Powers of the Fundamental Ideal in the Witt Ring

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Let $K$ be a field of characteristic different from 2. In the algebraic theory of quadratic forms, one studies the Witt ring $W(K)$ of equivalence classes of non-degenerate quadratic forms. The Witt ring has a filtration given by the powers $I^n(K)$ of the fundamental ideal $I(K)$ of even-dimensional forms. The ideal $I^n(K)$ is generated by the set $P_n(K)$ of $n$-fold Pfister forms, $\langle a_1, \ldots, a_n \rangle = \bigotimes_{i=1}^{n} \langle 1, -a_i \rangle$, where $a_i \in K^* = K \setminus \{0\}$. Many questions about this filtration and its quotients have arisen in the study of $W(K)$. The spectacular work of Voevodsky, together with his collaborative work with Orlov and Vishik, allows one to answer many old questions. The purpose of this paper is to indicate some of these solutions.

1. INTRODUCTION

Let $\omega$ be an anisotropic $k$-fold Pfister form over $K$. It is well known that the kernel of the natural morphism $W(K) \rightarrow W(K(\omega))$ is the principal ideal $W(K)\omega$ (cf. [1, Satz 1.3]). Also well known is the theorem attributed to both Pfister and Witt that the ann$_K(\omega)$ in $W(K)$ is the ideal $I_\omega(K)$ in $W(K)$ generated by all the $\langle 1, -t \rangle$, where $t$ lies in $D_h(\omega)$, the set of non-zero elements of $K$ represented by $\omega$ (cf. [18, 2,10.13]). For a long time it has been an open problem whether corresponding statements hold for the powers $I^n(K)$ of the fundamental ideal of $W(K)$ or for the quotients $I^n(K)/I^{n+1}(K)$. Recently, Voevodsky, together with Orlov and
Vishik, announced far-reaching results implying a positive answer for the quotients. We shall use this to get the same for the powers.

There is a well-known presentation of $W(K)$ due to Witt in terms of (the canonical) generators and relations. The results just mentioned give presentations for the quotients. They were conjectured by Milnor in [15]. Here we give presentations for the powers. They were originally suggested by the second author in his thesis [3].

We start by describing the results of Orlov, Vishik, and Voevodsky that we shall use. Let $k_n(K)$ be the $n$th Milnor $K$-group mod 2 and let $H^n(K, \mathbb{Z}/2\mathbb{Z}) := H^n(G_K, \mathbb{Z}/2\mathbb{Z})$, where $G_K$ is the absolute Galois group of $K$. Then the following are true:

**THEOREM 1.1** [20]. The natural morphisms $k_n(K) \to H^n(K, \mathbb{Z}/2\mathbb{Z})$ are isomorphisms.

**THEOREM 1.2** [16]. Let $\omega$ be an anisotropic $m$-fold Pfister form over $K$ and denote by $L(\omega)$ its class in $k_n(K)$. Then the kernel of $k_n(K) \to k_n(K(\omega)) = k_n(K) L(\omega)$, and the annihilator of $L(\omega)$ in $k_n(K)$ is generated by the $l(t) \in k_n(K)$, where $t$ runs through the non-zero elements in $K$ represented by $\omega$.

**COROLLARY 1.3** [16]. The natural epimorphisms $k_n(K) \to I^n(K)/I^{n+1}(K)$ are isomorphisms.

Rewriting the above in terms of the quotients of the powers of the fundamental ideal, we have

**COROLLARY 1.4.** Let $\omega$ be an anisotropic $k$-fold Pfister form over $K$ and let $n$ be a positive integer. Then

1. The kernel of the natural morphism
   
   $I^{n+k}(K)/I^{n+k+1}(K) \to I^{n+k}(K(\omega))/I^{n+k+1}(K(\omega))$

   equals the image of $I^n(\omega)$ in $I^{n+k}(K)/I^{n+k+1}(K)$

2. The kernel of the morphism
   
   $I^n(K)/I^{n+1}(K) \to I^{n+k}(K)/I^{n+k+1}(K)$

   given by multiplication by $\omega$ equals the image of $I^{n-1}(K)I_\omega(K)$ in $I^n(K)/I^{n+1}(K)$.

As $k_n(K)$ is defined in terms of generators and relations, Corollary 1.3 gives a presentation of $I^n(K)/I^{n+1}(K)$ in terms of generators and relations. In fact, the generators are the canonical ones, the classes of the $n$-fold Pfister forms in $I^n(K)/I^{n+1}(K)$.
As noted by Orlov, Vishik, and Voevodsky, an easy consequence of their work is the solution to a problem of Knebusch. For convenience, we give the easy proof. Suppose that $H_n$ is a non-hyperbolic form. Let $L/K$ be a field extension such that $H_n$ is not hyperbolic and whose anisotropic part has minimal dimension $d$. By [9, Proposition 6.1], the integer $d = 2^n$ for some $n$. The integer $n$ is called the degree of $H_n$ and is written $\deg(H_n)$. A hyperbolic space is defined to have infinite degree. The set $\mathcal{J}_n(K) = \{ \phi \mid \deg(\phi) \geq n \}$ is an ideal in $W(K)$ containing $I^n(K)$ (cf. [9, Theorem 6.4, Corollary 6.6]). The problem of Knebusch [9, Question 6.7] is to show that $\mathcal{J}_n(K) = I^n(K)$. This is easy to do for $n = 2$ and was known to be true for $n = 3$.

**Theorem 1.5.** If $K$ is a field then $J_n(K) = I^n(K)$ for all $n$.

**Proof.** Assume the result is false. Let $n$ be the minimal integer such that $J_n(K) \neq I^n(K)$ for some field $K$. Choose $\phi \in J_n(K) \setminus I^n(K)$.

The least integer $n$ such that $I^{n+1}(K) = 2I^n(K)$ (or infinity if no such integer exists) is called the stability index of $K$ and is denoted $\text{st}(K)$.

**Lemma 2.1.** Suppose that $K$ is finitely generated over its prime field $F$. Then $\text{st}(K)$ is finite and $I^n(K)$ is torsion free for all large $n$.

**Proof.** Let $k$ be the transcendence degree of $K$ over $F$. If the characteristic of $F$ is positive then $K$ is a $C_{k+1}$-field and the result is trivial. If

2. ANNIHILATORS

The main purpose of this section is to prove that the torsion $n$-fold Pfister forms generate the torsion subgroup $I^n_n(K)$ of $I^n(K)$. We are confident that the results in [16] do follow from [20]. We also want to note that, given the results of [20] and [16], some of our results may be extracted from [10] and [11]. We include them here because of completeness and because we feel that our presentation is clearer. Basic results and terminology in quadratic form theory can be found in [18] and in the theory of real closed fields in [13] or [18].
the characteristic of \( F \) is zero then the cohomological 2-dimension of \( K(\sqrt{-1}) \) is \( k + 2 \). This implies that \( \mathrm{st}(K) \leq k + 2 \) by [8]. The result follows easily, e.g., by [3, 4.1.13]. (By [11, Proposition 1], it follows in fact that \( I^{k+3}(K) \) is torsion free.)

**Remark.** The proof above needs [20] but not [16], as the triviality of \( k_1(K) \) is equivalent to the triviality of \( I^n(K) \). There should, however, be a proof of the lemma independent of [20] if no precise bound is needed.

Let \( X_K \) denote the space of orderings on \( K \). Let \( \hat{\cdot} : W(K) \to C(X_K, \mathbb{Z}) \) be the total signature map given by \( \varphi \mapsto \hat{\varphi} \) where \( \hat{\varphi}(P) = \text{sign}_P(\varphi) \), the signature of \( \varphi \) at \( P \). The kernel of this map is the ideal \( W_{\omega}(K) \) of torsion forms in \( W(K) \). We write \( \text{supp}(\varphi) \) for the support of \( \hat{\varphi} \).

We first show that a principal ideal in \( W(K) \) generated by a Pfister form filters compatibly through the powers of the fundamental ideal.

**Theorem 2.2.** Let \( \omega \) be a \( \mathit{k} \)-fold Pfister form over \( K \). Let \( \varphi \in I^{n+k}(K) \), with \( n \geq 1 \). If \( \varphi \in W(K)\omega \) then \( \varphi \in I^n(K)\omega \).

**Proof.** The hypothesis can be expressed using only finitely many elements of \( K \). It follows that it suffices to prove the statement in the case that \( K \) is finitely generated over its prime field.

Using [16], we see that we can write \( \varphi = \chi_1 + \varphi_1 \) with \( \chi_1 \in I^n(K)\omega \) and \( \varphi_1 \in I^{n+k+1}(K) \). Then also \( \varphi_1 \in W(K)\omega \). By induction, we see that for any \( m \geq 1 \), we can write \( \varphi = \chi_m + \varphi_m \) with \( \chi_m \in I^n(K)\omega \) and \( \varphi_m \in I^{n+k+m}(K) \cap W(K)\omega \). As \( K \) is finitely generated over its prime field, we can choose \( m \) such that \( n + m \geq \text{st}(K) \) and such that \( I^{n+m}(K) \) is torsion free. This shows that it suffices to prove the statement in the case that \( n \geq \text{st}(K) \) and \( I^n(K) \) is torsion free. (We call this the stable range of \( K \).

So we now assume that \( n \geq \text{st}(K) \) and \( I^n(K) \) is torsion free (cf. [11, Proposition 3]). As \( n \geq \text{st}(K) \), we can write \( \varphi = \langle 1, 1 \rangle \chi \) with \( \chi \in I^n(K) \).

Using that \( \text{sign}_P(\omega) = 2^k \) if \( \text{sign}_P(\omega) \neq 0 \) and that \( \text{sign}_P(\varphi) = 0 \) if \( \text{sign}_P(\omega) = 0 \) because \( \varphi \in W(K)\omega \), we see that \( \varphi \) and \( \chi\omega \) have the same total signature. As \( I^n(K) \) is torsion free, it follows that \( \varphi = \chi \omega \).

A theorem attributed to Pfister and Witt (cf. [18, 2.10.13]) says that if \( \omega \) is an anisotropic \( k \)-fold Pfister form then \( \text{ann}_K(\omega) = I_0(K) \). We show that this result filters compatibly through the powers of the fundamental ideal.

**Theorem 2.3.** Let \( \omega \) be a \( \mathit{k} \)-fold Pfister form over \( K \). Let \( \varphi \in I^n(K) \), with \( n \geq 1 \), satisfy \( \varphi \in I_{n+1}(K) \). Then \( \varphi \in I^{n-1}(K)I_{n+1}(K) \).

**Proof.** Using [16], we see as in the proof of Theorem 2.2 that it suffices to prove the statement in the case that \( n \geq \text{st}(K) \) and \( I^n(K) \) is torsion free.

So we now assume that \( n \geq \text{st}(K) \) and \( I^n(K) \) is torsion free (cf. [11, Corollary 2]). Using Fact 2.4 below on \( 2^{-n} \varphi \), it then follows that any
\( \varphi \in I^n(K) \) can be written as a \( \mathbb{Z} \)-linear combination \( \sum_{i=1}^{r} k_i \cdot \varphi_i \) of \( n \)-fold Pfister forms \( \varphi_i \) such that \( \text{supp}(\varphi_i) \subseteq \text{supp}(\varphi) \) for every \( i \). If \( \omega \varphi = 0 \), hence \( \text{supp}(\omega) \cap \text{supp}(\varphi) = \emptyset \), we see that \( \text{supp}(\omega) \cap \text{supp}(\varphi_i) = \emptyset \) for every \( i \). As \( I^{n+k}(K) \) is torsion free, this means that \( \omega \varphi_i = 0 \) for every \( i \). In this way we have reduced to the case where \( \varphi \) is a Pfister form.

As the case where \( \varphi \) is an \( n \)-Pfister form seems not to have been treated in the literature, we include a proof of this special case. We do so by inducting on \( n \). The case \( n = 1 \) is trivial. Write \( \varphi = (1, -a)\chi \) for some \((n-1)\)-fold Pfister form \( \chi \) and \( a \in K^* \). By induction, \( \chi = \sum \langle c_i \rangle \langle 1, -z_i \rangle \psi_i \), where \( c_i \in K^* \), \( \psi_i \) are \((n-2)\)-fold Pfister forms, and \( z_i \in D_K(1-a)\omega \). Write \( z_i = x_i - a y_i \) with \( x_i, y_i \in D_K(\omega) \cup \{0\} \). If \( x_i \neq 0 \) and \( y_i \neq 0 \) then

\[
\langle 1, -a \rangle \langle 1, -z_i \rangle = \langle y_i \rangle \langle 1, -ax_i y_i z_i \rangle \langle 1, -x_i \rangle + \langle 1, -z_i \rangle \langle 1, -y_i \rangle;
\]

hence \( \langle 1, -a \rangle \langle 1, -z_i \rangle \in I(K)I_\omega(K) \). If \( x_i = 0 \) or \( y_i = 0 \) then this is even easier to see. It follows that \( \varphi = (1, -a)\chi \) lies in \( I^{n-1}(K)I_\omega(K) \).

In the proof above we used the following fact.

**Fact 2.4.** Let \( f \in C(X_K, \mathbb{Z}) \). Then there is a positive integer \( n \) such that \( 2^nf \) can be written as a sum \( \sum_{i=1}^{r} k_i \cdot \varphi_i \) with integers \( k_i \) and \( n \)-fold Pfister forms \( \varphi_i \) such that \( \text{supp}(\varphi_i) \subseteq \text{supp}(f) \) for every \( i = 1, \ldots, r \).

Without the condition on the supports, this is well known. It follows at once from the Normality Theorem of [6], and, in fact, the proof there gives this stronger version.

Marshall has a proof, valid for any abstract Witt ring, that shows that we can even have \( \text{supp}(\varphi_i) \) and \( \text{supp}(\varphi_j) \) disjoint for \( i \neq j \) (cf. [14, Lemma 7.12]). Using Witt's trick with signs, Krüskemper also has a nice proof of this version (cf. [10, Lemma 1]). However, neither author states the result explicitly.

Combining Theorems 2.2 and 2.3 yields the following generalization of [4, Corollary 2.4].

**Corollary 2.5.** Let \( \omega \) be a \( k \)-fold Pfister form over \( K \) and let \( \varphi \) be an \( n \)-fold Pfister form over \( K \). If \( m \geq 1 \) then \( W(K)\varphi \cap \text{ann}_K(\omega) \cap I^{m+n}(K) \) is generated by \((m+n)\)-fold Pfister forms \( \rho \) in \( \text{ann}_K(\omega) \) that are divisible by \( \varphi \) (i.e., such that \( \rho = \varphi \sigma \) for some \( m \)-fold Pfister form \( \sigma \) over \( K \)).

**Proof.** Let \( \mu \in W(K)\varphi \cap \text{ann}_K(\omega) \cap I^{m+n}(K) \). By Theorem 2.2, we can write \( \mu = \varphi \tau \) with \( \tau \in I^m(K) \). So \( \tau \in \text{ann}_K(\varphi \omega) \cap I^m(K) = I^{m+n-1}(K)I_{\varphi \omega}(K) \) by Theorem 2.3. As \( \langle \langle -t \rangle \rangle \varphi \in \text{ann}_K(\omega) \) if \( t \in D_K(\varphi \omega) \), the result follows.

We can generalize Theorem 2.3 by using a directed set of Pfister forms instead of a single Pfister form. Here, a set of Pfister forms is said to be
directed if for any two forms in it, it also contains some common multiple. For a directed set $\Omega$ of Pfister forms over $K$ we let $I_\Omega(K)$ be the union of all $I_\omega(K)$, $\omega \in \Omega$. Then $I_\Omega(K)$ is an ideal in $W(K)$.

**Theorem 2.6.** Let $\Omega$ be a directed set of Pfister forms over $K$. Let $\varphi \in I^n(K)$, with $n \geq 1$, such that $\varphi \in I_\Omega(K)$. Then $\varphi \in I^{n-1}(K)I_\Omega(K)$.

**Proof.** This follows at once from Theorem 2.3.

For a preordering $T$ of $K$ we denote by $I_T(K)$ the ideal in $W_K$ generated by the $1, t$, where $t$ runs through the non-zero elements of $T$.

**Corollary 2.7.** Let $T$ be a preordering of $K$. Let $\varphi \in I^n(K)$, with $n \geq 1$, such that $\text{sign}_P(\varphi) = 0$ for every ordering $P$ of $K$ containing $T$. Then $\varphi \in I^{n-1}(K)I_T(K)$.

**Proof.** This follows from Theorem 2.6 and, e.g., [12, Theorem 1.26] by letting $\Omega$ be the set of all Pfister forms of the type $\langle -t_1, \ldots, -t_k \rangle$ with non-zero elements $t_1, \ldots, t_k$ of $T$.


**Corollary 2.8.** Assume that $K$ is formally real. Let $\varphi$ be a torsion element in $I^n(K)$, with $n \geq 1$. Then $\varphi \in I^{n-1}(K)I_{tor}(K)$.

**Proof.** This follows from Corollary 2.7 by letting $T$ be the set of sums of squares in $K$.

Let $W_{\text{red}}(K)$ be the reduced Witt ring of $K$ and let $I^n_{\text{red}}(K)$ be the image of $I^n(K)$ in $W_{\text{red}}(K)$. If $L = K(\sqrt{a})$ is a quadratic extension, let

$$W_{\text{red}}(L/K) := \{ \varphi + W_{\text{tor}}(K) \mid \varphi \in W(K), \text{sign}_a(\varphi) = 0 \}.$$  

If $a >_{\omega} 0$, $\alpha \in X(K)$.

In [7, Proposition 6.15], it was shown that we have an exact sequence

$$0 \rightarrow W_{\text{red}}(L/K) \rightarrow W_{\text{red}}(K) \rightarrow W_{\text{red}}(L) \rightarrow W_{\text{red}}(K),$$

where $W_{\text{red}}(K) \rightarrow W_{\text{red}}(L)$ is induced by the inclusion $K \subset L$ and $W_{\text{red}}(L) \rightarrow W_{\text{red}}(K)$ is induced by the $K$-functional $\sqrt{a} \mapsto 1$ and $1 \mapsto 0$. Then by Theorem 2.6, we have

**Corollary 2.9.** Let $L = K(\sqrt{a})$ be a quadratic extension. Then there is an exact sequence

$$0 \rightarrow I^n_{\text{red}}(L/K) \rightarrow I^n_{\text{red}}(K) \rightarrow I^n_{\text{red}}(L) \rightarrow I^n_{\text{red}}(K),$$
where \( I_{\text{reg}}^n(L/K) := \{ \varphi + W_{\text{tor}}(K) \mid \varphi \in I^n(K), \ \text{sign}_\alpha(\varphi) = 0 \text{ if } a >_\alpha 0, \ \alpha \in X(K) \} \).

3. GENERATORS

In this section, we determine a presentation for \( I^n(K) \).

For \( n \geq 1 \) let \( I_n(K) \) be the additive abelian group generated by the equivalence classes of \( n \)-fold Pfister forms \( \varphi \) over \( K \) subject to the relations

\[
\begin{align*}
(0) \quad [\varphi] &= 0 \text{ if } \varphi = 0 \text{ in } W(K), \\
(1) \quad [\langle a \rangle \otimes \rho] + [\langle b \rangle \otimes \rho] &= [\langle a + b \rangle \otimes \rho] + [\langle ab(a + b) \rangle \otimes \rho] \text{ if } a + b \neq 0, \\
(2) \quad [\langle ab, c \rangle \otimes \omega] + [\langle a, b \rangle \otimes \omega] &= [\langle ac, b \rangle \otimes \omega] + [\langle a, c \rangle \otimes \omega] \text{ if } n \geq 2.
\end{align*}
\]

Then there is a canonical epimorphism \( I_n(K) \to I^n(K) \).

For each \( n \), the natural map \( I_{n+1}(K) \to I_n(K) \) given by

\[
[\langle a, b \rangle \otimes \rho] \to [\langle a \rangle \otimes \rho] + [\langle b \rangle \otimes \rho] - [\langle ab \rangle \otimes \rho]
\]

is easily checked to be a well-defined homomorphism \( I_{n+1}(K) \to I_n(K) \) compatible with the inclusion \( I^{n+1}(K) \to I^n(K) \). The cokernel of this homomorphism is naturally isomorphic to \( k_n(K) \). Hence we have a commutative diagram,

\[
\begin{array}{cccccc}
I_{n+1}(K) & \longrightarrow & I_n(K) & \longrightarrow & k_n(K) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & I^{n+1}(K) & \longrightarrow & I^n(K) & \longrightarrow & I^n(K)/I^{n+1}(K) \longrightarrow 0,
\end{array}
\]

with exact rows. By [16] the right-hand vertical morphism is an isomorphism, and by construction the other two are epimorphisms. It easily follows that if \( I_{n+1}(K) \to I^n(K) \) is an isomorphism then \( I_n(K) \to I^n(K) \) is also an isomorphism. By induction we see that if \( I_n(K) \to I^n(K) \) is an isomorphism then \( I_m(K) \to I^n(K) \) is also an isomorphism for all \( n < m \).

**Theorem 3.1.** The canonical epimorphism \( I_n(K) \to I^n(K) \) is an isomorphism.

**Proof.** We have to show that if \( \Phi \) lies in the kernel of the canonical epimorphism then \( \Phi = 0 \). The hypothesis can be expressed using only finitely many elements of \( K \). As in the proof of Theorem 2.2, we therefore see, using the discussion above, that it suffices to prove the statement in the stable range. In the remainder of this section we shall actually prove a stronger result.
Remarks. (1) In [3] an equivalent system of generators was studied. The map above was shown to be an isomorphism for the case \( n = 2 \). (This was also shown independently in [19].) It was also shown in [3] that the above map is an isomorphism in the stable range when \( k_n(K) \to I^n(K)/I^{n+1}(K) \) is an isomorphism in the stable range.

(2) Relation (1) is only needed when \( n < 2 \). If \( n \geq 2 \) then Pfister forms and relations (0) and (2) on Pfister forms suffice to give a presentation for \( I^n(K) \).

Proof. As (1) only depends on the square classes of the elements and \( \langle r, s \rangle = \langle 1, rs \rangle \) if and only if \( \langle \langle r, s \rangle \rangle = 0 \), it follows that (1) is equivalent to

\[
[\langle \langle a \rangle \rangle \otimes \rho] + [\langle \langle ars \rangle \rangle \otimes \rho] = [\langle \langle as \rangle \rangle \otimes \rho] + [\langle \langle ar \rangle \rangle \otimes \rho]
\]

if \( \langle \langle r, s \rangle \rangle = 0 \).

Suppose that \( n \geq 2 \). Let \( \rho = \langle \langle c \rangle \rangle \otimes \omega \). Then by (2) we have

\[
[\langle \langle as \rangle \rangle \otimes \rho] - [\langle \langle a \rangle \rangle \otimes \rho] = [\langle \langle ac, s \rangle \rangle \otimes \omega] - [\langle \langle a, s \rangle \rangle \otimes \omega].
\]

Similarly,

\[
[\langle \langle ars \rangle \rangle \otimes \rho] - [\langle \langle ar \rangle \rangle \otimes \rho] = [\langle \langle arc, s \rangle \rangle \otimes \omega] - [\langle \langle ar, s \rangle \rangle \otimes \omega].
\]

So (1) can now be written as

\[
[\langle \langle a, s \rangle \rangle \otimes \omega] + [\langle \langle ar, s \rangle \rangle \otimes \omega] = [\langle \langle ac, s \rangle \rangle \otimes \omega] + [\langle \langle ar, s \rangle \rangle \otimes \omega]
\]

if \( \langle \langle r, s \rangle \rangle = 0 \).

But \( \langle \langle r, s \rangle \rangle = 0 \) implies \( \langle \langle ar, s \rangle \rangle = \langle \langle a, s \rangle \rangle \) and \( \langle \langle arc, s \rangle \rangle = \langle \langle ac, s \rangle \rangle \), so this holds. 

In what follows, we shall assume that \( n \) is a positive integer such that \( n \geq \text{st}(K) \) and \( I^n(K) \) is torsion free. It follows that if \( \rho \in P_n(K) \) and \( \omega \in P_m(K) \) for some integer \( m \geq 1 \) then there is a unique \( \sigma \in P_n(K) \) such that \( \omega \rho = \langle 1, 1 \rangle^m \sigma \).

We let \( M \) be the additive group on generators \( \rho, \rho \in P_n(K) \), and relations \( \rho = [\sigma] + [\tau] \) if \( \rho, \sigma, \tau \in P_n(K) \) satisfy \( \rho = \sigma + \tau \). In particular, \( \rho = 0 \) in \( M \) if \( \rho = 0 \). By construction, there is a canonical epimorphism \( M \to I^n(K) \). In fact, there is even a canonical epimorphism \( M \to J_n(K) \). To see this, it suffices to check that if \( \rho, \sigma, \tau \in P_n(K) \) satisfy \( \rho = \sigma + \tau \) then \( [\rho] = [\sigma] + [\tau] \) in \( J_n(K) \). By [5, Theorem 4.8] or [1, Satz 1.5], already \( \sigma + \tau - \rho \in I^{n+1}(K) \) implies that there is an \( \mu \in P_{n-1}(K) \) such that \( \sigma = \langle 1, -a \rangle \mu, \tau = \langle 1, -b \rangle \mu, \) and \( \rho = \langle 1, -ab \rangle \mu \) with \( a, b \in K^* \). But then \( \sigma + \tau - \rho = 0 \) means that \( \langle 1, -a \rangle \langle 1, -b \rangle \mu = 0 \). So \( [\rho] = [\sigma] + [\tau] \) in \( J_n(K) \) because \( J_{n+1}(K) \to J_n(K) \) is well defined.
Let \( \rho \in P_n(K) \) and \( d \in K^* \). Write \( \langle 1, -d \rangle \rho = \langle 1, 1 \rangle \sigma \) and \( \langle 1, d \rangle \rho = \langle 1, 1 \rangle \tau \) with \( \sigma, \tau \in P_n(K) \). Adding, we then get \( \langle 1, 1 \rangle \rho = \langle 1, 1 \rangle \sigma + \langle 1, 1 \rangle \tau \); hence \( \rho = \sigma + \tau \). It follows that \( [\rho] = [\sigma] + [\tau] \) in \( M \). We generalize this as follows:

**Lemma 3.2.** Suppose that \( n \) is a positive integer such that \( n \geq \text{st}(K) \) and \( I^n(K) \) is torsion free. Let \( \rho \in P_n(K) \) and \( d_1, \ldots, d_m \in K^* \). For every \( \xi = (\xi_1, \ldots, \xi_m) \in \{+, -\}^m \) write \( \langle \langle \xi_1 d_1, \ldots, \xi_m d_m \rangle \rangle \rho = \langle 1, 1 \rangle^{m_\xi} \sigma_\xi \) with \( \sigma_\xi \in P_n(K) \). Then \( [\rho] = \sum_\xi [\sigma_\xi] \) in \( M \).

**Proof.** By induction on \( m \): The case \( m = 1 \) is done above. So we assume that \( m > 1 \). For every \( \xi' = (\xi_2, \ldots, \xi_m) \in \{+, -\}^{m-1} \) write

\[
\langle \langle \xi_2 d_2, \ldots, \xi_m d_m \rangle \rangle \rho = \langle 1, 1 \rangle^{m-1} \tau_{\xi'}
\]

with \( \tau_{\xi'} \in P_n(K) \). By the induction hypothesis, we then have \( [\rho] = \sum_\xi [\tau_{\xi'}] \) in \( M \). It therefore suffices to show that \( [\tau_{\xi'}] = [\sigma_{\xi', +}] + [\sigma_{\xi', -}] \) for every \( \xi' \). But

\[
\langle 1, 1 \rangle^{m-1} \tau_{\xi'} = \langle 1, 1 \rangle \langle \langle \xi_2 d_2, \ldots, \xi_m d_m \rangle \rangle \rho = \langle 1, 1 \rangle \langle 1, -d_1 \rangle + \langle 1, d_1 \rangle \langle \langle \xi_2 d_2, \ldots, \xi_m d_m \rangle \rangle \rho = \langle 1, 1 \rangle^{m} \sigma_{\xi', +} + \langle 1, 1 \rangle^{m} \sigma_{\xi', -},
\]

hence \( \tau_{\xi'} = \sigma_{\xi', +} + \sigma_{\xi', -} \). Consequently, \( [\tau_{\xi'}] = [\sigma_{\xi', +}] + [\sigma_{\xi', -}] \).

**Proposition 3.3.** Suppose that \( n \) is a positive integer such that \( n \geq \text{st}(K) \) and \( I^n(K) \) is torsion free. Then every element in \( M \) can be written as a linear combination \( \sum_{i,j} l_{ij} [\sigma_{ij}] \) with the elements \( \sigma_{ij} \) in \( P_n(K) \) having pairwise disjoint supports in \( X_K \).

**Proof.** Let \( \mu = \sum_{i=1}^r k_i [\rho_i] \in M \). Write \( \rho_i = \langle \langle a_{i1}, \ldots, a_{in} \rangle \rangle \) for \( i = 1, \ldots, r \). For every matrix \( \xi = (\xi_k)_{k=1}^n \) in \( \{+, -\}^n \) let \( \omega_{\xi} = \prod_{k=1}^n \langle \langle \eta_k a_{ik} \rangle \rangle \) and write \( \omega_{\xi} \rho_i = \langle 1, 1 \rangle^{n_{\xi}} \sigma_{r, \xi} \) with \( \sigma_{r, \xi} \in P_n(K) \) for \( i = 1, \ldots, r \). By Lemma 3.2 we then have \( [\rho_i] = \sum_{\xi} [\sigma_{r, \xi}] \) in \( M \) for \( i = 1, \ldots, r \), hence \( \mu = \sum_{i=1}^r k_i [\rho_i] = \sum_{i=1}^r k_i \sum_{\xi} [\sigma_{r, \xi}] = \sum_{\xi} \sum_{i=1}^r k_i [\sigma_{r, \xi}] \) in \( M \).

For each \( \xi \) we write \( \omega_{\xi} = \langle 1, 1 \rangle^{n_{\xi}} \tau_{\xi} \) with \( \tau_{\xi} \in P_n(K) \). Clearly, the \( \omega_{\xi} \) have pairwise disjoint supports, and hence also the \( \tau_{\xi} \). Now look at a pair \( (i, \xi) \). If all the \( e_{ik}, k = 1, \ldots, r \), are + then \( \omega_{\xi} \rho_i = \omega_{\xi} \rho_i = \langle 1, 1 \rangle^{n_{\xi}} \tau_{\xi} \); hence \( \sigma_{r, \xi} = \tau_{\xi} \). If, however, some \( e_{ik}, k = 1, \ldots, r \), is − then \( \omega_{\xi} \rho_i = 0 \); hence \( \sigma_{r, \xi} = 0 \). It follows that for each \( \xi \) we have \( \sum_{i=1}^r k_i [\sigma_{r, \xi}] = l_{\xi} \tau_{\xi} \) for some integer \( l_{\xi} \). Consequently, \( \mu = \sum_{\xi} \sum_{i=1}^r k_i [\sigma_{r, \xi}] = \sum_{\xi} l_{\xi} \tau_{\xi} \).
COROLLARY 3.4. Suppose that $n$ is a positive integer such that $n \geq \text{st}(K)$ and $I^n(K)$ is torsion free. Then the canonical epimorphism $M \rightarrow I^n(K)$ is an isomorphism.

Proof. Use Proposition 3.3 on an element in the kernel. Composing with the total signature $I^n(K) \rightarrow C(X_K, 2^n\mathbb{Z})$, we see that the condition on the supports implies that all the coefficients $l_j$ are 0.

Suppose that $n \geq \text{st}(K)$ and $I^n(K)$ is torsion free. Then the isomorphism $M \rightarrow I^n(K)$ factors as $M \rightarrow I^n(K) \rightarrow I^n(K)$, where the first morphism is an epimorphism. It follows that both factors are isomorphisms. In particular, the proof of Theorem 3.1 is complete.

For possible future references we now rewrite Proposition 3.3 in light of Corollary 3.4.

COROLLARY 3.5. Suppose that $n$ is a positive integer such that $n \geq \text{st}(K)$ and $I^n(K)$ is torsion free. Then every element in $I^n(K)$ can be written as a linear combination $\sum_j l_j \cdot \sigma_j$ of elements $\sigma_1, \ldots, \sigma_\ell \in P_n(K)$ having pairwise disjoint supports in $X_K$.

We note that the proof of Corollary 3.4 above actually works in the setting of abstract Witt rings in the sense of Marshall. In particular, it works for the reduced Witt ring of a field $K$.

REFERENCES