_2^2 + n_3^2 = 1
$$

(17)
If \( T_1 \) and \( T_2 \) \((i = 1, 2, 3)\) are known and then \( T_3 \) are obtained using Eq. (3c), it follows from Eqs. (3b) and (17) that

\[
\cos \Phi = \frac{T_{11} + T_{22} + T_{33} - 1}{2} \tag{18a}
\]

\[
n_i = \pm \sqrt{\frac{T_{ii} - \cos \Phi}{1 - \cos \Phi}}, \quad n_i n_j = \frac{T_{ij} + T_{ji}}{2(1 - \cos \Phi)} \tag{18b}
\]

\[
\sin \Phi = \frac{T_{ii} - n_i n_j(1 - \cos \Phi)}{n_k}, \quad i \neq j \neq k \tag{18c}
\]

For Eq. (18c), \( i, j \) and \( k \) permute in a natural order. One can use Eq. (18a) to obtain \( \cos \Phi \), Eq. (18b) to obtain two sets of answers \( \pm(n_1, n_2, n_3) \), and then use Eq. (18c) to obtain \( \sin \Phi \). The sign of \( \pm(n_1, n_2, n_3) \) can be determined by assuming continuous increase/decrease of \( \Phi \) and/or \( \mathbf{n} \) with \( \mathbf{s} \). Hence, a unique set of values for \( \Phi, n_1, n_2, \) and \( n_3 \) can be obtained. The \([T]\) in Eq. (17) has no singular points. Moreover, the \([T]\) has three eigenvalues \( \lambda_i = 1 \pm \cos \Phi \pm \sin \Phi \), \( j \equiv \sqrt{-1} \), and the eigenvector corresponding to \( \lambda = 1 \) is actually the vector \( \mathbf{n} \) (Kane et al., 1983; Shuster, 1993).

Euler parameters \( \beta_i \) \((i = 0, 1, 2, 3)\) are defined as

\[
\beta_0 = \cos \frac{\Phi}{2}, \quad \beta_i = n_i \sin \frac{\Phi}{2}, \quad i \neq 0 \tag{19a}
\]

Eq. (12b) can be explained here. For a clamped end, it follows from Eqs. (12b) and (22b) that \( \delta \theta_1 = \delta \theta_2 = 0 \) because \( T_{22} = T_{33} = 1 \) and \( T_{31} = T_{13} = 0, \delta \theta_2 = -\sum_{i=1}^3 T_{ii} \delta \theta_{1i} = 0 \) in Eq. (12b) is equivalent to \( \delta \theta_{11} = 0 \) because \( T_{22} = 1 \) and \( T_{31} = T_{13} = 0 \). Similarly, \( \delta \theta_3 = \sum_{i=1}^3 T_{ii} \delta \theta_{1i} = 0 \) is equivalent to \( \delta \theta_{12} = 0 \) because \( T_{22} = 1 \) and \( T_{13} = T_{31} = 0 \). It means that \( T_{33} \) is known, and it is a nonlinear constraint equation involving several variables. If the influences of \( k \) and \( \varepsilon \) are neglected, it follows from Eq. (3a) that these boundary conditions reduce to \( \delta \theta_1 = \delta \theta_2 = 0 \) and \( \delta \theta_3 = \delta \theta_4 = 0 \), which are the same as linear cases. Other boundary conditions can be similarly determined.

As shown in Section 2, a beam's deformed geometry is fully described by \( u, v, w \) and \( \Phi \). Because of Eqs. (3c) and (3d), a beam's deformed geometry can be fully described by \( u, v, w, T_{11}, T_{12}, T_{13}, T_{21}, T_{22}, T_{23} \). Because of Eq. (3a), a beam's deformed geometry can also be fully described by \( u, v, w, T_{11}, T_{12}, T_{13}, T_{21}, T_{22}, T_{23} \). Similarly, a beam's deformed geometry can also be fully described by \( u, v, w, T_{11}, T_{12}, T_{13}, T_{21}, T_{22}, T_{23} \). If \( u, v, w, T_{11}, T_{12}, T_{13}, T_{21}, T_{22}, T_{23} \) are used as 7 degrees of freedom (DOFs) at a node in total-Lagrangian finite-element modeling, the \([T]\) in Eqs. (13a), (15a) and (16a) have singular points at \( T_{ij} = \pm 1 \). In programming, these singular points can be bypassed by subtracting a very small number \( e.g., \) sing(T)/ deem from \( T_{ij} \) when \( T_{ij} = \pm 1 \) happens. Although this may cause small errors in the obtained displacements at the singular points, these errors will not accumulate like those happen to updated-Lagrangian formulations because these are displacement-based total-Lagrangian beam theories. However, the torsional angle \( \varphi \) may be spatially discontinuous and causes problems when it is used as a nodal DOF, as shown later in Section 5.1 by examples. For a beam theory with a spatially discontinuous torsional variable, spatial discretization of \( \varphi \) using continuous polynomial shape functions in finite-element modeling is problematic.

Because of Eqs. 17, 19b, 20b and 21b, a beam's deformed geometry can also be fully described by \( u, v, w, T_{11}, T_{12}, T_{13}, T_{21}, T_{22}, T_{23} \). Because of Eq. (3a), a beam's deformed geometry can also be fully described by \( u, v, w, T_{11}, T_{12}, T_{13}, T_{21}, T_{22}, T_{23} \). If \( u, v, w, T_{11}, T_{12}, T_{13}, T_{21}, T_{22}, T_{23} \) are used as 7 nodal DOFs in a total-Lagrangian displacement-based finite-element formulation, the \([T]\) in Eq. (20b) has singular points. If \( u, v, w, T_{11}, T_{12}, T_{13}, T_{21}, T_{22}, T_{23} \) are used as 7 nodal DOFs, the \([T]\) in Eq. (21b) also has singular points. However, if \( u, v, w, T_{11}, T_{12}, T_{13}, T_{21}, T_{22}, T_{23} \) are used as 7 nodal DOFs, the \([T]\) in Eq. (17) or (19b) has no singular points and the continuity of displacements and rotations

\[
\begin{bmatrix}
1 + \gamma_1^2 - \gamma_2^2 - \gamma_3^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 + \gamma_2^2 - \gamma_1^2 - \gamma_3^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 + \gamma_3^2 - \gamma_1^2 - \gamma_2^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 + \gamma_1^2 - \gamma_2^2 - \gamma_3^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 + \gamma_2^2 - \gamma_3^2 - \gamma_1^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 + \gamma_3^2 - \gamma_1^2 - \gamma_2^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 + \gamma_1^2 - \gamma_3^2 - \gamma_2^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 + \gamma_2^2 - \gamma_1^2 - \gamma_3^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 + \gamma_3^2 - \gamma_2^2 - \gamma_1^2
\end{bmatrix}
\]
at nodes is guaranteed. Unfortunately, because the explicit forms of \((f_0, f_1, f_2, f_3)\) or \((T_1, T_2, T_3, \Theta)\) in terms of \((u', v', w')\) cannot be derived, nodal DOFs \((f_0, f_1, f_2, f_3)\) cannot be used in the derivation of shape functions for \(u, v\), and \(w\). With only \((u, v, w, f_0, f_1, f_2, f_3)\) at the two end nodes of a two-node beam element, the so-derived shape functions are linear polynomials for both displacement and rotational variables. This order inconsistency in shape functions may cause membrane locking and other numerical problems in finite element analysis (Bathe, 1996). Hence, at least one mid-point node with 3 DOFs (i.e., \(u, v, w\)) needs to be used in order to have the order of polynomial shape functions for \(u, v\), and \(w\) being at least one order higher than those for \(f_1\) or \(\gamma_i\). This ends up that at least 17 DOFs (or 15 DOFs if \(\gamma_i\) or \(v_i\) are used) are needed for a three-node beam element, comparing with 14 DOFs when \((u, v, w, u', v', w', \Theta)\) are used for a two-node beam element. More seriously, the first- and higher-order spatial derivatives of \(f_1\) or \(\gamma_i\) or \(v_i\) can be discontinuous (soft singularity, as shown later in Section 5.1), but commonly used polynomial shape functions or any other continuous shape functions cannot describe such discontinuity. Hence, \(f_1\) (or \(\gamma_i\) or \(v_i\)) cause numerical problems when they are used as nodal DOFs.

Hence, \((u, v, w, u', v', w', T_{21}, T_{23})\) is the most favorable set of nodal DOFs because the corresponding \([T]\) has no singular points and no rotation angles are used. Moreover, all these 8 DOFs are continuous mechanics-based physical variables, and they guarantee continuity of displacements and rotations at nodes. Because \(b_1 = \sum_{i=1}^{3} T_{1i} T_{23} = 0\) and \(b_2 = \sum_{i=1}^{3} T_{1i} T_{23} = 1\), one can express \(T_{21}\) and \(T_{22}\) in terms of \(T_{11}\) and \(T_{23}\) as

\[
T_{21} = -T_{11} T_{13} T_{23} + \left( T_{11} T_{13} T_{23}^2 + (T_{11} + T_{12}) (T_{12}^3 - T_{23}^2 + T_{11} T_{23}) \right) / T_{11}^2 + T_{12},
\]

\[
T_{22} = -T_{11} T_{21} - T_{13} T_{23} / T_{12} \tag{23}
\]

Hence, only 7 nodal DOFs \((u, v, w, u', v', w', T_{21}, T_{23})\) are needed for full description of large elastic deformation of highly flexible Euler–Bernoulli beams. The sign to be chosen for the first one of Eq. (23) can be determined by the condition of smooth continuity of \(T_{21}\) between neighboring points starting from any known boundary conditions on \(T_{21}\). However, reducing one variable complicates the formulation and programming, and the number of singular points to be monitored during computation increases from one \((T_{12} = 0\) in Eq. (3d)) to two \((T_{12} = T_{11} + T_{12} = 0\) in Eq. (23)). Hence, we will derive the finite element formulation in Section 3 and the multiple shooting formulation in Section 4 for the beam theory using the five dependent variables \((u, v, w, T_{21}, T_{23})\) and 8 nodal DOFs \((u, v, w, u', v', w', T_{21}, T_{23})\). Note that the 8 nodal DOFs have no singular points.

Because \(b_1 = 1, T_{21}, T_{22}\) and \(T_{23}\) are on a unit sphere. Hence, stereographic projection (Conway, 1978; Tsiorras and Longuski, 1995) can be used to reduce the number of variables by one by defining variables \(t_{21}\) and \(t_{23}\) and then presenting \(T_{21}\) in terms of these two variables as

\[
t_{21} = \frac{T_{21}}{T_{21} + T_{22}}, \quad t_{23} = \frac{T_{23}}{T_{21} + T_{22}}, \quad 1 + t_{21}^2 + t_{23}^2 = 2 / (1 + T_{21}),
\]

\[
T_{21} = \frac{2 t_{21}}{1 + t_{21}^2 + t_{23}^2}, \quad T_{23} = \frac{2 t_{23}}{1 + t_{21}^2 + t_{23}^2}, \quad T_{22} = \frac{1 - t_{21}^2 - t_{23}^2}{1 + t_{21}^2 + t_{23}^2} \tag{24}
\]

Hence, the nodal DOFs can be chosen to be \((u, v, w, u', v', w', T_{21}, T_{23})\). Then there are no singular points to be monitored during computation because \(1 + T_{21}^2 + T_{23}^2 > 0\). However, the variables \(t_{21}\) and \(t_{23}\) themselves can be singular when \(T_{22} = -1\). Hence, \(T_{21}\) and \(T_{23}\) are better than \(t_{21}\) and \(t_{23}\) for finite-element modeling and analysis.

3. Finite element formulation

The weak forms shown in Eqs. (7b), (8c) and (10c) can be used with Eqs. (9) and (6b) for the finite element formulation using \((u, v, w, u', v', w', T_{21}, T_{23})\) as nodal DOFs. It follows from Eq. (8c) that

\[
\delta \Pi = \int_0^1 \left\{ \delta \psi ight\}^T [D] \{\psi\} ds = \int_0^1 \left\{ \delta U \right\}^T [\Psi]^T [D] \{\psi\} ds
\]

\[
[\delta \psi] = \{[\Psi]\} [\delta U],
\]

\[
\{U\} = \{u, u', v', v', w, w', T_{21}, T_{23}, T_{23}\}^T
\]

The explicit forms of \(\Psi_{ij} = \partial \psi_i / \partial U_j\) can be derived from Eqs. (6b), (9) and (3d). Next we use two-node beam elements to discretize a beam into \(n_e\) elements. The continuous dependent variables of the \(i\)th element are discretized as

\[
\{d\} = \{u, v, w, T_{21}, T_{23}\}^T = \{N_i\} \{q_i\} \tag{25a}
\]

where \([N]\) is a \(5 \times 16\) matrix of shape functions (i.e., Hermitic cubic polynomials for \(u, v, w, T_{21}\), and \(T_{23}\)) and \(\{q\}\) is the element displacement vector of the \(i\)th element defined as

\[
\{q\} = \{u_1, v_1, w_1, u_2, v_2, w_2, T_{21,1}, T_{23,1}, T_{23,1}, T_{23,1}\}^T \tag{25b}
\]

It follows from Eqs. (25a) and (25) that

\[
\{U\} = \{\partial N\} \{q\}, \quad \{\partial N\} = \{\partial N_i\} \tag{26c}
\]

where \(\{\partial N\}\) is a \(13 \times 5\) matrix of differential operators. Then we obtain the variation of elastic energy as

\[
\delta \Pi = \int_0^{l_i} \left\{ \delta q_i \right\}^T \{N_i\}^T [\Psi]^T [D] \{\psi\} ds
\]

\[
= \int_0^{l_i} \left\{ \delta q_i \right\}^T \{K\} \{q\} \tag{27a}
\]

where \([K]\) is the stiffness matrix and \([q]\) is the displacement vector of \(\{q\}\) (or \[\{q\}\]) are nonlinearly coupled into a vector and cannot be separated in this fully nonlinear formulation.

Eqs. 7b, 7c, 6b and 6d show that the rotary inertial moments \(A_i\) are nonlinear functions of \(u, v, w, T_{21\text{ and }T_{23}}\) and their spatial derivatives. However, because rotary inertias \(J_k\) of highly flexible beams are negligibly small, the inertial moments \(A_i\) are negligible if the vibration frequency is low (e.g., lower than the 10th natural frequency). If inertial moments are neglected, Eq. (7b) reduces to

\[
\delta K_r = -\int_0^l \left( m \ddot{u} + m \dot{v} \dot{v} + m \ddot{w} \right) ds
\]

\[
= -\int_0^l \left\{ \delta q_i \right\}^T \{N_i\}^T \{m\} \{\dot{q}\} \{\dot{q}\}^T ds
\]

\[
= -\int_0^l \left\{ \delta q_i \right\}^T \{m\} \{\dot{q}\} \{\dot{q}\}^T \tag{28a}
\]

where \([m]\) is the global mass matrix and

\[
{[m]} = \int_0^l [N_i]^T [m] ds, \quad \{m\} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 \\ 0 & 0 & 0 & m & 0 \\ 0 & 0 & 0 & 0 & m \end{bmatrix} \tag{28b}
\]
The variation of non-conservative work due to six external distributed loads is obtained from Eqs. (10) and (22b) as

\[
\delta W_{nc} = \int_0^L \left( q_1 \delta u + q_2 \delta v + q_3 \delta w + q_4 \delta \theta_1 + q_5 \delta \theta_2 + q_6 \delta \theta_3 \right) ds
\]

Substituting Eq. (31a) into \{q\} gives

\[
\{ q \} = \begin{bmatrix} q_1 + \frac{T_{1x} t_{2x} - T_{1z} t_{2z}}{T_{1x} t_{2x} - T_{1z} t_{2z}} q_3 + \frac{T_{1z} t_{2z} - T_{1x} t_{2x}}{T_{1x} t_{2x} - T_{1z} t_{2z}} q_6 - \frac{T_{1x} t_{2x}}{T_{1x} t_{2x} - T_{1z} t_{2z}} q_5 + \frac{T_{1z} t_{2z}}{T_{1x} t_{2x} - T_{1z} t_{2z}} q_4, 0, \frac{T_{1z} t_{2z}}{T_{1x} t_{2x} - T_{1z} t_{2z}} q_4 + \frac{T_{1x} t_{2x}}{T_{1x} t_{2x} - T_{1z} t_{2z}} q_5, 0, 0 \end{bmatrix}^T
\]

where \( \{ R \} \) is the global nodal load vector and \( \{ R^{(0)} \} \) is the elemental nodal load vector given by

\[
\{ R^{(0)} \} \equiv \int_{t_i} \{ \dot{\omega} N \}^T \{ \dot{\omega} R \} ds
\]

Substituting Eqs. (27a), (28a) and (29a) into Eq. (5) yields the following equation of motion

\[
[M] \{ \ddot{q} \} + [C] \{ \dot{q} \} + [K] \{ q \} = \{ R \}
\]

where the damping matrix \([C]\) is added and can be obtained by using the concept of modal damping or proportional damping (Ewins, 2000).

For static problems, Eq. (30) can be solved by using an incremental/iterative method based on the modified Riks method (Riks, 1979). For dynamic problems, Eq. (30) can be solved by direct numerical integration using the Newmark-\( \beta \) method (Newmark, 1959). If \( q_a = q_b = q_c = 0 \) and/or the linear forms of \( \delta \theta \), are adopted, then \([M]\) and \([R]\) are constant and only \([K]\) is a function of displacements. To derive the incremental form of Eq. (27b) for static and dynamic analyses using any incremental/iterative methods, first we define

\[
\{ q^{(0)} \} = \{ q \} + \{ \Delta q^{(0)} \}, \quad \{ U \} = \{ \Omega \} + \{ \Delta U \}
\]

where \( \{ q \} \) denotes an equilibrium state and \( \{ \Delta q^{(0)} \} \) denotes a displacement increment vector when the loads increase and/or time proceeds. Substituting Eq. (31a) into \( \{ \dot{\phi} \} \) and \( \{ \dot{\psi} \} \) in Eq. (27b) yields

\[
\{ \ddot{\phi} \} = \{ \ddot{\phi} \} + \{ [T] \{ \dot{\phi} \} + [\dot{\psi}] \{ \Delta U \} \}
\]

where \( \{ \dot{\psi} \} = \frac{\partial \dot{\phi}}{\partial \dot{\psi}} \{ \Delta U \} \)

By direct expansion, one can show that (Pai, 2007)

\[
\{ \Theta \} = \frac{\partial^2 \psi_m}{\partial \dot{U}_i \partial U_j} D_{mn} \psi_n = \frac{\partial^2 \psi_m}{\partial \dot{U}_i \partial U_j} D_{mn} \psi_n = \Gamma_{ij}
\]

(32a)

\[
[k^{(0)}] \{ q^{(0)} \} = [k^{(0)}] \{ q^{(0)} \} + \{ k^{(0)} \} \{ \Delta q^{(0)} \}
\]

(32b)

(31a)

(31b)

Moreover, the unit vectors \( \{ i_{xyz} \} \) of the undeformed frame on element \#1 and the unit vectors \( \{ i_{xyz} \} \) of the deformed frame on element \#2 are related to the reference frame abc as

\[
\{ i_{xyz} \} = [\hat{T}^0] \{ i_{abc} \}, \quad \{ i_{xyz} \} = [\hat{T}^0] \{ i_{abc} \} = [\hat{T}^0] \{ i_{xyz} \}, \quad [\hat{T}^0] = [\hat{T}^0]^T[\hat{T}^0]^T
\]

(33)

If it is a rigid body between the two elements, \( \{ i_{123} \} = [\hat{T}^0] \{ i_{123} \} \) exists and hence we have

Fig. 3. Methods of connecting two beam elements: (a) an L-frame, (b) discrete connection, and (b) smooth connection.
\[ [\hat{T}] = [\hat{T}]_{0}[T]^T \iff [\hat{T}] = [\hat{T}]_{0}[T] \iff [\hat{T}] = [\hat{T}]_{0}[T]^T ]^{34b}\]

\[ T_{\gamma} = \sum_{m=1}^{3} \sum_{n=1}^{3} \hat{T}_{mn} T_{mn} \hat{T}_{mn}^{0}, \quad \gamma = 11, 12, 13, 21, 23 \quad (34c)\]

Moreover, the displacement vector \( D_0 \) at the junction node can be presented as

\[ D_0 = \{u, v, w\} | [i_{123}] = \{u, v, w\} | [i_{123}] = \{u, v, w\} | [i_{123}] \]

(35a)

Hence, we obtain from Eqs. (35a) and (33) that

\[ \{u, v, w\}^{T} = [\hat{T}]_{0}\{u, v, w\}^{T} \quad (35b)\]

Moreover, it follows from Eq. (3a) that

\[ \ddot{u} = -1 + k_3 - w k_2 + (1 + e)\ddot{T}_{111}, \]
\[ \ddot{v} = w k_3 + (1 + e)\ddot{T}_{122}, \]
\[ \ddot{w} = -w k_1 + (1 + e)\ddot{T}_{111}, \quad \ddot{w} = -w k_1 - \ddot{v} k_1 + (1 + e)\ddot{T}_{133}. \quad (35c)\]

Hence, when \( u, v, w, u', v', w', T_{121}, \) and \( T_{233} \) are known, \( u, v, w, u', v', w' \) can be obtained from Eqs. (35b), (35c) and (34c), \( u', v', w' \), and \( w \) need to be obtained by iteration because \( e \) is a function of \( u', v', w' \). However, the iteration can be easily started with \( e = 0 \) and it converges quickly within a few iterations.

For assembly of elements, the relations between \( \delta u, \delta v, \delta w, \delta u', \delta v', \delta w', \delta T_{122}, \) and \( \delta u, \delta v, \delta w, \delta u', \delta v', \delta w' \), \( \delta T_{122} \) and \( \delta T_{233} \) are also needed. It follows from Eqs. (35b) and (34c) that

\[ \delta T_{ij} = \sum_{m=1}^{3} \sum_{n=1}^{3} \hat{T}_{mn} \delta T_{mn} \hat{T}_{mn}^{0}, \quad \gamma = 11, 12, 13, 21, 23 \quad (36a)\]

Hence, we have

\[ A|\delta p| = [B]|\delta \rho| \iff [\delta \rho] = [A]^{-1}[B]|\delta p|, \]
\[ p = \{u, v, w, u', v', w', T_{121}, T_{233}\}^{T} \quad (36b)\]

where \( [A] \) and \( [B] \) are non-singular \( 8 \times 8 \) matrices that can be obtained from Eqs. (36a) and (33). Eq. (36b) works like a multiple-point constraint except that \( [A] \) and \( [B] \) are not constant matrices. Eqs. 34c, 35b, 35c and 36b give the exact relations between the nodal DOFs of two rigidly connected elements at a joint. It is obvious the computation is non-trivial because of many nonlinear terms. However, several approximate methods can be used, as discussed next.

For two beam elements smoothly connected at a node, their \( u, v, w, u', v', w', T_{121}, \) and \( T_{233} \) are the same, and hence no coordinate transformation is needed. Hence, one can use an initially curved small beam element to smoothly connect two oblique elements, as shown in Fig. 3(c). By the way, the actual beam is more close to the one shown in Fig. 3(c) than the one shown in Fig. 3(b) and described by Eqs. (33), (36b). However, small elements with sizes similar to the curved connecting element need to be used around the node in order to have accurate solutions. A simpler approach is that the rigid angle between two connected beam elements can be enforced by using one or two massless rigid truss elements to connect them at locations very close to the node. Another approach similar to the use of rigid truss elements is to use multiple-point constraints on a few points on each of the two elements at locations close to the node to enforce a rigid relative angle between the two elements.

### 3.2. Modeling of rigid components

This total-Lagrangian modeling method can also be used to model a rigid body as a straight beam element with two nodes connected to other flexible elements. Hence, a flexible multibody system can be modeled using the same approach for both flexible and rigid components.

For a rigid component, \( \delta \Pi = 0 \), the nonlinear rotary inertias terms \( A_{0} \) and \( \delta \theta_{0} \) shown in Eqs. (7b) and (7c) need to be fully accounted for, and the area integrations for rotary inertias need to be replaced with volume integrations. Eqs. (7b), (7c), (6b), (6d) and (22b) show that they are nonlinear functions of \( u, v, w, T_{23} \) and \( T_{23} \) and their spatial derivatives. However, the nodal DOFs of two end nodes (see Eq. (26b)) are rigidly related as

\[ \{u_{1}', v_{1}', w_{1}', T_{121}, T_{233}\} = \{u_{2}', v_{2}', w_{2}', T_{233}\} \]

where \( L_{5} \) is the rigid length between the two end nodes. Moreover, because \( e = k_{1} = 0 \), it follows from Eq. (3a) that

\[ u_{1}' = T_{111} - 1, \quad v_{1}' = T_{122}, \quad w_{1}' = T_{233} \quad (37b)\]

Hence, the DOFs of a rigid element are reduced to \( u_{5}, v_{5}, w_{5}, T_{121}, T_{233} \), and \( T_{233} \), which fully describe the location and orientation of the rigid component. However, \( T_{111} \) can be determined by using

\[ T_{111} = \pm \sqrt{1 - T_{122}^{2} - T_{133}^{2}} \approx \frac{1 - \nu k_{3} + w k_{2} + \Delta u/\Delta x}{1 + e} \quad (38)\]

where the second expression obtained from Eq. (3a) is proposed for determining the sign of \( T_{111} \) using spatial finite difference. Moreover, \( T_{121} \) can be obtained from Eq. (23). Hence, the six variables \( u_{5}, v_{5}, w_{5}, T_{121}, T_{133} \) and \( T_{233} \) can fully describe the motion of a rigid body.

### 4. Multiple shooting formulation

For nonlinear static problems or steady-state dynamic problems, the presented geometrically beam theory can be transformed into nonlinear ordinary differential equations (ODEs) with \( s \) as the only independent variable (Pai, 2007). If these governing equations can be put into a group of first-order nonlinear ODEs, they can be solved for numerically exact solutions using a multiple shooting algorithm based on direct numerical integration using the Runge–Kutta method or others (Pai, 2007). Because Eq. (12b) shows that there are six boundary conditions at each end, this is a 12th-order system. The governing Eqs. (12a) and (11b) can be arranged into the following 15 first-order nonlinear ODEs:

\[ F_{1} = \rho_{2} F_{2} - \rho_{2} F_{3} + T_{11}(m\ddot{u} - q_{1}) - T_{12}(m\ddot{v} - q_{2}) + T_{13}(m\ddot{w} - q_{3}) \]
\[ F_{2} = \rho_{2} F_{3} - \rho_{2} F_{1} + T_{21}(m\ddot{u} - q_{1}) - T_{22}(m\ddot{v} - q_{2}) + T_{23}(m\ddot{w} - q_{3}) \]
\[ F_{3} = \rho_{2} F_{1} - \rho_{2} F_{2} + T_{31}(m\ddot{u} - q_{1}) - T_{32}(m\ddot{v} - q_{2}) + T_{33}(m\ddot{w} - q_{3}) \]

\[ M_{1} = \rho_{3} M_{2} - \rho_{2} M_{3} - q_{4} \]
\[ M_{2} = \rho_{3} M_{3} - \rho_{2} M_{1} + (1 + e)F_{3} - q_{5} \]
\[ M_{3} = \rho_{3} M_{1} - \rho_{2} M_{2} - (1 + e)F_{2} - q_{6} \]
\[ T_{11} = \rho_{3} T_{21} - \rho_{2} T_{31} + T_{12} \quad T_{13} \]
\[ T_{12} = \rho_{3} T_{22} - \rho_{2} T_{32} + T_{12} \quad T_{13} \]
\[ T_{13} = \rho_{3} T_{23} - \rho_{2} T_{33} + T_{12} \quad T_{13} \]
\[ T_{21} = \rho_{3} T_{31} - \rho_{2} T_{11} + T_{12} \quad T_{13} \]
\[ T_{22} = \rho_{3} T_{32} - \rho_{2} T_{12} + T_{12} \quad T_{13} \]
\[ T_{23} = \rho_{3} T_{33} - \rho_{2} T_{13} + T_{12} \quad T_{13} \]
\[ u' = -1 + \nu k_{3} - w k_{2} + (1 + e)T_{11} \]
\[ v' = w k_{3} - w k_{1} + (1 + e)T_{12} \]
\[ w' = w k_{3} - w k_{1} + (1 + e)T_{13} \]

(39)
The equations of $T_{ij}$ are obtained from the alternative form \([T] = [K][T] - [T][k]\) of Eq. (4a), and the equations for $u'$, $v'$, and $w'$ are obtained from Eq. (3a). Because $e$ and $p_i$ are linear functions of $F_1$ and $M_1$ as shown in Eq. (8d), there are 15 unknowns in Eq. (39), i.e., $F_1$, $F_2$, $F_3$, $M_1$, $M_2$, $M_3$, $T_{11}$, $T_{12}$, $T_{13}$, $T_{21}$, $T_{22}$, $T_{23}$, $u$, $v$, and $w$. However, $T_{11}$ can be determined by using Eq. (38), $T_{22}$ can be determined using Eq. (3d), and $T_{21}$ can be obtained from Eq. (23). Hence, only 12 equations with 12 unknowns need to be solved with 12 boundary conditions, indicating a 12th-order system. However, the use of the 15 equations in Eq. (39) is easier for programming, assigning boundary conditions, and computation.

5. Numerical simulations

5.1. Discontinuity of rotation variables

To show the spatial discontinuity of rotational variables used in modeling of multibody systems, we consider an initially straight clamped-free inextensible thin beam having a length $L$ on the $x$ axis, as shown in Fig. 4(a). The beam is first rotated by $\alpha_2 = \pi$ w.r.t the $z$ axis, then bent into a circular arc of a total sectional angle $\hat{\alpha}_2 = 1.95\pi$ (i.e., $\psi = 0.05\pi$ in Fig. 4(b)) at $s = L$, and then uniformly twisted along the bent $\zeta$ axis by a total twisting angle $\phi = 1.95\pi$ at $s = L$, as shown in Fig. 4(b). The displacement field is given by

$$u = -s - r \sin(s/r), \quad v = 0, \quad w = r - r \cos(s/r), \quad r = \frac{L}{\hat{\alpha}_2}$$

$$T_{21} = \sin \phi \sin(s/r), \quad T_{22} = -\cos \phi, \quad T_{23} = \sin \phi \cos(s/r)$$

Fig. 5(a) shows the spatial distribution of $\phi(s)$ from the three geometrically exact beam theories, and Fig. 5(b) shows the $\Phi(s)$ and $n(s)$ from Euler's principal rotation theorem. The discontinuities of $\phi(s)$ of beam theories $\Theta_{321}$ and $\Theta_{132}$ at $s/r = 0.5\pi$ (i.e., $s/L \approx 0.25$) and $s/r = 1.5\pi$ (i.e., $s/L \approx 0.75$) are because these two theories are only valid for $|\alpha_2| \leq 0.5\pi$. The discontinuity of $\phi(s)$ of the beam theory $\Theta_{321}$ around $s = 0$ can be explained by using Fig. 4(a). Because the beam cross section at $s = 0$ is rotated by $2\pi = \pi(=\alpha_3)$ w.r.t. the $z$ axis to be aligned with the final deformed configuration, $\phi(0) = 0$. For other cross sections, they are rotated by $2\pi < \pi$ w.r.t. the $n_i$ axis. For the cross section at $s = 0'$, the $z$ axis is rotated to be almost along the $-z$ direction, and hence it needs $\phi(0') \approx -\pi$ to align the cross section with the final deformed configuration. This discontinuity of $\phi$ makes it impossible to use $\phi$ directly in the presented space without using continuous functions. For example, if the $\phi$ is used as a nodal DOF in total-Lagrangian finite-element modeling and analysis, it would result in an extremely high artificial torsional strain in the element that connects the beam root ($\phi = 0$) and its neighboring node ($\phi = -\pi$).

For this case, Fig. 5(b) shows that the $\Phi(s)$ and $n(s)$ from Euler's principal rotation theorem have no discontinuity. However, the sudden change of $\Phi(s)$ by $2\pi$ indicates that, if $\Phi$ is used as a nodal DOF, it needs to be allowed for continuous change beyond $-\pi < \Phi < \pi$. For free flying dynamic problems, the continuous increase of $\Phi$ may result in huge numerical values and numerical overflow problems. Moreover, because the $n(s)$ are not directly related to the orientation of the deformed cross section, they cannot be used for direct implementation of geometric boundary conditions on direction and rotation of boundary cross sections.

If the total twisting angle is increased from $\phi = 1.95\pi$ to $\phi = 2.0\pi$, Fig. 6(a) shows that the discontinuities of $\phi(s)$ from the three geometrically exact beam theories still exist. On the other hand, Fig. 6(b) shows that, although $\Phi(s)$ and $n(s)$ from Euler’s principal rotation theorem are always continuous, the $n_e$ experience severe discontinuous change of first-order derivatives (i.e., cusps) and have large localized gradients. If $\Phi$ and $n_e$ represent a set of solution, $-\Phi$ and $-n_e$ represent another set of solution. However, they actually represent the same solution in the physical space. Since $n_e$ may have large localized gradients, Euler parameters, Rodrigues parameters, and their variants can also have large localized gradients and are not really appropriate for use as nodal DOFs in finite-element modeling. If these variables are used as nodal DOFs, the high gradient would cause high artificial strains in beam elements, especially if truncated nonlinear strain–displacement relations are used. Some may argue that $\Phi(s)$ and $n(s)$ are geometry-based (instead of mechanics-based) variables, and hence they should not cause serious artificial strains if appropriate fully nonlinear strain–displacement relations are used. But they are definitely going to cause numerical difficulty in such stiff problems. On the other hand, $u$, $v$, $w$, $T_{21}$ and $T_{23}$ are all mechanics-based continuous variables and have no large localized gradients.

For a free flying rigid body, one just needs to replace the spatial variable $s$ with the time variable $t$. However, the localized high gradients of Euler and Rodrigues parameters would cause loss of solution accuracy during step-by-step time marching of a rigid body by direct time integration. This is particularly true when such a rigid-body system is required to perform fast angular maneuvers.

![Fig. 4. Large deformation of a flexible beam: (a) a flexible beam undergoing rotation, bending and twisting, and (b) a deformed 3D geometry and its three 2D projections.](image-url)
5.2. Inverse analysis

For forward nonlinear static analysis of beams, unknown displacements under known external loads are often obtained using an updated-Lagrangian formulation with linear or truncated nonlinear strain–displacement relations. For inverse calculation of the needed/unknown loads on a highly flexible beam with a designed/desired deformed geometry, it is possible only if the fully nonlinear explicit strain–displacement relations are available. It follows from Eqs. (12a) and (4a) that one can calculate the needed/unknown static distributed loads as

\[
\begin{bmatrix}
q_1 \\
q_2 \\
q_3 \\
q_4
\end{bmatrix} = [T]^T \begin{bmatrix}
F_1 \\
F_2 \\
F_3
\end{bmatrix} + \frac{\partial}{\partial s} \begin{bmatrix}
F_1 \\
F_2 \\
F_3
\end{bmatrix},
\]

where

\[
q_4 = -M_1' - M_3 \rho_2 + M_2 \rho_3
\]

(41)

Fig. 5. Discontinuity of rotation variables for the beam shown in Fig. 4b under a total twisting angle of 1.95 \(\pi\): (a) \(\phi\) from the three beam theories, and (b) \(\Phi\) and \(n_i\) from Euler’s principal rotation theorem.

Fig. 6. Discontinuity of rotation variables for the beam shown in Fig. 4b under a total twisting angle of 2.0 \(\pi\): (a) \(\phi\) from the three beam theories, and (b) \(\Phi\) and \(n_i\) from Euler’s principal rotation theorem.
where $F_2 = (-M_1 - M_3 \rho_1 + M_1 \rho_2)/(1 + e)$ and $F_3 = (M_2 - M_3 \rho_1 + M_1 \rho_2)/(1 + e)$ are obtained from Eq. (11b). Moreover, Eq. (8d) shows that $F_1$ and $M_i$ are linear functions of $e$ and $\rho_i$, Eqs. (3a) and (4b) show that $e$ and $q_i$ are nonlinear functions of $u, v, w$ and $T_{ij}$, and $u, v, w$ and $T_{ij}$ are known from the designed/desired deformed geometry.

Here we consider the static deformation of the clamped-free titanium alloy beam shown in Fig. 4(b) and Eq. (40). The beam’s Young’s modulus $E$, Poisson’s ratio $\nu$, mass density $\rho$, and dimensions are given below

$$E = 127 \text{ GPa}, \quad \nu = 0.36, \quad \rho = 4430 \text{ kg/m}^3$$

$$L \times b \times h = 800 \text{ mm} \times 10 \text{ mm} \times 0.45 \text{ mm} \quad (42)$$

Fig. 7(a) shows the calculated spatial distributions of required external loads $q_i$ where the unit of $q_1, q_2$ and $q_3$ is N/m, and the unit of $q_4$ is N. The needed external loads are non-zero because bending-torsion coupling due to geometric nonlinearity intends to buckle the beam away from the assumed deformed geometry and the external loads prevent it from happening. Fig. 7(b) shows that the internal stress resultants $F_1 = 0$ and $M_2 = 0.1055$ N-m$=1.95 \pi EI_1/L$ are constant everywhere, but $\sqrt{M_2^2 + M_3^2}$ is not constant indicating the existence of bending-torsion coupling.

If the beam has a circular cross section with a radius of 2 mm, the bending-torsion coupling does not exist and hence all $q_i$ are zero, indicating no external loads are needed in order to maintain the assumed, deformed beam geometry. Moreover, the internal twisting moment and the total internal bending moment are constant everywhere with $M_1 = 8.9861$ N-m$=1.95 \pi GI_{11}/L$ and $\sqrt{M_2^2 + M_3^2} = 12.221$ N-m$(-1.95 \pi EI=I_{33} = I_{zz})$. Furthermore, the direction of the total internal bending moment is always along the axis $y$. In other words, the deformed geometry can be obtained by applying a twisting moment of 8.9861 N-m and a bending moment of 12.2210 N-m at the beam end at $s = L$. If there is no twisting on the beam shown in Fig. 4(b), all external loads and internal stress resultants are zero except $M_2$, which is expected because it is equivalent to a cantilever subjected to an end moment $M_2 = 1.95 \pi EI_{zz}/L$ at $s = L$.

This inverse analysis method provides a valuable tool for fast design optimization of a highly flexible beam with a desired deformed geometry. However, the desired deformed geometry needs to be well described by a mathematical function in order to obtain analytical first- up to fourth-order spatial derivatives needed for the calculation. If the distributed external loads are also desired/required, the material properties, the cross section geometry, and/or the initial curvatures can be designed/adjusted to satisfy the required deformed geometry and external loads.

5.3. Multiple shooting analysis

We consider a circular ring twisted by an angle of $\alpha_3$ at the two ends of a diameter, as shown in Fig. 8. The ring has

$$E = 127 \text{ GPa}, \quad \nu = 0.36, \quad \rho = 4430 \text{ kg/m}^3$$

$$b = 3h = 10 \text{ mm}, \quad R \pi = 800 \text{ mm} \quad (43)$$

Fig. 7. Inverse analysis: (a) distributed external loads, and (b) internal stress resultants.

Fig. 8. Circular ring twisted by $\alpha_3$ at the two opposite ends of a diameter.
The boundary conditions are given by

\[ T_{11} = T_{22} = \cos \alpha_3, \quad T_{12} = -T_{21} = \sin \alpha_3, \]
\[ T_{11} = T_{22} = u = v = w = 0 \text{ at } s = 0 \]
\[ T_{13} = T_{23} = u = v = w = 0 \text{ at } s = R \pi \]
\[ T_{13} = T_{23} = u = v = w = 0 \text{ at } s = R \pi \]

Only one half of the ring needs to be analyzed because the deformation of the other half can be determined by using the symmetry of geometry and loading as

\[ u(2\pi - \theta) = -u(\theta), \quad v(2\pi - \theta) = -v(\theta), \quad w(2\pi - \theta) = w(\theta) \]
\[ T_{21}(2\pi - \theta) = T_{21}(\theta), \quad T_{22}(2\pi - \theta) = T_{22}(\theta), \]
\[ T_{23}(2\pi - \theta) = T_{23}(\theta) \]

Multiple shooting analysis is performed using the formulation shown in Eqs. (39) and (40) shooting points uniformly distributed along one half ring. Fig. 9(a) shows the internal forces \( F_i \) at \( \theta = \pi \) under different angles of twisting. Because the right end is free to

![Fig. 9. Internal forces and moments at \( s = R \pi \): (a) internal forces, and (b) internal moments.](image)

![Fig. 10. Different 3D deformed geometries (red) under different twisting angles and their 2D projections (blue). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)](image)
move along the $a$ axis, $F_2 = 0$ at any angle of twisting. Fig. 9(b) shows the internal moments $M_i$ at $\theta = \pi$ under different angles of twisting. In Fig. 9, all the curves are highly nonlinear and have multiple turning points. When $x_3 = \pi$, $F_1 = F_2 = F_3 = M_1 = M_2 = 0$, indicating a self-locked deformed configuration. Although $M_2 \neq 0$, $dM_2/dx_3 = 0$ reveals that it is a self-balanced configuration. Because there is a lower half ring, the required external twisting moment $M_1$ shown in Fig. 8 should be two times the internal bending moment $M_3$ shown in Fig. 9b. The $M_1$ curve agrees well with the numerical results obtained using the beam theory $\Theta_{21}$ (Pai, 2007) and with the experimental results obtained using a special experimental setup (Pai et al., 2000). However, the beam theory $\Theta_{21}$ suffers from convergence problems when $x_3 \to \pi$, and only very small increments can be used in order to obtain a converged solution using the multiple shooting method. Moreover, $|\phi| \to \infty$ can happen when the beam theory $\Theta_{21}$ is used, which causes numerical difficulty and the deformed state corresponding to $x_3 = \pi$ can never be obtained. On the other hand, because of the use of $T_{21}$ and $T_{23}$ in the new beam theory, the convergence problem does not exist and the deformed state when $x_3 = \pi$ can be actually obtained, as shown in Fig. 10. Note that the ring is twisted into three small rings when $x_3 = \pi$.

Fig. 10 shows different 3D deformed geometries and their 2D projections on the $ab$, $ac$, and $bc$ planes. These solutions are numerically exact because they are obtained from the exact nonlinear ordinary differential equations by direct numerical integrations. If boundary conditions are appropriately applied, the unity of $T_{11}^3 + T_{12}^3 + T_{13}^3 = 1$ and $T_{21}^3 + T_{22}^3 + T_{23}^3 = 1$ are automatically maintained without giving extra constraints during the iteration process because exact coordinate transformation and nonlinear coupling terms are used in the theory. Hence, we highly recommend the multiple shooting formulation shown in Eq. (39) because it is easy for programming, assigning boundary conditions, and computation.

5.4. Nonlinear finite element analysis

We consider an initially straight clamped-free titanium alloy beam on the $x$ axis with an initial angular speed $\omega = \pi$ rad/s w.r.t. the vertical $z$-axis, as shown in Fig. 11. The beam has $\rho = 4430$ kg/m$^3$, $E = 127$ GPa, and $L \times b \times h = 479$ mm $\times 50.8$ mm $\times 0.45$ mm. After the beam is rotated away from the $x$ axis, gravity causes the beam to vibrate. We set $u = v = w = T_{23} = 0$ for the clamped end. Using 15 equal beam elements based on the beam theory shown in Section 3 without damping, Fig. 12a shows the time-varying displacements $u$ and $w$ of nodes 6, 11 and 16 (i.e., at $s = L/3, 2L/3, L$), and Fig. 12(b) shows the time-varying elastic energy $\Pi$, kinetic energy $K_e$, gravitational potential energy $E_g$, and total energy ($=\Pi + K_e + E_g$). The Newmark-$\beta$ method with coefficients for constant acceleration is used here for direct numerical integration. Fig. 12(b) shows that the total energy keeps at the value of the beginning kinetic energy, indicating the beam theory and the finite element algorithm are energy conserved. The results shown in Fig. 12 agree well with those from the use of the beam theory $\Theta_{21}$ (Wu et al., 2011).

Because the proposed total-Lagrangian beam element is based on a geometrically exact beam theory that accounts for any arbitrarily large rigid–elastic displacements, when it is used to model a beam attached to a moving base, there is no need of extra equation derivations in order to account for the base’s motion. Hence, this beam element is convenient for modeling and analysis of flexible multibody systems. However, more numerical and experimental evaluations of the proposed beam element for finite-element analysis of flexible multibody systems and the use of relative constraints derived in Sections 3.1 and 3.2 will be separately reported later.

5.5. Discussions

For an Euler–Bernoulli beam, the governing Eqs. (12a) and (11b) can be rewritten by using Eq. (44a) as

$$
\begin{align*}
&\frac{d}{dt} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} + [K]^T \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{pmatrix},

&\begin{cases}
\frac{\ddot{u}}{w} = \begin{pmatrix} u \\ \dot{u} \end{pmatrix} \\
\frac{\ddot{v}}{w} = \begin{pmatrix} v \\ \dot{v} \end{pmatrix}
\end{cases},

&\begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{pmatrix} = \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{pmatrix} = \begin{pmatrix} A_{q1} \\ A_{q2} \\ A_{q3} \end{pmatrix}
\end{align*}
$$

where $\ddot{q}_i$ are distributed loads and $\dot{u}, \dot{v}$ and $\dot{w}$ are displacements defined w.r.t. the deformed coordinate system $\psi$. Eq. (8d) shows that the internal forces and moments $F_1$ and $M_i$ are linear functions of $e$ and $p_i$, and Eq. (7c) shows that the rotary inertial terms $A_e$ are direct functions of $\dot{\psi}_i (i = 1, 2, 3)$. These 10 variables are not really independent. By using the Kirchhoff– kinetic analogy (Hodges, 2006; Pai, 2007), four constraint equations can be derived to relate $\dot{e}$, $\dot{p}_1$, $\dot{p}_2$, $\dot{p}_3$ to $\dot{\psi}$, $\dot{\psi}'$, $\omega_1$, $\omega_2$, $\omega_3$. Then, the six nonlinear equations (not necessary only quadratic nonlinearities) in Eq. (44) with these 4 nonlinear constraint equations can be used to solve for the 10 variables without solving for the variables $u$, $v$, $w$ and $\phi$ (or other rotation variables). Of course, this approach is valid only if there are no geometric boundary conditions on $u$, $v$, $w$ and/or $\phi$, no deformation-dependent loads, and/or no relative geometric constraints for two misaligned beam elements. Hence, this approach only has limited applicability.

Moreover, obtaining the $\dot{u}$, $\dot{v}$, $\dot{w}$ and $\phi$ from the obtained $u$, $v$, $w$ and $\omega = \omega_i, j = 1, 2, 3$ by time integration and using Euler parameters, or from the obtained $e$ and $p_i (i = 1, 2, 3)$ by spatial integration is problematic. The four constraint equations used with the 6 governing equations are only valid for dynamic cases, and they require the use of Lagrange multipliers (user-dependent choice) in the solution process. Because the rotary inertias of flexible beams are often negligibly small, the accuracy of these constraints on enforcing the equality of these two sets of displacements is questionable. Moreover, because the four constraints only enforce equality on the strain level, integration constants due to rigid-body movement cannot be enforced. Furthermore, if the exact nonlinear strain–displacement relations for $e$ and $p_i (i = 1, 2, 3)$ are not known and used, one can only use finite difference or finite elements with polynomial shape functions to perform step-by-step integration to
obtain \( u, v, w \) and \( \phi \). This approach is essentially an updated-Lagrangian approach and the obtained solution cannot be claimed to be geometrically exact. The accumulation of numerical errors and other problems related to the use of updated Lagrangian formulations also exist in the deformed geometry obtained from this approach. If fully nonlinear strain–displacement relations are unknown, the inverse analysis shown in Section 5.2 cannot be performed at all by using this approach.

6. Concluding remarks

This paper presents a geometrically exact beam theory that uses no rotation variables and hence has no singular points in the spatial domain. The theory is used to reveal that rotation angles commonly used in modeling of geometrically nonlinear beams can be spatially discontinuous and/or sequence-dependent and are not appropriate for use as nodal DOFs in total-Lagrangian finite-element modeling. Such commonly used rotation angles include three Euler angles, two Euler angles, one principal rotation angle plus three direction cosines of the principal rotation axis, four Euler parameters, three Rodrigues parameters, and three modified Rodrigues parameters. The new theory fully accounts for geometric nonlinearities and initial curvatures by using Jaumann strains, exact coordinate transformations, and orthogonal virtual rotations. The derivations are presented in detail, a total-Lagrangian finite-element formulation is included, fully nonlinear governing equations and boundary conditions are presented, and the corresponding form for numerically exact analysis using multiple shooting methods is also derived. Numerical examples are used to reveal the problems of rotational variables and to illustrate the accuracy of the proposed geometrically exact beam theory.

Acknowledgments

This work is supported by the National Science Foundation under Grant CMMI 1039433. This support is gratefully acknowledged.

References