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High-order finite element methods for time-fractional partial differential equations[☆]

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ABSTRACT

The aim of this paper is to develop high-order methods for solving time-fractional partial differential equations. The proposed high-order method is based on high-order finite element method for space and finite difference method for time. Optimal convergence rate $O((\Delta t)^{2-\alpha} + N^{-r})$ is proved for the $(r-1)$th-order finite element method ($r \geq 2$).

1. Introduction

Fractional partial differential equations (PDEs) have wide applications in the real world (see e.g., in [1,2] and [3]) and thus the solutions of the equations become increasingly popular (see e.g., in [1,4–6] and the listed references). In this paper, we study one type of time-fractional PDEs, which can be obtained from the standard parabolic PDEs by replacing the first-order time derivative with a fractional derivative of order α, $0 < \alpha < 1$. More precisely, we consider

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t), \quad (x,t) \in [0,1] \times [0,T] \tag{1}$$

subject to the initial and boundary conditions:

$$u(x,0) = u_0(x), \quad x \in I = [0,1], \tag{2}$$

$$u(0,t) = u(1,t) = 0, \quad t \in (0,T], \tag{3}$$

where $0 < \alpha < 1$, f and u_0 are given smooth functions and $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ is Caputo fractional derivative defined by

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^\alpha}.$$

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The analytical solutions of the time-fractional PDEs are studied using Green's functions or Fourier–Laplace transforms (see e.g., in [1,7–9]). However, the references for the numerical methods are very limited. Most existing methods are lower-order methods, for example, Liu et al. [10] study the first-order finite difference methods; Scherer et al. [11] develop Grünwald–Letnikov's approach (a variant of finite difference method), analyze the stability and discuss the convergence rates.

As pointed out in paper [12], it is necessary to develop high-order methods due to the fractional term. High-order methods—spectral methods are studied by Lin and Xu [12]. Lin and Xu [12] (in Theorem 4.2, 4.3) show that the methods for α-order time-fractional partial differential equations with $0 < \alpha < 1$ have convergence rate $O(\Delta t^{2-\alpha} + N^{-m}/(\Delta t)^{\alpha})$, where m measures the regularity of the solution in space. Obviously the convergence rates in their paper are not optimal due to the impairment of the factor $(\Delta t)^{-\alpha}$. In this paper, we use high-order finite element methods to solve the same equation and prove an optimal convergence rate. Since the finite element methods use piecewise polynomial bases not like the spectral methods using global polynomial bases, the finite element methods are much easier to implement.

In the rest of the paper, we assume that the solution u is sufficiently smooth. We use the following norms: $\|v\| = \|v\|_{L^2(I)}$ and $\|v\|_r = \|v\|_{H^r(I)}$. C denotes a generic positive constant that is independent of mesh but depends on the smoothness of u.

2. High-order finite element methods

Let $\tau = T/L$ be the time meshsize, $t_n = n\tau, n = 0, 1, \ldots, L$ be mesh points and $t_{n-1/2} = \frac{t_{n-1}+t_n}{2}, n = 1, 2, \ldots, L$, be mid mesh points, where L is a positive integer. The time-fractional derivative $\frac{\partial^\alpha u(x,t)}{\partial^\alpha t}$ at t_n is estimated by

$$\begin{aligned}\frac{\partial^\alpha u(x, t_n)}{\partial t^\alpha} &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \frac{\partial u(x, s)}{\partial s} \frac{ds}{(t_n - s)^\alpha} \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{n} \frac{\partial}{\partial t} u(x, t_{k-1/2}) \int_{t_{k-1}}^{t_k} \frac{ds}{(t_n - s)^\alpha} + \gamma_n^{(1)}(x) \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \frac{\partial}{\partial t} u(x, t_{n-k-1/2}) \int_{t_k}^{t_{k+1}} \frac{ds}{s^\alpha} + \gamma_n^{(1)}(x) \\ &= \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} b_k \frac{\partial}{\partial t} u(x, t_{n-k-1/2}) + \gamma_n^{(1)}(x), \end{aligned} \tag{4}$$

where $b_k = (k+1)^{1-\alpha} - k^{1-\alpha}$ and

$$\gamma_n^{(1)}(x) = \frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \frac{\partial u(x, s)}{\partial s} \frac{ds}{(t_n - s)^\alpha} - \frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{n} \frac{\partial u(x, t_{k-1/2})}{\partial t} \int_{t_{k-1}}^{t_k} \frac{ds}{(t_n - s)^\alpha}.$$

Let $h = 1/N$ and use the uniform space mesh with mesh points

$$x_i = ih, \quad i = 0, 1, \ldots, N.$$

Denote S_h the set of piecewise polynomials of degree at most $r-1$ on mesh $\{x_i\}$. Define Ritz projection R_h from $H_0^1(I)$ into S_h by the orthogonal relation

$$a(R_h v, \chi) = a(v, \chi), \quad \forall \chi \in S_h,\ v \in H_0^1(I).$$

Define

$$\gamma_n^{(2)}(x) = \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} b_k \left(\frac{\partial}{\partial t} u(x, t_{n-k-1/2}) - \frac{R_h u(x, t_{n-k}) - R_h u(x, t_{n-k-1})}{\tau} \right).$$

Then combining with (4), we have

$$\frac{\partial^\alpha u(x, t_n)}{\partial t^\alpha} = \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} b_k \frac{R_h u(x, t_{n-k}) - R_h u(x, t_{n-k-1})}{\tau} + \gamma_n(x), \tag{5}$$

with $\gamma_n(x) = \gamma_n^{(1)}(x) + \gamma_n^{(2)}(x)$.

The weak form of (1)–(3) is given by

$$\left(\frac{\partial^\alpha}{\partial t^\alpha} u, \phi \right) + a(u, \phi) = (f, \phi), \quad \forall \phi \in H_0^1(I), \tag{6}$$

where $(\cdot,\cdot)$ is the inner product in $L^2(I)$, $a(u,\phi)=\int_I u'\phi'\mathrm{d}x$. Denoting

$$\partial R_h u(x,t_k)=\frac{R_h u(x,t_k)-R_h u(x,t_{k-1})}{\tau}$$

and using (5), we rewrite the weak form (6) at t_n as

$$\frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{k=0}^{n-1} b_k\left(\partial R_h u(x,t_{n-k}),\phi\right)+a(u(x,t_n),\phi)+(\gamma_n(x),\phi)=(f_n,\phi),\quad \forall\phi\in H_0^1(I), \tag{7}$$

where $f_n(x)=f(x,t_n)$. Moreover, since $S_h\subset H_0^1(I)$, we have

$$\frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{k=0}^{n-1} b_k\left(\partial R_h u(x,t_{n-k}),\chi\right)+a(R_h u(x,t_n),\chi)+(\gamma_n(x),\chi)=(f_n,\chi),\quad \forall\chi\in S_h. \tag{8}$$

Denote by $U^n\in S_h$ the approximation of $u(\cdot,t_n)$ and

$$\partial U^n=\frac{U^n-U^{n-1}}{\tau}.$$

Now we define the fully discrete finite element method by

$$\frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{k=0}^{n-1} b_k\left(\partial U^{n-k},\chi\right)+a(U^n,\chi)=(f_n,\chi),\quad \forall\chi\in S_h \tag{9}$$

or

$$(U^n,\chi)+\Gamma(2-\alpha)\tau^\alpha a(U^n,\chi)=\sum_{k=1}^{n-1}(b_{k-1}-b_k)(U^{n-k},\chi)+b_{n-1}(U^0,\chi)+\Gamma(2-\alpha)\tau^\alpha(f_n,\chi),\quad \forall\chi\in S_h. \tag{10}$$

From (8), we know that the truncation error is γ_n, which will be estimated in the following lemma.

Remark 2.1. The stability is stable and the proof is analogous to that in [12]. So the proof is omitted here.

Lemma 2.1. *The truncation error $\gamma_n(x)$ defined by Eq. (8) is bounded by*

$$\|\gamma_n(x)\|\le C(h^r+\tau^2+\tau^{2-\alpha}),$$

where C is dependent of T,α,u.

Proof. $\gamma_n^1(x)$ can be estimated by (see [12])

$$\|\gamma_n^{(1)}(x)\|\le\left|\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\frac{2s-(t_{k-1}-t_k)}{2(t_n-s)^\alpha}\mathrm{d}s\right|+C\tau^2\le C\tau^{2-\alpha} \tag{11}$$

with C being dependent of T,α,u. To estimate $\gamma_n^2(x)$, we estimate that

$$\frac{\partial}{\partial t}u(x,t_{k-1/2})-\partial R_h u(x,t_k)=(I-R_h)\frac{\partial}{\partial t}u(x,t_{k-1/2})+R_h\left(\frac{\partial}{\partial t}u(x,t_{k-1/2})-\frac{u(x,t_k)-u(x,t_{k-1})}{\tau}\right).$$

Hence we have

$$\left\|\frac{\partial}{\partial t}u(x,t_{k-1/2})-\frac{R_h u(x,t_k)-R_h u(x,t_{k-1})}{\tau}\right\|\le C(h^r+\tau^2),$$

and thus

$$\|\gamma_n^{(2)}(x)\|\le C(h^r+\tau^2), \tag{12}$$

where C is dependent of T,α,u. Combining the estimations on $\gamma_n^1(x)$ and $\gamma_n^2(x)$, we complete the proof of this lemma. □

The following lemma will be also used in the proof of the convergence rate of the finite element methods.

Lemma 2.2. *Let $\varepsilon^k\ge 0$, $k=0,1,\ldots,L$, satisfy*

$$\varepsilon^n\le\sum_{k=1}^{n-1}(b_{k-1}-b_k)\varepsilon^{n-k}+\gamma \tag{13}$$

with $\gamma > 0$. *Then*

$$\varepsilon^n \le b_{n-1}^{-1}\gamma. \tag{14}$$

Furthermore, we have

$$\varepsilon^n \le n^\alpha n^{-\alpha} b_{n-1}^{-1}\gamma \le C\tau^{-\alpha}\gamma, \quad n = 1, 2, \ldots, L, \tag{15}$$

with C *being dependent of* T, α *and* u.

Proof. This lemma and the proof are included in the proof of Theorem 3.2 in [12]. □

We are now ready to present and prove the main convergence theorem.

Theorem 2.1. *With* u *and* U^n *be the solutions of* (1) *and* (9). *Then*

$$\|u(\cdot, t_n) - U^n\| \le C(\tau^{2-\alpha} + h^r),$$

where positive constant C *is independent of* T, α, u.

Proof. Letting $\varepsilon^n = U^n - R_h u(x, t_n)$ and subtracting (9) and (8) give

$$\frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} b_k \left(\partial \varepsilon^{n-k}, \chi\right) + a(\varepsilon^n, \chi) = (\gamma_n(x), \chi) \tag{16}$$

or

$$(\varepsilon^n, \chi) + \Gamma(2-\alpha)\tau^\alpha a(\varepsilon^n, \chi) = \sum_{k=1}^{n-1}(b_{k-1} - b_k)(\varepsilon^{n-k}, \chi) + b_{n-1}(\varepsilon^0, \chi) + \Gamma(2-\alpha)\tau^\alpha(\gamma_n(x), \chi). \tag{17}$$

Taking $\chi = \varepsilon^n$ in (17) gives

$$\|\varepsilon^n\|^2 \le \sum_{k=1}^{n-1}(b_{k-1} - b_k)\|\varepsilon^{n-k}\| \, \|\varepsilon^n\| + b_{n-1}\|\varepsilon^0\| \, \|\varepsilon^n\| + \Gamma(2-\alpha)\tau^\alpha \|\gamma_n(x)\| \, \|\varepsilon^n\|. \tag{18}$$

By (11) and (12), $\|\gamma_n(x)\| \le C(\tau^{2-\alpha} + h^r)$, and we have

$$\|\varepsilon^n\| \le \sum_{k=1}^{n-1}(b_{k-1} - b_k)\|\varepsilon^{n-k}\| + b_{n-1}\|\varepsilon^0\| + C\tau^\alpha(\tau^{2-\alpha} + h^r), \quad n = 1, 2, \ldots, L. \tag{19}$$

By Lemma 2.2, we have $\|\varepsilon^n\| \le C(\tau^{2-\alpha} + h^r)$. That is,

$$\|R_h u(x, t_n) - U^n\| \le C(\tau^{2-\alpha} + h^r).$$

It is well known that (see e.g., in [13])

$$\|u(x, t_n) - R_h u(x, t_n)\| + h\|(u(x, t_n) - R_h u(x, t_n))'\| \le Ch^r, \quad n = 0, 1, \ldots, L. \tag{20}$$

Therefore, we obtain

$$\|u(x, t_n) - U^n\| \le \|u(x, t_n) - R_h u(x, t_n)\| + \|R_h u(x, t_n) - U^n\| \le C(\tau^{2-\alpha} + h^r).$$

The theorem is proved. □

3. Numerical examples

In this section, we use the following example to verify our theoretical finding. We consider the same equation as that in Lin and Xu [12]:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial^2 u(x,t)}{\partial^2 x} = f(x,t), \quad (x,t) \in [0,1] \times [0,1] \tag{21}$$

with

$$f(x,t) = \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} \sin(2\pi x) + 4\pi^2 t^2 \sin(2\pi x),$$

initial condition $u_0(x) = 0$ and homogeneous boundary conditions. The exact solution of the problem is given by $u = t^2 \sin(2\pi x)$. The spatial and temporal meshes are taken uniform. That is, $h = 1/N$, $\tau = 1/L$, where N and L are the

Table 1
$\alpha = 0.2, L = 10\,000$.

N	10	15	20	25
$\max_n \|u^n - U^n\|_\infty$	8.0204×10^{-5}	1.5861×10^{-5}	4.9956×10^{-6}	2.0696×10^{-6}
$\max_n \|u^n - U^n\|$	1.8782×10^{-5}	3.6930×10^{-6}	1.1665×10^{-6}	4.7731×10^{-7}
Rate		4.0113	4.0059	4.0046
N	30	35	40	
$\max_n \|u^n - U^n\|_\infty$	1.0004×10^{-6}	5.3949×10^{-7}	3.1550×10^{-7}	
$\max_n \|u^n - U^n\|$	2.2997×10^{-7}	1.2399×10^{-7}	7.2570×10^{-8}	
Rate	4.0051	4.0073	4.0115	

Table 2
$\alpha = 0.2, N = 200$.

L	1000	3000	5000	7000
$\max_n \|u^n - U^n\|_\infty$	2.2699×10^{-8}	3.5862×10^{-9}	1.6865×10^{-9}	1.1038×10^{-9}
$\max_n \|u^n - U^n\|$	1.5657×10^{-8}	2.1762×10^{-9}	8.6694×10^{-10}	4.7265×10^{-10}
Rate		1.7962	1.8017	1.8028
L	9000	11 000	13 000	
$\max_n \|u^n - U^n\|_\infty$	8.5336×10^{-10}	7.1606×10^{-10}	6.3280×10^{-10}	
$\max_n \|u^n - U^n\|$	3.0059×10^{-10}	2.0933×10^{-10}	1.5487×10^{-10}	
Rate	1.8010	1.8032	1.8036	

Table 3
$\alpha = 0.5, L = 20\,000$.

N	10	15	20	25
$\max_n \|u^n - U^n\|_\infty$	8.0187×10^{-5}	1.5856×10^{-5}	4.9917×10^{-6}	2.0658×10^{-6}
$\max_n \|u^n - U^n\|$	1.8778×10^{-5}	3.6905×10^{-6}	1.1640×10^{-6}	4.7488×10^{-7}
Rate		4.0124	4.0108	4.0181
N	30	35	40	
$\max_n \|u^n - U^n\|_\infty$	9.9666×10^{-7}	5.3572×10^{-7}	3.1174×10^{-7}	
$\max_n \|u^n - U^n\|$	2.2754×10^{-7}	1.2156×10^{-7}	7.0151×10^{-8}	
Rate	4.0353	4.0665	4.1174	

numbers of meshes in space and time. The finite element method using third-order piecewise polynomials is used for the space and the scheme for time described in previous sections is used in this example. The set of piecewise polynomials of degree at most 3 is constructed as

$$S_h = \left\{ \sum_{k=1}^{3N-1} v_k \phi_{k/3};\ v_k \in R, k = 1, \ldots, 3N-1 \right\},$$

where $\phi_{k/3}$, $k = 1, \ldots, 3N-1$ are basis functions defined in the following way: let $x_{k+1/3}, x_{k+2/3}$ be three-equal-division points of the interval $[x_k, x_{k+1}]$, denote $l_{k,j}(x)$ $(j = 0, 1, 2, 3)$ the basis functions of the cubic Lagrange interpolation with respect to points $x_k, x_{k+1/3}, x_{k+2/3}, x_{k+1}$ and

$$\phi_k = \begin{cases} l_{k,0}(x), & x \in [x_k, x_{k+1}] \\ l_{k-1,3}(x), & x \in [x_{k-1}, x_k] \\ 0, & \text{otherwise} \end{cases} \quad k = 1, \ldots, N-1;$$

$$\phi_{k+j/3} = \begin{cases} l_{k,j}(x), & x \in [x_k, x_{k+1}] \\ 0, & \text{otherwise} \end{cases} \quad j = 1, 2, k = 0, 1, \ldots, N-1.$$

The rates of the convergence are computed by

$$\text{Rate for space} = \left| \frac{\ln(\|\text{Error on finer grid}\| / \|\text{Error on coarser grid}\|)}{\ln(N \text{ of finer grid} / N \text{ of coarser grid})} \right|,$$

$$\text{Rate for time} = \left| \frac{\ln(\|\text{Error on finer grid}\| / \|\text{Error on coarser grid}\|)}{\ln(L \text{ of finer grid} / L \text{ of coarser grid})} \right|.$$

The L_2-norm and L_∞-norm are used for space in this example. From the numerics in Tables 1–6, we can see that the convergence rate for space is fourth order and the convergence rate for time is $\tau^{2-\alpha}$. The numerical results are consistent with our theoretical results in Theorem 2.1.

Table 4
$\alpha = 0.5, N = 80$.

L	1000	3000	5000	7000
$\max_n \|u^n - U^n\|_\infty$	3.8371×10^{-7}	8.6701×10^{-8}	4.8715×10^{-8}	3.5630×10^{-8}
$\max_n \|u^n - U^n\|$	2.5717×10^{-7}	4.9093×10^{-8}	2.2700×10^{-8}	1.3650×10^{-8}
Rate		1.5074	1.5100	1.5116
L	9000	11 000	13 000	
$\max_n \|u^n - U^n\|_\infty$	2.9368×10^{-8}	2.5812×10^{-8}	2.3568×10^{-8}	
$\max_n \|u^n - U^n\|$	9.3328×10^{-9}	6.8876×10^{-9}	5.3476×10^{-9}	
Rate	1.5129	1.5139	1.5148	

Table 5
$\alpha = 0.8, L = 60\,000$.

N	10	15	20	25
$\max_n \|u^n - U^n\|_\infty$	8.0142×10^{-5}	1.5824×10^{-5}	4.9606×10^{-6}	2.0345×10^{-6}
$\max_n \|u^n - U^n\|$	1.8756×10^{-5}	3.6702×10^{-6}	1.1439×10^{-6}	4.5479×10^{-7}
Rate		4.0232	4.0524	4.1334
N	30	35	40	
$\max_n \|u^n - U^n\|_\infty$	9.6537×10^{-7}	5.0447×10^{-7}	2.8557×10^{-7}	
$\max_n \|u^n - U^n\|$	2.0759×10^{-7}	1.0188×10^{-7}	5.0984×10^{-8}	
Rate	4.3014	4.6173	5.1845	

Table 6
$\alpha = 0.8, N = 40$.

L	1000	3000	5000	7000
$\max_n \|u^n - U^n\|_\infty$	5.0671×10^{-6}	1.5391×10^{-6}	9.4844×10^{-7}	7.1645×10^{-7}
$\max_n \|u^n - U^n\|$	3.3741×10^{-6}	8.9630×10^{-7}	4.8330×10^{-7}	3.2157×10^{-7}
Rate		1.2066	1.2091	1.2108
L	9000	11 000	13 000	
$\max_n \|u^n - U^n\|_\infty$	5.9503×10^{-7}	5.2121×10^{-7}	4.7196×10^{-7}	
$\max_n \|u^n - U^n\|$	2.3712×10^{-7}	1.8587×10^{-7}	1.5174×10^{-7}	
Rate	1.2122	1.2134	1.2145	

4. Concluding remarks

In this paper, we studied high-order finite element methods for solving a class of time-fractional partial differential equations. The convergence rate of the method was proved to be optimal. Moreover, the theoretical results in this paper are also valid when the finite element methods are used to solve two-dimensional time-fractional partial differential equations. In the future we will investigate the finite element methods on moving meshes and simulate the blow-up solutions of the time-fractional equations with nonlinear source term (see e.g., in [14]).

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