Journal of Mathematical Analysis and Applications **252,** 790–803 (2000)
doi:10.1006/jmaa.2000.7122, available online at http://www.idealibrary.com on IDEAL®

Attractors for Partly Dissipative Reaction Diffusion Systems in $\mathbb{R}^n$

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Submitted by William F. Ames

Received April 23, 1999

In this paper, we study the asymptotic behavior of solutions for the partly dissipative reaction diffusion equations in $\mathbb{R}^n$. We prove the asymptotic compactness of the solutions and then establish the existence of the global attractor in $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. © 2000 Academic Press

Key Words: global attractor; asymptotic compactness; partly dissipativeness; reaction diffusion equation.

1. INTRODUCTION

In this paper, we investigate the asymptotic behavior of solutions for the partly dissipative reaction diffusion equations which are concerned with

[1] Partially supported by DGES PB96-0648.
[2] Partially supported by Postdoctorship of Ministerio de Educacion y Cultura (Spain).

0022-247X/00 $35.00

two unknown functions u and v defined in $\mathbb{R}^n$,

$$\frac{\partial u}{\partial t} - \nu\Delta u + \lambda u + h(u) + \alpha v = f, \tag{1.1}$$

$$\frac{\partial v}{\partial t} + \delta v - \beta u = g, \tag{1.2}$$

where ν, λ, and δ are positive constants, α, $\beta \in \mathbb{R}$, f and g are given functions, and h is a nonlinear function satisfying some growth conditions. System (1.1)–(1.2) describes the signal transmission across axons and is a model of FitzHugh–Nagumo equations in neurobiology; see [1–3] and the references therein.

The long time behavior of solutions for problem (1.1)–(1.2) in a bounded domain has been studied by several authors; see, e.g., [4–6]. Since the dynamical system associated with problem (1.1)–(1.2) is not compact even in the case of a bounded domain, Marion [4] used a splitting technique and proved the existence of the global attractor in this case. More precisely, the author decomposed the semigroup into two parts such that one part asymptotically tends to zero and the other part is compact. Then the existence of the global attractor follows from a standard result (see, e.g., [7–9]). However, to prove the compactness of one part of the decomposition, the author used the fact that Sobolev embeddings are compact when domains are bounded.

For the case of unbounded domains, the lack of compactness of Sobolev embeddings introduces then some extra difficulties. In fact, even for the single reaction diffusion equation in an unbounded domain, the existence of the global attractor in usual Sobolev spaces is not easy to prove and does not follow from the same arguments as for the bounded domain case. To overcome this difficulty and regain some kind of compactness, there have been two different approaches. First, some authors have used weighted Sobolev spaces with weights of the form $(1 + |x|^2)^\alpha$ for some either positive or negative α, which forces the functions to decrease or grow sufficiently fast at infinity. For this approach the nonhomogeneous forcing terms are also assumed to be in a weighted space and existence of global attractors in $L^2(\mathbb{R}^n)$ is not achieved; see Remark 2.14 in [10]. On the other hand, some authors have used spaces of bounded continuous functions together with suitably smooth and fast decreasing super-solutions; see, e.g., [10–15] and the references therein.

Recently, in [16] the existence of the global attractor was proved in $L^2(\mathbb{R}^n)$ for a single reaction diffusion equation in an unbounded domain. Here, using a similar idea, we shall study the existence of the global attractor for the partly dissipative reaction diffusion system (1.1)–(1.2) in

$L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. The key point to our method relies on the exact estimates on solutions which show that the solutions are uniformly small at infinity for large time; see Lemma 2.3 below. This combined with suitable estimates and the compactness of Sobolev embeddings in bounded domains allows us to prove the existence of the global attractor.

This paper is organized as follows. In the next section, we derive some estimates on the solutions which will be used in the proof of existence of the global attractor. In Section 3, we present the proof of the asymptotic compactness and then conclude our result by an abstract theorem. Finally Section 4 contains some other situations in which our proofs work the same.

2. PRELIMINARIES

In this section, we consider the following partly dissipative reaction diffusion system,

$$\frac{\partial u}{\partial t} - \nu \Delta u + \lambda u + h(u) + \alpha v = f \qquad \text{in } \mathbb{R}^n \times \mathbb{R}^+, \tag{2.1}$$

$$\frac{\partial v}{\partial t} + \delta v - \beta u = g \qquad \text{in } \mathbb{R}^n \times \mathbb{R}^+, \tag{2.2}$$

with the initial data

$$u(0, x) = u_0(x), \qquad v(0, x) = v_0(x) \text{ in } \mathbb{R}^n. \tag{2.3}$$

where ν, λ, $\delta > 0$, f, $g \in L^2(\mathbb{R}^n)$ given, α and β are numbers satisfying

$$\alpha\beta > 0, \tag{2.4}$$

while h is a nonlinear function satisfying the conditions

$$h(s)s \geq -C_0 s^2, \qquad h(0) = 0, \qquad h'(s) \geq -C, \qquad s \in \mathbb{R}, \tag{2.5}$$

for some $0 \leq C_0 < \lambda$,

$$|h'(s)| \leq C(1 + |s|^r), \qquad s \in \mathbb{R}, \tag{2.6}$$

with $r \geq 0$ for $n \leq 2$ and $r \leq \min(\frac{4}{n}, \frac{2}{n-2})$ for $n \geq 3$, and C is a positive constant. Note that since $0 \leq C_0 < \lambda$, adding a linear term to h and relabeling the constant λ, we can assume that $\lambda > 0$, $C_0 = 0$. Note that conditions (2.5) and (2.6) are usual conditions for the study of the asymptotic behavior of solutions in an unbounded domain; see, e.g., [9, 10].

In view of (2.1)–(2.2), it is clear that if (u, v) is a solution for the data (α, β, f, g), then $(u, -v)$ is a solution of (2.1)–(2.2) for the data $(-\alpha, -\beta, f, -g)$. Using this and condition (2.4) we will assume throughout this paper and without loss of generality that α and β are both positive.

In the sequel, we denote by $H^s(\mathbb{R}^n)$ the standard Sobolev spaces and $H = L^2(\mathbb{R}^n)$. We also use $\|\cdot\|$ and $(\cdot,\cdot)$ for the usual norm and inner product of $L^2(\mathbb{R}^n)$. For any $1 \le p \le \infty$, we denote by $\|\cdot\|_p$ the norm of $L^p(\mathbb{R}^n)$ and in particular, for $p = 2$ we denote $\|\cdot\|_2 = \|\cdot\|$. In general, $\|\cdot\|_X$ denotes the norm of any Banach space X.

By standard methods, it is easy to prove that if (2.4)–(2.6) hold and $f, g \in H$, then problem (2.1)–(2.3) is well-posed in $H \times H$; that is, for any $(u_0, v_0) \in H \times H$, there exists a unique solution $(u, v) \in C(\mathbb{R}^+, H \times H)$. This establishes the existence of a dynamical system $\{S(t)\}_{t \ge 0}$ which maps $H \times H$ into $H \times H$ such that $S(t)(u_0, v_0) = (u(t), v(t))$, the solution of problem (2.1)–(2.3); see [10].

In what follows, we shall formally derive a priori estimates on the solutions which hold for smooth functions and will become rigorous by a limiting process.

Hereafter, we shall denote by C any positive constants which may change from line to line.

We begin with the estimates in $H \times H$.

LEMMA 2.1. *Assume that* (2.4)–(2.6) *hold and* $f, g \in H$. *Then the solution* (u, v) *of problem* (2.1)–(2.3) *satisfies*

$$\begin{aligned} &\|u(t)\| + \|v(t)\| \le M, \qquad t \ge T_1, \\ &\int_t^{t+1} \|\nabla u(\tau)\|^2 \, d\tau \le M, \qquad t \ge T_1, \end{aligned} \tag{2.7}$$

where M is a constant depending only on the data $(\nu, \lambda, \delta, \alpha, \beta, f, g)$ and T_1 depends on the data $(\nu, \lambda, \delta, \alpha, \beta, f, g)$ and R when $\|(u_0, v_0)\|_{H\times H} \le R$.

Proof. Taking the inner product of (2.1) with βu in H, we find that

$$\begin{aligned} &\frac{1}{2}\beta\frac{d}{dt}\|u\|^2 + \beta\nu\|\nabla u\|^2 + \beta\lambda\|u\|^2 \\ &\qquad + \beta\int_{\mathbb{R}^n} h(u)u + \beta\alpha\int_{\mathbb{R}^n} uv = \beta\int_{\mathbb{R}^n} fu. \end{aligned} \tag{2.8}$$

Similarly, taking the inner product of (2.2) with αv in H, we find

$$\frac{1}{2}\alpha\frac{d}{dt}\|v\|^2 + \alpha\delta\|v\|^2 - \alpha\beta\int_{\mathbb{R}^n} uv = \alpha\int_{\mathbb{R}^n} gv. \tag{2.9}$$

Summing up (2.8) and (2.9), we have

$$\frac{1}{2}\frac{d}{dt}\left(\beta\|u\|^2+\alpha\|v\|^2\right)+\beta\nu\|\nabla u\|^2+\beta\lambda\|u\|^2+\alpha\delta\|v\|^2$$

$$+\beta\int_{\mathbb{R}^n}h(u)u=\beta\int_{\mathbb{R}^n}fu+\alpha\int_{\mathbb{R}^n}gv. \tag{2.10}$$

We now majorize the right-hand side of (2.10) as follows. First, we have

$$\left|\beta\int_{\mathbb{R}^n}fu\right|\le\beta\|f\|\,\|u\|\le\frac{1}{2}\beta\lambda\|u\|^2+\frac{\beta}{2\lambda}\|f\|^2, \tag{2.11}$$

and

$$\left|\alpha\int_{\mathbb{R}^n}gv\right|\le\alpha\|g\|\,\|v\|\le\frac{1}{2}\alpha\delta\|v\|^2+\frac{\alpha}{2\delta}\|g\|^2. \tag{2.12}$$

By (2.10)–(2.12) and (2.5), we get

$$\frac{d}{dt}\left(\beta\|u\|^2+\alpha\|v\|^2\right)+2\beta\nu\|\nabla u\|^2+\beta\lambda\|u\|^2$$

$$+\alpha\delta\|v\|^2\le\frac{\beta}{\lambda}\|f\|^2+\frac{\alpha}{\delta}\|g\|^2. \tag{2.13}$$

Let $\sigma=\min\{\lambda,\delta\}$. Then we find

$$\frac{d}{dt}\left(\beta\|u\|^2+\alpha\|v\|^2\right)+\sigma\left(\beta\|u\|^2+\alpha\|v\|^2\right)\le\frac{\beta}{\lambda}\|f\|^2+\frac{\alpha}{\delta}\|g\|^2.$$

Thus, the Gronwall lemma gives, when $\|(u_0,v_0)\|_{H\times H}\le R$,

$$\begin{aligned}&\beta\|u(t)\|^2+\alpha\|v(t)\|^2\\&\quad\le e^{-\sigma t}\left(\beta\|u(0)\|^2+\alpha\|v(0)\|^2\right)+\frac{\beta}{\sigma\lambda}\|f\|^2+\frac{\alpha}{\sigma\delta}\|g\|^2\\&\quad\le(\alpha+\beta)R^2e^{-\sigma t}+\frac{\beta}{\sigma\lambda}\|f\|^2+\frac{\alpha}{\sigma\delta}\|g\|^2\le\frac{2\beta}{\sigma\lambda}\|f\|^2+\frac{2\alpha}{\sigma\delta}\|g\|^2,\end{aligned} \tag{2.14}$$

for $t\ge T_1:=\frac{1}{\sigma}\ln(\sigma\lambda\delta(\alpha+\beta)R^2/\beta\delta\|f\|^2+\alpha\lambda\|g\|^2)$. Integrating (2.13) between t and $t+1$, by (2.14) we get for $t\ge T_1$,

$$2\beta\nu\int_t^{t+1}\|\nabla u(\tau)\|\,d\tau\le\frac{\beta}{\lambda}\|f\|^2+\frac{\alpha}{\delta}\|g\|^2+\beta\|u(t)\|^2+\alpha\|v(t)\|^2\le C. \tag{2.15}$$

Then, (2.14) and (2.15) conclude the proof of Lemma 2.1. ∎

Note that from (2.14)–(2.15) if $f = 0 = g$ then all solutions converge to the trivial solution $(u, v) = (0, 0)$ as time goes to infinity. Hence in the lemma above one can take M any positive number and T_1 depends on $(\nu, \lambda, \delta, \alpha, \beta)$ and on M. Since this is not a very interesting case, for the remainder of this section we will assume that f and g are not simultaneously zero.

In what follows, we denote by B the ball,

$$B = \{(u, v) \in H \times H : \|(u, v)\|_{H \times H} \leq M\}, \tag{2.16}$$

where M is the constant in (2.7). Then it follows from Lemma 2.1 that B is an absorbing set for $S(t)$ in $H \times H$. By the boundedness of B and Lemma 2.1 again, we know that there exists a constant $T(B)$ depending only on the data $(\nu, \lambda, \delta, \alpha, \beta, f, g)$ and B such that

$$S(t)B \subset B, \qquad t \geq T(B). \tag{2.17}$$

We now establish the estimates in the space $H^1(\mathbb{R}^n)$.

LEMMA 2.2. *Assume that* (2.4)–(2.6) *hold and* $f, g \in H$. *Then any solution* (u, v) *of problem* (2.1)–(2.3) *satisfies*

$$\|u(t)\|_{H^1} \leq C, \qquad t \geq T_1 + 1, \tag{2.18}$$

where C *depends only on the data* $(\nu, \lambda, \delta, \alpha, \beta, f, g)$ *and* T_1 *is the constant in Lemma* 2.1.

Proof. Taking the inner product of (2.1) with $-\Delta u$ in H, we get

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + \nu\|\Delta u\|^2 + \lambda\|\nabla u\|^2$$

$$= \int_{\mathbb{R}^n} h(u)\Delta u + \alpha\int_{\mathbb{R}^n} v\Delta u - \int_{\mathbb{R}^n} f\Delta u. \tag{2.19}$$

We now handle each term on the right-hand side of (2.19). By (2.5) we have

$$\int_{\mathbb{R}^n} h(u)\Delta u = -\int_{\mathbb{R}^n} h'(u)|\nabla u|^2 \leq C\|\nabla u\|^2. \tag{2.20}$$

Also, we find that

$$\left|\int_{\mathbb{R}^n} f\Delta u\right| \leq \|f\|\,\|\Delta u\| \leq \frac{1}{4}\nu\|\Delta u\|^2 + \frac{1}{\nu}\|f\|^2, \tag{2.21}$$

and, by (2.7), we get for $t \geq T_1$,

$$\left|\alpha\int_{\mathbb{R}^n} v\Delta u\right| \leq \alpha\|v\|\,\|\Delta u\| \leq C\|\Delta u\| \leq \frac{1}{4}\nu\|\Delta u\|^2 + C. \tag{2.22}$$

It follows from (2.19)–(2.22) that for $t \geq T_1$,

$$\frac{d}{dt}\|\nabla u\|^2 + \nu\|\Delta u\|^2 + 2\lambda\|\nabla u\|^2 \leq C\big(1 + \|\nabla u\|^2\big),$$

which implies

$$\frac{d}{dt}\|\nabla u\|^2 \leq C\big(1 + \|\nabla u\|^2\big). \tag{2.23}$$

By (2.23), Lemma 2.1, and the uniform Gronwall lemma in [9], we get that

$$\|\nabla u(t)\|^2 \leq C, \qquad t \geq T_1 + 1, \tag{2.24}$$

which concludes the proof of Lemma 2.2. ∎

We are now in a position to derive the estimates on solutions for large time and space variables. These estimates will be used when we prove the asymptotic compactness of the dynamical system $S(t)$. In order to overcome the non-compactness of Sobolev embeddings in $\mathbb{R}^n$, we decompose the whole space $\mathbb{R}^n$ into a bounded ball and its complement. Then the asymptotic compactness of $S(t)$ will follow from the compact Sobolev embeddings in the bounded ball and the estimates in its complement.

The following lemma will play a crucial role in the proof of our result.

LEMMA 2.3. *Assume that* (2.4)–(2.6) *hold*, $f, g \in H$, *and* $(u_0, v_0) \in B$, *the bounded absorbing set in* (2.16). *Then for every* $\varepsilon > 0$, *there exist* $T(\varepsilon)$ *and* $k(\varepsilon)$ *depending on* ε *such that the following holds, for* $t \geq T(\varepsilon)$ *and* $k \geq k(\varepsilon)$,

$$\int_{|x|\geq k}\big(|u(t)|^2 + |v(t)|^2\big)\,dx \leq \varepsilon. \tag{2.25}$$

Proof. Choose a smooth function θ such that $0 \leq \theta(s) \leq 1$ for $s \in \mathbb{R}^+$, and

$$\theta(s) = 0 \text{ for } 0 \leq s \leq 1; \qquad \theta(s) = 1 \text{ for } s \geq 2,$$

and there exists a constant C such that $|\theta'(s)| \leq C$ for $s \in \mathbb{R}^+$.

Multiplying (2.1) by $\beta\theta(|x|^2/k^2)u$, and then integrating the resulting identity, we find

$$\frac{1}{2}\beta\frac{d}{dt}\int_{\mathbb{R}^n}\theta\left(\frac{|x|^2}{k^2}\right)|u|^2 - \beta\nu\int_{\mathbb{R}^n}\theta\left(\frac{|x|^2}{k^2}\right)u\Delta u + \beta\lambda\int_{\mathbb{R}^n}\theta\left(\frac{|x|^2}{k^2}\right)|u|^2$$
$$= -\beta\int_{\mathbb{R}^n}\theta\left(\frac{|x|^2}{k^2}\right)h(u)u - \beta\alpha\int_{\mathbb{R}^n}\theta\left(\frac{|x|^2}{k^2}\right)uv + \beta\int_{\mathbb{R}^n}\theta\left(\frac{|x|^2}{k^2}\right)uf. \tag{2.26}$$

Similarly, multiplying (2.2) by $\alpha\theta(|x|^2/k^2)v$, and then integrating the resulting identity, we get

$$\frac{1}{2}\alpha\frac{d}{dt}\int_{\mathbb{R}^n}\theta\left(\frac{|x|^2}{k^2}\right)|v|^2 + \alpha\delta\int_{\mathbb{R}^n}\theta\left(\frac{|x|^2}{k^2}\right)|v|^2$$
$$= \alpha\beta\int_{\mathbb{R}^n}\theta\left(\frac{|x|^2}{k^2}\right)uv + \alpha\int_{\mathbb{R}^n}\theta\left(\frac{|x|^2}{k^2}\right)vg. \tag{2.27}$$

Summing up (2.26) and (2.27) we find

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\theta\left(\frac{|x|^2}{k^2}\right)\left(\beta|u|^2 + \alpha|v|^2\right)$$
$$+ \beta\lambda\int_{\mathbb{R}^n}\theta\left(\frac{|x|^2}{k^2}\right)|u|^2 + \alpha\delta\int_{\mathbb{R}^n}\theta\left(\frac{|x|^2}{k^2}\right)|v|^2$$
$$= \beta\nu\int_{\mathbb{R}^n}\theta\left(\frac{|x|^2}{k^2}\right)u\Delta u - \beta\int_{\mathbb{R}^n}\theta\left(\frac{|x|^2}{k^2}\right)h(u)u$$
$$+\beta\int_{\mathbb{R}^n}\theta\left(\frac{|x|^2}{k^2}\right)uf + \alpha\int_{\mathbb{R}^n}\theta\left(\frac{|x|^2}{k^2}\right)vg. \tag{2.28}$$

We now majorize each term on the right-hand side of (2.28). We start with

$$\beta\nu\int_{\mathbb{R}^n}\theta\left(\frac{|x|^2}{k^2}\right)u\Delta u = -\beta\nu\int_{\mathbb{R}^n}\theta\left(\frac{|x|^2}{k^2}\right)|\nabla u|^2$$
$$- \beta\nu\int_{\mathbb{R}^n}\theta'\left(\frac{|x|^2}{k^2}\right)u\left(\frac{2x}{k^2}\cdot\nabla u\right). \tag{2.29}$$

Note that the first term above has a negative sign so it can be neglected, so we just need to bound the second term. By (2.17) and (2.24) we have

$$\left|-\beta\nu\int_{\mathbb{R}^n}\theta'\left(\frac{|x|^2}{k^2}\right)u\left(\frac{2x}{k^2}\cdot\nabla u\right)\right| \le C\int_{k\le|x|\le\sqrt{2}k}\frac{|x|}{k^2}|u|\,|\nabla u|$$
$$\le \frac{C}{k}\int_{k\le|x|\le\sqrt{2}k}|u|\,|\nabla u| \le \frac{C}{k}\int_{\mathbb{R}^n}|u|\,|\nabla u|$$
$$\le \frac{C}{k}\|u\|\,\|\nabla u\| \le \frac{C}{k}, \qquad t \ge T(B) + 1, \tag{2.30}$$

when C is independent of k. So, given $\varepsilon > 0$, setting $k_t(\varepsilon) = \frac{2C}{\varepsilon}$, then for $k \geq k_1$, by (2.29) and (2.30) we get

$$\beta\nu\int_{\mathbb{R}^n}\theta\left(\frac{|x|^2}{k^2}\right)u\Delta u \leq \frac{\varepsilon}{2}, \qquad t \geq T(B)+1. \tag{2.31}$$

Note that the second term in the right hand side of (2.28) is negative and can be then neglected. For the third term we have

$$\begin{aligned}
\beta\int_{\mathbb{R}^n}\theta\left(\frac{|x|^2}{k^2}\right)uf &\leq \left|\beta\int_{|x|\geq k}\theta\left(\frac{|x|^2}{k^2}\right)uf\right| \\
&\leq \beta\left(\int_{|x|\geq k}|f|^2\right)^{\frac{1}{2}}\left(\int_{|x|\geq k}\theta^2\left(\frac{|x|^2}{k^2}\right)|u|^2\right)^{\frac{1}{2}} \\
&\leq \beta\left(\int_{|x|\geq k}|f|^2\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^n}\theta\left(\frac{|x|^2}{k^2}\right)|u|^2\right)^{\frac{1}{2}} \qquad (\text{using } 0 \leq \theta \leq 1) \\
&\leq \frac{1}{2}\beta\lambda\int_{\mathbb{R}^n}\theta\left(\frac{|x|^2}{k^2}\right)|u|^2 + \frac{\beta}{2\lambda}\int_{|x|\geq k}|f|^2.
\end{aligned} \tag{2.32}$$

Similarly, we can also deduce that

$$\left|\alpha\int_{\mathbb{R}^n}\theta\left(\frac{|x|^2}{k^2}\right)vg\right| \leq \frac{1}{2}\alpha\delta\int_{\mathbb{R}^n}\theta\left(\frac{|x|^2}{k^2}\right)|v|^2 + \frac{\alpha}{2\delta}\int_{|x|\geq k}|g|^2. \tag{2.33}$$

By (2.28) and (2.31)–(2.33), using (2.5) we get for $\varepsilon > 0$,

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n}\theta\left(\frac{|x|^2}{k^2}\right)\left(\beta|u|^2 + \alpha|v|^2\right) + \beta\lambda\int_{\mathbb{R}^n}\theta\left(\frac{|x|^2}{k^2}\right)|u|^2 + \alpha\delta\int_{\mathbb{R}^n}\theta\left(\frac{|x|^2}{k^2}\right)|v|^2 \\
&\quad\leq \varepsilon + \frac{\beta}{\lambda}\int_{|x|\geq k}|f|^2 + \frac{\alpha}{\delta}\int_{|x|\geq k}|g|^2.
\end{aligned} \tag{2.34}$$

Since f and g belong to H, for given $\varepsilon > 0$, there exists $k_2(\varepsilon)$ such that for $k \geq k_2(\varepsilon)$

$$\frac{\beta}{\lambda}\int_{|x|\geq k}|f|^2 + \frac{\alpha}{\delta}\int_{|x|\geq k}|g|^2 \leq \varepsilon. \tag{2.35}$$

Letting $\sigma = \min\{\lambda, \delta\}$ and $k(\varepsilon) = \max\{k_1(\varepsilon), k_2(\varepsilon)\}$, then from (2.34) and (2.35) we find for $t \geq T(B)+1$ and $k \geq k(\varepsilon)$,

$$\frac{d}{dt}\int_{\mathbb{R}^n}\theta\left(\frac{|x|^2}{k^2}\right)\left(\beta|u|^2 + \alpha|v|^2\right) + \sigma\int_{\mathbb{R}^n}\theta\left(\frac{|x|^2}{k^2}\right)\left(\beta|u|^2 + \alpha|v|^2\right) \leq 2\varepsilon.$$

By (2.17), it follows from the Gronwall lemma that

$$\begin{aligned}\int_{\mathbb{R}^n} \theta\left(\frac{|x|^2}{k^2}\right)\left(\beta|u(t)|^2 + \alpha|v|^2\right) &\\ \leq e^{-\sigma(t-T(B)-1)} \int_{\mathbb{R}^n} \theta\left(\frac{|x|^2}{k^2}\right) &\\ \times\left(\beta|u(T(B)+1)|^2 + \alpha|v(T(B)+1)|^2\right) + \frac{2\varepsilon}{\sigma} &\\ \leq (\alpha+\beta)M^2 e^{-\sigma(t-T(B)-1)} + \frac{2\varepsilon}{\sigma}.\end{aligned}$$

Setting $T(\varepsilon) = \frac{1}{\sigma}\ln(\sigma(\lambda+\beta)M^2/\varepsilon) + T(B) + 1$, then for $t \geq T(\varepsilon)$ and $k \geq k(\varepsilon)$, we have

$$\int_{|x|\geq 2k}\left(\beta|u(t)|^2 + \alpha|v(t)|^2\right) \leq \int_{\mathbb{R}^n} \theta\left(\frac{|x|^2}{k^2}\right)\left(\beta|u(t)|^2 + \alpha|v(t)|^2\right) \leq \frac{3\varepsilon}{\sigma},$$

which concludes the proof of Lemma 2.3. ∎

3. GLOBAL ATTRACTORS

In this section, we present our main result, that is, the existence of the global attractor for $S(t)$ in $H \times H$. To this end, we first need to establish the asymptotic compactness for the dynamical system $S(t)$. Once this kind of compactness is obtained, the existence of the global attractor follows from a standard result which can be stated as follows (see [7–9, 17] for similar results).

PROPOSITION 3.1. *Assume that X is a metric space and $\{S(t)\}_{t\geq 0}$ is a semigroup of continuous operators in X. If $\{S(t)\}_{t\geq 0}$ has a bounded absorbing set and is asymptotically compact, then $\{S(t)\}_{t\geq 0}$ possesses a global attractor which is a compact invariant set and attracts every bounded set in X.*

Below, we present a general result which concerns the asymptotic compactness for the functions defined in unbounded domains. For every $k \geq 0$, we denote by $\Omega_k = \{x \in \mathbb{R}^n : |x| \leq k\}$ and for a function w defined on $\mathbb{R}^n$, we denote by $w\,|_{\Omega_k}$ the restriction of w to Ω_k. Then we have the following theorem.

THEOREM 3.1. *Let Z be a set in $H^r(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$ with $r, s \geq 0$. Then Z is precompact in $H^r(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$ if the following conditions are satisfied:*

(i) *For every* $\varepsilon > 0$, *there exists a constant* $K(\varepsilon)$ *depending on* ε *such that for every* $w \in Z$,

$$\|w\|_{H^r(\mathbb{R}^n \setminus \Omega_{K(\varepsilon)}) \times H^s(\mathbb{R}^n \setminus \Omega_{K(\varepsilon)})} < \varepsilon.$$

(ii) *For every* $k \geq 0$, *the set of restrictions* $Z_k = \{w \mid_{\Omega_k}, w \in Z\}$ *is precompact in* $H^r(\Omega_k) \times H^s(\Omega_k)$.

Proof. We prove that for every $\varepsilon > 0$ there exists a finite covering of Z of balls of radii at most 3ε. Let $\varepsilon > 0$ and $K(\varepsilon)$ be as in (i). Since $Z_{K(\varepsilon)}$ is precompact in $H^r(\Omega_{K(\varepsilon)}) \times H^s(\Omega_{K(\varepsilon)})$, then $Z_{K(\varepsilon)} \subset \bigcup_{i=1}^m B(z_i, \varepsilon_i)$ where $z_i \in Z_{K(\varepsilon)}$ and $\varepsilon_i \leq \varepsilon$.

Now we claim that $Z \subset \bigcup_{i=1}^m B(z_i, 3\varepsilon)$. In fact, if $z \in Z$ then for some $i = 1, \ldots, m$, $\|z - z_i\|_{H^r(\Omega_{K(\varepsilon)}) \times H^s(\Omega_{K(\varepsilon)})} \leq \varepsilon_i$ and then

$$\begin{aligned}\|z - z_i\|_{H^r(\mathbb{R}^n) \times H^s(\mathbb{R}^n)} &\leq \|z - z_i\|_{H^r(\mathbb{R}^n \setminus \Omega_{K(\varepsilon)} \times H^s(\mathbb{R}^n \setminus \Omega_{K(\varepsilon)})} \\ &\quad + \|z - z_i\|_{H^r(\Omega_{K(\varepsilon)}) \times H^s(\Omega_{K(\varepsilon)})}\end{aligned}$$

which is less than $2\varepsilon + \varepsilon_i \leq 3\varepsilon$. ∎

We now prove the asymptotic compactness for the dynamical system $S(t)$ generated by problem (2.1)–(2.3).

THEOREM 3.2. *Assume that* (2.4)–(2.6) *hold and* $f, g \in H$. *Then the semigroup* $S(t)$ *is asymptotically compact, that is, if* $\{u_m, v_m\}$ *is bounded in* $H \times H$ *and* $t_m \to +\infty$, *then* $\{S(t_m)(u_m, v_m)\}$ *is precompact in* $H \times H$.

Proof. Let $Z = \{S(t_m)(u_m, v_m) : m = 1, 2, \ldots\}$ and we check below that Z satisfies the conditions in Theorem 3.1 with $r = s = 0$.

First, since $\{(u_m, v_m)\}$ is bounded in $H \times H$, we can assume that there exists R such that $\|(u_m, v_m)\|_{H \times H} \leq R$. Then by Lemma 2.1 we see that there exists a constant $T_1(R)$ depending on R such that

$$(u^m(t), v^m(t)) = S(t)(u_m, v_m) \subset B, \qquad t \geq T_1(R), \tag{3.1}$$

for every $m \in \mathbb{N}$, where B is the absorbing set given in (2.16). Since $t_m \to +\infty$, there exists $M_1(R)$ such that if $m \geq M_1(R)$, then $t_m \geq T_1(R)$, and hence

$$S(t_m)(u_m, v_m) = S(t_m - T_1)\big(S(T_1)(u_m, v_m)\big). \tag{3.2}$$

Using (3.1) and Lemma 2.3, we find that for every $\varepsilon > 0$, there exist $T(\varepsilon)$ and $K(\varepsilon)$ such that

$$\|S(t)\big(S(T_1)(u_m, v_m)\big)\|_{L^2(\mathbb{R}^n \setminus \Omega_{K(\varepsilon)}) \times L^2(\mathbb{R}^n \setminus \Omega_{K(\varepsilon)})} < \varepsilon, \qquad t \geq T(\varepsilon). \tag{3.3}$$

Again, since $t_m \to +\infty$, there exists $M_2(\varepsilon)$ such that if $m \geq M_2(\varepsilon)$, then $t_m - T_1 \geq T(\varepsilon)$. Therefore, by (3.2) and (3.3) we get for $m \geq \max\{M_1(\varepsilon), M_2(\varepsilon)\}$,

$$\begin{aligned} &\| S(t_m)(u_m, v_m) \|_{L^2(\mathbb{R}^n \setminus \Omega_{K(\varepsilon)}) \times L^2(\mathbb{R}^n \setminus \Omega_{K(\varepsilon)})} \\ &\quad = \| S(t_m - T_1)(S(T_1)(u_m, v_m)) \|_{L^2(\mathbb{R}^n \setminus \Omega_{K(\varepsilon)}) \times L^2(\mathbb{R}^n \setminus \Omega_{K(\varepsilon)})} < \varepsilon, \quad (3.4) \end{aligned}$$

which verifies the condition (i) in Theorem 3.1.

On the other hand, assume $k > 0$ is given. Using now Lemma 2.2 we have, like in (3.2),

$$(u^m(t_m), v^m(t_m)) = S(t_m)(u_m, v_m) = S(t_m - T_2)(S(T_2)(u_m, v_m)) \quad (3.5)$$

with $T_2 = T_1 + 1$, for all sufficiently large m. Since the embedding $H^1(\Omega_k) \subset L^2(\Omega_k)$ is compact, from the estimate (2.18) it is clear that $\{u^m(t_m)\}$ lies in a compact set in $L^2(\Omega_k)$.

On the other hand, for the solutions of (2.1)–(2.3), one can decompose $v(t) = v_1(t) + v_2(t)$ where $v_i(t)$ solves, respectively,

$$\frac{\partial c_1}{\partial t} + \delta v_1 = 0, v_1(0) = v_0, \qquad \frac{\partial v_2}{\partial t} + \delta v_2 - \beta u = g, v_2(0) = 0$$

that is,

$$v_1(x,t) = v_0(x) e^{-\delta t},$$

$$v_2(x,t) = \beta \int_0^t e^{-\delta(t-s)} u(x,s)\, ds + g(x) \frac{(1 - e^{-\delta t})}{\delta}.$$

Using (3.5), the estimate (2.18), and the above with $v(t) = v^m(t + T_2) = v_1^m(t + T_2) + v_2^m(t + T_2)$, $u(x,s) = u^m(x, s + T_2)$, $t = t_m - T_2$, and $v_0 = v^m(T_2)$, we obtain that the $H^1(\Omega_k)$ norm of the first term in $v_2^m(t_m)$ is uniformly bounded in m. On the other hand, the second term lies in the compact set in $L^2(\Omega_k)$, $\{\theta g(x), \theta \in [0, \frac{1}{\delta}]\}$. Therefore, $v_2^m(t_m)$ lies in a compact set in $L^2(\Omega_k)$. Since $v_1^m(t_m)$ decays to zero as $t_m \to \infty$ we get that $v^m(t_m)$ also lies in a compact set of $L^2(\Omega_k)$.

Hence, we find that the set $\{S(t_m)(u_m, v_m) \mid_{\Omega_k} : m = 1, 2 \cdots\}$ is precompact in $L^2(\Omega_k) \times L^2(\Omega_k)$ which verifies the hypothesis (ii) in Theorem 3.1. Then the proof is complete. ∎

We are now in a position to state our main result.

THEOREM 3.3. *Assume that* (2.4)–(2.6) *hold and* $f, g \in H$. *Then problem* (2.1)–(2.3) *has a global attractor which is a compact invariant set and attracts every bounded set in* $H \times H$.

Proof. We note that we have established the existence of a bounded absorbing set for $S(t)$ in Lemma 2.1 and the asymptotic compactness in Theorem 3.2, so the existence of the global attractor follows from Proposition 3.1. ∎

As observed after Lemma 2.1, if the forcing terms f and g are both zero, then all solutions converge to $(0,0)$. In this trivial case the global attractor reduces to the single point $(0,0)$.

4. FINAL REMARKS

The arguments in the previous sections can be applied to more general partly dissipative reaction diffusion systems of the form

$$\frac{\partial u}{\partial t} - \nu\Delta u + \lambda u + h(u,v) = f \qquad \text{in } \mathbb{R}^n \times \mathbb{R}^+, \tag{4.1}$$

$$\frac{\partial v}{\partial t} + \delta v + j(u,v) = g \qquad \text{in } \mathbb{R}^n \times \mathbb{R}^+, \tag{4.2}$$

with initial data

$$u(0,x) = u_0(x), \qquad v(0,x) = v_0(x) \text{ in } \mathbb{R}^n, \tag{4.3}$$

where ν, λ, $\delta > 0$ and $f, g \in L^2(\mathbb{R}^n)$ are given.

On the nonlinear terms we assume that they satisfy some natural growth assumptions such that (4.1)–(4.3) is well posed in $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

In order for Lemma 2.1 to remain true, observe that it is enough that for some $\alpha, \beta > 0$ one has

$$\beta h(u,v)u + \alpha j(u,v)v \geq -C_0|u|^2 - C_1|v|^2 \tag{4.4}$$

for some $0 \leq C_0 < \lambda$ and $0 \leq C_1 < \delta$. With this the estimate derived from (2.10) can be carried over.

On the other hand, the proof of Lemma 2.2 can be repeated with no major changes provided for example that for some positive C

$$\begin{gathered} h(u,v) = h_0(u) + h_1(u,v), \qquad h_0(0) = 0, \qquad h_0'(u) \geq -C, \\ |h_1(u,v)| \leq C(|u| + |v|). \end{gathered} \tag{4.5}$$

On the other hand, condition (4.4) allows us to repeat the proof of Lemma 2.3 along the same lines as above, since one can derive an estimate very similar to (2.28).

Finally, the proof of Theorem 3.2 can be carried out provided for example that

$$j = j(u), \qquad |j'(u)| \leq C(1 + |u|^r)$$

with $r \geq 0$ for $n \leq 2$ and $r < \frac{2}{n-2}$ for $n \geq 3$, and C a positive constant since in this case, for any bounded set Ω, the mapping $j: H^1(\Omega) \to L^2(\Omega)$ is compact. Therefore, the argument using the splitting $v(t) = v_1(t) + v_2(t)$ runs as before.

REFERENCES

1. R. FitzHugh, Impulses and physiological states in theoretical models of nerve membrane, *Biophys. J.* **1** (1961), 445–466.
2. J. Nagumo, S. Arimoto, and S. Yosimzawa, An active pulse transmission line simulating nerve axon, *Proc. J. R. E.* **50** (1964), 2061–2070.
3. J. Bell, Some threshold results for models of myelinated nerves, *Math. Biosci.* **54** (1981), 181–190.
4. M. Marion, Finite-dimensional attractors associated with partly dissipative reaction-diffusion systems, *SIAM J. Math. Anal.* **20** (1989), 816–844.
5. M. Marion, Inertial manifolds associated to partly dissipative reaction-diffusion systems, *J. Math. Anal. Appl.* **143** (1989), 295–326.
6. Z. Shao, Existence of inertial manifolds for partly dissipative reaction diffusion systems in higher space dimensions, *J. Differential Equations* **144** (1998), 1–43.
7. A. V. Babin and M. I. Vishik, "Attractors of Evolution Equations," North-Holland, Amsterdam, 1991.
8. J. K. Hale, "Asymptotic Behavior of Dissipative Systems," Amer. Math. Soc., Providence, 1988.
9. R. Temam, "Infinite Dimensional Dynamical Systems in Mechanics and Physics," 2nd ed., Springer-Verlag, New York, 1997.
10. A. V. Babin and M. I. Vishik, Attractors of partial differential evolution equations in an unbounded domain, *Proc. Roy. Soc. Edinburgh Sect. A* **116** (1990), 221–243.
11. F. Abergel, Existence and finite dimensionality of the global attractor for evolution equations on unbounded domains, *J. Differential Equations* **83** (1990), 85–108.
12. E. Feireisl, Ph. Laurencot, F. Simondon, and H. Toure, Compact attractors for reaction-diffusion equations in $\mathbb{R}^n$, *C. R. Acad. Sci. Paris Ser. I* **319** (1994), 147–151.
13. E. Feireisl, Ph. Laurencot, and F. Simondon, Global attractors for degenerate parabolic equations on unbounded domains, *J. Differential Equations* **129** (1996), 239–261.
14. S. Merino, On the existence of the compact global attractor for semilinear reaction diffusion systems on $\mathbb{R}^n$, *J. Differential Equations* **132** (1996), 87–106.
15. D. Daners and S. Merino, Gradient-like parabolic semiflows on BUC($\mathbb{R}^n$), *Proc. Roy. Soc. Sect. A Edinburgh*, **128** (1998), 1281–1291.
16. B. Wang, Attractors for reaction diffusion equations in unbounded domains, *Phys. D* **128** (1999), 41–52.
17. J. M. Ball, Continuity properties and global attractors of generalized semiflows and the Navier–Stokes equations, *J. Nonlinear Sci.* **7** (1997), 475–502.