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Basic generated universal fuzzy measures

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Abstract

The concept of basic generated universal fuzzy measures is introduced. Special classes and properties of basic generated universal fuzzy measures are discussed, especially the additive, the symmetric and the maxitive case. Additive (symmetric) basic universal fuzzy measures are shown to correspond to the Yager quantifier-based approach to additive (symmetric) fuzzy measures. The corresponding fuzzy integral-based aggregation operators are introduced, including the generated OWA operators.
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1. Introduction

Denote by $\mathbb{N}$ the set $\{1, 2, \ldots\}$. General aggregation operator $\mathbf{A} : \bigcup_{n\in\mathbb{N}}[0, 1]^n \to [0, 1]$ as introduced in [7,8,2] with specific properties can be characterized as a fuzzy integral with respect to a system of fuzzy measures $(m_n)_{n\in\mathbb{N}}$, $m_n : \mathscr{P}(X_n) \to [0, 1]$, where $X_n = \{1, 2, \ldots, n\}$. For example the only additive symmetric aggregation operator M (arithmetic mean) is related to the system $(m_n)_{n\in\mathbb{N}}$, given by $m_n(A) = \frac{|A|}{n}$ for $A \subset X_n$. To

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capture and study properties of the system $(m_n)_{n\in\mathbb{N}}$, universal fuzzy measures were introduced in [12] and studied in [23].

Definition 1. Denote $\mathscr{A} = \{(n,A) \mid n \in \mathbb{N}, A \subset X_n\}$. A mapping $M : \mathscr{A} \to [0,1]$ is called a universal fuzzy measure whenever for each $n \in \mathbb{N}$, $M(n,\cdot)$ is non-decreasing (with respect to the set inclusion of the second argument), $M(n,\emptyset) = 0$ and $M(n,X_n) = 1$.

The idea of universal fuzzy measures is related to the fuzzy measures in the same way as the extended aggregation operators (i.e., aggregation operators without fixed arity, acting on arbitrary finitely many inputs) are related to n-ary aggregation operators. For example, in several multicriteria decision problems we meet the problem of not knowing a priori the number of applied criteria, however, we need to know the weights of groups of criteria which will vary when adding some new criteria. Such complex knowledge about normalized weights of group of criteria can be summarized in a relevant universal fuzzy measure. Note that the name *universal fuzzy measure* reflects the fact that it describes the situation on an arbitrary finite space $\{1,2,\dots,n\}$ independently of n, similarly as for example the arithmetic means is a universal operator on real functions with finite domain.

Standard properties of fuzzy measures (additivity, symmetry, maxitivity, submodularity, subadditivity, belief, etc.) are extended to universal fuzzy measures in a natural way, requiring the relevant property to be satisfied by any fuzzy measure $m_n = M(n,\cdot),\ n \in \mathbb{N}$. These properties are inherited in the case of universal fuzzy measure M induced by a fuzzy measure μ on $\mathbb{N}$, $\mu(\{1\}) > 0$, $M(n,A) = \frac{\mu(A)}{\mu(X_n)}$.

Another class of universal fuzzy measures can be determined by means of a priori given weights.

Two subsequent properties establish some connections between fuzzy measures $m_n = M(n,\cdot),\ n \in \mathbb{N}$.

Definition 2. Let $M : \mathscr{A} \to [0,1]$ be a universal fuzzy measure. Then M is called natural whenever for each $n,k \in \mathbb{N},\ A \subset X_n$, it holds $M(n,A) \geqslant M(n+k,A)$. Moreover, M is called proportional whenever for each $n,k \in \mathbb{N},\ A \subset X_n$, it holds $M(n,A) \cdot M(n+k,X_n) = M(n+k,A)$.

Note that the decrease of a weight of a group of criteria in multi criteria decision making when taking into account some new criteria is a natural property reflected by the above introduced naturality of universal fuzzy measures. Proportionality of universal fuzzy measures strengthen the naturality, modelling the preservation of the ratio of weights of groups of criteria when adding some new criteria.

Evidently, each proportional universal fuzzy measure M is also natural, but not vice versa.

In [21], OWA operators induced by weight generator (quantifier) $g : [0,1] \to [0,1]$, g non-decreasing, $g(0) = 0,\ g(1) = 1$, were introduced as follows:

$$\mathrm{OWA}_g(x_1,\dots,x_n) = \sum_{i=1}^{n} \left(g\left(\frac{i}{n}\right) - g\left(\frac{i-1}{n}\right)\right) x'_i, \tag{1}$$

where $(x'_1,\dots,x'_n)$ is a non-decreasing permutation of the input vector $(x_1,\dots,x_n)$. As noticed in [5], each OWA operator on $[0,1]^n$ can be represented as the Choquet integral with respect to a symmetric fuzzy measure $m_n : \mathscr{P}(X_n) \to [0,1],\ m_n(A) = \mathrm{OWA}(\mathbf{1}_A(1),\dots,\mathbf{1}_A(n))$. Thus OWA_g can be represented as the Choquet integral with respect to the universal fuzzy measure $M : \mathscr{A} \to [0,1]$, $M(n,A) = \mathrm{OWA}_g(\mathbf{1}_A(1), \ldots, \mathbf{1}_A(n)) = \sum_{i=1}^{|A|} \left(g\left(\frac{i}{n}\right) - g\left(\frac{i-1}{n}\right)\right) = g\left(\frac{|A|}{n}\right)$.

Similarly, universal weighted mean generated by a weight generator g and denoted by W_g is a general aggregation operator given by

$$W_g(x_1, \ldots, x_n) = \sum_{i=1}^{n} \left(g\left(\frac{i}{n}\right) - g\left(\frac{i-1}{n}\right)\right)x_i.$$

This general aggregation operator can be seen as the Lebesgue integral with respect to the universal fuzzy measure $M : \mathscr{A} \to [0,1]$, $M(n,A) = W_g(\mathbf{1}_A(1), \ldots, \mathbf{1}_A(n)) = \sum_{i \in A}\left(g\left(\frac{i}{n}\right) - g\left(\frac{i-1}{n}\right)\right)$.

Inspired by these facts and by [8–10], we have introduced generator-based universal fuzzy measures in [13], generalizing both above mentioned universal fuzzy measures linked to the weight generator g. The aim of this paper is to continue in the study of special generated universal fuzzy measures and related fuzzy integrals. In the next section, we introduce basic generated universal fuzzy measures. In Section 3, special classes of basic generated universal fuzzy measures are studied, while Section 4 is devoted to the study of universal fuzzy measures representable as basic generated universal fuzzy measures. In Section 5 the naturality and the proportionality of basic generated universal fuzzy measures is discussed. In Section 6 we give a look on fuzzy integrals based on basic generated universal fuzzy measures. Finally, some conclusions are included.

2. Basic generated universal fuzzy measures

Definition 3. Let $g : [0,1] \to [0,1]$ be a non-decreasing mapping satisfying $g(0) = 0, g(1) = 1$, and let $h : \mathscr{A} \to \mathscr{P}(\mathbb{N})$ be a mapping non-decreasing in the second argument such that $h(n, \emptyset) = \emptyset$ and $h(n, X_n) = X_n$ for all $n \in \mathbb{N}$. Then the mapping g is called a weight generator, the mapping h is called a set generator, and a mapping $M_{h,g} : \mathscr{A} \to [0,1]$ given by $M_{h,g}(n,A) = \sum_{i \in h(n,A)}\left(g\left(\frac{i}{n}\right) - g\left(\frac{i-1}{n}\right)\right)$ (with convention $M_{h,g}(n,A) = 0$ if $h(n,A) = \emptyset$) is called a generated universal fuzzy measure. We denote by $\mathscr{G}$ the set of all weight generators and by $\mathscr{H}$ the set of all set generators.

Observe that the sets $\mathscr{G}$ and $\mathscr{H}$ are complete lattices with the weakest and strongest elements g_*, h_* and g^*, h^*, respectively (on $\mathscr{G}$ the standard partial order of functions is considered, while on $\mathscr{H}$ we consider the inclusion partial order of set-valued mappings),

$$g_*(x) = \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{else}, \end{cases} \qquad g^*(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{else}, \end{cases}$$
$$h_*(n,A) = \begin{cases} X_n & \text{if } A = X_n, \\ \emptyset & \text{else}, \end{cases} \qquad h^*(x) = \begin{cases} \emptyset & \text{if } A = \emptyset, \\ X_n & \text{else}. \end{cases}$$

Moreover, $\mathscr{G}$ is a convex set (i.e., it is closed under convex combinations of its members).

Note also that $M_{h_*,g}$ is the weakest universal fuzzy measure independently of g (i.e., each corresponding m_n is the weakest fuzzy measure on X_n). Similarly, $M_{h^*,g}$ is the strongest universal fuzzy measure.

Inspired by the two examples from introduction, we are interested in the case when the set generator $h : \mathscr{A} \to \mathscr{P}(\mathbb{N})$ is independent on n, i.e., when $h(n,A) = h(n+k,A)$ for all finite subsets $A \subset \mathbb{N}$, $n = \max A$ and for all $k \in \mathbb{N}$.

Definition 4. Assume mapping $h \in \mathscr{H}$ which is independent of n (we will denote it by γ, i.e., $\gamma(A) = h(n, A)$ for all $n \geqslant \max A$). Then the generated universal fuzzy measure $M_{\gamma,g}(n, A) = \sum_{i \in \gamma(A)} \left(g\left(\frac{i}{n}\right) - g\left(\frac{i-1}{n}\right)\right)$ will be called a basic generated universal fuzzy measure and the set of all set generators γ with the above property will be denoted by Γ. Moreover, $\gamma \in \Gamma$ will be called a basic set generator.

Evidently, a set generator $\gamma : \mathscr{F} \to \mathscr{F}$, where $\mathscr{F} = \{A \subset \mathbb{N} | A \text{ is finite}\}$, is a basic set generator if and only if it is non-decreasing and $\gamma(X_n) = X_n$ for all $n \in \mathbb{N}$, $\gamma(\emptyset) = \emptyset$. The class Γ is a complete lattice induced by a partial ordering $\preceq$, $\gamma_1 \preceq \gamma_2$ if and only if $\gamma_1(A) \subseteq \gamma_2(A)$ for all $A \in \mathscr{F}$. The following result shows that Γ is a bounded lattice.

Proposition 1. *Let* $\gamma_*(A) = X_{\max\{k, X_k \subseteq A\}} = X_{\min(\mathbb{N} \setminus A) - 1}$ *and* $\gamma^*(A) = X_{\max A}$, *with convention* $X_0 = \emptyset$. *Then* $\gamma_*, \gamma^* \in \Gamma$ *and for each* $\gamma \in \Gamma$ *it holds* $\gamma_*(A) \subseteq \gamma(A) \subseteq \gamma^*(A)$, $A \subset \mathbb{N}$, *i.e.,* γ_* *is the minimal basic set generator and* γ^* *is the maximal basic set generator.*

The proof is straightforward and therefore omitted.

Remark 1

(i) Note that a set generator $h \in \mathscr{H}$ can be derived from a family $(\gamma_n)_{n \in \mathbb{N}}$ of basic set generators, $h(n, A) = \gamma_n(A)$, if and only if $h(n, X_k) = X_k$ for each $n, k \in \mathbb{N}$, $k \leqslant n$.

(ii) Observe that the strongest basic generated universal fuzzy measure $M_{\gamma^*,g}$ based on a weight generator $g \in \mathscr{G}$ is given by the formula

$$M_{\gamma^*,g}(n, A) = g\left(\frac{\max A}{n}\right) = \bigvee_{i \in A} g\left(\frac{i}{n}\right),$$

and thus for each $n \in \mathbb{N}$, the fuzzy measure $m_n = M_{\gamma^*,g}(n, \cdot)$ is a possibility measure [25] on X_n with the possibility distribution $\pi_n : X_n \to [0, 1]$ given by $\pi_n(i) = g\left(\frac{i}{n}\right)$. Similarly, the weakest basic generated universal fuzzy measure $M_{\gamma_*,g}$ related to $g \in \mathscr{G}$ is given by the formula

$$M_{\gamma_*,g}(n, A) = g\left(\frac{\min \mathbb{N} \setminus A}{n}\right) = \bigwedge_{i \notin A} g\left(\frac{\min(i-1, n)}{n}\right).$$

Moreover, for each $n \in \mathbb{N}$, the corresponding fuzzy measure $m_n = M_{\gamma_*,g}(n, \cdot)$ is a necessity measure on X_n (its dual m_n^d is a possibility measure on X_n with the possibility distribution $\pi_n^d : X_n \to [0, 1]$ given by $\pi_n^d(i) = 1 - g\left(\frac{i-1}{n}\right)$).

3. Special classes of basic generated universal fuzzy measures

Recall that a universal fuzzy measure M is called additive if and only if $M(n, A \cup B) = M(n, A) + M(n, B)$ for all $n \in \mathbb{N}$, $A, B \subset X_n$, $A \cap B = \emptyset$. Similarly, M is called symmetric (maxitive) whenever $M(n, A) = M(n, B)$ for $A, B \subset X_n$, $|A| = |B| (M(n, A \cup B) = \max(M(n, A), M(n, B))$ for all $n \in \mathbb{N}$, $A, B \subset X_n$).

In the following proposition we describe three special types of basic set generators γ.

Proposition 2. *A basic generated universal fuzzy measure* $M_{\gamma,g}$ *is*

(i) *symmetric for each weight generator* $g \in \mathscr{G}$ *if and only if* $\gamma(A) = X_{|A|}$ *for* $A \subset \mathbb{N}$ *(*γ *with this property will be denoted by* γ_s*).*

(ii) *additive for each weight generator* $g \in \mathscr{G}$ *if and only if* $\gamma(A) = A$ *for* $A \subset \mathbb{N}$ *(*γ *with this property will be denoted by* γ_a*).*
(iii) *maxitive for each weight generator* $g \in \mathscr{G}$ *if and only if* $\gamma(A) = X_{\max A}$*, i.e., if* $\gamma = \gamma^*$.

Proof. The sufficiency in all three cases is straightforward (and in the case (iii) it was already discussed in Remark 1). In the necessity parts of the proof for each item (i)-(iii) we will deal with weight generators $g_n \in \mathscr{G}$, $n \in \mathbb{N}$, determined by $g_n\left(\frac{i}{n}\right) = \frac{2^i-1}{2^n-1}$ for $i \in \{1, \ldots, n-1\}$ (for example each g_n can be a continuous piecewise linear weight generator). Then $M_{\gamma,g_n}(n,A) = M_{\gamma,g_n}(n,B)$ imply $\gamma(A) = \gamma(B)$. For a fixed $\gamma \in \Gamma$,

(i) Let $M_{\gamma,g}$ be a symmetric universal fuzzy measure for each $g \in \mathscr{G}$. Fix $A \subset X_n$. Due to the symmetry of M_{γ,g_n}, $M_{\gamma,g_n}(n,A) = M_{\gamma,g_n}(n,X_{|A|})$ and thus $\gamma(a) = \gamma(X_{|A|}) = X_{|A|}$.
(ii) Let $M_{\gamma,g}$ be an additive universal fuzzy measure for each $g \in \mathscr{G}$. Because of the additivity, for any fixed $A \subset X_n$, $M_{\gamma,g_n}(n,A) = \sum_{i\in A} M_{\gamma,g_n}(n,\{i\})$ and $\gamma(\{i\}) \cap \gamma(\{j\}) = \emptyset$ for all $i,j \in X_n$. However, then $\gamma(\{1\}) = \{1\}$ (this holds for any $\gamma \in \Gamma$), $\gamma(\{2\}) = \{2\}$ because of $1 = M_{\gamma,g_2}(2,X_2) = M_{\gamma,g_2}(2,\{1\}) + M_{\gamma,g_2}(2,\{2\}) = g_2\left(\frac{1}{2}\right) + M_{\gamma,g_2}(2,\{2\})$, i.e.,$M_{\gamma,g_2}(2,\{2\}) = 1 - g_2\left(\frac{1}{2}\right) = g_2\left(\frac{2}{2}\right) - g_2\left(\frac{1}{2}\right) = \sum_{i\in\gamma(\{2\})} g_2\left(\frac{i}{n}\right) - g_2\left(\frac{i-1}{n}\right)$, and similarly by induction we get $\gamma(\{i\}) = \{i\}$, $i \in \mathbb{N}$. However, then $M_{\gamma,g_n}(n,A) = \sum_{j\in\gamma(A)}\left(g_n\left(\frac{i}{n}\right) - g_n\left(\frac{i-1}{n}\right)\right) = \sum_{i\in A} M_{\gamma,g_n}(n,\{i\}) = \sum_{i\in A}\left(g_n\left(\frac{i}{n}\right) - g_n\left(\frac{i-1}{n}\right)\right)$ imply $\gamma(A) = A$.
(iii) Let $M_{\gamma,g}$ be a maxitive universal fuzzy measure for each $g \in \mathscr{G}$. Evidently, $\gamma(\{1\}) = \{1\}$. Moreover, $1 = M_{\gamma,g_2}(2,X_2) = \bigvee_{i\in X_2} M_{\gamma,g_2}(2,\{i\}) = g_2\left(\frac{1}{2}\right) \vee \left(\bigvee_{i\in\gamma(\{2\})}\left(g_2\left(\frac{i}{2}\right) - g_2\left(\frac{i-1}{2}\right)\right)\right)$ imply $\bigvee_{i\in\gamma(\{2\})}\left(g_2\left(\frac{i}{2}\right) - g_2\left(\frac{i-1}{2}\right)\right) = 1$, i.e., $\gamma(\{2\}) = X_2$. Similarly, $\gamma(\{i\}) = X_i$ for each $i \in \mathbb{N}$. The monotonicity of γ ensures for each finite subset A of $\mathbb{N}$ that $X_{\max A} = \gamma(\{\max A\}) \subseteq \gamma(A) \subseteq \gamma(X_{\max A}) = X_{\max A}$, i.e., $\gamma(A) = X_{\max A}$ and thus $\gamma = \gamma^*$. □

Remark 2. Similarly we can show that $M_{\gamma,g}$ is a minitive universal fuzzy measure for each $g \in \mathscr{G}$, i.e., $M_{\gamma,g}(n, A \cap B) = M_{\gamma,g}(n,A) \wedge M_{\gamma,g}(n,B)$ for all $n \in \mathbb{N}$, $A, B \in \mathscr{P}(X_n)$, if and only if $\gamma = \gamma_*$.

Remark 3. A symmetric basic universal fuzzy measure generated by a weight generator $g \in \mathscr{G}$ is given by $M_{\gamma_s,g}(n,A) = g\left(\frac{|A|}{n}\right)$, and an additive basic universal fuzzy measure generated by a weight generator $g \in \mathscr{G}$ is given by $M_{\gamma_a,g}(n,A) = \sum_{i\in A}\left(g\left(\frac{i}{n}\right) - g\left(\frac{i-1}{n}\right)\right)$. These two types of generated universal fuzzy measures were exploited and discussed in [1,21].

4. Universal fuzzy measures representable as basic generated universal fuzzy measures

In the following proposition we describe conditions for additive and symmetric universal fuzzy measures to be basic generated universal fuzzy measures. First, let us note that for a universal fuzzy measure $M_{h,g}$ generated by a weight generator $g \in \mathscr{G}$ it is enough to know the values of the weight generator g only on $[0,1] \cap \mathbb{Q}$. Due to the required monotonicity of weight generator g, for irrational $x \in]0,1[$ we can always assume $g(x) = \sup\{g(r) | r \in [0,x] \cap \mathbb{Q}\}$.

Proposition 3

(i) *A symmetric universal fuzzy measure* M *is a basic generated universal fuzzy measure if and only if for all* $n, k, t \in \mathbb{N}$, $t \leqslant n$ *it holds* $M(n, X_t) = M(kn, X_{kt})$.

(ii) *An additive universal fuzzy measure M is a basic generated universal fuzzy measure if and only if for all* $n,k,t \in \mathbb{N}$, $t \leqslant n$ *it holds* $M(n,X_t) = M(kn,X_{kt})$ *(or equivalently if* $M(n,\{t\}) = M(kn,X_{kt} \setminus X_{k(t-1)})$*).*

Proof

(i) Let a symmetric universal fuzzy measure M be a basic generated universal fuzzy measure, i.e., $M(n,A) = g(\frac{|A|}{n})$. Then

$$M(n,X_t) = g\left(\frac{t}{n}\right) = g\left(\frac{kt}{kn}\right) = M(kn,X_{kt}).$$

Vice versa, let a symmetric universal fuzzy measure M fulfils

$$M(n,X_t) = M(kn,X_{kt})$$

for all $n,k,t \in \mathbb{N}$, $t \leqslant n$. Let us define a mapping $g : [0,1] \cap \mathbb{Q} \to [0,1]$ by putting $g\left(\frac{p}{q}\right) = M(q,X_p)$. We will show that g is a weight generator of universal fuzzy measure M. The mapping g is well-defined since for $\frac{p}{q} = \frac{r}{s}$ we have $g\left(\frac{p}{q}\right) = M(q,X_p) = M(sq,X_{sp}) = M(sq,X_{qr}) = M(s,X_r) = g(\frac{r}{s})$. Moreover, $g\left(\frac{0}{q}\right) = M(q,\emptyset) = 0$ and $g(\frac{1}{1}) = M(1,X_1) = 1$ and g is non-decreasing since for $\frac{p}{q} \leqslant \frac{r}{s}$ we have $g\left(\frac{p}{q}\right) = M(q,X_p) = M(sq,X_{sp}) \leqslant M(sq,X_{qr}) = M(s,X_r) = g(\frac{r}{s})$. Using the facts that $[0,1] \cap \mathbb{Q}$ is dense in [0,1] and that g is non-decreasing we can extend the mapping g to the whole interval [0,1]. We get $M(n,A) = M(n,X_{|A|}) = g\left(\frac{|A|}{n}\right)$, i.e., g is a weight generator of universal fuzzy measure M, $M = M_{\gamma_s,g}$.

(ii) First we will show that for $n,k,t \in \mathbb{N}$, $t \leqslant n$ conditions $M(n,X_t) = M(kn,X_{kt})$ and $M(n,\{t\}) = M(kn,X_{kt} \setminus X_{k(t-1)})$ are equivalent. Let M be an additive universal fuzzy measure such that

$$M(n,X_t) = M(kn,X_{kt}).$$

Then because of additivity of M we have $\sum_{i=1}^{t} M(n,\{i\}) = M(n,X_t) = M(kn,X_{kt}) = \sum_{j=1}^{kt} M(kn,\{j\})$.For $t = 1$ we have $M(n,\{1\}) = \sum_{j=1}^{k} M(kn,\{j\})$.For $t = 2$ we have

$$M(n,\{1\}) + M(n,\{2\}) = \sum_{j=1}^{k} M(kn,\{j\}) + \sum_{j=k+1}^{2k} M(kn,\{j\}),$$

i.e., we get $M(n,\{2\}) = \sum_{j=k+1}^{2k} M(kn,\{j\})$ and analogically $M(n,\{t\}) = \sum_{j=(t-1)k+1}^{tk} M(kn,\{j\})$.Vice versa, let M be an additive universal fuzzy measure such that

$$M(n,\{t\}) = M(kn,X_{kt} \setminus X_{k(t-1)}).$$

Then $M(n,X_t) = \sum_{i=1}^{t} M(n,\{i\}) = \sum_{i=1}^{t} \sum_{j=(i-1)k+1}^{ik} M(kn,\{j\}) = \sum_{j=1}^{kt} M(kn,\{j\}) = M(kn, X_{kt})$.

Let M be an additive basic universal fuzzy measure generated by a weight generator $g \in \mathcal{G}$, i.e., $M(n,A) = \sum_{i \in A} \left(g(\frac{i}{n}) - g(\frac{i-1}{n})\right)$. Then

$$M(n,X_t) = \sum_{i=1}^{t} \left(g\left(\frac{i}{n}\right) - g\left(\frac{i-1}{n}\right)\right) = g\left(\frac{t}{n}\right) = g\left(\frac{kt}{kn}\right)$$

$$= \sum_{i=1}^{kt} \left(g\left(\frac{i}{kn}\right) - g\left(\frac{i-1}{kn}\right)\right) = M(kn,X_{kt}).$$

Vice versa, let M be an additive basic universal fuzzy measure such that $M(n,X_t) = M(kn,X_{kt})$. Let us assume the mapping g from part (i) of this proof. We will show that g generates M. We have $M(n,A) = \sum_{i\in A} M(n,\{i\}) = \sum_{i\in A}(M(n,X_i) - M(n,X_{i-1})) = \sum_{i\in A}\left(g\left(\frac{i}{n}\right) - g\left(\frac{i-1}{n}\right)\right)$, i.e., $M = M_{\gamma_a,g}$. $\square$

Similarly, we can show that a maxitive universal fuzzy measure M is a maxitive basic generated universal fuzzy measure if $M(n,A) = M(n,X_{\max A})$ and $M(n,X_t) = M(kn,X_{kt})$ for all $n,k,t \in \mathbb{N}$, $t \leqslant n$, $A \subset X_n$.

5. Naturality and proportionality of basic generated universal fuzzy measures

In the following propositions we describe the properties of a weight generator which generates an additive (symmetric) universal fuzzy measure which is natural (proportional).

Proposition 4. *Each symmetric basic generated universal fuzzy measure $M_{\gamma_s,g}$ is natural.*

Proof. Let $M_{\gamma_s,g}$ be a symmetric basic universal fuzzy measure generated by the weight generator g, $n_1 \leqslant n_2$ and $A \subset X_{n_1}$. Then $\frac{|A|}{n_1} \geqslant \frac{|A|}{n_2}$ and since g is non-decreasing we get $g\left(\frac{|A|}{n_1}\right) \geqslant g\left(\frac{|A|}{n_2}\right)$, i.e., $M_{\gamma_s,g}(n_1,A) \geqslant M_{\gamma_s,g}(n_2,A)$. Consequently, $M_{\gamma_S,g}$ is natural. $\square$

Proposition 5. *Let $g \in \mathcal{G}$ be continuously differentiable on $]0,1[$. Then the following are equivalent*:

(i) *$x \cdot g'(x)$ is non-decreasing on $]0,1[$.*
(ii) *The additive basic generated universal fuzzy measure $M_{\gamma_a,g}$ is natural.*
(iii) *The basic generated universal fuzzy measure $M_{\gamma,g}$ is natural for an arbitrary $\gamma \in \Gamma$.*

Proof. (i) $\Rightarrow$ (ii) Let $x \cdot g'(x)$ be non-decreasing on $]0,1[$. Then for all $t \in \left[\frac{i}{n+1},\frac{i}{n}\right]$ we have

$$t\cdot g'(t) \geqslant \frac{i}{n+1} g'\left(\frac{i}{n+1}\right),$$

$$g'(t) \geqslant \frac{i}{n+1} g'\left(\frac{i}{n+1}\right)\frac{1}{t},$$

$$\int_{\frac{i}{n+1}}^{\frac{i}{n}} g'(t)\,\mathrm{d}t \geqslant \frac{i}{n+1} g'\left(\frac{i}{n+1}\right)\int_{\frac{i}{n+1}}^{\frac{i}{n}} \frac{1}{t}\,\mathrm{d}t$$

$$\int_{\frac{i}{n+1}}^{\frac{i}{n}} g'(t)\,\mathrm{d}t \geqslant \frac{i}{n+1} g'\left(\frac{i}{n+1}\right)\ln\frac{n+1}{n}.$$

Similarly for all $t \in \left[\frac{i-1}{n+1},\frac{i-1}{n}\right]$ we have

$$\frac{i-1}{n} g'\left(\frac{i-1}{n}\right) \geqslant t\cdot g'(t),$$

$$\frac{i-1}{n} g'\left(\frac{i-1}{n}\right)\ln\frac{n+1}{n} \geqslant \int_{\frac{i-1}{n+1}}^{\frac{i-1}{n}} g'(t)\,\mathrm{d}t.$$

Since $x \cdot g'(x)$ is non-decreasing we have:

$$\frac{i}{n+1} g'\left(\frac{i}{n+1}\right) \ln\frac{n+1}{n} \geqslant \frac{i-1}{n} g'\left(\frac{i-1}{n}\right) \ln\frac{n+1}{n}$$

and together we get

$$\int_{\frac{i}{n+1}}^{\frac{i}{n}} g'(t)\,\mathrm{d}t \geqslant \int_{\frac{i-1}{n+1}}^{\frac{i-1}{n}} g'(t)\,\mathrm{d}t,$$

i.e.,

$$g\left(\frac{i}{n}\right) - g\left(\frac{i}{n+1}\right) \geqslant g\left(\frac{i-1}{n}\right) - g\left(\frac{i-1}{n+1}\right)$$

for all $n \in \mathbb{N}$, $i = 1, \ldots, n$. Then $M_{\gamma_a,g}(n,\{i\}) \geqslant M_{\gamma_a,g}(n+1,\{i\})$ and from additivity of $M_{\gamma_a,g}$ it follows that $M_{\gamma_a,g}$ is natural.(ii) $\Rightarrow$ (i) Let $M_{\gamma_a,g}$ be natural. Then $M_{\gamma_a,g}(n,A) \geqslant M_{\gamma_a,g}(n+1,A)$ for all $n \in \mathbb{N}$ and $A \subset X_n$. For any $x, y \in]0,1[\cap \mathbb{Q}, x > y$ there are $n, i, j \in \mathbb{N}$, $j < i < n$ such that $x = \frac{i}{n}$ and $y = \frac{j}{n}$. Then also for all $k \in \mathbb{N}$ we have $x = \frac{ki}{kn}$ and $y = \frac{kj}{kn}$. From the naturality of $M_{\gamma_a,g}$ we have

$$M_{\gamma_a,g}(kn, X_{ki} \setminus X_{kj}) \geqslant M_{\gamma_a,g}(kn+1, X_{ki} \setminus X_{kj}),$$

i.e.,

$$g\left(\frac{ki}{kn}\right) - g\left(\frac{kj}{kn}\right) \geqslant g\left(\frac{ki}{kn+1}\right) - g\left(\frac{kj}{kn+1}\right)$$
$$g\left(\frac{ki}{kn}\right) - g\left(\frac{ki}{kn+1}\right) \geqslant g\left(\frac{kj}{kn}\right) - g\left(\frac{kj}{kn+1}\right)$$
$$g(x) - g\left(x - \frac{x}{kn+1}\right) \geqslant g(y) - g\left(y - \frac{y}{kn+1}\right)$$
$$x\frac{g(x) - g\left(x - \frac{x}{kn+1}\right)}{\frac{x}{kn+1}} \geqslant y\frac{g(y) - g\left(y - \frac{y}{kn+1}\right)}{\frac{y}{kn+1}}$$

and since $\lim_{k\to\infty} x\frac{g(x)-g\left(x-\frac{x}{kn+1}\right)}{\frac{x}{kn+1}} = x \cdot g'(x)$ and $\lim_{k\to\infty} y\frac{g(y)-g\left(y-\frac{y}{kn+1}\right)}{\frac{y}{kn+1}} = y \cdot g'(y)$ we get $x \cdot g'(x) \geqslant y \cdot g'(y)$. However, $]0,1[\cap \mathbb{Q}$ is dense in $]0,1[$ and g is continuously differentiable on $]0,1[$ and thus $x \cdot g'(x) \geqslant y \cdot g'(y)$ for all $x, y \in]0,1[, x > y$, i.e., $x \cdot g'(x)$ is non-decreasing on $]0,1[$.

(ii)$\Rightarrow$ (iii) Let $M_{\gamma_a,g}$ be natural and let $\gamma \in \Gamma$. Then $M_{\gamma_a,g}(n,\{i\}) \geqslant M_{\gamma_a,g}(n+1,\{i\})$ for all $n \in \mathbb{N}$ and $i \in X_n$, i.e.,

$$g\left(\frac{i}{n}\right) - g\left(\frac{i-1}{n}\right) \geqslant g\left(\frac{i}{n+1}\right) - g\left(\frac{i-1}{n+1}\right)$$

and thus

$$\sum_{i\in\gamma(A)} \left(g\left(\frac{i}{n}\right) - g\left(\frac{i-1}{n}\right)\right) \geqslant \sum_{i\in\gamma(A)} \left(g\left(\frac{i}{n+1}\right) - g\left(\frac{i-1}{n+1}\right)\right)$$

for all $n \in \mathbb{N}$, $A \subset X_n$, i.e., $M_{\gamma,g}(n,A) \geqslant M_{\gamma,g}(n+1,A)$, i.e., $M_{\gamma,g}$ is natural.

(iii) $\Rightarrow$ (ii) This implication is obvious. □

Proposition 6. *For any basic set generator $\gamma \in \Gamma$, the basic generated universal fuzzy measure $M_{\gamma,g}$ with a weight generator $g \in \mathcal{G}$ is proportional if and only if g is a power function, i.e., $g(x) = x^r, r > 0$.*

Proof. Let $M_{\gamma,g}$ be a basic universal fuzzy measure generated by a weight generator $g \in \mathcal{G}$. Let $g(x) = x^r, r > 0$. Then

$$\begin{aligned}\frac{M_{\gamma,g}(n+k,A)}{M_{\gamma,g}(n+k,X_n)} &= \frac{\sum_{i\in\gamma(A)}\left(g\left(\frac{i}{n+k}\right) - g\left(\frac{i-1}{n+k}\right)\right)}{\sum_{i=1}^{n}\left(g\left(\frac{i}{n+k}\right) - g\left(\frac{i-1}{n+k}\right)\right)} \\ &= \frac{\sum_{i\in\gamma(A)}\left(\left(\frac{i}{n+k}\right)^r - \left(\frac{i-1}{n+k}\right)^r\right)}{\left(\frac{n}{n+k}\right)^r} = \sum_{i\in\gamma(A)}\left(\left(\frac{i}{n}\right)^r - \left(\frac{i-1}{n}\right)^r\right) \\ &= \sum_{i\in\gamma(A)}\left(g\left(\frac{i}{n}\right) - g\left(\frac{i-1}{n}\right)\right) = M_{\gamma,g}(n,A),\end{aligned}$$

i.e., $M_{\gamma,g}$ is proportional.

Vice versa, let $M_{\gamma,g}$ be proportional. Then for all $k, n \in \mathbb{N}$, $A \subset X_n$ we have

$$\frac{M_{\gamma,g}(n+k,A)}{M_{\gamma,g}(n+k,X_n)} = M_{\gamma,g}(n,A),$$

$$\frac{\sum\limits_{i\in\gamma(A)}\left(g\left(\frac{i}{n+k}\right) - g\left(\frac{i-1}{n+k}\right)\right)}{\sum\limits_{i=1}^{n}\left(g\left(\frac{i}{n+k}\right) - g\left(\frac{i-1}{n+k}\right)\right)} = \sum_{i\in\gamma(A)}\left(g\left(\frac{i}{n}\right) - g\left(\frac{i-1}{n}\right)\right),$$

$$\sum_{i\in\gamma(A)}\left(g\left(\frac{i}{n+k}\right) - g\left(\frac{i-1}{n+k}\right)\right) = g\left(\frac{n}{n+k}\right)\sum_{i\in\gamma(A)}\left(g\left(\frac{i}{n}\right) - g\left(\frac{i-1}{n}\right)\right).$$

Let $A = X_t, t \in \{0, 1, \ldots, n\}$. Then we obtain

$$g\left(\frac{t}{n+k}\right) = g\left(\frac{n}{n+k}\right)g\left(\frac{t}{n}\right)$$

for all $k, n \in \mathbb{N}$ and $t \in \{0, 1, \ldots, n\}$. This means that the function g is a solution of the Cauchy functional equation $g(ab) = g(a)g(b)$ for all $a, b \in [0, 1] \cap \mathbb{Q}$. Since $[0, 1] \cap \mathbb{Q}$ is dense in [0,1] and g is bounded and non-decreasing function on [0,1] we get $g(x) = x^r, r > 0$. □

6. Fuzzy integrals based on basic generated universal fuzzy measures

As already mentioned, fuzzy integrals [11,17] with respect to universal fuzzy measures allow to construct several types of aggregation operators.

For example, the Choquet integral [3,11,16,17,24] with respect to an additive basic generated universal fuzzy measure $M_{\gamma_a,g}$ is just the weighted mean W related to the weighting triangle $\Delta = (w_{in} \mid n \in \mathbb{N}, i \in \{1, \ldots, n\})$, $w_{in} = g\left(\frac{i}{n}\right) - g\left(\frac{i-1}{n}\right)$. Moreover, the Choquet integral with respect to a symmetric basic generated universal fuzzy measure $M_{\gamma_s,g}$ yields the

OWA operator [20,22] OWA_g, see (1). The next results concerning the basic universal fuzzy measures and the Choquet integral can be derived directly from our previous results and the formula for the discrete Choquet integral, see e.g., [3,5,6,16,18,24].

Proposition 7. *Let $M_{\gamma,g}$ be a basic generated fuzzy measure. Then the corresponding Choquet integral-based aggregation operator $C_{\gamma,g} : \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ is given by*

$$C_{\gamma,g}(x_1,\dots,x_n) = \sum_{i=1}^{n}\left(x'_i \cdot \sum_{j\in A_i}\left(g\left(\frac{j}{n}\right) - g\left(\frac{j-1}{n}\right)\right)\right), \tag{2}$$

where $A_i = \gamma(\{j \in X_n | x_j \geqslant x'_i\}) \setminus \gamma(\{j \in X_n | x_j \geqslant x'_{i+1}\})$, and $x'_{n+1} = 2$ by convention.

Let $\sigma : \{1,\dots,n\} \to \{1,\dots,n\}$ be a permutation such that $x'_i = x_{\sigma(i)}$. Then for the extremal basic set generators γ_*, γ^* we have:

1. $C_{\gamma_*,g}(x_1,\dots,x_n) = \sum_{i=1}^{n} x'_i\left(g\left(\frac{s(i)}{n}\right) - g\left(\frac{s(i+1)}{n}\right)\right)$, where $s(1) = n, s(j) = \min(\sigma(1),\dots,\sigma(j-1)) - 1$ for $j = 2,\dots,n$.
2. $C_{\gamma^*,g}(x_1,\dots,x_n) = \sum_{i=1}^{n} x'_i\left(g\left(\frac{r(i)}{n}\right) - g\left(\frac{r(i+1)}{n}\right)\right)$, where $r(i) = \max(\sigma(i),\dots,\sigma(n))$.

Note that for the extremal set generators h_* and h^* we have $C_{h_*,g} = \min$ and $C_{h^*,g} = \max$, independently of the weight generator g.

Similarly, we can introduce several results concerning the basic universal fuzzy measures and the Sugeno integral (for its discrete version see, e.g., [4,6,7,16–18,24]).

Proposition 8. *Let $M_{\gamma,g}$ be a basic generated fuzzy measure. Then the corresponding Sugeno integral-based aggregation operator $S_{\gamma,g} : \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ is given by*

$$S_{\gamma,g}(x_1,\dots,x_n) = \bigvee_{i=1}^{n}\left(x'_i \wedge \left(\sum_{j\in\gamma(\{\sigma(i),\dots,\sigma(n)\})}\left(g\left(\frac{j}{n}\right) - g\left(\frac{j-1}{n}\right)\right)\right)\right).$$

Moreover, using the same notation as in Proposition 7 it holds

$$S_{\gamma_*,g}(x_1,\dots,x_n) = \bigvee_{i=1}^{n}\left(x'_i \wedge g\left(\frac{s(i)}{n}\right)\right),$$

$$S_{\gamma^*,g}(x_1,\dots,x_n) = \bigvee_{i=1}^{n}\left(x'_i \wedge g\left(\frac{r(i)}{n}\right)\right).$$

Note that similarly as in the case of the Choquet integral, also for the Sugeno integral it holds $S_{h_*,g} = \min$ and $S_{h^*,g} = \max$, independently of g.

7. Conclusion

We have introduced and discussed basic generated universal fuzzy measures. The corresponding fuzzy integrals yield special aggregation functions, including the generated OWA operators and the generated weighted means. We expect several applications of our concept of basic generated universal fuzzy measures. First steps in this direction were already done in [1] when fitting the weight generators of generated OWA operators to real data. Another promising application is the theory of asymptotic densities of subsets of $\mathbb{N}$, see e.g., [14,19]. Here the first step in this direction was done in [15].

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