The Residue Complex of a Noncommutative Graded Algebra

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0. INTRODUCTION

Suppose \( A \) is a finitely generated commutative algebra over a field \( k \). According to Grothendieck duality theory, there is a canonical complex \( \mathfrak{H}_A^q \) of \( A \)-modules, called the residue complex. It is characterized as the Cousin complex of the twisted inverse image \( \pi^*k \), where \( \pi : X = \text{Spec } A \to k \) is the structural morphism. \( \mathfrak{H}_A \) has the decomposition

\[
\mathfrak{H}_A^q = \bigoplus_{x \in X_q/X_{q-1}} \mathfrak{H}_A(x)
\]

where \( X_q/X_{q-1} \subseteq X \) is the set of points of dimension \( q \) (the \( q \)-skeleton) and \( \mathfrak{H}_A(x) \) is an injective hull of the residue field \( k(x) \). The coboundary operator \( \delta : \mathfrak{H}_A(x) \to \mathfrak{H}_A(y) \) is nonzero precisely when \( y \) is an immediate specialization of \( x \). For a discussion of the commutative theory see [R.D] and [Y.e2].

In this paper we propose a definition of the residue complex \( R^* \) of a noncommutative Noetherian graded \( k \)-algebra \( A = k \oplus A_1 \oplus A_2 \oplus \cdots \).

We begin, in Section 1, with the generalized Auslander–Gorenstein (A-G) condition. This condition can be checked whenever \( A \) has a dualizing complex; if \( A \) is Gorenstein (i.e., has finite injective dimension) it reduces

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to the usual A-G condition. The generalized A-G condition is necessary
for the existence of a residue complex (see below) and seems to be a
reasonable requirement if \( A \) is expected to have any geometry associated
to it. We generalize a result of Bjork and Levasseur to the effect that the
canonical dimension \( \text{Cdim} := -j \), where \( j(M) \) is the grade of the module
\( M \), is a finitely partitive exact dimension function (Theorem 1.3). We also
extend results of [ATV2] regarding normalization of Cohen–Macaulay
modules of dimension 1 (Theorem 1.9).

In Section 2 we define a strong residue complex over \( A \) (Definition 2.3).
This is a refinement of the notion of balanced dualizing complex which
appeared in [Ye1]. The strong residue complex \( R \) is unique, up to an
isomorphism of complexes of graded bimodules (Theorem 2.4). So when it
exists, \( R \) is a new invariant of \( A \). The algebraic structure of \( R \) should
carry some “geometric information” about \( A \), in analogy to the commuta-
tive case. Existence is proved in two general circumstances: (i) \( A \) is finite
over its center; and (ii) \( A \) is the twisted homogeneous coordinate ring of a
triple \((X, \sigma, \mathcal{O})\) (Propositions 2.11, 2.8). In Section 3 we prove existence
for a three-dimensional Sklyanin algebra (see below).

There is evidence that many important algebras, including some four-di-
mensional A-S (Artin–Schelter) regular algebras, do not have strong
module complexes [ASZ]. Guided by this evidence we devised the defini-
tion of weak residue complex (Def. 2.14). However, we do not have a single
example of an algebra which admits a weak residue complex but not a
strong one. We show that the existence of a weak residue complex implies
the generalized A-G condition (Theorem 2.18).

Section 3 is devoted to proving that a three-dimensional Sklyanin
algebra (see [ST, ATV1]) has a strong residue complex. Let \((E, \sigma, \mathcal{O})\) be
the triple defining \( A \); so \( E \) is an elliptic curve, and the automorphism \( \sigma \) is
a translation. We show that \( A \) is localizable at every \( \sigma \)-orbit on \( E \)
(Proposition 3.5). This fact is used to show that the minimal left graded-in-
jective resolution \( I^\cdot \) of \( A \) is also the minimal right resolution. According
to [Aj3] the modules \( I^\cdot \) have the correct GK dimensions. Therefore by
tensoring with the dualizing bimodule \( \omega \) we obtain the residue complex
\( R^\cdot = \omega \otimes_{A} I^\cdot \) (Theorem 3.13, Corollary 3.14).

1. The Generalized Auslander–Gorenstein Condition

In [Ye1] some ideas of Grothendieck duality theory were extended to
noncommutative rings, and we shall briefly review them here. Suppose
\( A = k \oplus A_1 \oplus A_2 \oplus \cdots \) is a Noetherian graded algebra over a field \( k \). It
follows that \( A \) is a finitely generated algebra. By default an \( A \)-module will
mean a graded left module. Let $\text{GrMod}(A)$ be the abelian category of graded left $A$-modules with degree 0 homomorphisms, and let $\text{GrMod}_f(A)$ be the subcategory of finite (that is, finitely generated) modules. We write $\text{Hom}^g_A(M, N)$, for the group of degree $i$ homomorphisms between graded left $A$-modules, so

$$\text{Hom}^g_A(M, N) = \text{Hom}_{\text{GrMod}(A)}(M, N(i)),$$

where $N(i)$ is the shifted module. Define

$$\text{Hom}^g_M(N, N) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}^g_A(M, N)_i \in \text{GrMod}(k).$$

Note that if $M$ is finite then $\text{Hom}^g_A(M, N) = \text{Hom}_A(M, N)$.

We denote by $A^o$ the opposite ring, and $A^e := A \otimes_k A^o$. A right module (resp. a bimodule) is regarded as a left $A^o$ (resp. $A^e$) module.

**Remark 1.1.** Most definitions, operations, and conditions in this paper have a left–right symmetry, expressible by interchanging $A$ and $A^o$. For instance, if $M, N \in \text{GrMod}(A^o)$ we get $\text{Hom}^g_A(M, N) \in \text{GrMod}(A^e)$.

Denote by $D(\text{GrMod}(A))$ the derived category of the abelian category $\text{GrMod}(A)$. Let $D^b(\text{GrMod}(A))$ be the subcategory of bounded complexes with finite cohomologies. Recall that a complex $R \in D^+(\text{GrMod}(A^o))$ is called dualizing if $R^*$ has finite injective dimension over $A$ and $A^o$; each $H^q R^*$ is finite over $A$ and $A^o$; and the natural morphisms $A \to R \text{Hom}^g_A(R^*, R)$ and $A \to R \text{Hom}^g_A(R^*, R^*)$ are isomorphisms in $D(\text{GrMod}(A^o))$. Then the functors $R \text{Hom}^g_A(-, R)$ and $R \text{Hom}^g_A(-, R^*)$ are anti-equivalences between $D^b(\text{GrMod}(A))$ and $D^b(\text{GrMod}(A^o))$. The dualizing complex $R^*$ is unique in the following sense: any other dualizing complex is isomorphic in $D(\text{GrMod}(A^o))$ to $R^* \otimes_A L[n]$, for some invertible bimodule $L$ and integer $n$ (see [Ye1, Theorem 3.9]).

Let $\mathfrak{m}$ be the augmentation ideal of $A$. Write $\Gamma_\mathfrak{m}$ (resp., $\Gamma_\mathfrak{m}^o$) for the functor of left (resp. right) $\mathfrak{m}$-torsion. A dualizing complex $R^*$ is called balanced if there are isomorphisms $R \Gamma_\mathfrak{m}^o R^* \cong R \Gamma_\mathfrak{m} R^* \cong A^e$ in $D(\text{GrMod}(A^o))$. Here $A^e := \text{Hom}^g_A(A, k)$, the graded-injective hull of the trivial module $k$. The balanced dualizing complex $R^*$ is unique up to isomorphism in $D(\text{GrMod}(A^o))$. For example, a Noetherian Artin–Schelter regular algebra $A$ of dimension $n$ has an invertible bimodule $\omega$ s.t. $\omega[n]$ is a balanced dualizing complex (see [Ye1, Cor. 4.14]).

Suppose $R^*$ is a dualizing complex over $A$. Given a finite graded $A$-module $M$, its grade number w.r.t. $R^*$ is defined to be

$$j_{A, R}(M) := \inf \{ q | \text{Ext}^q_A(M, R^*) \neq 0 \} \in \mathbb{Z} \cup \{ \infty \}.$$
Note that if $A$ is Gorenstein (i.e. it has finite injective dimension) and $R^\sim = A$ we recover the usual grade number.

**Definition 1.2.** We say $A$ satisfies the generalized Auslander–Gorenstein (A-G) condition if for every $M \in \text{GrMod}_A$, integer $q$ and graded submodule $N \subseteq \text{Ext}^q_A(M, R)$, one has $\text{Ext}^q_A(N) \geq q$, and if the same holds with $A, A^\sim$ interchanged.

It is easily seen that this definition does not depend on the particular dualizing complex $R$. Indeed, if we take any other complex $R'$, then it is isomorphic in $D(\text{GrMod}(A^\circ))$ to $R' \otimes_A A[n]$, and these twists will cancel out. The condition is clearly left–right symmetric (cf. Remark 1.1). In Section 2 we will relate the generalized A-G condition with residue complexes.

The next theorem generalizes results of Bjork [Bj] and Levasseur [Le].

**Theorem 1.3.** Suppose $A$ satisfies the generalized Auslander–Gorenstein condition. Then $M \mapsto -\text{Ext}^q_A(M)$ is a finitely partitve exact dimension function on $\text{GrMod}_A(A)$ (see [M R., Sects. 6.8, 8.3]).

**Proof.** According to [Ye1, Prop. 2.4], we can assume $R$ is a bounded complex of bimodules and each $R^q$ is graded-injective over $A$ and $A^\sim$. Then the adjunction homomorphism $M \mapsto H$, where

$$H := \text{Hom}_A^G(\text{Hom}_A^G(M, R), R)$$

is a quasi-isomorphism. Pick a positive integer $d$ large enough so that $R^q \neq 0$ only if $|q| \leq d$. Consider the decreasing filtration on $H$ given by the subcomplexes

$$F^p H := \text{Hom}_A^G(\text{Hom}_A^G(M, R), R \geq p).$$

Then $F$ is an exhaustive filtration, and there is a convergent spectral sequence

$$E_2^{p, q} = \text{Ext}^q_A^{p, q}(\text{Ext}^q_A(M, R^q), R) \Rightarrow M. \quad (1.1)$$

The corresponding decreasing filtration

$$M = F^{-d} M \supset F^{-d+1} M \supset \cdots \supset F^{d+1} M = 0$$

is called the b-filtration in [Le].

The generalized A-G condition tells us that $E_2^{p, q} = 0$ if $p < -q$. So the spectral sequence lives in a bounded region of the $(p, q)$ plane: $p \geq -q$ and $|p|, |q| \leq d$. We conclude from formula (1.1) that for every $|p| \leq d$
there is an exact sequence of graded $A$-modules

\[ 0 \to \frac{F^pM}{F^{p+1}M} \to E^{p-q}_p \to Q^p \to 0 \]

with $Q^p$ a subquotient of $\bigoplus_i E^{p+1-i,-p+i}_p$. Therefore $j_{A;R}(F^pM/F^{p+1}M) \geq p$ (cf. [Bj, Thm. 1.3] and [Le, Thm. 2.2]).

From here the proof continues just like in [Bj, Propositions 1.6, 1.8] and [Le, Sects. 2–4].

From here to the end of this section we will assume $A$ satisfies the generalized A-G condition, and also that it has some balanced dualizing complex $R$: The uniqueness of $R^*$ in $D(\text{GrMod}(A^g))$ justifies the following definition.

**Definition 1.4.** The canonical dimension of a finite graded $A$-module $M$ is

\[ \text{CDim} M := -j_{A;R}(M) \in \mathbb{Z} \cup \{ -\infty \}. \]

**Corollary 1.5.** Any finite $A$-module $M$ has a critical composition series w.r.t. CDim.

*Proof.* See [Le, (4.6.4)] or [MR, Lemma 6.2.10 and Prop. 6.2.20].

**Proposition 1.6.** Let $M$ be a finite graded $A$-module.

1. One has

\[ \text{CDim} M \in \{ -\infty, 0, 1, \ldots, \text{CDim} A \}, \]

$\text{CDim} M \leq 0$ iff $M$ is finite-torsion, and $\text{CDim} M = -\infty$ iff $M = 0$.

2. If $\text{Ext}_A^q(M, R) \neq 0$ then $-\text{CDim} M \leq q \leq 0$.

*Proof.* (1) Suppose $M$ has finite length. Since $R^*$ is balanced, $\text{RHom}_A^g(M, R^*) \cong M^*$, so $\text{CDim} M \in \{ -\infty, 0 \}$. Now suppose $M$ is a critical module. Then either $M \cong k$, or $M$ has a nonzero finite length quotient $\overline{M}$, in which case $\text{CDim} M > \text{CDim} \overline{M} = 0$. But any module $M$ has a critical composition series.

(2) The inequality $q \geq -\text{CDim} M$ is trivial. By the generalized A-G condition and part 1 we have $-q \geq \text{CDim} \text{Ext}_A^q(M, R^*) \geq 0$.

Let us finish off this section with an application, due to Artin. It is a generalization of [ATV2, Propositions 6.3 and 6.6].

**Definition 1.7.** We say a finite graded $A$-module $M$ is Cohen–Macaulay (C-M) if $\text{RHom}_A^g(M, R^*) \cong M^\vee[n]$ for some $A^g$-module $M^\vee$ and integer $n$. 

The $A^n$-module $M^\vee$ is called the dual module of $M$, and it is also C-M: $(M^\vee)^\vee = M$. Of course, $n = \text{CDim } M = \text{CDim } M^\vee$.

We shall abbreviate the dualizing functors as follows: $D := \text{RHom}_A^a(-, R')$ and $D^* := \text{RHom}_A^a(-, R')$. Fix for the remainder of the section an isomorphism $R \Gamma_m R \cong A^*$ in $\text{DGrMod}(A^e)$ (a rigidification of $R'$). This determines an isomorphism $R \Gamma_m R \cong A^*$ such that $D^*k = k \cong (k^*)^*$ (see [Ye1, Remark 5.7]).

**Proposition 1.8.** Suppose $A$ satisfies the generalized A-G condition.

1. Let $M$ be a finite graded $A$-module with $\text{CDim } M = 1$. Then $M$ is C-M iff it is $\varpi$-torsion free.

2. Suppose $\phi : M' \to M$ is a homomorphism between C-M modules of dimension 1, which is an isomorphism modulo $\varpi$-torsion. Then $\phi^\vee : M^\vee \to (M')^\vee$ is also an isomorphism modulo $\varpi$-torsion. To be precise, there is a natural exact sequence of $A^e$-modules

$$0 \to M^\vee \xrightarrow{\phi^\vee} (M')^\vee \to \text{Coker}(\phi)^* \to 0.$$ 

**Proof.** 1. First assume $M$ is $\varpi$-torsion free. Set $N^{-1} := H^{-1}DM$ and $N^0 := H^0DM$. Let $\sigma_{\leq q}$ and $\sigma_{>q}$ be the truncation functors of [R.D., Chap. 1, Sect. 7]. Since $\sigma_{\leq -1}DM = N^{-1}$ [1] and $\sigma_{<0}\sigma_{>1}DM = N^0$ we get a triangle

$$N^{-1}[1] \to DM \to N^0 \to N^{-1}[2] \tag{1.2}$$

in $\text{D}^b(\text{GrMod}(A^e))$. By the generalized A-G condition the module $N^0$ has finite length, so $D^*N^0 = (N^0)^*$. Because $\text{CDim } N^{-1} \leq 1$ it follows that $H^qD^*N^{-1} = 0$ only for $q = -1, 0$. Therefore $H^0D(N^{-1}[2]) = 0$. Applying $H^0D^*$ to the triangle (1.2) we get $0 \to (N^0)^* \to M$. The conclusion is that $N^0 = 0$, so $M$ is C-M with dual $M^\vee = N^{-1}$.

Conversely, suppose $M$ is C-M, so $DM = M^\vee[1]$. Let $T := \Gamma_m M$, $\overline{M} := M/T$. The triangle $T \to M \to \overline{M} \to T[1]$ gives an exact sequence

$$H^0DM \to H^0DT \to H^1D\overline{M}.$$ 

Since $M$ is C-M we have $H^0DM = 0$. By Proposition 1.6, $H^1D\overline{M} = 0$. Therefore $T^* = H^0DT = 0$, so $M$ is $\varpi$-torsion free.

2. Let $N := \text{Coker}(\phi)$. Since $M'$ is $\varpi$-torsion free, it follows that $\text{Ker}(\phi) = 0$, so there is a triangle $M' \to M \to N \to M'[1]$. Apply $H^0D$ to this triangle, and use the fact that $DN \cong N^*$. ■

**Theorem 1.9.** Suppose $A$ has a balanced dualizing complex and satisfies the generalized Auslander–Gorenstein condition. Let $M$ be a Cohen–Macaulay $A$-module with $\text{CDim } M = 1$. Then there is an $A$-module $\text{Norm } M$, which is
functorial in $M$. There is a natural exact sequence of $A$-modules

$$0 \to M \to \text{Norm } M \to (M^\vee)^* \to 0. \tag{1.3}$$

If $M \to \tilde{M}$ is an isomorphism modulo $m$-torsion then $\text{Norm } M \to \text{Norm } \tilde{M}$ is an isomorphism. The module $\text{Norm } M$ is $m$-torsion free. There is a natural isomorphism $(\text{Norm } M)^* \cong \text{Norm}(M^\vee)$.

Proof. For $n \geq 0$ define $A$-modules $M'_n := M_{\geq n} \subseteq M$ and $M''_n := M/M'_n$. So $M'_n$ is a $C$-$M$ module and $M''_n$ is of finite length. The triangle

$$DM''_n \to DM \to DM'_n \to (DM''_n)[1]$$

gives an exact sequence

$$0 \to M ^\vee \to (M'_n) ^\vee \to (M''_n)^* \to 0. \tag{1.4}$$

Taking $k$-linear duals we obtain an inverse system

$$0 \to M''_n \to ((M'_n) ^\vee)^* \to (M ^\vee)^* \to 0 \tag{1.5}$$

and in the limit we get the sequence (1.3), where $\text{Norm } M := \lim_{n \to \infty} (M'_n)^*$. Clearly this construction is functorial for $A$-linear homomorphisms $\phi : M \to \tilde{M}$ between $C$-$M$ modules. If $\ker(\phi)$ and $\coker(\phi)$ are $m$-torsion then $\phi : M'_n \to M'_n$ is bijective for $n \gg 0$, so $\text{Norm}(\phi)$ is also bijective.

Next we shall prove that $N := \text{Norm } M$ is $m$-torsion free. For any integer $m$ consider the $A^e$-submodule $(M ^\vee)'_m := (M ^\vee)_{\geq m} \subseteq M ^\vee$ and the quotient $(M ^\vee)'_m := M ^\vee/(M ^\vee)'_m$. Set $M_{-m} := ((M ^\vee)'_m)^\vee$. The exact sequence

$$0 \to (M ^\vee)'_m \to M ^\vee \to (M ^\vee)'_m^\vee \to 0,$$

when dualized, gives, according to Proposition 1.8, an exact sequence

$$0 \to M \to \tilde{M}_{-m} \to ((M ^\vee)'_m)^* \to 0. \tag{1.6}$$

Since $N \cong \text{Norm } M_{-m}$ we get injections $M \to \tilde{M}_{-m} \to N$. On comparing the size of cokernels in formulas (1.3) and (1.6) we conclude that $\lim_{n \to \infty} \tilde{M}_{-m} \cong N$. Therefore $N$ is $m$-torsion free.

Now consider the $C$-$M$ $A^e$-module $M ^\vee$. In the construction of $\text{Norm}(M ^\vee)$, the sequence corresponding to (1.4) is (1.6), so

$$(\text{Norm}(M ^\vee))^* = \lim_{m \to \infty} \tilde{M}_{-m} \equiv N.$$
2. RESIDUE COMPLEXES—DEFINITIONS AND PROPERTIES

Let \( A = k \oplus A_1 \oplus A_2 \oplus \cdots \) be a Noetherian graded algebra over a field \( k \). Suppose \( \dim \) is an exact dimension function for \( A \)-modules (in the sense of [M R, Sect. 6.8]). Really we need two such functions, \( \dim_A : \text{GrMod}_A(A) \to \mathbb{N} \cup \{-\infty\} \) and \( \dim_{A^e} : \text{GrMod}_{A^e}(A^e) \to \mathbb{N} \cup \{-\infty\} \), but we will try to keep this fact invisible, when possible.

**Definition 2.1.** Let \( M \) be a left graded \( A \)-module and \( q \) an integer. Define \( G_M \) to be the sum of all finite submodules \( M' \subseteq M \) with \( \dim \ M' \leq q \). Let \( M_q \subseteq \text{GrMod}(A) \) be the subcategory whose objects are the modules \( M \) satisfying \( \Gamma^q_M = M \). For a right module \( N \) we write \( \Gamma^q_{M^e} N \subseteq N' \) and the corresponding category is \( M_q \subseteq \text{GrMod}(A^e) \).

One should think of \( \Gamma^q_M \) as the submodule of elements “supported on \( M \)”, in analogy to commutative algebraic geometry. For any module \( M \) there is a filtration

\[
0 = \Gamma^q_{M-d-1} M \subseteq \Gamma^q_{M-d} M \subseteq \cdots \subseteq \Gamma^q_{M} M = M
\]

where \( d = \dim_A \).

The subquotients are

\[
\Gamma^q_{M-d-1}M/\Gamma^q_{M-d}M := \Gamma^q_{M-d} M/\Gamma^q_{M-d-1} M.
\]

We get additive functors \( \Gamma^q_{M-q} \) and \( \Gamma^q_{M-q}/\Gamma^q_{M-q-1} \) on the category of graded left modules. If \( M \) is a bimodule then for any \( a \in A \), right multiplication by \( a \) preserves \( \Gamma^q_M \). Hence the functors \( \Gamma^q \) and \( \Gamma^q/\Gamma^q_{-1} \) send bimodules to bimodules.

**Definition 2.2.**

1. A nonzero (graded left) \( A \)-module \( M \) is said to be pure of dimension \( q \) w.r.t. \( \dim \) if \( \Gamma^q_M = M \) and \( \Gamma^q_{M-1} M = 0 \).

2. An \( A \)-module \( M \) is said to be essentially pure of dimension \( q \) if there is an essential submodule \( M' \subseteq M \) which is pure of dimension \( q \).

3. The algebra \( A \) is called pure if every essentially pure graded \( A \)-module or \( A^e \)-module is pure.

**Definition 2.3.** A strong residue complex over \( A \) w.r.t. \( \dim \) is a complex of bimodules \( R^q \) satisfying:

(i) Each bimodule \( R^q \) is a graded-injective module over \( A \) and \( A^e \).

(ii) Each bimodule \( R^q \) is pure of dimension \( -q \) over \( A \) and \( A^e \).

(iii) \( R^q \) is a balanced dualizing complex.
It is immediate to see that the complex $R'$ is bounded; in fact, $R^q \neq 0$ only for $-d \leq q \leq 0$, where $d = \min\{\dim A, \dim A'\}$.

**Theorem 2.4.** A strong residue complex is unique. Specifically, if $R'$ and $\tilde{R}'$ are two strong residue complexes, then there is an isomorphism of complexes of graded bimodules $\phi : R' \rightarrow \tilde{R}'$, and $\phi$ is unique up to a constant in $k^*$.

The proof is given after some preparatory results.

**Lemma 2.5.** The functors $\Gamma_{M_q}$ and $\Gamma_{M_q/M_{q+1}}$ have derived functors

$$R \Gamma_{M_q}, R \Gamma_{M_q/M_{q+1}} : D^+(\text{GrMod}(A^e)) \rightarrow D^+(\text{GrMod}(A^e)).$$

If $I \in D^+(\text{GrMod}(A^e))$ is a complex with each $I^p$ a graded-injective $A$-module, then $R \Gamma_{M_q} I = \Gamma_{M_q} I$ and $R \Gamma_{M_q/M_{q+1}} I = \Gamma_{M_q/M_{q+1}} I$.

**Proof.** The proof is based on that of [Ye1, Theorem 1.2], which in turn relies on [RD, Chap. I, Theorem 5.1]. Any complex $M' \in D^+(\text{GrMod}(A^e))$ is quasi-isomorphic to some complex $I'$ as above (see [Ye1, Lemma 1.1]). Thus it suffices to prove that if $I'$ is such a complex which is acyclic, then the complexes $\Gamma_{M_q} I'$ and $\Gamma_{M_q/M_{q+1}} I'$ are also acyclic.

Denote by $\delta$ the coboundary operator of $I'$. Suppose $x \in \Gamma_{M_q} I^p$, $\delta x = 0$. Let $L \subseteq A$ be the annihilator of $x$, so $\dim A/L \leq q$. Since the complex $\text{Hom}^q_A(A/L, I')$ is acyclic, there is some $y \in \Gamma_{M_q} I^{p-1}$ with $\delta y = x$. This proves the acyclicity of $\Gamma_{M_q} I'$. From the exact sequence of complexes

$$0 \rightarrow \Gamma_{M_q/M_{q+1}} I' \rightarrow \Gamma_{M_q} I' \rightarrow \Gamma_{M_q/M_{q+1}} I' \rightarrow 0$$

we see that $\Gamma_{M_q/M_{q+1}} I'$ is also acyclic.

**Lemma 2.6.** Suppose $R'$ is a strong residue complex w.r.t. dim. Then the generalized $A$-$G$ condition holds and $\dim = \text{Cdim}$ (for $A$ and $A'$).

**Proof.** If $\dim M < -q$ then $\text{Hom}_A(M, R^q) = 0$, and therefore $\text{Ext}_A^q(M, R^q) = 0$. This means that $\text{Cdim} M \leq \dim M$.

Take any surjection $\bigoplus_{i=1}^m M(n_i) \rightarrow M$ in GrMod($A$). Then the $A^e$-module $\text{Ext}_A^q(M, R^q)$ is a subquotient of $\bigoplus R^q(-n_i)$, and hence $\dim \text{Ext}_A^q(M, R^q) \leq -q$. At this point we have proved the generalized $A$-$G$ condition. Next, the convergence of the spectral sequence (1.1) implies that $\dim M \leq \max(\dim E_2^{q, -q})$. But $\dim E_2^{q, -q} \leq -p$, and $E_2^{q, -q} \neq 0$ implies $-p \leq q \leq \text{Cdim} M$.

**Proof of Theorem 2.4.** The proof is an adaptation of ideas found in [RD, Chap. IV]. First observe that by Lemma 2.6, both $R'$ and $\tilde{R}'$ are strong residue complexes w.r.t. Cdim. We define $\Gamma_{M_q}$ using this dimension function. Let $M'$ be any complex in $D^+(\text{GrMod}(A^e))$. Replace $M'$ by a
quasi-isomorphic complex $I^*$ as in Lemma 2.5. Define a decreasing filtration on $I^*$ by $F^p I^* := \Gamma_{M_{-p}} I^*$. This filtration gives the usual spectral sequence of a filtered complex, and after identifying terms we obtain

$$E^p,q = H^{p+q} (F^p I^*/F^{p+1} I^*) = H^{p+q} R\Gamma_{M_{-p}/M_{-p-1}} M^* 
\Rightarrow H^{p+q} M^*$$

(see [ML, Chap. XI, Sect. 8]). Define the (left) Cousin complex of $M^*$ to be the complex $(EM)^p := E^{p,0}$ with operator $d^{p,0} : E^{p,0} \to E^{p+1,0}$. The result is a functor $E : D^+(\text{GrMod}(A^e)) \to \text{C(GrMod}(A^e))$, where the latter is the (abelian) category of complexes of graded bimodules.

If $R^q$ is a strong residue complex, then $\Gamma_{M_{-p}/M_{-p-1}} R^q = R^q$ if $q = p$ and 0 otherwise. Therefore $ER^q \cong R^q$ as complexes.

Now according to [Ye1, Sect. 4], balanced dualizing complexes are unique up to isomorphism in $D^+(\text{GrMod}(A^e))$. Choose such an isomorphism $\psi : R^q \to R^q$, which is known to be unique up to a constant. Then $\phi = E(\psi) : R^q \to R^q$ is the desired isomorphism.

The next proposition is a generalization of [Aj3, Theorem 3.14].

**Proposition 2.7.** If $A$ has a strong residue complex then it is a pure algebra.

**Proof.** Let $M$ be a finite $A$-module and $M' \subseteq M$ an essential submodule, pure of dimension $q$. It will suffice to produce an injection $M' \to (R^{-q})^i$ for some $i$. Suppose $N \subseteq M'$ is critical. By the generalized A-G condition there is a nonzero homomorphism $\phi : N \to R^{-q}$, which by purity must be injective. Since every nonzero $A$-module has a critical submodule (cf. Corollary 1.5) it follows that there is an essential submodule $N_1 \oplus \cdots \oplus N_i \subseteq M'$ with all $N_i$ critical. Choose injective homomorphisms $\phi_i : N_i \to R^{-q}$ and let $\psi : M' \to (R^{-q})^i$ be any extension of $\bigoplus \phi_i$. Then $\psi$ is necessarily injective.

When we can associate with $A$ a sufficiently rich geometry, e.g., when the projective spectrum $\text{Proj} A$ is a classical projective scheme (in the terminology of [AZ]), one would expect that $A$ would have a strong residue complex. The propositions below justify this expectation. First consider the twisted homogeneous coordinate ring of a triple $(X, \sigma, \mathcal{L})$, where $X$ is a proper scheme, $\sigma$ is an automorphism, and $\mathcal{L}$ is a $\sigma$-ample invertible sheaf (cf. [AV]).

**Proposition 2.8.** Suppose $A$ is a twisted homogeneous coordinate ring. Then $A$ has a strong residue complex, w.r.t. to $\text{dim} = \text{Kdim (Krull dimension)}$.

**Proof.** A balanced dualizing complex $R^*$ exists by [Ye1, Theorem 7.3]. It is the cone over the natural homomorphism of complexes $\Gamma_{\mathcal{L}} A_X \to A^e$.
arising from Grothendieck duality. Here $\mathcal{R}_X$ is the residue complex of $X$. For each $q$, $R^q$ is a graded-injective module over $A$ and $A^\circ$.

Since $R^0 \equiv A^\circ$, it has $K \dim = 0$. For $q < 0$ we have $R^q \cong \Gamma_* \mathcal{R}_X^{q+1}$. Because of the equivalence of categories between $\text{GrMod}(A)$ modulo $\text{m}$-torsion and quasi-coherent $\mathcal{O}_X$-modules, it follows that for any nonzero coherent sheaf $\mathcal{M}$, $K \dim \Gamma_* \mathcal{M} = \dim \text{Supp} \mathcal{M} + 1$. It is known that the quasi-coherent sheaf $\mathcal{R}_X^{q+1}$ is pure of dimension $-q - 1$ (by this we mean that each nonzero coherent subsheaf $\mathcal{M} \subset \mathcal{R}_X^{q+1}$ has $\dim \text{Supp} \mathcal{M} = -q - 1$). Hence $R^q_A$ is pure of $K \dim = -q$. All this works for right modules too.

**Remark 2.9.** One can show that if some positive power of $L^e$ over $L$ is in the identity component $\text{Pic}^0 X$ of the Picard scheme of $X$, then for each graded $A$-module $M$ one has the equality $G \dim M = K \dim M$. On the other hand, in [AV, Example 5.18] we see a twisted homogeneous coordinate ring $A$ with $G \dim A = 5$ and $K \dim A = 3$.

The decomposition $\mathcal{R}_X = \bigoplus_{x \in X} \mathcal{R}_X(x)$ (cf. formula (0.1)) induces a bimodule decomposition $R^r = (\bigoplus_T R(T)) \otimes A^\ast$, where $T$ runs through the $\sigma$-orbits in $X$ and $R(T) := \bigoplus_{x \in T} \Gamma_* \mathcal{R}_X(x)$. It is known that $\mathcal{R}_X(x)$ is an indecomposable injective in $\text{QCoh}(X)$, so $\Gamma_* \mathcal{R}_X(x)$ is indecomposable in $\text{GrMod}(A)$.

**Problem 2.10.** Is $R(T)$ an indecomposable bimodule?

The second general situation to consider is an algebra finite over its center.

**Proposition 2.11.** If $A$ is finite over its center then it has a strong residue complex, w.r.t. $\dim = K \dim = G \dim$.

**Proof.** There is a finite centralizing homomorphism $C \to A$, where $C = k[t_1, \ldots, t_d]$ is a (commutative) polynomial ring, and the variables $t_i$ all have degree $e \geq 1$. The algebra $C$ has a residue complex $R^r_C$. If $e = 1$ use Prop. 2.8 with $X = P^{d-1}$; if $e > 1$ simply take the same complex as for $e = 1$ and change the grading. Let $R^r_A := \text{Hom}^C(A, R^r_C)$. According to [Ye1, Theorem 5.4] this is a balanced dualizing complex over $A$. Each $R^r_A$ is graded-injective on both sides. Since as a $C$-module $R^r_A$ embeds into a finite direct sum of twists of $R^r_C$, it is pure of $G \dim$ dimension $-q$.

Here again commutative geometry says there is a bimodule decomposition $R^r = \bigoplus \nu, R(\nu)$, where $\nu$ runs over the graded primes of the center of $A$.

**Problem 2.12.** Is $R(\nu)$ an indecomposable bimodule?

**Remark 2.13.** Let $A_q$ be the multiparameter quantum deformation of the polynomial ring $A = k[t_1, \ldots, t_d]$, depending on a $d \times d$ matrix $q = ...
We do not know whether, for all $q$, $A_q$ admits a strong residue complex. The problem is that localization destroys the $\mathbb{Z}^d$-grading which is used to deform $A$-modules into $A_q$-modules, so the residue complex $R_A$ cannot be deformed.

In Section 3 we shall prove that a three-dimensional Sklyanin algebra has a strong residue complex. Recent work of Ajitabh et al. [ASZ] shows that some four-dimensional Artin–Schelter regular algebras do not admit strong residue complexes. They actually find an algebra $A$ such that in the minimal graded-injective resolution $0 \to A \to I^{-4} \to I^{-3} \to \cdots$, each $I^q$ is essentially pure of dimension $-q$ (w.r.t. Cdim = GKdim), but $I^{-1}$ is not pure. Influenced by this result we make the next definition, even though we have no example (so far) of an algebra with a weak residue complex but no strong residue complex.

**Definition 2.14.** A weak residue complex w.r.t. dim is a complex of bimodules $R^q$ satisfying:

(i) Each bimodule $R^q$ is a graded-injective module over $A$ and $A^o$.

(ii) Each bimodule $R^q$ is essentially pure of dimension $-q$ over $A$ and $A^o$, and there is equality $\Gamma_{M^q} R = \Gamma_{M^o} R \subseteq R$.

(iii) $R^q$ is a balanced dualizing complex.

Let $J^q$ be a complex of graded-injective $A$-modules. We say $J^q$ is a minimal injective complex if for every $q$, $\text{Ker} (\delta : J^q \to J^{q+1}) \subseteq J^q$ is an essential submodule. Any complex $M^q \in D^-(\text{GrMod}(A))$ admits a quasi-isomorphism to a minimal injective complex $J^q$; and one can easily check that this $J^q$ is unique up to isomorphism (cf. [Ye1, Lemma 4.2]). Observe that minimality has nothing to do with a dimension function, nor is it functorial.

**Lemma 2.15.** Suppose $J^q$ is a complex of graded-injective $A$-modules with $J^q$ essentially pure of dimension $-q$. Then $J^q$ is minimal.

**Proof.** Pick an integer $q$. Let $M := \text{Ker} (\delta : J^q \to J^{q+1})$ and let $I$ be a graded-injective hull of $M$. So $J^q \cong I \oplus I'$ and $\delta : I' \to J^{q+1}$ is an injection. By the purity assumption we get $I' = 0$.

We conclude:

**Proposition 2.16.** If $R^q$ and $\tilde{R}^q$ are weak residue complexes, then they are isomorphic as complexes of $A$-modules and as complexes of $A^o$-modules. In particular, if one is a strong residue complex then so is the other.

**Problem 2.17.** Is it possible for an algebra $A$ to admit two weak residue complexes $R^q$ and $\tilde{R}^q$ which are not isomorphic as complexes of graded bimodules? (Of course $A$ cannot be pure.)
At this point we wish to relate residue complexes to the generalized A-G condition.

**Theorem 2.18.** Let \( \dim \) be an exact dimension function for \( A \). Suppose that either condition holds:

(i) \( A \) admits a strong residue complex.

(ii) \( A \) admits a weak residue complex, and every finite left or right graded \( A \)-module has a \( \dim \) critical composition series.

Then \( A \) satisfies the generalized A-G condition, and \( \dim = \text{Cdim} \).

**Lemma 2.19.** Say \( R \) is the residue complex in condition (ii) of the theorem. Let \( M \) be a critical finite module with \( \dim M < d \), and a homomorphism \( \overline{M} \rightarrow M \) is surjective.

**Proof.** Write \( E(M) = \text{Ext}^3_{\mathcal{A}}(M, R) \). Say \( \phi \in E(M) \) is represented by \( \phi : M \rightarrow R^d \). Because \( M \) is critical (and therefore pure of dimension \( d \)) and \( R^d \) is essentially pure of dimension \( -q \), \( \phi \) cannot be injective. So \( \overline{M}_\phi := \text{Im}(\phi) \) has \( \dim \overline{M}_\phi < d \) and \( \{ \phi \} \in \text{Im}(\overline{M}_\phi \rightarrow E(M)) \). Now choose \( \{ \phi_1 \}, \ldots, \{ \phi_m \} \) which generate \( E(M) \) over \( \mathcal{A} \). Then \( \overline{M} := \bigoplus \overline{M}_{\phi_i} \) has the required properties.

**Lemma 2.20.** Let \( M \) be a finite \( A \)-module. Assume \( \dim M = d \). Then in the situation of condition (ii) of the theorem:

1. \( \dim \text{Ext}^3_{\mathcal{A}}(M, R') \leq d \) for all \( q \).
2. \( \dim \text{Ext}^3_{\mathcal{A}}(M, R') < d \) for all \( q > -d \).
3. \( \text{Ext}^3_{\mathcal{A}}(M, R') = 0 \) for all \( q < -d \).

**Proof.** Say \( \bigoplus_{i=1}^n A(n_i) \rightarrow M \) is a presentation of \( M \). Then

\[
\text{Hom}_A(M, R^q) \subseteq \text{Im}_A \left( \bigoplus R^q(-n_i) \right) = \Gamma_{M, \mathcal{A}} \left( \bigoplus R^q(-n_i) \right).
\]

Since \( \text{Ext}^3_{\mathcal{A}}(M, R') \) is a subquotient of \( \text{Hom}_A(M, R^q) \) this implies part 1. If moreover \( d < -q \) then \( \Gamma_{M, \mathcal{A}} R^q = 0 \), giving part 3.

Let us prove part 2. We may assume \( M \) is critical. Then the assertion is a consequence of Lemma 2.19 and part 1 applied to \( \overline{M} \).

Note that the two lemmas work also for right modules (exchange \( \mathcal{A} \) and \( \mathcal{A}^\circ \)).

**Proof of Theorem 2.18.** We need only consider condition (ii) of the theorem (cf. Lemma 2.6). Say \( \dim M = d \). By part 3 of Lemma 2.20 we
have $\text{Cdim } M \leq d$. Suppose $\text{Cdim } M < d$. Then by parts 1 and 2 of the lemma all the terms in the spectral sequence (1.1) have $\dim < d$, which is impossible. The conclusion is $\text{Cdim } M = \dim M$.

To prove the generalized A-G condition it suffices to check that $\dim \text{Ext}_A^d(M, R') \leq -q$. We will do so by induction on $d = \dim M$. For $d \leq -q$ this is part 1 of Lemma 2.20. For $d > -q$ and $M$ critical, the module $\hat{M}$ of Lemma 2.19 has $\dim \hat{M} < d$ so we can use induction. For other modules this is true by looking at a critical composition series.

**Problem 2.21.** Is it true that every algebra which satisfies the generalized A-G condition admits a weak residue complex? It was proved in [Le] and [TV] that Sklyanin algebras of all dimensions satisfy the A-G condition, yet it is not known even whether every four-dimensional Sklyanin algebra admits a weak residue complex.

Let us finish this section with the Cohen–Macaulay case.

**Corollary 2.22.** Assume the hypotheses of Theorem 2.18. Furthermore, assume $R' \cong \omega[d]$ in $\text{D}(\text{GrMod}(A^e))$ for some bimodule $\omega$ and some integer $d$. Then $d = \text{Cdim } A$, and

$$0 \to \omega \to R^{-d} \to \cdots \to R^0 \to 0$$

(2.2)

is a minimal graded-injective resolution of $\omega$, both as left and right module.

**Proof.** The isomorphism $R\text{Hom}_A^d(A, R') = R' \cong \omega[d]$ means that $A$ is a C-M $A$-module with $\text{Cdim } A = d$ and dual module $A^\vee = \omega$. Hence $R^{-d-1} = 0$ and we deduce the exact sequence (2.2). By Lemma 2.19 it is a minimal resolution. 

**3. The Residue Complex of a Three-Dimensional Sklyanin Algebra**

In this section $k$ is an algebraically closed field. We assume $A$ is a three-dimensional Sklyanin algebra (see [ST]), which is the same as a type A three-dimensional regular algebra with three generators (in the classification of [ATV1]). The triple $(E, \sigma, \mathcal{L})$ consists of a smooth elliptic curve $E \subseteq \mathbb{P}^2_k$, an invertible sheaf $\mathcal{L} = \mathcal{O}_E(1)$, and a translation $\sigma$ by some point of $E(k)$. We shall prove that $A$ is localizable at any $\sigma$-orbit $T \subseteq E(k)$. Such a result was obtained in [Aj2] for twisted homogeneous coordinate rings of $\mathbb{P}^1_k$ by another method.

Let $B$ be the twisted homogeneous coordinate ring of the triple $(E, \sigma, \mathcal{L})$. Then $B \cong A/(g)$ where $g$ is a central element of $A$ of degree
3. An $\mathcal{O}_E$-module $\mathcal{M}$ defines a left graded $B$-module

$$
\Gamma_* \mathcal{M} := \bigoplus_{n \in \mathbb{Z}} \Gamma(E, \mathcal{O}^{(1-\alpha^n)/(1-\alpha)} \otimes \mathcal{M}^{*^n}),
$$

where the exponents are in the integral group ring $\mathbb{Z}(\alpha)$ and $\mathcal{M}^{*^n} := \alpha^n \mathcal{M}$.

If $\mathcal{M}$ is equivariant w.r.t. $\alpha$ then $\Gamma_* \mathcal{M}$ is actually a $B$-$B$-bimodule, and if $\mathcal{A}$ is an equivariant $\mathcal{O}_E$-algebra, then $\Gamma_* \mathcal{A}$ is a graded $k$-algebra with an algebra homomorphism $B \to \Gamma_* \mathcal{A}$ (cf. [AV] and [Ye1]).

Given a point $p \in E(k)$ let $\mathcal{J}(p) := k(E) / \mathcal{O}_{E,p}$, considered as a quasi-coherent sheaf. So $\mathcal{J}(p)$ is an injective hull of the residue field $k(p)$, and there is an exact sequence

$$
0 \to \mathcal{O}_E \to k(E) \to \bigoplus_{p \in E(k)} \mathcal{J}(p) \to 0. \quad (3.1)
$$

Let $B_E := \Gamma_* k(E)$, a graded $k$-algebra, and $I_B(p) := \Gamma_* \mathcal{J}(p)$, a graded left $B$-module. Recall that the point module $N_p$ is the module $(\Gamma_* k(p))_{\geq 0}$.

**Lemma 3.1.** $B_E \cong \text{Frac}^g B$, the graded total ring of fractions. $I_B(p)$ is a left graded-injective hull of $N_p$. Applying $\Gamma_*$ to the sequence (3.1) we get an exact sequence of graded left $B$-modules

$$
0 \to B \to B_E \to \bigoplus_{p \in E(k)} I_B(p).
$$

It is the beginning of a minimal graded-injective resolution, and the only missing term is $B^* = \text{Hom}^g(B, k)$.

**Proof.** By [Ye1, Theorem 7.3], plus the fact that $\mathcal{O}_E \equiv \omega_E$ (noncanonically).

Fix a $\alpha$-orbit $T \subseteq E(k)$. Then $\bigoplus_{p \in T} k(p)$ is an equivariant sheaf, and hence $\bigoplus_{p \in T} N_p$ is a $B$-$B$-bimodule. Let

$$
I_B(T) := \bigoplus_{p \in T} I_B(p) \equiv \Gamma_* \left( \bigoplus_{p \in T} \mathcal{J}(p) \right).
$$

This too is a bimodule, and is also a graded-injective hull of $\bigoplus_{p \in T} N_p$ on both sides. Define the $\mathcal{O}_E$-subalgebra $\mathcal{O}_{E,T} \subseteq k(E)$ by

$$
\Gamma(U, \mathcal{O}_{E,T}) := \bigcap_{p \in T \cap U} \mathcal{O}_{E,p}
$$

for $U \subseteq E$ open. Then we get a $\alpha$-equivariant exact sequence

$$
0 \to \mathcal{O}_{E,T} \to k(E) \to \bigoplus_{p \in T} \mathcal{J}(p) \to 0, \quad (3.2)
$$
from which we see that $\mathcal{O}_{E,T}$ is a $\sigma$-equivariant quasi-coherent sheaf. Let $B_T := \Gamma_s \mathcal{O}_{E,T}$, a graded subalgebra of $B_E$. Define a multiplicative set

$$S_T := B \cap \{\text{homogeneous units of } B_T\}. \quad (3.3)$$

**Proposition 3.2.** The sequence of $B_T$-$B_T$-bimodules

$$0 \to B_T \to B_E \overset{\delta_{E,T}}{\to} I_B(T) \to 0, \quad (3.4)$$

given by applying $\Gamma_s$ to (3.2), is exact. $S_T$ is a left and right denominator set in $B$, and $B_T = S_T^{-1}B = BS_T^{-1}$.

To prove the proposition we first need two lemmas.

**Lemma 3.3.** Given $p \in E(k) - T$ there is some $b \in B_1$, s.t. $b(p) = 0$ but $b(q) \neq 0$ for every $q \in T$.

**Proof.** Say $\sigma$ is translation by $y \in E(k)$ and $T = q_0 + \langle r \rangle$ in the group structure of $E(k)$. Given any nonzero $b \in B_1 = \Gamma(E, \mathcal{L})$ (which is the same as a line $\{b = 0\}$ in $\mathbb{P}^1$) its divisor of zeroes is $\{p_1, p_2, p_3\}$, and these points satisfy $p_1 + p_2 + p_3 = 0$. Consider a line through $p_1 = p$; then $p_2$ is in the $\sigma$-orbit $T' := -p - q_0 + \langle r \rangle$. Now $E(k)$ being a divisible group, the cyclic subgroup $\langle r \rangle$ has infinite index. Hence in $E(k)$ there are infinitely many $\sigma$-orbits, and so there are infinitely many lines through $p$ which do not intersect $T$ at all.

**Lemma 3.4.** Consider the left $B$-module $BS_T^{-1} \subseteq B_T$. Then $BS_T^{-1} = \lim_{\rightarrow} B_s^{-1}$, the limit over $s \in S_T$.

**Proof.** We have to prove that given $s_1, s_2 \in S_T$ there is some $s \in S_T$ s.t. $B_s^{-1} = B_1^{-1}B_2^{-1}$ for any nonzero $s \in B$ let $\mathcal{R}(s) \subseteq \mathcal{O}_{E,T}$ be the sheaf associated to the free module $BS_s^{-1}$; so $B_s^{-1} \equiv \Gamma_s \mathcal{R}(s)$. It therefore suffices to prove that for some $s$, $\mathcal{R}(s_1) \cap \mathcal{R}(s_2) \subseteq \mathcal{R}(s)$.

Now $\mathcal{R}(s) = \mathcal{O}_{E}(D)$ for some effective divisors $D$, supported on $E - T$. Let $D := D_1 + D_2$, so $\mathcal{R}(s) \subseteq \mathcal{O}_{E}(D)$. Say $D = \sum_{i=1}^n p_i$ (with repetition). By Lemma 3.3 we can find $b_i \in B_1$ s.t. $b_i(\sigma^{-1}(p_i)) = 0$ but for all $q \in T$, $b_i(q) \neq 0$. Then taking $s := b_1 \cdots b_n \in S_T$ we get $\mathcal{O}_{E}(D) \subseteq \mathcal{R}(s)$.

**Proof of Proposition 3.2.** First observe that (3.2) is a $\sigma$-equivariant sequence of $\mathcal{O}_{E,T}$-modules, so (3.4) is a sequence of graded $B_T$-$B_T$-bimodules.

Choose any affine open set $U \subseteq E$ containing $T$. This is possible since $T \neq E(k)$ (cf. Lemma 3.3) and we can take $U = E - \{p'\}$ for some $p' \notin T$. Since $H^1(U, \mathcal{O}_E) = 0$, it follows that $k(E) \to \Gamma(U, \bigoplus_{p \in E(k)} \mathcal{R}(p))$ is surjective. But $I_B(T)$ is a direct summand of $\Gamma(U, \bigoplus_{p \in E(k)} \mathcal{R}(p))$. This proves that $\delta_{E,T}$ is surjective in degree 0. For other degrees just twist everything.
To prove the second assertion it suffices, by Lemma 3.4, to prove that \( B_T = BS_T^{-1} \) (cf. [M R, Chap. 3.1]). Now \( BS_T^{-1} \) is a graded left \( B \)-module.

Let \( \mathcal{R} \) be the sheaf on \( E \) associated to \( BS_T^{-1} \), so \( \mathcal{O}_E \subseteq \mathcal{R} \subseteq \mathcal{O}_{E,T} \). If \( p \in T \) then the stalk \( \mathcal{O}_{E,p} = (\mathcal{O}_{E,T})_p \), so a fortiori \( \mathcal{R}_p = (\mathcal{O}_{E,T})_p \). If \( p \notin T \) then by Lemma 3.3 we may find some \( a_i, b_i \in B_1 = \Gamma(E, \mathcal{R}) \), \( i \geq 1 \), s.t. \( a_i(\sigma^{-1}(p)) \neq 0 \), \( b_i(\sigma^{-1}(p)) = 0 \) and for all \( q \in T \), \( b_i(q) \neq 0 \). Since \( b_i \in S_T \) we get

\[
c_n = a_1 \cdots a_n b_n^{-1} \cdots b_1^{-1} \in (BS_T^{-1})_0 \subseteq \Gamma(E, \mathcal{R}),
\]

so \( c_n \in \mathcal{R}_p \subseteq k(E) \) has a pole of order at least \( n \). This implies that \( \mathcal{R}_p = k(E) \). Since \( BS_T^{-1} = \lim \to \limits_{s \to 1} B_s^{-1} \) and \( BS_T^{-1} \approx \Gamma_{\mathcal{R}}(s) \) it follows that \( BS_T^{-1} \approx \Gamma_\mathcal{R} \mathcal{R} = B_T \).

Finally, by the left-right symmetry of \( \Gamma_\mathcal{R} \) for equivariant sheaves (cf. [Ye1, Prop. 6.17]) we also get \( S_T^{-1}B = B_T \).

Define

\[
\tilde{S}_T := \{ s \in A \mid s \text{ is homogeneous and } s + (g) \in S_T \subseteq B \}
\]

which is clearly a multiplicative set. Let \( Q := \text{Frac}_A \), the graded total ring of fractions.

**Proposition 3.5.** \( \tilde{S}_T \) is a left and right denominator set, with ring of fractions \( \Lambda := \text{AS}_{\tilde{S}_T} \subseteq Q \).

**Proof.** Copy the proof of [A1, Chap. III, Prop. 3.6].

From here to Corollary 3.14 we will assume the automorphism \( \sigma \) has infinite order.

Consider a minimal graded-injective resolution of \( A \) as a left module:

\[
0 \to A \to I^{-3} \xrightarrow{\delta} I^{-2} \xrightarrow{\delta} I^{-1} \xrightarrow{\delta} I^0 \to 0 \quad (3.5)
\]

(with unusual numbering). Inside \( Q \) there are the two subrings \( \Lambda := A[g^{-1}] \) and

\[
A_E = A_{(g)} := \{ as^{-1} \mid s \text{ is homogeneous, } s \notin (g) \}
\]

(cf. [A3]). Define an \( A_E \)-\( A_E \)-bimodule \( I_A(E) := Q/A_E \). For every \( p \in E(k) \) let \( I_p(E) \) be a graded-injective hull of \( N_p \) (as an \( A \)-module). In [A3] Ajitabh proves the following:

**Theorem 3.6.** The left \( A \)-module \( I^{-q} \) is pure of GK dimensions \( q \). Moreover,

\[
I^{-3} \cong Q
\]
\[ I^{-2} \equiv \frac{Q}{\Lambda} \oplus I_A(E) \]

\[ I^{-1} \equiv \bigoplus_{p \in E(k)} I_A(p) \]

\[ I^0 \equiv A^*(3) \]

as graded left \(A\)-modules. The homomorphism \(\delta: I^{-2} \to I^{-1}\) is the sum of the two projections \(Q \to Q/\Lambda\) and \(Q \to I_A(E)\).

The algebra \(Q\) is filtered by the “fractional ideals” \(A_{E,g}^n, n \in \mathbb{Z}\) (the \("(g)\)-adic valuation") and we denote by \(\text{gr}^{(g)} Q\) the resulting graded algebra. Note that this algebra carries two gradings. Since \(g\) is a central regular element, we see that \(\text{gr}^{(g)} Q = B_E[\bar{g}, g^{-1}]\), where \(\bar{g}\) is the symbol of \(g\), and this algebra is isomorphic to a Laurent polynomial algebra over \(B_E\) in the central indeterminate \(\bar{g}\). Similarly, we have \(\text{gr}^{(g)} A = B[\bar{g}]\).

Suppose \(M\) is a \((g)\)-torsion left \(A\)-module. Then we write

\[ M_{-n} := \text{Hom}_A(A/(g^{n+1}), M) \subseteq M. \]

This defines a decreasing exhaustive filtration on \(M\), with \(M_1 = 0\). Denote by \(\text{gr}^{(g)} M\) the associated graded module.

**Lemma 3.7.** The left \(A\)-modules \(I_A(E), I^{-1},\) and \(I^0\) are \((g)\)-torsion. The modules \(\text{gr}^{(g)} I_A(E), \text{gr}^{(g)} I^{-1},\) and \(\text{gr}^{(g)} I^0\) are \(B[\bar{g}^{-1}]\)-modules; in fact, writing \(M\) for either of these modules we get a bijection

\[ B[\bar{g}^{-1}] \otimes_0 M_0 \cong \text{gr}^{(g)} M. \]

**Proof.** According to Theorem 3.6, \(I^{-1}\) has GK dimension 1. Since no power of \(\sigma\) fixes the class of \([\mathcal{L}]\) in \(\text{Pic } E\) it follows that \(I^{-1}\) is \((g)\)-torsion (see [ATV2, Prop. 7.8]). The other two modules are trivially \((g)\)-torsion. Almost by definition multiplication by \(\bar{g}\) is injective on \(\text{gr}^{(g)} M, n > 0\). Since \(M\) is a graded-injective \(A\)-module it is \(g\)-divisible, and so \(\text{gr}^{(g)} M\) is uniquely \(\bar{g}\)-divisible.

The class \(g^{-1}\) of \(g^{-1}\) in \(I_A(E) = Q/A_E\) is killed by \((g)\). Thus it induces a degree 0 \(B\)-module homomorphism \(B(3) \xrightarrow{\sigma^{-1}} I_A(E)_0\).

**Lemma 3.8.** 1. The sequence

\[ 0 \to B(3) \xrightarrow{\sigma^{-1}} I_A(E)_0 \xrightarrow{\delta} I^{-1}_0 \xrightarrow{\delta} I^0_0 \to 0 \]

is a minimal left graded-injective resolution of \(B(3)\) as a \(B\)-module.
2. The sequence

\[ 0 \to B\left[\sigma^{-1}\right](3) \to \text{gr}^{(\sigma)} I_{A}(E) \to \text{gr}^{(\sigma)} I_{A}(E) \to \text{gr}^{(\sigma)} I_{A}(E) \to 0 \]

of \(B[\sigma^{-1}]\)-modules is exact.

Proof. Since \(Q/\Lambda\) has no \((g)\)-torsion, it follows that

\[ \text{Hom}_{A}(B, I) = \left( 0 \to I_{A}(E) \to I_{A}(E) \to 0 \right). \]

But \(\text{Ext}^{3}_{B}(A, I) = 0\), unless \(q = 1\), in which case it is isomorphic to \(B(3)\).

Hence the sequence is exact. Clearly each \(\text{Hom}_{A}(B, I)\) is a graded-injective \(B\)-module. By Theorem 3.6 and Lemma 2.20 we see that the resolution is minimal.

2. Use Lemma 3.7.

Now fix an \(s\)-orbit \(T \subseteq E(k)\). Set

\[ I_{A}(T) := \bigoplus_{p \in T} I_{A}(p) \subseteq I^{-1} \]

and let \(\delta_{E, T} : I_{A}(E) \to I_{A}(T)\) be the homomorphism

\[ \delta_{E, T} : I_{A}(E) \to I^{-1} \xrightarrow{\delta} I^{-1} \to I_{A}(T). \]

PROPOSITION 3.9. \(\delta_{E, T}\) is surjective.

Proof. We shall prove by induction on \(n\) that \((\delta_{E, T})_{-n} : I_{A}(E)_{-n} \to I_{A}(T)_{-n}\) is surjective. For \(n = 0\) this is done in Prop. 3.2 (in view of Lemma 3.8 part 1, and Lemma 3.1). Therefore by Lemma 3.8, part 2, \(\text{gr}^{(\sigma)}(\delta_{E, T})\) is surjective. Now suppose \(x \in I_{A}(T)_{-(n+1)} - I_{A}(T)_{-n}\) with symbol \([x] \in \text{gr}^{(\sigma)}(I_{A}(T)_{-(n+1)})\). Then \([x] = \text{gr}^{(\sigma)}(\delta_{E, T})(y)\) for some \(y \in I_{A}(E)_{-(n+1)}\). But then \(x - \delta_{E, T}(y) \in I_{A}(T)_{-n}\), and we can use the induction hypothesis.

LEMMA 3.10. \(I_{A}(T)\) is a left \(A_{T}\)-module, and \(\delta_{E, T} : I_{A}(E) \to I_{A}(T)\) is \(A_{T}\)-linear.

Proof. It suffices to prove that for all \(n \geq 0\), \(A_{T} \otimes_{A} I_{A}(T)_{-n} \cong I_{A}(T)_{-n}\). For \(n = 0\) this is done in Proposition 3.2 \((B_{T} : T_{A} \to B_{T})\). To prove the claim for \(n > 0\) it is enough to show that every \(s \in S_{T}\) acts invertibly on \(I_{A}(T)_{-n}\). Look at the commu-
Now by induction the two extreme vertical arrows are bijective. Therefore so is the middle one. 

**Proposition 3.11.** The kernel of $\delta_{E,T}$ is

$$\bigcup_{n \geq 1} A_T g^{-n} = \frac{A_T [g^{-1}] + A_E}{A_E} \subseteq \frac{Q}{A_E} = I_A(E).$$

In particular, it is a sub-$A_T - A_T$-bimodule of $I_A(E)$.

**Proof.** We shall prove by induction on $n$ that $\ker((\delta_{E,T})_{-n}) : I_A(E)_{-n} \to I_A(T)_{-n}$ is the submodule $A_T g^{-n}$. For $n = 0$ this is done in Proposition 3.2. Now for any $n$, $\delta_{E,T}(g^{-n}) = 0$, since we can start with $g^{-n} \in Q = I^{-3}$, and then corresponding to the sequence

$$Q \xrightarrow{\delta} \frac{Q}{A} \oplus I_A(E) \xrightarrow{\delta} I^{-1} = \bigoplus_{T'} I_A(T')$$

(sum on all orbits $T'$) we get

$$g^{-n} \mapsto (g^{-n}, g^{-n}) \mapsto (0, g^{-n}) \mapsto 0 = \sum_{T'} \delta_{E,T}(g^{-n}).$$

By Lemma 3.10, $\ker(\delta_{E,T})$ is an $A_T$-module, so $A_T g^{-n} \subseteq \ker((\delta_{E,T})_{-n})$.

Now let $x \in I_A(E)_{-(n+1)} - I_A(E)_{-n}$, $\delta_{E,T}(x) = 0$. Since $\ker(\text{gr}(\delta_{E,T})) = B_T[g^{-1}]$ (cf. Lemma 3.7), there is some $a \in A_T$ st. $x - ag^{-(n+1)} \in I_A(E)_{-n}$ and $\delta_{E,T}(x - ag^{-(n+1)}) = 0$. Here we can use induction. 

**Remark 3.12.** Observe the similarity to the proof of [Ye2, Theorem 4.3.13], the main step in constructing the residue complex on a scheme. In both instances the surjection from a generic component to a special component is used to parametrize the special component.

**Theorem 3.13.** Suppose $A$ is a three-dimensional Sklyanin algebra over an algebraically closed field, and the automorphism $\sigma$ has infinite order. Then there is an exact sequence of graded $A - A$-bimodules

$$0 \to A \to I^{-3} \xrightarrow{\delta} I^{-2} \xrightarrow{\delta} I^{-1} \xrightarrow{\delta} I^0 \to 0$$
which is a minimal graded-injective resolution of $A$, both as a left and right module. Moreover, $I^{-q}$ is pure of GK dimension $q$, both as left and right module.

**Proof.** By Theorem 3.6, $I^{-3}$, $I^{-2}$ are bimodules, which are graded-injective modules on both sides, and $\delta : I^{-3} \to I^{-2}$ is a bimodule map. According to Prop. 3.11, for every orbit $T$ the kernel $\text{Ker}(\delta_{E,T}) \subseteq I_A(E)$ is a sub-bimodule. Furthermore this same kernel occurs in the minimal right resolution of $A$. Since $\delta_{E,T}$ is surjective (Prop. 3.9) this endows $I_A(T)$ with a bimodule structure. Now $I^{-1} = \bigoplus_I I_A(T)$. We conclude that $0 \to A \to I^{-3} \to I^{-2} \to I^{-1}$ is a bimodule complex which is at the same time the beginning of a left and the beginning of a right minimal resolution. Since the sequence (3.5) is exact we see that $\text{Coker}(\delta : I^{-2} \to I^{-1}) \cong I^0$. This puts a bimodule structure on $I^0$, and necessarily $I^0 \cong A^*(3)$ as right modules. 

Finally, we have

**Corollary 3.14.** Let $A$ be a three-dimensional Sklyanin algebra. Then $A$ has a strong residue complex w.r.t. GKdim.

**Proof.** If $\sigma$ has finite order then $A$ is finite over its center, so we can apply Proposition 2.11. Otherwise let $\omega$ be the dualizing bimodule of $A$, namely the bimodule s.t. $\omega[3]$ is a balanced dualizing complex. (Actually in this special case $\omega$ is just $A(-3)$.) Taking the complex $I$ of the theorem we see that $R : \omega \otimes_A I^*$ is a strong residue complex.

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