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Maximum (g, f)-factors of a general graph

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Abstract

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This paper presents a characterization of maximum (g, f)-factors of a general graph in which multiple edges and loops are allowed. An analogous characterization of the minimum (g, f)-factors of a general graph is also presented. In addition, we obtain a transformation theorem for any two general graphs on the same vertex set. As special cases, we have the transformation theorems for both maximum (g, f)-factors and minimum (g, f)-factors. Our results generalize some of C. Berge's results on maximum matchings and maximum c-matchings of a multiple graph.

1. Introduction

In general, we follow the notation and terminology of [1] and [5]. A general graph G is a pair $(V(G), E(G))$, where $V(G)$ is a finite non-empty set of elements called vertices, and $E(G)$ is a finite collection of unordered pairs of the elements of $V(G)$, called edges. We allow multiple edges and loops in a general graph. A general graph without loops is called a multigraph. A simple graph is a general graph having neither loops nor multiple edges.

Let G be a general graph, and g, f and c be integer-valued functions defined on $V(G)$. We denote by $d_G(v)$ the valency of vertex v in G, and assume that a loop on v contributes 2 to the valency of v. A spanning subgraph H and G is said to be a (g, f)-factor of G if $g(v) \leq d_H(v) \leq f(v)$ for every vertex $v \in V(G)$. A c-matching of G is defined as a (g, f)-factor of G with $g(v) = 0$ and $f(v) = c(v)$ for every $v \in V(G)$. If $c(v) = 1$ for every $v \in V(G)$, then a c-matching reduces to a matching of G. A (g, f)-factor of G with maximum (minimum) number of edges is called a maximum (minimum) (g, f)-factor of G. The maximum matching and maximum c-matching are defined analogously (see [5]).

Lovász [8] has given a necessary and sufficient condition for the existence of a (g, f)-factor in a general graph. In this paper, we shall consider the problem of characterizing maximum (g, f)-factors of G if G contains (g, f)-factors. We obtain such a characterization in terms of augmenting chains which generalizes the well-known theorems of C. Berge on maximum matchings and maximum c-matchings. We also give a characterization of minimum (g, f)-factors.

Definition 1.1. For a given subgraph M of a general graph G, an *M-alternating chain* is defined as a chain of G whose edges are alternately in $E(M)$ and $E(G)\backslash E(M)$. Here, a chain is not permitted to use the same edge more than once, but it may visit the same vertex several times.

Definition 1.2. Given a (g, f)-factor F of G, an *F-augmenting chain* is defined as an F-alternating chain $x_1x_2 \dots x_k$ of G satisfying the following conditions:

(1) (x_1, x_2) and (x_{k-1}, x_k) belong to $E(G)\backslash E(F)$.

(2) If x_1 and x_k are the same vertex, then $d_F(x_1) \leqslant f(x_1) - 2$; otherwise, $d_F(x_1)$ and $d_F(x_k)$ are less than $f(x_1)$ and $f(x_k)$ respectively.

Similarly, an F-reducing chain in G is defined as follows.

Definition 1.3. Given a (g, f)-factor of G, an *F-reducing chain* is defined as an F-alternating chain $x_1x_2 \dots x_k$ of G satisfying the following conditions:

(1) (x_1, x_2) and (x_{k-1}, x_k) belong to $E(F)$.

(2) If x_1 and x_k are the same vertex, then $d_F(x_1) \geqslant g(x_1) + 2$; otherwise, $d_F(x_1)$ and $d_F(x_k)$ are greater than $g(x_1)$ and $g(x_k)$ respectively.

In this paper, we prove that a (g, f)-factor F of G is maximum if and only if G has no F-augmenting chains. Another problem about maximum c-matchings considered by Berge is the relationship between any two maximum c-matchings. Here we obtain a transformation theorem for any two general graphs on the same vertex set. As special cases, we have transformation theorems for both maximum (g, f)-factors and minimum (g, f)-factors, and Berge's transformation theorem.

2. A characterization of maximum (g, f)-factors

In this section, we give the following characterization of maximum (g, f)-factors of a general graph.

Theorem 2.1. *Let G be a general graph and F be a (g, f)-factor of G. Then F is a maximum (g, f)-factor of G if and only if G has no F-augmenting chains.*

Corollary 2.2 (Berge [3]). *Let G be a multigraph and F be a c-matching of G. Suppose the edges of F are denoted by heavy lines. If we add a new vertex x_0, linked to x_i by $d_F(x_i)$ light edges and $c_i - d_F(x_i)$ heavy edges, F is a maximum c-matching, if and only if no alternating chain leaves x_0 by a heavy edge and comes back to x_0 by a heavy edge.*

Corollary 2.3 (Berge [2]). *Let F be a matching of a multigraph G. Then F is a maximum matching if and only if there is no F-augmenting chain in G.*

As a dual of Theorem 2.1, we have the following theorem.

Theorem 2.4. *Suppose F is a (g, f)-factor of G. Then F is a minimum (g, f)-factor of G if and only if G has no F-reducing chains.*

Here we only give the proof of Theorem 2.1, because Theorem 2.4 can be proved similarly. To this end, we need the following definitions and lemmas. For two general graphs F and H on the same vertex set, we shall use $F \oplus H$ to denote the symmetric difference of F and H, namely, the general graph obtained from F and H by removing their common edges and putting all the remaining edges together. Strictly speaking, let F and G be two general graphs on $V = \{v_1, v_2, \ldots, v_n\}$, and f_i and h_{ij} be the number of edges joining v_i and v_j in F and G respectively, then $F \oplus H$ is the general graph on V with $|f_{ij} - h_{ij}|$ edges joining v_i and v_j.

Definition 2.5. Let F and H be two general graphs on the same vertex set. An F-alternating chain $x_1x_2 \ldots x_k$ $(k \geq 2)$ in $F \oplus H$ is said to be an *F-augmenting chain with respect to H* if the following conditions are satisfied:

(1) (x_1, x_2) and (x_{k-1}, x_k) belong to $E(H)$.

(2) If x_1 and x_k are the same vertex, then $d_H(x_1) \geq d_F(x_1) + 2$; otherwise, $d_H(x_1)$ and $d_H(x_k)$ are greater than $d_F(x_1)$ and $d_F(x_k)$ respectively.

Let H be a graph containing an edge (x, y). We denote by $H \backslash (x, y)$ the graph obtained from H by deleting the edge (x, y).

Lemma 2.6. *Let F and H be two general graphs on the same vertex set, and let x, y and z be vertices of G (here x, y and z are not necessarily distinct). Suppose $(x, y) \in E(H)$, $(y, z) \in E(F)$ and $d_F(z) > d_H(z)$. Denote by H' and F' the graphs $H \backslash (x, y)$ and $F \backslash (y, z)$ respectively. If there exists an F'-augmenting chain C with respect to H', then C is also an F-augmenting chain with respect to H.*

Proof. We first consider the case when x, y and z are distinct.

Suppose $C = x_1x_2 \ldots x_k$ is an F'-augmenting chain with respect to H'. So C is also an F-alternating chain in $F \oplus H$ with (x_1, x_2), $(x_{k-1}, x_k) \in E(H)$. Therefore, we only need to prove that x_1 and x_k satisfy condition (2) in Definition 2.5.

Case 1: $x_1 \neq x_k$.

Suppose $d_F(x_1) \geq d_H(x_1)$. Since $d_{H'}(x_1) > d_{F'}(x_1)$, $F = F' \cup (y, z)$ and $H = H' \cup (x, y)$, we must have $x_1 = z$. Hence

$$d_F(z) = d_{F'}(z) + 1 \leq d_{H'}(z) = d_H(z).$$

This contradicts the condition $d_F(z) > d_H(z)$. Therefore, we have $d_H(x_1) > d_F(x_1)$. Similarly, we have $d_H(x_k) > d_F(x_k)$.

Case 2: $x_1 = x_k$.

Suppose $d_H(x_1) < d_F(x_1) + 2$. Since $d_{H'}(x_1) \geq d_{F'}(x_1) + 2$, we may have $x_1 = z$ as in Case 1. Hence

$$d_F(z) = d_{F'}(z) + 1 \leq d_{H'}(z) - 1 = d_H(z) - 1,$$

a contradiction again. Thus, we must have $d_H(x_1) \geq d_F(x_1) + 2$.

We have now shown that the lemma is true for the case when x, y and z are distinct. Next we consider the case when x, y and z are not distinct. We have the following four cases: (1) $x = y = z$. (2) $x = z$, $x \neq y$. (3) $x = y$, $x \neq z$. (4) $x \neq y$, $y = z$. Now for each of the above four cases, we can show that the lemma is true. This is accomplished by using a similar argument for the case when x, y and z are distinct. □

Lemma 2.7. *Let F and H be two general graphs on the same vertex set. If $|E(F)| < |E(H)|$, then there exists an F-augmenting chain in $F \oplus H$.*

Proof. We apply induction on $|E(F)|$, and we shall simply write $|F|$ for $|E(F)|$.

When $|F| = 0$, i.e., $d_F(v) = 0$ for any $v \in V$, any edge or loop of H is an F-augmenting chain in $F \oplus H$. So we assume the lemma is true for $|F| \leq k - 1$ ($k \geq 1$).

Now suppose $|F| = k$. Since $|H| > |F|$, there exists a vertex x_1 such that $d_H(x_1) > d_F(x_1)$. Suppose x_2 is a vertex which is adjacent to x_1 in H. Therefore, we may assume that $x_1 x_2 \ldots x_k$ is a maximum F-alternating chain in $F \oplus H$ with initial edge (x_1, x_2).

Case 1: $(x_{k-1}, x_k) \in E(F)$.

Note that from the maximality of $x_1 x_2 \ldots x_k$, we have $d_F(x_k) > d_H(x_k)$. Since $d_H(x_1) > d_F(x_1)$, we have $x_1 \neq x_k$. Given the fact that $(x_{k-2}, x_{k-1}) \in E(H)$, let $F' = F \backslash (x_{k-1}, x_k)$ and $H' = H \backslash (x_{k-2}, x_{k-1})$.

Now since $|F'| = k - 1$ and $|F'| < |H'|$, we have by the inductive hypothesis, that there exists an F'-augmenting chain in $F' \oplus H'$. Applying Lemma 2.6, it follows that there exists an F-augmenting chain in $F \oplus H$ with respect to H.

Case 2: $(x_{k-1}, x_k) \in E(H)$.

Because $x_1 x_2 \ldots x_k$ is a maximum F-alternating chain in $F \oplus H$ with initial edge (x_1, x_2), we have:

(1) If $x_1 = x_k$, then $d_H(x_1) \geq d_F(x_1) + 2$.

(2) If $x_1 \neq x_k$, then $d_H(x_1) > d_F(x_1)$ and $d_H(x_k) > d_F(x_k)$.

Hence there exists an F-augmenting chain in $F \oplus H$ with respect to H. This completes the proof. □

Proof of Theorem 2.1. The necessity of Theorem 2.1 is clear. Now we proceed to prove the sufficiency. Suppose F is not a maximum (g, f)-factor of G, and assume there exists a (g, f)-factor H of G such that $|H| > |F|$. From Lemma 2.7, we know that there exists an F-augmenting chain C with respect to H. By inspection, we can see that C is an F-augmenting chain in G. This completes the proof. □

3. A transformation theorem

In this section, we shall give a transformation theorem for any two general graphs on the same vertex set $V = \{x_1, x_2, \ldots, x_n\}$. Let F and H be two general graphs on V. Consider the alternating chains $C = x_1x_2 \ldots x_k$ in the graph $F \oplus H$. Then the end vertex x_1 in the chain C is said to be negative if $(x_1, x_2) \in E(H)$, and $d_H(x_1) > d_F(x_1)$ when $x_1 \neq x_k$, or $d_H(x_1) \geqslant d_F(x_1) + 2$ when $x_1 = x_k$; similarly, x_1 is said to be positive if $(x_1, x_2) \in E(F)$, and $d_F(x_1) > d_H(x_1)$ when $x_1 \neq x_k$, or $d_F(x_1) \geqslant d_H(x_1) + 2$ when $x_1 = x_k$. The sign of the end vertex x_k is defined analogously.

Definition 3.1. An alternating chain C in $F \oplus H$ is said to be an *alternating cycle* if it is also an even cycle, and it is said to be a *signed chain* if it is not an alternating cycle and its end vertices are signed.

Then we have the following theorem.

Theorem 3.2. *Let F and H be two general graphs on V. Then $F \oplus H$ can be decomposed into a set of alternating cycles and signed chains.*

Proof. Let $D = F \oplus H$. If D is a balanced graph, i.e., $d_H(x) = d_F(x)$ for every vertex x, then by a theorem of [7] D can be decomposed into a set of alternating cycles (see also [6]). Otherwise, we may assume without loss of generality that there is a vertex x_1 such that $d_H(x_1) > d_F(x_1)$. Now suppose that $C = x_1x_2 \ldots x_k$ is a maximum alternating chain in D such that $(x_1, x_2) \in E(H)$. Then we have the following two cases:

Case 1: $(x_{k-1}, x_k) \in E(H)$.

If $x_1 = x_k$, we must have that $d_H(x_1) \geqslant d_F(x_1) + 2$. Otherwise, we have $d_H(x_k) > d_F(x_k)$. Therefore, C is a signed chain.

Case 2: $(x_1, x_k) \in E(F)$.

From the maximality of C, we have $d_F(x_k) > d_H(x_k)$. Hence, $x_1 \neq x_k$. So C is a signed chain.

Remove the edges in the above signed chain from F and H, and denote the resulting graphs by F_1 and H_1 respectively. Let $D_1 = F_1 \oplus H_1$. If D_1 is a balanced graph, then the proof is complete. Otherwise, we may repeat the above procedure to obtain a signed chain C_1 in D_1. If this procedure is repeated, then we may obtain a sequence of signed chains $C, C_1, \ldots C_t$ in a sequence $D, D_1, \ldots D_{t+1}$ of graphs such that $D_i = F_i \oplus H_i$, F_i and H_i are obtained by deleting the edges in C_{i-1} from F_{i-1} and H_{i-1}, and D_{t+1} is a balanced graph.

We now proceed to prove that the sign of the end vertices of C_i for D_i is the same as that for D. We use induction on i. When $i = 1$ the result is obvious. Now suppose that the assertion is true for the subsequence $C, C_1, \ldots C_{i-1}$ (here we regard C as C_0). Let x be an end vertex of C_i. Without loss of generality we may assume that x is negative for C_i with respect to D_i. Let $\Delta_i = d_{H_i}(x) - d_{F_i}(x)$ and let $\Delta = d_H(x) - d_F(x)$. Consider the change from Δ to Δ_i. Note that only the end vertices of C_j $(0 \leq j \leq i-1)$ which are identical to x could affect Δ_i. If there is no end vertex of C_j which is identical to x and also positive with respect to D_j, then the proof is complete; otherwise we may assume that x is an end vertex of some C_j $(0 \leq j \leq i-1)$ which is positive for C_j with respect to D_j, then, from the inductive hypothesis, x must be a positive vertex for all the signed chains C_j $1 \leq j \leq i-1$ which have x as and end vertex with respect to D_j. Choose the largest j such that C_j has x as an end vertex and $0 \leq j \leq i-1$. Consider the following two cases:

Case 1: *The two end vertices of* C_j *coincide with* x.

Thus, we have $d_{F_j}(x) \geq d_{H_j}(x) + 2$. Since j is the largest, it follows that $d_{F_i}(x) = d_{F_j}(x) - 2$ and $d_{H_i}(x) = d_{H_j}(x)$. Thus, we have $d_{F_i}(x) \geq d_{H_i}(x)$. However, this contradicts the assumption that x is a negative vertex of C_i with respect to D_i.

Case 2: *The two end vertices of* C_j *are distinct.*

Similar to Case 1, we have $d_{F_j}(x) > d_{H_j}(x)$, $d_{F_i}(x) = d_{F_j}(x) - 1$ and $d_{H_i}(x) = d_{H_j}(x)$. Thus, we have $d_{F_i}(x) \geq d_{H_i}(x)$, a contradiction again.

Therefore, by induction we have proved that x is a negative vertex of C_i with respect to D. □

Once we have proved Theorem 3.2, we may see that the order of signed chains in the proof is not important. Suppose F and H are both maximum (g, f)-factors of a general graph G. Thus, neither an F-augmenting chain with respect to H nor an H-augmenting chain with respect to F can exist. Then we may obtain the following transformation theorem for maximum (g, f)-factors of G which generalizes Berge's transformation theorem for maximum c-matchings.

For a (g, f)-factor H of G, we construct a colored graph $R(H)$ as follows: Add to G a vertex x_0, called *origin* that is joined to each vertex v_i by $f(x_i) - g(x_i)$ edges for all vertices x_i. Let the edges of H be represented by dark lines, and let $f(x_i) - d_H(x_i)$ edges from x_0 to x_i also be represented by dark lines for all x_i. Then other edges of $R(H)$ are represented by light lines. A transfer along a dark/light alternating chain is defined as the interchange of the dark and light coloring along the chain.

Theorem 3.3. *Let F and H be two maximum (g, f)-factors of G. Then F can be obtained from H by transfers along dark/light alternating cycles of $R(H)$ that are pairwise edge disjoint (but not necessarily elementary).*

Note that the above theorem also holds for minimum (g, f)-factors.

Corollary 3.4 (Berge [3]). *If E_0 and E_1 are two maximum c-matchings of a multigraph G, then E_1 can be obtained from E_0 by transfers along dark/light alternating cycles of $R(E_0)$ that are pairwise edge disjoint (but not necessarily elementary).*

Let $\mathcal{P}$ be the set of maximum (g, f)-factors of G. A *free edge* of $\mathcal{P}$ is defined to be any edge of G that is contained in some graph in $\mathcal{P}$ but not in every graph in $\mathcal{P}$. We have the following corollary as an extension to a theorem in [3] (see also [5, p. 152]).

Corollary 3.5. *An edge e is a free edge of $\mathcal{P}$, if and only if, given a (g, f)-factor H of G in $\mathcal{P}$, e lies on a dark/light alternating cycle of $R(H)$.*

Finally, we remark that Theorem 2.1 can also be proved by using the above transformation theorem for two general graphs on the same vertex set.

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