Some Constructions over Graded Rings: Applications

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Communicated by Kent R. Fuller

March 27, 1987

INTRODUCTION

Let $G$ be a group with identity element "1", $R = \bigoplus_{\sigma \in G} R_{\sigma}$ a $G$-graded ring. If we fix $\sigma \in G$, we can consider the exact functor $(-)_{\sigma} : R_{-}\text{-mod} \rightarrow R_{\sigma}\text{-mod}$ given by $A \mapsto M_{\sigma}$, where $M = \bigoplus_{\lambda \in G} M_{\lambda}$ is a graded left $R$-module. To this functor, we can associate the localizing subcategory $\mathcal{C}_{\sigma}$ of $R$-gr:

$\mathcal{C}_{\sigma} = \left\{ M = \bigoplus_{\lambda \in G} M_{\lambda} \text{ such that } M_{\sigma} = 0 \right\}$.

A well-known result of Dade [7; Theorem 2.8] states that $R$ is a strongly graded ring $\Leftrightarrow$ the functor $(-)_{\sigma}$ is faithful $\Leftrightarrow$ $\mathcal{C}_{\sigma} = \{0\}$, $\forall \sigma \in G$.

After presenting in Section 0 some definitions and results about modules over graded rings, we construct in Section 1 the induced functor Ind: $R_{1}\text{-mod} \rightarrow R_{-}\text{-gr}$ and the coinduced functor Coind: $R_{1}\text{-mod} \rightarrow R_{-}\text{-gr}$. The main result of this section is Theorem 1.1, where properties of these functors are given; we focus our attention on the study of the functor Coind. The Coind functor is then used for the construction of the gr-injective envelope of a graded module. We remark that the construction of the Coind functor appears in the proof of Theorem 2.1 of [14] and also in [11], where it is used for the study of Morita duality of graded rings.

In Section 2, the graded rings of finite support are studied. The main results of this section are Theorem 2.1, where the gr-injective modules are studied (some consequences are also given), and Theorem 2.2 (a theorem of incomparability for prime ideals), which extends Theorem 3.3 of [6].

In Section 3, Dade's result (Theorem 3.1) is extended using the notion of quotient category and the localizing subcategories $\mathcal{C}_{\sigma}$. Some applications of Theorem 3.1 are then given (Corollaries 3.1, 3.2, 3.3), extending some results of Menini and Năstăescu [12]. Finally, Theorem 3.2 is a generalization of Theorem 3.1, but the proof is of a different nature.
0. Notation and Preliminaries

All rings considered in this paper will be unitary. If \( R \) is a ring, by an \( R \)-module we will mean a left \( R \)-module, and we will denote the category of \( R \)-modules by \( \text{R-mod} \).

Let \( G \) be a multiplicative group with identity element "1". A \( G \)-graded ring \( R \) is a ring with identity 1, together with a direct sum decomposition \( R = \bigoplus_{\sigma \in G} R_{\sigma} \) (as additive subgroups) such that

\[
R_{\sigma}R_{\tau} \subseteq R_{\sigma \tau} \quad \text{for all } \sigma, \tau \in G. \tag{1}
\]

It is well-known [16] that \( R_1 \) is a subring of \( R \), and \( 1 \in R_1 \). Clearly, \( R_{\sigma} \) is an \( R_1-R_1 \)-bimodule, \( \forall \sigma \in G \). By a left \( G \)-graded \( R \)-module we understand a left \( R \)-module \( M \), plus an internal direct sum decomposition \( M = \bigoplus_{\sigma \in G} M_{\sigma} \), where \( M_{\sigma} \) are subgroups of the additive group of \( M \) such that \( R_{\sigma}M_{\tau} \subseteq M_{\sigma \tau} \) for all \( \sigma, \tau \in G \). We denote by \( R\text{-gr} \) the category of left \( G \)-graded \( R \)-modules. In this category, if \( M = \bigoplus_{\sigma \in G} M_{\sigma} \) and \( N = \bigoplus_{\sigma \in G} N_{\sigma} \) are two objects, then \( \text{Hom}_{R\text{-gr}}(M, N) \) is the set of morphisms in the category \( R\text{-gr} \) from \( M \) to \( N \), i.e.,

\[
\text{Hom}_{R\text{-gr}}(M, N) = \{ f: M \to N \mid f \text{ is } R\text{-linear and } f(M_{\sigma}) \subseteq N_{\sigma}, \forall \sigma \in G \}.
\]

It is well-known [16] that \( R\text{-gr} \) is a Grothendieck category. In particular, \( R\text{-gr} \) has enough injective objects. Then if \( M \in R\text{-gr} \), we denote by \( E^s(M) \) the injective envelope of \( M \) in \( R\text{-gr} \), and by \( E(M) \) the injective envelope of \( M \) in \( \text{R-mod} \).

If \( M = \bigoplus_{\lambda \in G} M_{\lambda} \) is a graded \( R \)-module and \( \sigma \in G \), then \( M(\sigma) \) is the graded module obtained by \( M \) by putting \( M(\sigma)_{\lambda} = M_{\sigma \lambda} \); the graded module \( M(\sigma) \) is called the \( \sigma \)-suspension of \( M \). It is well-known [16] that the mapping \( M \mapsto M(\sigma) \) defines a functor \( T_{\sigma}: R\text{-gr} \to R\text{-gr} \), which is an equivalence of categories.

If in (1) we have equality, i.e., \( R_{\sigma}R_{\tau} = R_{\sigma \tau} \), then \( R \) is called a strongly graded ring. It is well-known [16] that \( R \) is a strongly graded ring \( \iff \) \( R_{\sigma}R_{\tau^{-1}} = R_1 \) for any \( \sigma \in G \Rightarrow R_{\sigma}M_{\tau} = M_{\sigma \tau} \) for any \( \sigma, \tau \in G \), where \( M = \bigoplus_{\sigma \in G} M_{\sigma} \) is an arbitrary left graded \( R \)-module.

If \( R \) is a strongly graded ring, the connection between the categories \( R\text{-gr} \) and \( R_1\text{-mod} \) is given by the following result of Dade:

**Theorem P** [7, Theorem 2.8; 16, Theorem 1.3.4]. *Let \( R = \bigoplus_{\sigma \in G} R_{\sigma} \) be a \( G \)-graded ring. Then the following assertions are equivalent:*

1. \( R \) is strongly graded.
2. The functor \( R \otimes_{R_1} - : R_1\text{-mod} \to R\text{-gr} \) given by \( M \mapsto R \otimes_{R_1} M \),
where \( M \in R_1\text{-mod} \) and \( R \otimes R_1 M \) is a graded left \( R \)-module by the grading 
\((R \otimes R_1 M)_\sigma = R_\sigma \otimes R_1 M\), is an equivalence.

(3) The functor \((-)_1 : R\text{-gr} \to R_1\text{-mod} \) given by \( M \mapsto M_1 \), where 
\( M = \bigoplus_{\sigma \in G} M_\sigma \in R\text{-gr} \), is an equivalence.

Now we briefly recall some ideas connected with the notion of quotient category. Let \( \mathcal{A} \) be a Grothendieck category. If \( \mathcal{C} \) is a nonempty subclass of objects of \( \mathcal{A} \), then \( \mathcal{C} \) is called a Serre class (or Serre subcategory) of \( \mathcal{A} \) if 
\( \mathcal{C} \) is closed under subobjects, quotient objects, and extensions. If moreover 
\( \mathcal{C} \) is closed under arbitrary direct sums, then we say that \( \mathcal{C} \) is a localizing 
subcategory of \( \mathcal{A} \).

If \( \mathcal{C} \) is a localizing subcategory of \( \mathcal{A} \), then for any \( M \in \mathcal{A} \) we can 
consider the greatest subobject \( t_\mathcal{C}(M) \) of \( M \) belonging to \( \mathcal{C} \). If \( t_\mathcal{C}(M) = 0 \), 
then \( M \) is said to be a \( \mathcal{C} \)-torsion free object. The mapping \( M \mapsto t_\mathcal{C}(M) \) 
defines a left exact functor \( t_\mathcal{C} : \mathcal{A} \to \mathcal{A} \).

Following Gabriel [9] (see also [1, 17]), if \( \mathcal{C} \) is a localizing subcategory 
of \( \mathcal{A} \), we can define the quotient category \( \mathcal{A}/\mathcal{C} \), which is also a Grothen- 
dieck category. We denote by \( \mathcal{T}_\mathcal{C} : \mathcal{A} \to \mathcal{A}/\mathcal{C} \), \( \mathcal{S}_\mathcal{C} : \mathcal{A}/\mathcal{C} \to \mathcal{A} \) the canonical 
functors (see [9, Chap. III]). It is well-known [17] that \( \mathcal{T}_\mathcal{C} \) is an exact 
functor and \( \mathcal{S}_\mathcal{C} \) is a right adjoint for \( \mathcal{T}_\mathcal{C} \). Moreover, \( \mathcal{S}_\mathcal{C} \) is a left exact 
functor. If \( \phi : \mathcal{T}_\mathcal{C} \circ \mathcal{S}_\mathcal{C} \to \mathcal{I}_{\mathcal{A}/\mathcal{C}} \) and \( \psi : \mathcal{I}_{\mathcal{A}} \to \mathcal{S}_\mathcal{C} \circ \mathcal{T}_\mathcal{C} \) 
are the natural transformations of functors, then \( \phi \) is an isomorphism. Further, if \( M \in \mathcal{A} \), then we 
have the exact sequence

\[
0 \to \text{Ker}(\psi(M)) \to M \to (\mathcal{S}_\mathcal{C} \circ \mathcal{T}_\mathcal{C})(M) \to \text{Coker}(\psi(M)),
\]

where \( \text{Ker}(\psi(M)) \) and \( \text{Coker}(\psi(M)) \) belong to \( \mathcal{C} \).

1. Induced and Coinduced Functors

Let \( R = \bigoplus_{\sigma \in G} R_\sigma \) be a \( G \)-graded ring, and \( N \in R_1\text{-mod} \). We consider 
the graded left \( R \)-module \( M = R \otimes R_1 N \), where \( M \) has the grading 
\( M_\sigma = R_\sigma \otimes R_1 N \). The graded \( R \)-module \( M = \bigoplus_{\sigma \in G} M_\sigma \) is called the induced 
\( R \)-module by the \( R_1 \)-module \( N \), and we denote this module by \( \text{Ind}(N) \). It is 
obvious that the mapping \( N \mapsto \text{Ind}(N) \) defines a covariant functor 
\( \text{Ind} : R_1\text{-mod} \to R\text{-gr} \), which is called the induced functor. This functor is 
right exact. Moreover, if \( R \) is a flat right \( R_1 \)-module (i.e., \( R_\sigma \) is a flat right 
\( R_1 \)-module for any \( \sigma \in G \)), then the functor \( \text{Ind} \) is exact.

Since \( R \) is an \( R_1 \)-\( R \)-bimodule, we can consider the left \( R \)-module 
\( M = \text{Hom}_{R_1}(R, N) \). If \( f \in \text{Hom}_{R_1}(R, N) \) and \( a \in R \), the multiplication \( af \) 
is given by the equality

\[
(af)(x) = fx(xa), \quad x \in R.
\]
For any \( \sigma \in G \), we define the set
\[
M'_\sigma = \{ f \in \text{Hom}_R(R, N) \mid f(R_{\sigma'}) = 0 \text{ for any } \sigma' \neq \sigma^{-1} \}.
\]

It is obvious that \( M'_\sigma \) is a subgroup of \( M' \) (in fact \( M'_\sigma \cong \text{Hom}_R(R_{\sigma^{-1}}, N) \)). The sum \( M^* = \sum_{\sigma \in G} M'_\sigma \) is a direct sum. Indeed, if \( f \in M'_\sigma \cap \sum_{\tau \neq \sigma} M'_\tau \), we have that \( f \in M'_\sigma \) and \( f = \sum_{\tau \neq \sigma} f'_\tau \), \( f'_\tau \in M'_\tau \). Thus if \( x \in R_{\sigma^{-1}} \) we have
\[
f(x) = \sum_{\tau \neq \sigma} f'_\tau(x) = 0, \quad \text{so } f(R_{\sigma^{-1}}) = 0. \]
Since \( f(R_{\tau}) = 0 \) for any \( \tau \neq \sigma^{-1} \), we obtain that \( f = 0 \). Now we prove that \( R_{\sigma} M^*_\tau \subseteq M^*_\sigma \) for any \( \sigma, \tau \in G \). Indeed, if \( a \in R_{\sigma} \) and \( f \in M^*_\tau \) we have for any \( x \in R_{\lambda} \), where \( \lambda \neq (\sigma \tau)^{-1} = \tau^{-1} \sigma^{-1} \),
\[
(af)(x) = f(xa) = 0, \quad \text{since } xa \in R_{\lambda \sigma} \text{ and } \lambda \sigma \neq \tau^{-1}. \]
Therefore, \( af \in M^*_\tau \). Consequently, \( M^* = \sum_{\sigma \in G} M'_\sigma \) is an object from the category \( R\text{-gr} \). This object is called the \textit{coinduced module} by \( N \), and is denoted by \( \text{Coind}(N) \).

It is obvious that the mapping \( N \mapsto \text{Coind}(N) \) defines a covariant functor \( \text{Coind}: R_1\text{-mod} \to R\text{-gr} \), which is called the \textit{coinduced functor}. It is obvious that \( \text{Coind} \) is a left exact functor. Furthermore, if \( R \) is a projective left \( R_1 \)-module, the \( \text{Coind} \) is an exact functor.

Now if \( \sigma \in G \) is fixed, we can define the functor
\[
(-)_{\sigma}: R\text{-gr} \to R_1\text{-mod} \quad \text{given by } M \mapsto M_{\sigma},
\]
where \( M = \bigoplus_{\tau \in G} M_{\tau} \in R\text{-gr} \). It is obvious that \( (-)_{\sigma} \) is an exact functor.

We recall that by \( T_{\sigma}: R_1\text{-mod} \to R\text{-gr} \) we have denoted the \( \sigma \)-suspension functor.

We are now in a position to state and prove the main result of this section.

**Theorem 1.1.** With the above notation we have the following assertions:

(a) The functor \( T_{\sigma^{-1}} \circ \text{Ind} \) is a left adjoint functor of the functor \( (-)_{\sigma} \). Moreover, \( (-)_{\sigma} \circ T_{\sigma^{-1}} \circ \text{Ind} \simeq 1_{R_1\text{-mod}} \).

(b) The functor \( T_{\sigma^{-1}} \circ \text{Coind} \) is a right adjoint functor of the functor \( (-)_{\sigma} \). Moreover, \( (-)_{\sigma} \circ T_{\sigma^{-1}} \circ \text{Coind} \simeq 1_{R_1\text{-mod}} \).

**Proof.** (a) We define the functorial morphisms
\[
\text{Hom}_{R\text{-gr}}(T_{\sigma^{-1}} \circ \text{Ind}, (-)_{\sigma}) \xrightarrow{\alpha} \text{Hom}_{R_1}((-), (-)_{\sigma})
\]
as follows: if \( N \in R_1\text{-mod} \) and \( M \in R\text{-gr} \), then \( \alpha(N, M): \text{Hom}_{R\text{-gr}}(\text{Ind}(N) (\sigma^{-1}), M) \to \text{Hom}_{R_1}(N, M_{\sigma}) \) is defined by \( \alpha(N, M)(u)(x) = u(1 \otimes x) \), where \( u: (R \otimes_{R_1} N)(\sigma^{-1}) \to M \) is a morphism in \( R\text{-gr} \). Clearly \( u(1 \otimes x) \in M_{\sigma} \), since \( 1 \otimes x \in (R \otimes_{R_1} N)(\sigma^{-1})_{\sigma} = (R \otimes_{R_1} N)_{\sigma} = R_1 \otimes_{R_1} N \simeq N \).

Now we define \( \beta(N, M): \text{Hom}_{R_1}(N, M_{\sigma}) \to \text{Hom}_{R\text{-gr}}(\text{Ind}(N) (\sigma^{-1}), M) \) as follows: if \( v \in \text{Hom}_{R_1}(N, M_{\sigma}) \), we put \( \beta(N, M)(v): (R \otimes_{R_1} N)(\sigma^{-1}) \to M \),

\[
\text{Hom}_{R\text{-gr}}(T_{\sigma^{-1}} \circ \text{Ind}, (-)_{\sigma}) \xrightarrow{\beta} \text{Hom}_{R_1}((-), (-)_{\sigma}).
\]
defined by $\beta(N, M)(v)(\lambda \otimes x) = \lambda v(x)$. We remark that if $\lambda \in R_{\sigma^{-1}}$, then $\lambda \otimes x \in (R \otimes R_1)(N)(\sigma^{-1})$. But in this case we have $\lambda v(x) \in R_{\sigma^{-1}} M_{\sigma} \subseteq M_{\tau}$, and therefore $\beta(N, M)(v) \in \text{Hom}_{\text{gr}}(\text{Ind}(N)(\sigma^{-1}), M)$. It is easy to see that $\alpha$ and $\beta$ are functorial morphisms. Now we have $(\beta(N, M) \circ \alpha(N, M))(u)(\lambda \otimes x) = \lambda \alpha(N, M)(u)(x) = \lambda u(1 \otimes x) = u(\lambda(1 \otimes x)) = u(\lambda \otimes x)$, and therefore $(\beta(N, M) \circ \alpha(N, M))(u) = u$, i.e., $\beta(N, M) \circ \alpha(N, M) = 1$.

Conversely, $(\alpha(N, M) \circ \beta(N, M))(v)(x) = \alpha(N, M)(\beta(N, M)(v))(x) = \beta(N, M)(v)(\lambda \otimes x) = u(x)$, and therefore $(\alpha(N, M) \circ \beta(N, M))(v) = v$, i.e., $\alpha(N, M) \circ \beta(N, M) = 1$. Consequently, $T_{\sigma^{-1}} \circ \text{Ind}$ is a left adjoint of $(\cdot)_{\sigma}$. The last assertion of (a) is obvious.

(b) We can define the functorial morphisms

$$\text{Hom}_{R_1}((-), -) \xrightarrow{\gamma} \text{Hom}_{R_{\text{gr}}}((-), T_{\sigma^{-1}} \circ \text{Coind})$$

as follows: if $N \in R_{\tau} \text{-mod}$ and $M \in R_{\text{gr}}$, then $\gamma(M, N): \text{Hom}_{R_1}(M_{\sigma}, N) \to \text{Hom}_{R_{\text{gr}}}(M, \text{Coind}(N)(\sigma^{-1}))$ is defined by putting for each $u \in \text{Hom}_{R_1}(M_{\sigma}, N)$ and $x_\tau \in M_\tau$, $\gamma(M, N)(u)(x_\tau): R \to N, \gamma(M, N)(u)(x_\tau)(a) = u(a_{\sigma^{-1}} x_\tau)$, where $a = \sum_{\mu \in \mathbb{G}} a_\mu \in R$. Thus we have for $a \in R_{\mu}$

$$(M, N)(u)(x_\tau)(a) = \begin{cases} u(ax_\tau) & \text{if } \mu = \sigma \lambda^{-1} \\ 0 & \text{if } \mu \neq \sigma \lambda^{-1}. \end{cases}$$

We note that $\gamma(M, N)(u)(x_\tau) \in \text{Coind}(N)(\sigma^{-1})_\lambda = \text{Coind}(N)_{\lambda \sigma^{-1}}$ since $\gamma(M, N)(u)(x_\tau)(R_\mu) = 0$ for any $\mu \neq (\lambda \sigma^{-1})^{-1} = \sigma \lambda^{-1}$. Therefore, the map $\gamma(M, N)$ is well-defined. Conversely, if $u \in \text{Hom}_{R_{\text{gr}}}(M, \text{Coind}(N)(\sigma^{-1}))$, we define $\delta(M, N)(v): M_{\tau} \to N$ by $\delta(M, N)(v)(x_\tau) = v(x_\tau)(1)$. We remark that this correspondence is well-defined because $v(x_\tau) \in \text{Coind}(N)(\sigma^{-1})_{\sigma} = \text{Coind}(N)_{1} = \{ f \in \text{Hom}_{R_1}(R, N) \mid f(R_\tau) = 0 \text{ for any } \tau \neq 1 \}$. Now if $u \in \text{Hom}_{R_1}(M_{\mu}, N)$, we have, for any $x_\mu \in M_{\mu}$, $\delta(M, N) \circ \gamma(M, N)(u)(x_\mu) = \delta(M, N)(\gamma(M, N)(u)(x_\mu))(a) = (\gamma(M, N)(u)(x_\mu)(1) = u(1 \cdot x_\mu) = u(x_\mu)$. Hence $(\delta(M, N) \circ \gamma(M, N))(u) = u$, i.e., $\delta(M, N) \circ \gamma(M, N) = 1$. Conversely, if $v \in \text{Hom}_{R_{\text{gr}}}(M, \text{Coind}(N)(\sigma^{-1}))$ and $x_\lambda \in M_\lambda$ and $a \in R$ we obtain that $(\gamma(M, N) \circ \delta(M, N))(v)(x_\lambda)(a) = (\gamma(M, N)(\delta(M, N)(v))(x_\lambda)(a) = (\delta(M, N)(v)(a_{\sigma^{-1}} x_\lambda)(1) = (a_{\sigma^{-1}} v(x_\lambda)(1) = v(x_\lambda)(a_{\sigma^{-1}})$. On the other hand, $v(x_\lambda)(a) = v(x_\lambda)(\sum_{\tau \in \mathbb{G}} a_\tau) = \sum_{\tau \in \mathbb{G}} v(x_\lambda)(a_\tau) = v(x_\lambda)(a_{\sigma^{-1}})$, since if $\tau \neq \sigma^{-1} = (\lambda \sigma^{-1})^{-1}$, we have $v(x_\lambda)(a_\tau) = 0$. Hence $\gamma(M, N) \circ \delta(M, N)(v) = v$, i.e., $\gamma(M, N) \circ \delta(M, N) = 1$. Consequently, the functor $T_{\sigma^{-1}} \circ \text{Coind}$ is a right adjoint of the functor $(\cdot)_{\sigma}$. The last assertion of (b) is obvious.

Now if $Q \in R_{\text{gr}}$ is an injective object in the category $R_{\text{gr}}$, then we say that $Q$ is gr-injective. It is well-known [16] that if $Q \in R_{\text{gr}}$ is injective in $R_{\tau} \text{-mod}$, then $Q$ is gr-injective, but the converse is not true in general.
COROLLARY 1.1. The following assertions hold:

(1) If \( N \) is an injective \( R_1 \)-module, then \( \text{Coind}(N) \) is gr-injective.

(2) If \( M = \bigoplus_{\sigma \in G} M_{\sigma} \) is gr-injective, and for any \( \sigma \in G \), \( R_{\sigma} \) is a flat right \( R_1 \)-module, then \( M_{\sigma} \) is an injective \( R_1 \)-module for each \( \sigma \in G \).

Proof. (1) By Theorem 1.1, the functor \( \text{Coind}: R_1\text{-mod} \to R\text{-gr} \) is a right adjoint of the functor \((-)_1 \). Since \((-)_1 \) is an exact functor, it is well-known from the theory of adjoint functors [17] that \( \text{Coind}(N) \) is an injective object in \( R\text{-gr} \).

(2) By Theorem 1.1, the functor \((-)_\sigma \) is a right adjoint of the functor \( T_{\sigma^{-1}} \circ \text{Ind} \). Since \( R \) is a flat right \( R_1 \)-module, then \( \text{Ind} \) is an exact functor, and therefore \( T_{\sigma^{-1}} \circ \text{Ind} \) is an exact functor. Thus \( M_{\sigma} \) is an injective \( R_1 \)-module.

Now if \( M \in R\text{-gr} \), the functorial isomorphisms \( \alpha \) and \( \beta \) define the canonical graded functorial morphism

\[
\mu(M): \text{Ind}(M_{\sigma})(\sigma^{-1}) \to M, \quad \mu(M)(\lambda \otimes x) = \lambda x, \quad \lambda \in R, \quad x \in M_{\sigma}.
\]  

Analogously, the functorial isomorphisms \( \gamma \) and \( \delta \) define the canonical graded functorial morphism

\[
\nu(M): M \to \text{Coind}(M_{\sigma})(\sigma^{-1}), \quad \nu(M)(x_{\lambda})(a) = a_{\sigma \lambda^{-1}}x_{\lambda},
\]

\[x_{\lambda} \in M_{\lambda}, \quad a = \sum_{\tau \in G} a_{\tau} \in R.\]  

PROPOSITION 1.1. With the above notation, \( \text{Im}(\nu(M)) \) is an essential submodule of \( \text{Coind}(M_{\sigma})(\sigma^{-1}) \).

Proof. Let \( f \in \text{Coind}(M_{\sigma})(\sigma^{-1}) \), \( f \neq 0 \), for some \( \lambda \in G \). Then \( f \in \text{Hom}_{R_1}(R, M) \) such that \( f(R_{\tau}) = 0 \) for any \( \tau \neq (\sigma \lambda^{-1})^{-1} = \sigma \lambda^{-1} \). Thus there exists \( a_{\sigma \lambda^{-1}} \in R_{\sigma \lambda^{-1}} \) such that \( f(a_{\sigma \lambda^{-1}}) \neq 0 \). If we denote \( x_{\sigma} = f(a_{\sigma \lambda^{-1}}) \), we have \( \nu(M)(x_{\sigma})(b) = b_{1}x = b_{1}f(a_{\sigma \lambda^{-1}}) = f(b_{1}a_{\sigma \lambda^{-1}}) \), where \( b = \sum_{\tau \in G} b_{\tau} \) is an arbitrary element from \( R \). On the other hand, \( (a_{\sigma \lambda^{-1}}f)(b) = f(ba_{\sigma \lambda^{-1}}) = \sum_{\tau \in G} f(b_{\tau}a_{\sigma \lambda^{-1}}) = f(b_{1}a_{\sigma \lambda^{-1}}) \), since \( f(b_{1}a_{\sigma \lambda^{-1}}) = 0 \) if \( \tau \neq 1 \). Therefore, \( a_{\sigma \lambda^{-1}} = \nu(M)(x_{\sigma}) \in \text{Im}(\nu(M)) \). Since \( \nu(M)(x_{\sigma})(1) = x_{\sigma} \neq 0 \), then \( \nu(M)(x_{\sigma}) \neq 0 \). Now we can apply Lemma 1.2.8 of [16].

We fix now \( \sigma \in G \), and we define the subclass of \( R\text{-gr} \):

\[\mathcal{C}_{\sigma} = \left\{ M \in R\text{-gr} \mid M = \bigoplus_{\tau \in G} M_{\tau}, \text{ such that } M_{\sigma} = 0 \right\}.\]

Following [11], if \( M = \bigoplus_{\tau \in G} M_{\tau} \) is a graded \( R \)-module, then we say that \( M \) is \( \sigma \text{-faithful} \) if for every \( x_{\tau} \in M_{\tau}, x_{\tau} \neq 0 \), we have \( R_{\sigma \tau^{-1}}x_{\tau} \neq 0 \). If \( M \) is
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σ-faithful for every σ ∈ G, we say that $M$ is faithful. We say that the graded ring $R$ is left σ-faithful if the left module $_R R$ is σ-faithful. The ring $R$ is said to be left faithful if the module $_R R$ is faithful. In [5] an example of a ring $R$ which is left and right faithful is given, but it is not a strongly graded ring [5, Example 2.7].

**Proposition 1.2.** The following assertions hold:

(a) For every σ ∈ G, $\mathcal{C}_\sigma$ is a localizing subcategory of $R$-gr which is closed under arbitrary direct products.

(b) If $M = \bigoplus_{\sigma \in G} M_\sigma$, then $M \in \mathcal{C}_\sigma$ if and only if for every $x_\tau \in M_\tau$, $R_{\sigma^{-1} \tau} x_\tau = 0$.

(c) If $M = \bigoplus_{\sigma \in G} M_\sigma$ is a non-zero graded module, then $M$ is $\mathcal{C}_\sigma$-torsion free if and only if $M$ is σ-faithful if and only if every non-zero submodule of $M$ intersects $M_\sigma$ non-trivially.

(d) $M$ is faithful if and only if $M$ is $\mathcal{C}_\sigma$-torsion free for any σ ∈ G.

(e) If $\mu(M)$ and $v(M)$ are the morphisms of (3) and (4), then $\ker(\mu(M))$, $\ker(v(M))$, $\coker(\mu(M))$, and $\coker(v(M))$ belong to $\mathcal{C}_\sigma$. Moreover, $\ker(v(M)) = t_{\mathcal{C}_\sigma}(M)$ and $\im(\mu(M))$ is the smallest graded submodule $L$ of $M$, such that $M/L \in \mathcal{C}_\sigma$.

**Proof.** (a) The fact that $\mathcal{C}_\sigma$ is a localizing subcategory of $R$-gr follows from the fact that $(-)_\sigma$ is an exact functor. Now let $(M_i)_{i \in I}$ be a family of objects from $\mathcal{C}_\sigma$.

(b) This is routine.

(c) Assume that $M$ is $\mathcal{C}_\sigma$-torsion free and let $x_\tau \in M_\tau$, $x_\tau \neq 0$. Then $Rx_\tau$ is also $\mathcal{C}_\sigma$-torsion free, and therefore $(Rx_\tau)_\sigma \neq 0$. But $(Rx_\tau)_\sigma = R_{\sigma^{-1} x_\tau}$. Hence $R_{\sigma^{-1} x_\tau} \neq 0$; i.e., $M$ is σ-faithful.

Conversely, assume that $M$ is σ-faithful. If $t_{\mathcal{C}_\sigma}(M) \neq 0$, then there exists $x_\tau \in (t_{\mathcal{C}_\sigma}(M))_\tau$, $x_\tau \neq 0$, for some $\tau \in G$. Thus $R_{\sigma^{-1} x_\tau} \neq 0$. On the other hand, $(t_{\mathcal{C}_\sigma}(M))_\sigma = 0$, and since $R_{\sigma^{-1} x_\tau} \in (t_{\mathcal{C}_\sigma}(M))_\sigma$, we obtain that $R_{\sigma^{-1} x_\tau} = 0$, a contradiction. Hence $t_{\mathcal{C}_\sigma}(M) = 0$; i.e., $M$ is $\mathcal{C}_\sigma$-torsion free.

The equivalence $M$ is σ-faithful if and only if every non-zero graded submodule of $M$ intersecting $M_\sigma$ non-trivially is obvious.

(d) This follows directly from (c).

(e) We denote by $K = \ker(\mu(M))$. Since $K$ is a graded submodule of $\text{Ind}(M)(\sigma^{-1})$, then we have that $K_\sigma \subseteq \text{Ind}(M)(\sigma^{-1})_\sigma = \text{Ind}(M_\sigma) = R_1 \otimes_R M_\sigma \simeq M_\sigma$. Thus $K_\sigma = 0$, and hence $K \in \mathcal{C}_\sigma$. Now we remark that $\im(\mu(M)) = RM_\sigma$, and since $(M/RM_\sigma)_\sigma = 0$, then $M/RM_\sigma \in \mathcal{C}_\sigma$. 

Now if $L$ is a graded submodule of $M$ such that $M/L \in \mathcal{C}_\sigma$, then $(M/L)_\sigma = 0$, and hence $L_\sigma = M_\sigma$, so $RM_\sigma = RL_\sigma \subseteq L$.

Let $P = \text{Ker}(\nu(M))$. If $x_\sigma \in P_\sigma$, then $\nu(M)(x_\sigma) = 0$, so $\nu(M)(x_\sigma)(1) = x_\sigma = 0$. Hence $P_\sigma = 0$, and therefore $P \subseteq t_{\sigma_0}(M)$. Conversely, let $x_\lambda \in (t_{\sigma_0}(M))_\lambda$. Hence $(Rx_\lambda)_\sigma = 0$, so $R_{\sigma_\lambda^{-1}}x_\lambda = 0$. Therefore, $\nu(M)(x_\lambda)(a) = a_{\sigma_\lambda^{-1}}x_\lambda = 0$ for any $a \in R$. Consequently, $\nu(M)(x_\lambda) = 0$, i.e., $x_\lambda \in P$, and hence $P = t_{\sigma_0}(M)$. Because $\text{Coind}(M_\sigma)(\sigma^{-1})_\sigma = \text{Coind}(M_\sigma)_1 = \{f \in \text{Hom}_{R_1}(R, M) \mid f(R_\tau) = 0 \text{ for any } \tau \neq 1\}$, if we consider $f \in \text{Coind}(M_\sigma)_1$, then we denote by $x_\sigma = f(1)$. We remark that $\nu(M)(x_\sigma) = f$, and therefore $(\text{Coker} \nu(M))_\sigma = 0$, i.e., $\text{Coker}(\nu(M)) \in \mathcal{C}_\sigma$.

**Corollary 1.2.** Let $Q = \bigoplus_{\sigma \in G} Q_\sigma$ be a gr-injective module. If $Q$ is $\sigma$-faithful, then $Q_\sigma$ is an injective $R_1$-module, and $Q \simeq \text{Coind}(Q_\sigma)(\sigma^{-1})$.

**Proof.** Since $Q$ is $\sigma$-faithful, then $t_{\sigma_0}(Q) = 0$, and therefore the canonical morphism $\nu(Q): Q \rightarrow \text{Coind}(Q_\sigma)(\sigma^{-1})$ is a monomorphism. By Proposition 1.1, $\nu(Q)$ is an isomorphism.

We prove now that $Q_\sigma$ is an injective $R$-module. Let $E(Q_\sigma)$ denote the injective envelope of $Q_\sigma$ in $R$-mod. Since $\text{Coind}$ is a left exact functor, we have the monomorphism $\text{Coind}(Q_\sigma) \simeq \text{Coind}(E(Q_\sigma))$. Since $\text{Coind}(Q_\sigma) \simeq Q(\sigma)$, then $\text{Coind}(Q_\sigma)$ is $\sigma$-injective, and therefore $\text{Coind}(E(Q_\sigma)) = \text{Coind}(Q_\sigma) \oplus X$, for some $X \in R$-gr. In particular, we have $\text{Coind}(E(Q_\sigma))_1 = \text{Coind}(Q_\sigma)_1 \oplus X_1$, so $E(Q_\sigma) - Q_\sigma \oplus X_1$, and therefore $X_1 = 0$. Hence $Q_\sigma = E(Q_\sigma)$; i.e., $Q_\sigma$ is an injective $R_1$-module.

**Corollary 1.3.** Let $M = \bigoplus_{\sigma \in G} M_\sigma$ be a graded $R$-module. If $M$ is $\sigma$-faithful, then

$$E^g(M) \simeq \text{Coind}(E(M_\sigma))(\sigma^{-1})$$

(recall that $E^g(M)$ denotes the injective envelope of $M$ in $R$-gr).

**Proof.** Since $M$ is $\sigma$-faithful, and $E^g(M)$ is an essential extension of $M$, it follows that $E^g(M)$ is $\sigma$-faithful too. By Corollary 1.2, we have that $E^g(M) \simeq \text{Coind}(E^g(M)_\sigma)(\sigma^{-1})$, and $E^g(M)_\sigma$ is an injective $R_1$-module. But since $E^g(M)$ is $\sigma$-faithful, then $M_\sigma$ is an essential $R_1$-submodule of $E^g(M)_\sigma$, and therefore $E(M_\sigma) = E^g(M)_\sigma$.

If $M = \bigoplus_{\sigma \in G} M_\sigma$ is a left graded $R$-module, we define the support of $M$ by

$$\text{supp}(M) = \{ \sigma \in G \mid M_\sigma \neq 0 \}.$$

If $M$ is a simple object in the category $R$-gr, then we say that $M$ is gr-simple.
**Corollary 1.4.** Let \( M = \bigoplus_{\sigma \in G} M_{\sigma} \) be a non-zero gr-simple module. Then for every \( \sigma \in \text{supp}(M) \), \( M \) is \( \sigma \)-faithful, and in this case \( E^\sigma(M)(\sigma) \simeq \text{Coind}(E(M_\sigma)) \).

**Proof.** We have \( M_\sigma \neq 0 \). If \( t_{\phi_\sigma}(M) \neq 0 \), then \( M = t_{\phi_\sigma}(M) \), so \( M_\sigma = 0 \), a contradiction. Apply now Corollary 1.3 to finish the proof.

An object \( Q \) in \( R\text{-gr} \) is called gr-\( \Sigma \)-injective if for any family \( (\sigma_i)_{i \in I} \) of elements of \( G \), the module \( \bigoplus_{i \in I} Q(\sigma_i) \) is gr-injective.

**Corollary 1.5.** If \( M = \bigoplus_{\sigma \in G} M_{\sigma} \) is faithful, then for any \( \sigma \in G \) we have
\[
E^\sigma(M)(\sigma) \simeq \text{Coind}(E(M_\sigma)).
\]
Moreover, if the ring \( R \) has the property that \( R_1 \) is left noetherian and for every \( \sigma \in G \), \( R_\sigma \) is a finitely generated \( R_1 \)-module, then \( E^\sigma(M) \) is gr-\( \Sigma \)-injective.

**Proof.** The first assertion follows from Corollary 1.3. The second assertion easily follows from the fact that if every \( R_\sigma \) is a finitely generated \( R_1 \)-module, then the functor \( \text{Coind} \) commutes with arbitrary direct sums.

**2. Graded Rings of Finite Support**

If a graded \( R \)-module \( M \) has the property that \( \text{supp}(M) \) is a finite set (see the definition of the support after Corollary 1.3), then we say that \( M \) is a graded module of finite support, and we write \( \text{supp}(M) < \infty \). We will denote by \( \mathcal{C}_f \) the class of all left graded \( R \)-modules of finite support.

**Proposition 2.1.** Assume that \( R \) is a graded ring of finite support. Then

1. \( \mathcal{C}_f \) is a Serre subcategory of \( R\text{-gr} \).
2. If \( M \in R\text{-gr} \) is finitely generated, then \( M \in \mathcal{C}_f \).
3. \( M \in \mathcal{C}_f \) if and only if there exists a finite set \( \sigma_1, \sigma_2, \ldots, \sigma_s \in G \) such that \( \bigcap_{i=1}^s t_{\phi_{\sigma_i}}(M) = 0 \). In particular, if \( M \) is \( \sigma \)-faithful for some \( \sigma \in G \), then \( M \in \mathcal{C}_f \).
4. If \( M \in R\text{-gr} \) is gr-artinian (i.e., it is an artinian object in \( R\text{-gr} \)), then \( M \in \mathcal{C}_f \).

**Proof:**

1. This is obvious.
2. This follows from the fact that \( _RR \in \mathcal{C}_f \).
3. Let \( \text{supp}(M) = \{\sigma_1, \sigma_2, \ldots, \sigma_s\} \). If we denote by \( K = \bigcap_{i=1}^s t_{\phi_{\sigma_i}}(M) \), we have \( K_{\sigma_i} = 0 \) for any \( 1 \leq i \leq s \). Since \( K_\sigma \subseteq M_\sigma \), then \( K_\sigma = 0 \) for any...
$\sigma \in \text{supp}(M)$. Consequently, $K_\sigma = 0$ for every $\sigma \in G$, so $K = 0$. Conversely, suppose that \( \bigcap_{i=1}^s t_{\varphi_i}(M) = 0 \). Then we have the canonical monomorphism

$$0 \to M \to \bigoplus_{i=1}^s M/t_{\varphi_i}(M).$$

Since $M/t_{\varphi_i}(M)$ is $\sigma_i$-faithful, then it is sufficient to prove that if $M$ is $\sigma$-faithful, then $M$ has finite support. Assume that $\text{supp}(R) = \{\tau_1, \tau_2, \ldots, \tau_r\}$. We prove that for every $\tau \notin \{\tau_1^{-1}\sigma, \ldots, \tau_r^{-1}\sigma\}$, we have $M_\tau = 0$. Indeed, if $M_\tau \neq 0$, there exists $x_\tau \in M_\tau$, $x_\tau \neq 0$. Since $M$ is $\sigma$-faithful, we have $R_{\tau^{-1}}x_\tau \neq 0$. But $\sigma_\tau^{-1} \notin \{\tau_1, \ldots, \tau_r\}$ and hence $R_{\tau^{-1}} = 0$, so $R_{\tau^{-1}}x_\tau = 0$, a contradiction. Hence $\text{supp}(M) \subset \{\tau_1^{-1}\sigma, \ldots, \tau_r^{-1}\sigma\}$.

(4) Obviously $\bigcap_{\sigma \in G} t_{\varphi}(M) = 0$. Since $M$ is gr-artinian, there exists $\sigma_1, \ldots, \sigma_s \in G$ such that $\bigcap_{i=1}^s t_{\varphi_i}(M) = 0$. Now we can apply assertion (3).

We prove now one of the main results of this section:

**Theorem 2.1.** Assume that $R$ is a graded ring with finite support. Then

(a) If $M \in R$-gr is $\sigma$-faithful, then the graded module $\text{Coind}(E(M_\sigma)) = \text{Hom}_R(R, E(M_\sigma))$ is the injective envelope of $M$ in $R$-mod.

(b) If $Q$ has finite support and $Q$ is gr-injective, then $Q$ is injective in $R$-mod.

(c) If $M \in \mathcal{C}_{\text{fs}}$, then a minimal injective resolution of $M$ in $R$-gr

$$0 \to M \to E_0^g \to E_1^g \to \cdots \to E_n^g \to \cdots$$

is a minimal injective resolution of $M$ in $R$-mod. Moreover, $E_n^g \in \mathcal{C}_{\text{fs}}$ for any $n \geq 0$.

(d) $\mathcal{C}_{\text{fs}}$ is closed under taking injective envelopes.

**Proof.** (a) Since $R$ has finite support, then the functor $\text{Coind}(-) = \text{Hom}_R(R, -)$. By Corollary 1.3 we have that $E^g(M) \simeq \text{Hom}_R(R, E(M_\sigma))$ ($\sigma^{-1}$). On the other hand, it is well-known that $\text{Hom}_R(R, E(M_\sigma))$ is an injective $R$-module. Therefore, $E^g(M)$ is injective in $R$-mod. Since $E^g(M)$ is an essential extension of $M$, then $E(M)$ is an essential extension of $E^g(M)$. Hence $E(M) = E^g(M)$.

(b) By Proposition 2.1 there exist $\sigma_1, \ldots, \sigma_s \in G$ such that

$$0 \to Q \to \bigoplus_{i=1}^s Q/t_{\varphi_i}(Q).$$
Since $Q/t_{a_i}(Q)$ is $\sigma_i$-faithful, then $E^g(Q/t_{a_i}(Q)) = E(Q/t_{a_i}(Q))$. It is obvious that we have the monomorphism in $R$-gr

$$0 \to Q \to \bigoplus_{i=1}^s E^g(Q/t_{a_i}(Q)).$$

Since $Q$ is gr-injective, then $Q$ is isomorphic to a direct summand of $\bigoplus_{i=1}^s E^g(Q/t_{a_i}(Q))$. Since $\bigoplus_{i=1}^s E^g(Q/t_{a_i}(Q))$ is injective in $R$-mod, then we obtain that $Q$ is injective in $R$-mod.

(c) We apply assertions (a) and (b).

(d) This follows from (c).

If $M \in R$-gr, we denote by gr-inj. dim$(M)$ (resp. inj. dim$(M)$) the injective dimension of $M$ in $R$-gr (resp. in $R$-mod).

**Corollary 2.1.** Assume that $R$ has finite support. If $M \in R$-gr has finite support, then gr-inj. dim$(M) =$ inj. dim$(M)$.

*Proof.* We apply (c) of Theorem 2.1.

**Corollary 2.2 [15].** If $G$ is a finite group and $Q \in R$-gr is gr-injective, then $Q$ is injective in $R$-mod.

*Remark.* Assertion (a) of Theorem 2.1 extends Proposition 4.3 of [11].

We recall [16] that a graded ring $R$ is said to be gr-noetherian if any ascending chain of graded left ideals of $R$ is stationary. If $R = \bigoplus_{\alpha \in G} R_\alpha$ is a graded ring of finite support, then we have the following equivalences: $R$ is left gr-noetherian $\iff R_1$ is left noetherian and for any $\sigma \in G$, $R_\sigma$ is a finitely generated left $R_1$-module $\iff R$ is left noetherian.

**Corollary 2.3.** Let $R$ be a graded ring of finite support. Assume that $R$ is a left noetherian ring and let $Q \in R$-gr. Then

$Q$ is gr-injective $\iff Q$ is injective in $R$-mod.

*Proof.* Since $R$ is a left noetherian ring, then $R$ is a left gr-noetherian ring. Then every injective object of $R$-gr is a direct sum of injective indecomposable objects of $R$-gr. So $Q = \bigoplus_{i \in I} Q_i$, where $Q_i$ is gr-injective and gr-indecomposable. Pick $x_i \in Q_i$, $x_i \neq 0$, $x_i$ a homogeneous element. Since $Q_i$ is gr-injective and gr-indecomposable, then $Q_i$ is an essential extension of $Rx_i$. But $Rx_i$ is finitely generated, and therefore $Rx_i$ has finite support. By Theorem 2.1, $Q_i$ has finite support and $Q_i$ is injective in $R$-mod too. Since $R$ is left noetherian, it follows that $\bigoplus_{i \in I} Q_i$ is injective in $R$-mod, and therefore $Q$ is injective in $R$-mod.
Corollary 2.4. Let $R$ be a graded ring of finite support, and assume that $R$ is left noetherian. If $Q$ is gr-injective and has finite support, then there exist $\sigma_1, ..., \sigma_s \in G$ such that $Q = Q_1 \oplus Q_2 \oplus \cdots \oplus Q_s$, and each $Q_i$ is $\sigma_i$-faithful.

Proof. Since $Q$ has finite support, there exist the elements $\tau_1, ..., \tau_s \in G$ such that $\bigcap_{i=1}^s t_{\varphi_{\tau_i}}(Q) = 0$. We have the canonical monomorphism

$$0 \to Q \to \bigoplus_{i=1}^s Q/t_{\varphi_{\tau_i}}(Q),$$

where $Q/t_{\varphi_{\tau_i}}(Q)$ is $\tau_i$-faithful. If we denote by $E_i = E^s(Q/t_{\varphi_{\tau_i}}(Q))$, then $E_i$ is $\tau_i$-faithful, and $Q$ is isomorphic to a direct summand of $E_1 \oplus \cdots \oplus E_r$. Since $R$ is left noetherian, we can write $Q = \bigoplus_{\lambda \in A} I_{\lambda}$ is gr-indecomposable. Now using the Krull–Remak–Schmidt Theorem, it follows that each $I_{\lambda}$ is isomorphic to a direct summand of some $E_k$, so $I_{\lambda}$ is $\tau_k$-faithful. Now the assertion is obvious.

Remark. There are numerous examples of graded rings of type $\mathbb{Z}$ having finite support: (1) Let $A$ be a ring and $\varphi : A \times M_A$ and $A-A$-bimodule. Assume that $\varphi = [-, -] : M \otimes_A M \to A$ is an $A-A$-bilinear map satisfying $[m_1, m_2] m_3 = m_1 [m_2, m_3]$ for all $m_1, m_2, m_3 \in M$. We define a multiplication on the abelian group $A \times M$ by $(a, m)(a', m') = (aa' + [m, m'], am' + ma')$. In this way, $A \times M$ becomes a ring which will be denoted by $A \times_{\varphi} M$, called the semi-trivial extension of $A$ by $M$ and $\varphi$. The ring $R = A \times_{\varphi} M$ can be considered a graded ring of type $\mathbb{Z}_2$ by putting $R_0 = A \times \{0\}$, $R_1 = \{0\} \times M$.

Important special cases of semi-trivial extensions are the generalized matrix rings: let $(R, R M_S, S N_R, S)$ be a Morita context with maps $(-, -) : M \otimes_S N \to R$ and $[\cdot, \cdot, \cdot] : N \otimes_R M \to S$ (see for example [4, p. 62]). We consider the matrix ring

$$T = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$$

in which the multiplication is defined by means of the mappings $(-, -)$ and $[\cdot, \cdot, \cdot]$. The ring $T$ may be graded as follows:

$$T_0 = \begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix}, \quad T_1 = \begin{pmatrix} 0 & M' \\ 0 & 0 \end{pmatrix}, \quad T_{-1} = \begin{pmatrix} 0 & 0 \\ N & 0 \end{pmatrix},$$

$$T_i = 0 \quad \text{for any } i \neq -1, 0, 1.$$ 

It is easy to see that $T$ is a particular case of semi-trivial extension.

(2) Another particular case of semi-trivial extension is when
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$M = A A$ and $\varphi = [ - , - ] = 0$. In this case, $R = A$ is a free $A$-module with basis $\{ 1, e \}$, where $e = (0, 1)$, $e^2 = 0$, and $e$ is a central element of $R$. This ring is called the "algebra of dual numbers over $A$" (see [4, p. AX. 27]). It is easy to see that the category $R$-gr is equivalent with the category of $A$-chain complexes.

We make now some remarks on the maximal quotient ring of a graded ring with finite support. Let $R = \bigoplus_{\sigma \in G} R_{\sigma}$ be a graded ring of finite support, and $Q \in R$-gr a gr-injective module with finite support. We have seen that $Q$ is injective in $R$-mod too. We denote this by $F_Q = \{ I \text{ left ideal of } R \mid \text{Hom}_{R}(R/I, Q) = 0 \}$. It is well-known [1] that $F_Q$ is an additive topology on the ring $R$. We denote by $t_Q$ the kernel functor corresponding to the topology $F_Q$, i.e., if $M \in R$-mod, then $t_Q(M) = \{ x \in M \mid \text{Ann}_{R}(x) \in F_Q \}$. It is well-known [1] that the class of all left modules $M$ such that $M = t_Q(M)$ is a localizing subcategory of $R$-mod.

**Proposition 2.2.** Let $R = \bigoplus_{\sigma \in G} R_{\sigma}$ be a graded ring of finite support. If $Q \in R$-gr is gr-injective and $Q$ is $\sigma$-faithful for some $\sigma \in G$, then

(a) $I \in F_Q \iff \text{Hom}_{R_i}(R/I, Q_\sigma) = 0$.
(b) $F_Q$ possesses a cofinal set of graded left ideals.
(c) If $M \in R$-gr, then $t_Q(M)$ is a graded $R$-submodule of $M$.

**Proof.** (a) By Corollary 2.2, $Q$ is an injective $R$-module, and $Q(\sigma) \cong \text{Coind}(Q_\sigma) = \text{Hom}_{R_i}(R, Q_\sigma)$. Since $\text{Hom}_{R_i}(R/I, Q) = \text{Hom}_{R_i}(R/I, \text{Hom}_{R_i}(R, Q_\sigma)) \cong \text{Hom}_{R_i}(R \otimes_R R/I, Q_\sigma) = \text{Hom}_{R_i}(R/I, Q_\sigma)$, the assertion follows.

(b) Since $0 \to R_\tau/I \cap R_\tau \subseteq R/I$, we have $\text{Hom}_{R_i}(R_\tau/I \cap R_\tau, Q_\sigma) = 0$ for any $\tau \in G$. We denote by $J = \sum_{\tau \in G} I \cap R_\tau$. It is obvious that $J$ is a left graded ideal of $R$ such that $J \subseteq I$. Furthermore, $\text{Hom}_{R_i}(R/J, Q_\sigma) = \text{Hom}_{R_i}(\bigoplus_{\tau \in G} R_\tau/J \cap R_\tau, Q_\sigma) = \prod_{\tau \in G} \text{Hom}_{R_i}(R_\tau/J \cap R_\tau, Q_\sigma) = 0$, so $J \in F_Q$.

(c) This follows from (b).

Let $M, N \in R$-gr. An $R$-linear map $f: M \to N$ is said to be a morphism of degree $\tau$, $\tau \in G$, if $f(M_{\sigma}) \subseteq N_{\tau \sigma}$ for any $\sigma \in G$. Morphisms of degree $\tau$ form an additive subgroup of $\text{Hom}_{R}(M, N)$, which we will denote by $\text{HOM}_{R}(M, N)$. It is clear that $\text{HOM}_{R}(M, N) = \sum_{\tau \in G} \text{HOM}_{R}(M, N)_{\tau}$ is a graded abelian group of type $G$ (see [16]). It is clear that $\text{HOM}_{R}(M, N) = \text{HOM}_{R}(M, N)_{\tau}$. The abelian group $\text{HOM}_{R}(M, N)$ is a subgroup of $\text{Hom}_{R}(M, N)$. The following result is well-known [16, Lemma I.2.10.]: The
subgroup $\text{HOM}_R(M, N)$ consists of all $f \in \text{Hom}_R(M, N)$ for which there exists a finite subset $F$ of $G$, such that

$$f(M_\sigma) \subseteq \sum_{\varphi \in F} N_{\sigma \varphi} \quad \text{for all } \sigma \in G.$$ 

In particular, it follows that if $M$ is finitely generated, or $G$ is a finite group, then $\text{HOM}_R(M, N) = \text{Hom}_R(M, N)$. We will need the following result:

**Lemma 2.1.** Let $R$ be a graded ring. If $M, N \in R\text{-gr}$ have finite support, then

$$\text{HOM}_R(M, N) = \text{Hom}_R(M, N).$$

**Proof.** Assume that $\text{supp}(M) = \{\sigma_1, ..., \sigma_s\}$ and $\text{supp}(N) = \{\tau_1, ..., \tau_r\}$. We denote by $F = \{\sigma_i^{-1} \tau_j | i = 1, ..., s; j = 1, ..., r\}$. If $f \in \text{Hom}_R(M, N)$ and $\sigma \in G$, then if $\sigma \notin \text{supp}(M)$, we have $f(M_\sigma) = 0$; if $\sigma = \sigma_k$, then $f(M_{\sigma_k}) \subseteq N = \sum_{\varphi \in F} N_{\sigma_k \varphi}$. Hence we can apply the above result, and the assertion follows.

We denote by $L_g(R)$ the lattice of all left graded ideals of $R$. Let $M \in R\text{-gr}$, and put $M_Q = \varprojlim_{I \in F_Q} \text{Hom}_R(I, M/I_Q(M))$. $M_Q$ is an $R$-module and it is called the module of quotients of $M$ with respect to $F_Q$ [17]. Suppose now that $R$ and $M$ have both finite support, and $Q$ is gr-injective and $\sigma$-faithful. Since $F_Q$ possesses a cofinal set of left graded ideals, $M_Q$ is, in a natural way, endowed with a graded structure. Indeed, $M_Q = \varprojlim_{I \in F_Q \cap L_g(R)} \text{Hom}_R(I, M/I_Q(M)) = \varprojlim_{I \in F_Q \cap L_g(R)} \text{HOM}_R(I, M/I_Q(M))$ where $M_Q$ has the grading

$$(M_Q)_\sigma = \varprojlim_{I \in F_Q \cap L_g(R)} (\text{HOM}_R(I, M/I_Q(M)))_\sigma \quad \text{for all } \sigma \in G.$$ 

In particular, $R_Q = \varprojlim_{I \in F_Q} \text{Hom}_R(I, R/I_Q(R))$ is a graded ring. $M_Q$ is a graded $R_Q$-module in a natural way. $R_Q$ is called the ring of quotients of $R$ with respect to $F_Q$. We remark that $R_Q$ and $M_Q$ have finite support too. We assume now that $R$ is left $1$-faithful, i.e., for every $r_\sigma \in R_\sigma$, $r_\sigma \neq 0$, $R_\sigma \div r_\sigma = 0$. This condition is equivalent with the fact that the inner product $(x, y) \mapsto (xy)_1$, $x, y \in R$, is left non-degenerate (see [5, 6]). In this case, if $R$ has finite support, $E(R) = E^R(R)$, and $E(R) = E(R_1)$ (Corollary 1.3 and Theorem 2.1). In this case, the ring of quotients $R_E(R)$ is called the left maximal quotient ring of $R$, and is denoted by $Q_{max}(R)$. Hence we obtained the following result:

**Proposition 2.3.** Let $R$ be a graded ring of finite support, and assume that $R$ is left $1$-faithful (i.e., left non-degenerate). Then the left maximal
quotient ring \( Q_{\text{max}}(R) \) is, in a natural way, endowed with a graded structure. Moreover, \( Q_{\text{max}}(R) \) has finite support too.

As a consequence, we get the following incomparability theorem, which extends Theorem 3.3 of [6], given there for the case when \( G \) is an abelian group. An incomparability theorem was proven in [10] for crossed products \( R \ast G \), with \( G \) a finite group, and then in [5], for a graded ring \( R = \bigoplus_{\sigma \in G} R_{\sigma} \), where \( G \) is finite.

**Theorem 2.2.** Let \( R = \bigoplus_{\sigma \in G} R_{\sigma} \) be a graded ring of finite support. Let \( P \) be a graded ideal of \( R \) which is a prime ideal of \( R \). If \( P \subseteq I \), \( I \) any ideal of \( R \), then \( P \subseteq (I)_{g} = P \cap R_{1} \subseteq I \cap R_{1} \) (\( (I)_{g} \) is the largest graded ideal contained in \( I \)).

**Proof.** We pass to \( R/P \). Since \( P \) is a prime ideal of \( R \), we may assume that \( R \) is a prime ring. By Proposition 1.2 of [6], the grading of \( R \) is non-degenerate and therefore \( R \) is left 1-faithful. Let \( Q = E(R, R) \). Since \( I \neq 0 \), then we have \( \text{Hom}_{R}(R/I, Q) = 0 \). Indeed, if \( \text{Hom}_{R}(R/I, Q) \neq 0 \), then there exists a non-zero \( R \)-morphism \( f: R/I \to Q \). Thus \( f(1) = x \in Q \), and \( x \neq 0 \). But there exists \( a \in R \), such that \( ax \in R \) and \( ax \neq 0 \). Since \( Ix = 0 \), then \( I(ax) \subseteq Ix = 0 \). Since \( R \) is a prime ring we obtain \( I = 0 \), a contradiction.

Now if we apply Proposition 2.2, we have \( (I)_{g} \subseteq F_Q \), so \( \text{Hom}_{R}(R/(I)_{g}, Q) = 0 \). Hence \( (I)_{g} \neq 0 \). By Proposition 1.2 of [6], \( I \cap R_{1} \neq 0 \), which ends the proof.

**Corollary 2.5.** Let \( R = \bigoplus_{\sigma \in G} R_{\sigma} \) be a graded ring of finite support, and suppose that the group \( G \) is ordered. Let \( P \) be a prime ideal of \( R \), and \( I \) an ideal of \( R \) such that \( P \subseteq I \). Then \( P \cap R_{1} \subseteq I \cap R_{1} \).

**Proof.** Since \( G \) is an ordered group, then \( (P)_{g} \) is also a prime ideal [16]. Since \( (P)_{g} \subseteq P \subseteq I \), we obtain that \( (P)_{g} \cap R_{1} \subseteq I \cap R_{1} \) by Theorem 2.2. But \( (P)_{g} \cap R_{1} = P \cap R_{1} \), and therefore \( P \cap R_{1} \subseteq I \cap R_{1} \).

### 3. A General Version of Theorem P: Applications

In this section we use the notion of a quotient category in order to provide a general version of Theorem P (see Section 1).

Let \( R = \bigoplus_{r \in G} R_{r} \) be a graded ring, and fix \( \sigma \in G \). We consider the quotient category \( R-\text{gr}/\mathcal{C}_{\sigma} \), and we denote by

\[
R-\text{gr} \xrightarrow{F_{\sigma}} S_{\sigma} \xrightarrow{S_{\sigma}} R-\text{gr}/\mathcal{C}_{\sigma}
\]
the canonical functors. We consider the functors

\[ U: R_1\text{-mod} \to R\text{-gr}/\mathcal{E}_\sigma, \quad U = F_\sigma \circ T_{\sigma}^{-1} \circ \text{Ind}. \]

\[ V: R\text{-gr}/\mathcal{E}_\sigma \to R_1\text{-mod}, \quad V = (-)_\sigma \circ S_\sigma. \]

The main result of this section is the following:

**Theorem 3.1.** With the above notation, we have \( V \circ U \simeq 1_{R_1\text{-mod}} \), and \( U \circ V \simeq 1_{R\text{-gr}/\mathcal{E}_\sigma} \); i.e., \( R_1\text{-mod} \) and \( R\text{-gr}/\mathcal{E}_\sigma \) are equivalent categories.

**Proof.** We denote by \( \psi: 1_{R\text{-gr}} \to S_\sigma \circ T_\sigma \) and \( \phi: T_\sigma \circ S_\sigma \to 1_{R\text{-gr}/\mathcal{E}_\sigma} \) the canonical functorial morphisms, where \( \phi \) is a functorial isomorphism. If \( N \in R_1\text{-mod} \), we have the morphism \( \psi(\text{Ind}(N)(\sigma^{-1})): \text{Ind}(N)(\sigma^{-1}) \to (S_\sigma \circ T_\sigma)\text{Ind}(N)(\sigma^{-1}) \). If we denote by \( \Theta(N) = (\psi(\text{Ind}(N)(\sigma^{-1})))_\sigma \), since \( \text{Ind}(N)(\sigma^{-1}) = \text{Ind}(N)(\sigma^{-1}) \), then \( \Theta(N): N \to (V \circ U)(N) \).

Since \( \text{Ker}(\psi(\text{Ind}(N)(\sigma^{-1}))) \) and \( \text{Coker}(\psi(\text{Ind}(N)(\sigma^{-1}))) \) belong to \( \mathcal{E}_\sigma \), then \( \text{Ker}(\Theta(N)) = \text{Coker}(\Theta(N)) = 0 \), and therefore \( \Theta(N) \) is an isomorphism. On the other hand, it is easy to see that \( \Theta \) is a functorial morphism.

Now let \( X \in R\text{-gr}/\mathcal{E}_\sigma \). We have the canonical morphism

\[ \mu(S_\sigma(X)): \text{Ind}(S_\sigma(X))_\sigma(\sigma^{-1}) \to S_\sigma(X), \]

where \( \text{Ker}(\mu(S_\sigma(X))) \) and \( \text{Coker}(\mu(S_\sigma(X))) \) belong to \( \mathcal{E}_\sigma \). Thus it follows that \( F_\sigma(\mu(S_\sigma(X))) \) is an isomorphism in the category \( R\text{-gr}/\mathcal{E}_\sigma \). If we denote by

\[ \varepsilon(X) = \phi(X) \circ F_\sigma(\mu(S_\sigma(X))), \]

we have that \( \varepsilon(X): (U \circ V)(X) \to X \). Since \( \phi(X) \) is an isomorphism, then \( \varepsilon(X) \) is an isomorphism. On the other hand, it is easy to see that \( \varepsilon \) is a functorial morphism. Thus the assertion follows.

**Remarks.** (1) Fix \( \sigma \in G \). Then we have the equivalence \( R \) is strongly graded \( \iff \mathcal{E}_\sigma = 0 \). Indeed, the implication \( \text{"\Rightarrow"} \) is obvious. Assume that \( \mathcal{E}_\sigma = 0 \), so \( M_\sigma \neq 0 \) for every non-zero graded \( R \)-module \( M \). If \( M \in R\text{-gr} \) is nonzero, then \( M(\sigma^{-1}\tau) \neq 0 \) for any \( \tau \in G \), and therefore \( M(\sigma^{-1}\tau)_\sigma = M_\tau \neq 0 \). Therefore, if \( M \neq 0 \), then \( M_\tau \neq 0 \) for any \( \tau \in G \). Since \( (M/RM_\tau)_\tau = 0 \), it follows that \( M/RM_\tau = 0 \), i.e., \( RM_\tau = M \). Hence \( R_\sigma M_\tau = M_\sigma \) for any \( \sigma, \tau \in G \). Therefore, \( R \) is a strongly graded ring.

(2) If \( \mathcal{E}_\sigma = 0 \), then Theorem 3.1 is exactly Theorem P.

We give now some applications of Theorem 3.1, but first, we need some preliminaries.
Let $\mathcal{A}$ be a Grothendieck category, and $\mathcal{C}$ a localizing subcategory of $\mathcal{A}$. We denote by

$$\mathcal{A} \xrightarrow{T} \mathcal{A}/\mathcal{C}$$

the canonical functors. A non-zero object $M \in \mathcal{A}$ is called $\mathcal{C}$-critical if $M$ is $\mathcal{C}$-torsion free and for any non-zero subobject $M'$ of $M$ we have $M/M' \in \mathcal{C}$. It is obvious that $M$ is $\mathcal{C}$-critical $\iff$ $M$ is $\mathcal{C}$-torsion free, and $T(M)$ is a simple object in $\mathcal{A}/\mathcal{C}$.

For a category $\mathcal{A}$, we denote by $\Omega_{\mathcal{A}}$ the set of all isomorphism classes of simple objects from $\mathcal{A}$, i.e.,

$$\Omega_{\mathcal{A}} = \{ [S] \mid S \text{ is a simple object from } \mathcal{A} \}$$

and $[S] = \{ S' \in \mathcal{A} \mid S' \simeq S \}$. It is obvious that if $S \in \mathcal{A}$ is a simple object, and $\mathcal{C}$ is a localizing subcategory of $\mathcal{A}$, then $S \in \mathcal{C}$ or $S$ is $\mathcal{C}$-torsion free. We denote this by

$$\Omega_{\mathcal{A}, \mathcal{C}} = \{ [S] \mid S \text{ is a simple object from } \mathcal{A}, \text{ and } S \text{ is } \mathcal{C}\text{-torsion free} \}.$$ 

If $\mathcal{A} = A\text{-mod}$, where $A$ is a ring, then we denote $\Omega_{A\text{-mod}}$ by $\Omega_A$.

**Proposition 3.1.** Assume that $\mathcal{C}$ is closed under arbitrary direct products. Then

1. If $M \in \mathcal{A}$ is $\mathcal{C}$-critical, then $M$ contains a unique simple subobject.
2. There exists a bijective correspondence between the sets $\Omega_{\mathcal{A}/\mathcal{C}}$ and $\Omega_{\mathcal{A}, \mathcal{C}}$.

**Proof.** (1) We denote by $(N_i)_{i \in I}$ the set of all non-zero subobjects of $M$. If $\bigcap_{i \in I} N_i = 0$, then we have the canonical monomorphism

$$0 \to M \to \prod_{i \in I} M/N_i.$$ 

Since $M/N_i \in \mathcal{C}$, then $\prod_{i \in I} M/N_i \in \mathcal{C}$, and therefore $M \in \mathcal{C}$, a contradiction. Hence $N = \bigcap_{i \in I} N_i \neq 0$. It is obvious that $N$ is a simple subobject of $M$, and that $N$ is unique.

(2) We define the map $\varphi: \Omega_{\mathcal{A}/\mathcal{C}} \to \Omega_{\mathcal{A}, \mathcal{C}}$, $\varphi([X]) = [Y]$, where $Y$ is the unique simple subobject of $S(X)$. It is well-known that $S(X)$ is $\mathcal{C}$-torsion free and since $T(S(X)) \simeq X$, it follows that $S(X)$ is $\mathcal{C}$-critical. Now it is easy to see that $\varphi$ is bijective.

**Corollary 3.1.** Assume that $R\text{-gr}$ is equivalent with the category $R_1\text{-mod}$ (the equivalence of categories is not necessarily canonical). If
\(|\Omega_{R_1}| < \infty\), then \(R\) is a strongly graded ring (\(|\Omega_{R_1}|\) denotes the cardinality of the set \(\Omega_{R_1}\)).

Proof. Fix \(\sigma \in G\). By Theorem 3.1, we have that \(R_1\text{-mod}\) is equivalent with the category \(R\text{-gr}/\mathcal{C}_\sigma\). By the hypothesis, and by Proposition 3.1, we obtain \(|\Omega_{R_1}| = |\Omega_{R\text{-gr},\mathcal{C}_\sigma}|\) and \(|\Omega_{R,\text{gr}}| = |\Omega_{R_1}|\), and therefore \(|\Omega_{R\text{-gr},\mathcal{C}_\sigma}| = |\Omega_{R\text{-gr}}| < \infty\). If \(\mathcal{C}_\sigma \neq 0\), then it is obvious that \(\mathcal{C}_\sigma\) contains a simple object \(S\), and \(S\) is not \(\mathcal{C}_\sigma\)-torsion free, a contradiction. Hence \(\mathcal{C}_\sigma = 0\), and consequently \(R\) is a strongly graded ring.

**Corollary 3.2** [12, Corollary 3.14]. Assume that \(R\text{-gr}\) is equivalent with the category \(R_1\text{-mod}\). If \(R_1\) is a semi-local ring, i.e., \(R_1/J(R_1)\) is a semi-simple artinian ring, then \(R\) is a strongly graded ring (\(J(R_1)\) denotes the Jacobson radical of the ring \(R_1\)).

**Corollary 3.3.** Suppose that there exists a ring \(A\) such that \(R\text{-gr}\) is equivalent with \(A\text{-mod}\). If \(|\Omega_A| = 1\), then \(R\) is a strongly graded ring.

Proof. Since \(|\Omega_A| = 1\), then \(|\Omega_{R\text{-gr}}| = 1\). Since \(R\text{-gr}/\mathcal{C}_\sigma\) is equivalent with \(R_1\text{-mod}\), we have \(|\Omega_{R\text{-gr},\mathcal{C}_\sigma}| \geq 1\). If \(\mathcal{C}_\sigma \neq 0\), then \(\mathcal{C}_\sigma\) contains a simple object, and therefore \(|\Omega_{R\text{-gr}}| \geq 2\), a contradiction. Hence \(\mathcal{C}_\sigma = 0\), and so \(R\) is a strongly graded ring.

Remark. In particular, if \(A\) is a local ring, i.e., \(A/J(A)\) is a simple artinian ring, it follows that \(R\) is a strongly graded ring [12, Corollary 3.13].

We now make some final remarks. If \(F \subseteq G\) is a non-empty subset of \(G\), we can define the class of graded \(R\)-modules:

\[
\mathcal{C}_F = \left\{ M = \bigoplus_{\tau \in G} M_\tau \in R\text{-gr} \mid M_\sigma = 0 \forall \sigma \in F \right\}.
\]

It is obvious that \(\mathcal{C}_F\) is a localizing subcategory of \(R\text{-gr}\) which is closed under arbitrary direct products.

Let \(P = \bigoplus_{\sigma \in F} R(\sigma^{-1}) \in R\text{-gr}\). Then \(P\) is a projective object in \(R\text{-gr}\). If \(M \in R\text{-gr}\), we have \(\text{Hom}_{R\text{-gr}}(P, M) \cong \prod_{\sigma \in F} \text{Hom}_{R\text{-gr}}(R(\sigma^{-1}), M) \cong \prod_{\sigma \in F} M_\sigma\). Thus we obtain that \(M \in \mathcal{C}_F \iff \text{Hom}_{R\text{-gr}}(P, M) = 0\).

We recall that if \(\mathcal{A}\) is a Grothendieck category with generator \(U\), then \(U\) is said to be small [17] if the functor \(\text{Hom}_{\mathcal{A}}(U, -) : \mathcal{A} \to \text{Ab}\) commutes with direct sums. The classical result of B. Mitchell is well-known (see, e.g., [17]): if \(\mathcal{A}\) is a Grothendieck category with a small projective generator \(U\), then \(\mathcal{A}\) is equivalent with the category of modules \(A\text{-mod}\), where \(A = \text{End}_{\mathcal{A}}(U)\).

Using this result, we give now a more general version of Theorem 3.1:
Theorem 3.2. With the above notation, the following assertions hold:

1. If we denote by $R\text{-gr} \cong \mathcal{F}$ the canonical functors, then $U = T_F(P)$ is a projective generator in the category $R\text{-gr}/\mathcal{C}_F$.

2. The canonical morphism $f \mapsto T_f(f)$, from $\text{End}_{R\text{-gr}}(P)$ to $\text{End}_{R\text{-gr}/\mathcal{C}_F}(U)$, is a ring isomorphism.

3. If moreover $F$ is a finite set, then $U$ is a small generator of $R\text{-gr}/\mathcal{C}_F$, and thus $R\text{-gr}/\mathcal{C}_F$ is equivalent with the category $\mathcal{A}\text{-mod}$, where $\mathcal{A} = \text{End}_{R\text{-gr}/\mathcal{C}_F}(U)$.

Proof. For (1) and (2), the proof is identical to the proof of the theorem of [13] (see also [1, p. 96, Proposition 8.6]).

For (3) let $(X_i)_{i \in I}$ be an arbitrary family of objects of $R\text{-gr}/\mathcal{C}_F$. The canonical morphism $\alpha_i: X_i \to \bigoplus_{i \in I} X_i$ yields the morphism

$$S_F(\alpha_i): S_F(X_i) \to S_F\left(\bigoplus_{i \in I} X_i\right)$$

and therefore we obtain the canonical morphism

$$\alpha: \bigoplus_{i \in I} S_F(X_i) \to S_F\left(\bigoplus_{i \in I} X_i\right)$$

such that $\alpha \circ \beta_i = S_F(\alpha_i)$ for all $i \in I$, where $\beta_i: S_F(X_i) \to \bigoplus_{i \in I} S_F(X_i)$ are the canonical morphisms. Since the functor $T_F$ commutes with direct sums, then it is easy to see that $T_F(\alpha)$ is an isomorphism, and therefore $\text{Ker}(\alpha)$ and $\text{Coker}(\alpha)$ belong to $\mathcal{C}_F$. Let $f: U \to \bigoplus_{i \in I} X_i$ be an arbitrary morphism. We have the diagram

$$\begin{array}{ccc}
0 & \longrightarrow & \text{Ker}(\alpha) \\
& \downarrow S_F(f) & \\
0 & \longrightarrow & \bigoplus_{i \in I} S_F(X_i) \longrightarrow S_F\left(\bigoplus_{i \in I} X_i\right) \longrightarrow \text{Coker}(\alpha) \longrightarrow 0
\end{array}$$

Since $S_F(U) = S_F(T_F(P))$, we have the canonical morphism $\psi(P): P \to S_F(T_F(P))$. Since $\text{Coker}(\alpha) \in \mathcal{C}_F$, then $\pi \circ S_F(f) \circ \psi(P) = 0$, and therefore the morphism $S_F(f) \circ \psi(P)$ factors by $\text{Im}(\alpha)$. Since $P$ is projective, then there exists $g: P \to \bigoplus_{i \in I} S_F(X_i)$ such that $\alpha \circ g = S_F(f) \circ \psi(P)$. Since $P$ is finitely generated, there exists a finite set $J \subseteq I$, such that $\text{Im}(g) \subseteq \bigoplus_{j \in J} S_F(X_j)$. Thus we have $T_F(\alpha) \circ T_F(g) = T_F(S_F(f)) \circ T_F(\psi(P))$, where $\text{Im}(T_F(g)) \subseteq T_F(\bigoplus_{j \in J} S_F(X_j)) = \bigoplus_{j \in J} X_j$. Since $T_F(\alpha)$ and $T_F(\psi(P))$ are isomorphisms, then $\text{Im}(T_F(S_F(f))) \subseteq \bigoplus_{j \in J} X_j$. Since $T_F \circ S_F \cong 1_{R\text{-gr}/\mathcal{C}_F}$, then we have $\text{Im}(f) \subseteq \bigoplus_{j \in J} X_j$, and therefore the functor $\text{Hom}_{R\text{-gr}/\mathcal{C}_F}(U, -)$ commutes with direct sums.
Remark. If $F = \{ \sigma \}$, then $P = R(\sigma^{-1})$ and $\text{End}_{R, \text{gr}}(P) \simeq \text{End}_{R, \text{gr}}(R) \simeq R_1$, and therefore we obtain Theorem 3.1.

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