Interpolation by weighted Paley–Wiener spaces associated with the Dunkl transform

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A B S T R A C T

Given $\alpha > -\frac{1}{2}$, $\sigma > 0$ and $1 \leq p < \infty$, we study the interpolation problem in the space $PW_{p,\alpha}$ of entire functions $f : \mathbb{C} \to \mathbb{C}$ of exponential type $\leq \sigma$ for which $\int |f(x)|^p \cdot |x|^{2\alpha + 1} dx < \infty$, with nodes of interpolation at $s_j/\sigma$, $j \in \mathbb{Z}$, where $\{s_j; j \in \mathbb{N}\}$ is the increasing sequence of all positive roots of the Bessel function $f_{\nu+1}(z)$ of order $\alpha + 1$, and $s_j = -s_{-j}$ for all $j \in \mathbb{Z}$. We prove that if $\frac{4(\alpha+1)}{2\alpha+\sigma} := p_1(\alpha) < p < p_2(\alpha) := \frac{4(\alpha+1)}{2\alpha+1}$, the interpolation problem

$$f \in PW_{p,\alpha} \text{ and } f(\sigma^{-1} s_j) = c_j \text{ for all } j \in \mathbb{Z}$$

has a unique solution for every sequence $\{ c_j \}$ of complex numbers satisfying $\sum_{j \in \mathbb{Z}} |c_j|^p (1 + |j|)^{2\alpha + 1} < \infty$, and that if $p \leq p_1(\alpha)$, the corresponding interpolation problem may not have a solution, and that the solution, if exists, is unique if and only if $p \leq p_2(\alpha)$. Finally, we show that

$$\int \frac{|f(x)|^p}{|x|^{2\alpha + 1}} dx \sim \sigma^{-2\alpha - 2} \sum_{j \in \mathbb{Z}} |f(s_j,\sigma^{-1})|^p (1 + |j|)^{2\alpha + 1},$$

with the constant of equivalence depending only on $p$ and $\alpha$, holds for all entire functions $f$ of exponential type $\leq \sigma$ if and only if $p_1(\alpha) < p < p_2(\alpha)$.

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1. Introduction

Given $\alpha > -\frac{1}{2}$ and $1 \leq p \leq \infty$, let $L^p(\mathbb{R}, d\mu_\alpha)$ denote the $L^p$ space defined with respect to the measure $d\mu_\alpha(x) = (2^{\alpha+1} \Gamma(\alpha + 1))^{-1} |x|^{2\alpha + 1} dx$ on $\mathbb{R}$, and let $\| \cdot \|_{p,\alpha}$ denote the norm of $L^p(\mathbb{R}, d\mu_\alpha)$. Recall that a function $f : \mathbb{C} \to \mathbb{C}$ is of exponential type $\leq \sigma$ if for any $\epsilon > 0$, there exists a constant $C_{f,\epsilon} > 0$ such that $|f(z)| \leq C_{f,\epsilon} e^{\sigma |z|^\epsilon}$ for all $z \in \mathbb{C}$. For $\sigma > 0$, we denote by $PW_{p,\alpha}$ the space of entire functions of exponential type $\leq \sigma$ whose restrictions to $\mathbb{R}$ belong to $L^p(\mathbb{R}, d\mu_\alpha)$. The space of $PW_{p,\alpha}$ can be equivalently characterized in terms of the distributional Dunkl transform $F_\alpha$ of order $\alpha$, whose precise definition will be given in Section 2. In fact, according to the Paley–Wiener theorem proved by Anderson and de Jeu [1], a continuous function $f : \mathbb{R} \to \mathbb{R}$ can be extended to an entire function in the space $PW_{p,\alpha}$ if

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and only if it belongs to $L^p(\mathbb{R}, d\mu_\omega)$ and its distributional Dunkl transform $\mathcal{F}_\omega f$ (or, equivalently, its distributional Fourier transform $\mathcal{F} f$) is supported in $[-\sigma, \sigma]$ (see Proposition 2.6 and Remark 2.7 for details).

Let $j_{\alpha}(z)$ denote the revised Bessel function of order $\alpha$ given by

$$j_{\alpha}(z) = J'(\alpha + 1) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(\alpha + 1)} \left(\frac{z}{2}\right)^{2n}, \quad z \in \mathbb{C},$$  \hspace{1cm} (11)

and let $\{s_{\alpha+1,k}\}_{k=1}^{\infty}$ denote the sequence of all positive zeros of the function $j_{\alpha+1}(x)$ ordered so that $0 = s_{\alpha+1,0} < s_{\alpha+1,1} < \cdots < s_{\alpha+1,k} < s_{\alpha+1,k+1} < \cdots \rightarrow \infty$. Set $s_{\alpha+1,-k} = -s_{\alpha+1,k}$ for $k \in \mathbb{Z}$. For the rest of this paper, unless otherwise stated, $\alpha$ will denote a fixed constant bigger than $-\frac{1}{2}$. For simplicity, we shall write $s_k$ for $s_{\alpha+1,k}$. Now define a sequence $\{U_k: k \in \mathbb{Z}\}$ of entire functions on $\mathbb{C}$ by

$$U_0(z) = j_{\alpha+1}(z), \quad \text{and} \quad U_k(z) := \frac{z j_{\alpha+1}(z)}{2(\alpha + 1) j_{\alpha}(s_k)(z - s_k)} \quad \text{for } k \neq 0.$$  \hspace{1cm} (1.2)

It is known (see [4] and Proposition 2.6) that $\{U_k: k \in \mathbb{Z}\}$ is a sequence of functions in $PW^{2,\alpha}_1$ with

$$U_j(s_k) = \delta_{k,j} = \begin{cases} 1, & \text{if } k = j, \\ 0, & \text{otherwise.} \end{cases}$$  \hspace{1cm} (1.3)

More importantly, Ciaurri and Varona [4] proved that if $f \in PW^{2,\alpha}_\sigma$, $\sigma > 0$ and $\alpha > -\frac{1}{2}$, then

$$f(z) = \sum_{j \in \mathbb{Z}} f(s_j/\sigma) U_j(\sigma z)$$  \hspace{1cm} (1.4)

where the series converges in $L^2(\mathbb{R}, d\mu_\omega)$-norm, and uniformly on any compact subset of $\mathbb{R}$. The significance of this formula lies in the fact that it provides an interpolation formula at the nodes $s_j/\sigma$, $j \in \mathbb{Z}$ for functions in $PW^{2,\alpha}_\sigma$.

Now let’s recall several related classical results in the unweighted case (i.e. $\alpha = -\frac{1}{2}$). For simplicity, we write $PW^\beta_\sigma$ for $PW^{\beta,\alpha}_\sigma$. The classical Whittaker–Shannon–Kotel’nikov sampling theorem asserts that a function $f \in PW^\beta_\sigma$ is uniquely determined by sampling $f$ at the points $\pi n \sigma^{-1}$, $n \in \mathbb{Z}$, with reconstruction formula

$$f(z) = \sum_{n \in \mathbb{Z}} f \left(\frac{\pi n}{\sigma}\right) \frac{\sin(\sigma z - \pi n)}{\sigma z - \pi n}, \quad z \in \mathbb{C},$$  \hspace{1cm} (1.5)

where the symmetric partial sums of this series converge to $f$ in the $L^2(\mathbb{R})$-norm, and uniformly in every compact subset of $\mathbb{C}$. Clearly, formula (1.4) can be seen as a weighted generalization of (1.5). On the other hand, for $f \in PW^\beta_\sigma$ with $1 < p < \infty$, a classical result of Plancherel and Pólya [12] asserts that

$$c^{-1} \|f\|_{L^p(\mathbb{R})} \leq \left(\sigma^{-1} \sum_{n \in \mathbb{Z}} |f(\sigma^{-1} \pi n)|^p\right)^{\frac{1}{p}} \leq c \|f\|_{L^p(\mathbb{R})}$$  \hspace{1cm} (1.6)

with the constant $c$ depending only on $p$. This fact was further applied in [12] to show that the series (1.5) converges to $f$ in $L^p(\mathbb{R})$-norm for all $f \in PW^\beta_\sigma$ with $1 < p < \infty$, and that given any sequence $(a_n) \in \ell^p(\mathbb{Z})$ with $1 < p < \infty$, the resulting series $\sum_{n \in \mathbb{Z}} a_n \prod_{k \neq n} \frac{\sin(\sigma z - \pi n)}{\sigma z - \pi n}$ represents a unique function $F(z)$ in $PW^\beta_\sigma$ with samples $(F(\sigma^{-1} \pi n) = a_n: n \in \mathbb{Z})$.

Our goal in this paper is studying the following interpolation problem:

$$f \in PW^{\beta,\alpha}_\sigma \quad \text{and} \quad f(s_n/\sigma) = a_n \quad \text{for all } n \in \mathbb{Z}$$  \hspace{1cm} (1.7)

for a suitably given sequence $(a_n: n \in \mathbb{Z})$ of complex numbers. We shall establish a weighted analogue of the Plancherel–Pólya inequality (1.6), and use it to study problems of existence and uniqueness of the solution of the interpolation problem, as well as the convergence of (1.4) in the $L^p(\mathbb{R}, d\mu_\omega)$-norm. Of course, our main interest will be the case of $p \neq 2$ since the case $p = 2$ has been fully solved in [4].

For convenience, we denote by $\ell^{p,\alpha}(\mathbb{Z})$ the space of all sequences $\{a_k: k \in \mathbb{Z}\}$ of complex numbers with $\|\{a_k\}\|_{\ell^{p,\alpha}(\mathbb{Z})} < \infty$, where $\|\{a_k\}\|_{\ell^{p,\alpha}} := \left(\sum_{k \in \mathbb{Z}} |a_k|^p (1 + |k|)^{2\alpha + 1}ight)^{\frac{1}{p}}$ for $0 < p < \infty$, and $\|\{a_k\}\|_{\ell^\infty} := \sup_{k \in \mathbb{Z}} |a_k|$ for $p = \infty$.

Our main result can be stated as follows:

Theorem 1.1. Assume that $\sigma > 0$, $\alpha > -\frac{1}{2}$ and $1 \leq p < \infty$. Let $p_1(\alpha) := \frac{4(\alpha+1)}{2\alpha+1}$ and $p_2(\alpha) := \frac{4(\alpha+1)}{4\alpha+1}$. Then the following statements hold:
Given any sequence \( \{a_k; \ k \in \mathbb{Z}\} \in \ell^p(\mathbb{Z}) \) with \( p_1(\alpha) < p < p_2(\alpha) \), the series
\[
f(z) = \sum_{k \in \mathbb{Z}} a_k U_k(\sigma z)
\]
converges uniformly on every compact subset of \( \mathbb{C} \), and unconditionally in the \( L^p(\mathbb{R}, d\mu(z)) \)-norm to a function \( f \in WP_{\sigma,\alpha}^p \), which is the unique solution of the interpolation problem (1.7). Conversely, for any function \( f \in WP_{\sigma,\alpha}^p \) with \( p_1(\alpha) < p < p_2(\alpha) \), the sequence \( \{f(s_j^{-1})\}; \ j \in \mathbb{Z}\) belongs to \( \ell^p(\mathbb{Z}) \), and there exists a positive constant \( c \) depending only on \( p \) and \( \alpha \) such that
\[
c^{-1} \| f \|_{p,\alpha} \leq \sigma^{-2(2\alpha+2)/p} \|\{f(s_j^{-1})\}\|_{\ell^p(\mathbb{Z})} \leq c\|f\|_{p,\alpha}.
\]
We point out that the classical approaches of proving (1.6) use either a complex technique (see, for instance, [12], [11, pp. 149–153]) or the boundedness of a discrete Hilbert transform (see [5]). These techniques, however, seem to be difficult to apply to the weighted case discussed here. Our proof of Theorem 1.1 uses a different, but relatively simpler method. The main tool in our proof will be the Dunkl transform and the Paley–Wiener theorem related to the Dunkl transform (see [1] or the boundedness of a discrete Hilbert transform (see [5]). These techniques, however, seem to be difficult to apply to the weighted case discussed here. Our proof of Theorem 1.1 uses a different, but relatively simpler method. The main tool in our proof will be the Dunkl transform and the Paley–Wiener theorem related to the Dunkl transform (see [1] and [10]).

The proof of Theorem 1.1 relies on the following theorem, which asserts that a function \( f \in WP_{\sigma,\alpha}^p \) is uniquely determined by its values on a “denser” grid \( \{s_j/\sigma; \ j \in \mathbb{Z}\} \) with \( \delta \in (0, 1) \) for every \( 1 \leq p \leq \infty \).

**Theorem 1.2.** Given \( \delta \in (0, 1) \), there exists a sequence \( \{\Phi_j; \ j \in \mathbb{Z}\} \) of entire functions of exponential type \( \leq \delta^{-1} \) whose restrictions to \( \mathbb{R} \) are Schwartz functions such that
\[
f(x) = \sum_{j \in \mathbb{Z}} f(s_j\delta^{-1})\Phi_j(\sigma x), \quad x \in \mathbb{R},
\]
for all \( f \in WP_{\sigma,\alpha}^p \) with \( \alpha > 0 \) and \( 1 \leq p \leq \infty \), where the series converges to \( f \) uniformly on every compact subset of \( \mathbb{R} \), and unconditionally in the \( L^p(\mathbb{R}, d\mu(z)) \)-norm when \( p < \infty \). Furthermore, given any \( \delta \in (0, 1) \), there exists a positive constant \( c \) depending only on \( \delta \) and \( \alpha \) such that for all \( f \in WP_{\sigma,\alpha}^p \) with \( 1 \leq p \leq \infty \), we have
\[
c^{-1} \| f \|_{p,\alpha} \leq \sigma^{-2(2\alpha+2)/p} \|\{f(\delta s_j/\sigma)\}\|_{\ell^p(\mathbb{Z})} \leq c\|f\|_{p,\alpha}.
\]
Finally, we point out that in general, given any \( 1 \leq p \leq \infty \), \( f \in WP_{\sigma,\alpha}^p \) cannot be uniquely determined by its values on a “larger” grid \( \{ s_j/\alpha \}; \ j \in \mathbb{Z} \) with \( \alpha' < \alpha \). To see this, consider the following example. Let \( f(z) = \frac{1}{z^{a+1}}(\sigma z)^{a+2}(\alpha - \sigma)^2 \). Then \( f \in WP_{\sigma,\alpha}^p \) for all \( 1 \leq p \leq \infty \), and \( f(s_j/\alpha') = 0 \) for all \( j \in \mathbb{Z} \), however, \( f \neq 0 \) since both \( j_{a+1} \) and \( j_{a+2} \) have countably many real zeros only.

We organize this paper as follows. Section 2 contains several important results and concepts related to the Dunkl transform, which will be needed in the proofs of our main results. The proof of Theorem 1.2 is relatively easier, and will be given first in Section 3. Finally, we prove the main theorem, Theorem 1.1, in the final section, Section 4, using the results established in Sections 2 and 3.

Throughout the paper, the letters \( C, c, c_1, c_1, c_2, c_2, \ldots \) denote some positive universal constants which depend on the parameters indicated as subscripts, and whose values may change from one appearance to the next.

2. Preliminaries

2.1. Bessel’s functions and the Dunkl transform

Let \( J_\alpha \) denote the Bessel function of the first kind of order \( \alpha \geq -\frac{1}{2} \) given by
\[
J_\alpha(z) = \left(\frac{z}{2}\right)^\alpha \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\alpha+1)} \left(\frac{z}{2}\right)^{2n}.
\]
Clearly,
\[ j\alpha(z) = \begin{cases} 2^n \Gamma(n+1)z^{-\alpha} j\alpha(z), & \text{for } z \neq 0, \\ 1, & \text{for } z = 0. \end{cases} \]

The following lemma collects some known facts on Bessel’s functions, which can be found in [2].

**Lemma 2.1.**

(i) For \( x \in \mathbb{R} \) and \( \alpha, \beta \geq 0 \),
\[
\int_{0}^{\infty} \frac{j_{\alpha+\beta+2n+1}(t) j_{\alpha}(xt)}{t^{\alpha+\beta+1}} t^{2\alpha+1} dt = \frac{\Gamma(n+1)}{2^n \Gamma(\beta+n+1)} (1-x^2)^{\beta} P_n^{(\alpha,\beta)} (1-2x^2) \chi_{[0,1]}(x), \quad n = 0, 1, \ldots, \tag{2.2}
\]
where \( P_n^{(\alpha,\beta)} \) denotes the usual Jacobi polynomial of indices \( \alpha, \beta \) and degree \( n \).

(ii) Let \( \beta \geq -\frac{1}{2} \). Then
\[
j_{\beta}(x) = 2^\beta \Gamma(\beta + 1) \sqrt{2\pi x} |x|^{-\frac{1}{2} - \beta} \cos \left( |x| - \frac{\beta \pi}{2} - \frac{\pi}{4} \right) + O(|x|^{-\frac{3}{2} - \beta}) \tag{2.3}
\]
for \( x \in \mathbb{R} \) and \( |x| \geq 1 \), whereas
\[
|j_{\beta}(z)| \leq C (1 + |z|)^{-\frac{1}{2} - \frac{\beta}{2}} e^{\| \text{Im} z \|}, \quad z \in \mathbb{C}. \tag{2.4}
\]

(iii) Let \( S_0 = 0 \), and let \( S_k = S_{k+1}, k \neq 0 \) denote the sequence of all roots of \( j_{\alpha+1}(z) \) ordered so that \( S_k < S_{k+1} \) for all \( k \in \mathbb{Z} \). Then \( S_{k+1} - S_k \geq \varepsilon_\alpha \) for all \( k \in \mathbb{Z} \) and some \( \varepsilon_\alpha > 0 \), and
\[
s_k = \left( |k| + \frac{\alpha}{2} + \frac{1}{4} \right) \pi + O(|k|^{-1}), \quad |k| \geq 1. \tag{2.5}
\]

Next, we give the definition of the Dunkl transform of order \( \alpha \geq -\frac{1}{2} \). Define the function \( E_\alpha : \mathbb{C} \to \mathbb{C} \) by \( E_\alpha(z) = j_\alpha(iz) + \frac{z}{\alpha(\alpha+1)} j_{\alpha+1}(iz) \). As is well known (see, for instance, [13, Corollary 5.4]), the function \( E_\alpha \) is an entire function on \( \mathbb{C} \) satisfying
\[
|E_\alpha^{(r)}(z)| \leq e^{\text{Re} z}, \quad z \in \mathbb{C}, \ r = 0, 1, \ldots, \tag{2.6}
\]
which in turn implies
\[
|\partial_y^r E_\alpha(ixy)| \leq |y|^r, \quad x, y \in \mathbb{R}, \ r = 0, 1, \ldots. \tag{2.7}
\]

The Dunkl transform of order \( \alpha \geq -\frac{1}{2} \) is defined by
\[
\mathcal{F}_\alpha f(x) := \int_{\mathbb{R}} f(y) E_\alpha(-ixy) d\mu_\alpha(y), \quad x \in \mathbb{R} \tag{2.8}
\]
for \( f \in L^1(\mathbb{R}, d\mu_\alpha) \). This integral makes sense for all \( f \in L^1(\mathbb{R}, d\mu_\alpha) \) and
\[
\|\mathcal{F}_\alpha f\|_\infty \leq \|f\|_{1,\alpha} \tag{2.9}
\]
because of (2.7). In the unweighted case (i.e., \( \alpha = -\frac{1}{2} \)), \( \mathcal{F}_\alpha f \) coincides with the classical Fourier transform \( \mathcal{F} f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(y) e^{-ixy} dy \).

The next lemma collects some known properties of the Dunkl transform.

**Lemma 2.2.**

(i) \( \mathcal{F}_\alpha \) is a continuous, linear isomorphism from the space \( S(\mathbb{R}) \) of Schwartz functions onto itself satisfying \( \mathcal{F}_\alpha^2 f(x) = f(-x) \) for all \( f \in S \) and \( x \in \mathbb{R} \).

(ii) If \( f \) and \( \mathcal{F}_\alpha f \) are both in \( L^1(\mathbb{R}, d\mu_\alpha) \) then the following inverse formula holds:
\[
f(x) = \int_{\mathbb{R}} \mathcal{F}_\alpha f(y) E_\alpha(ixy) d\mu_\alpha(y), \quad x \in \mathbb{R}. \]
(iii) $\mathcal{F}_\alpha$ extends to an isometric isomorphism on $\mathcal{L}^2(\mathbb{R}, d\mu_\alpha)$ so that $\|f\|_{2, \alpha} = \|\mathcal{F}_\alpha f\|_{2, \alpha}$ for all $f \in \mathcal{L}^2(\mathbb{R}, d\mu_\alpha)$. Moreover,

$$
\int_\mathbb{R} \mathcal{F}_\alpha(f(x)g(x))d\mu_\alpha(x) = \int_\mathbb{R} f(x)\mathcal{F}_\alpha(g)(x)d\mu_\alpha(x)
$$

(2.10)

for all $f, g \in L^p(\mathbb{R}, d\mu_\alpha)$ with $p = 1$ or 2.

(iv) If $1 \leq p \leq 2$ and $f \in L^p(\mathbb{R}, d\mu_\alpha)$ then $\mathcal{F}_\alpha f \in L^p(\mathbb{R}, d\mu_\alpha)$ and $\|\mathcal{F}_\alpha f\|_{p, \alpha} \leq \|f\|_{p, \alpha}$, where $\frac{1}{p} + \frac{1}{p} = 1$.

(v) If $f_\varepsilon(x) = e^{-\varepsilon^2|x|^2}(\varepsilon^2 x)\mathcal{F}_\alpha f$ for $f \in L^1(\mathbb{R}, d\mu_\alpha)$ and some $\varepsilon > 0$, then $\mathcal{F}_\alpha f_\varepsilon(\xi) = \mathcal{F}_\alpha f(\varepsilon \xi)$.

**Proof.** That $\mathcal{F}_\alpha : S \rightarrow S$ is continuous follows directly from (2.7) and (2.8). The remaining part of (i), as well as (ii), (iii), (v) were all proved by de Jeu [9]. Finally, (iv) follows directly from (2.9), (iii) and the Riesz–Thorin theorem. □

We conclude this subsection with the following multiplier theorem proved in [3] that will be useful in our proof of Theorem 1.1.

**Lemma 2.3.** (See [3].) Let $\alpha \geq -\frac{1}{2}$ and let $S_\alpha$ be the multiplier operator defined by

$$
\mathcal{F}_\alpha(S_\alpha f)(\xi) = \chi_{[-1, 1]}(\xi)\mathcal{F}_\alpha f(\xi), \quad \xi \in \mathbb{R}, \quad f \in L^2(\mathbb{R}, d\mu_\alpha).
$$

Then $S_\alpha$ extends to a bounded operator on $L^p(\mathbb{R}, d\mu_\alpha)$ if and only if $\frac{4(\alpha + 1)}{\alpha + 1} < p < \frac{4(\alpha + 1)}{\alpha + 1}$.

2.2. Generalized translations and convolutions

The material in this subsection can be found in [16, Section 3] and [17]. Given $y \in \mathbb{R}$, the generalized translation operator $T^y$ is defined by

$$
\mathcal{F}_\alpha(T^y f)(\xi) = E_\alpha(-iy)\mathcal{F}_\alpha f(\xi), \quad \xi \in \mathbb{R}, \quad f \in L^2(\mathbb{R}, d\mu_\alpha).
$$

It is easily seen from this definition that $T^y$ maps Schwartz functions to Schwartz functions, and $T^y f(x) = T^{-y}f(-y)$ for $f \in S$ and $x, y \in \mathbb{R}$. In general, $T^y$ is not positive, but can be extended to a bounded operator on $L^p(\mathbb{R}, d\mu_\alpha)$ with

$$
\|T^y f\|_{p, \alpha} \leq \|f\|_{p, \alpha}, \quad 1 \leq p \leq \infty.
$$

(2.12)

Furthermore, in the case when $f$ is an even function on $\mathbb{R}$, $T^y f(x)$ has the following integral representation:

$$
T^y f(x) = c_\alpha \int_{-1}^{1} f(\sqrt{x^2 + y^2 - 2xyt})(1 - t^2)^{\alpha - \frac{1}{2}}(1 + t^2)^{\frac{1}{2}} dt, \quad x \in \mathbb{R},
$$

(2.13)

where $c_\alpha = (\int_{-1}^{1}(1 - t^2)^{\alpha - \frac{1}{2}}(1 + t^2) dt)^{-1}$.

The generalized convolution of $f \in L^p(\mathbb{R}, d\mu_\alpha)$ and $g \in L^1(\mathbb{R}, d\mu_\alpha)$ with $1 \leq p \leq \infty$ is defined by

$$
f \ast_\alpha g(x) := \int_\mathbb{R} f(y) T^y g(y) d\mu_\alpha(y), \quad x \in \mathbb{R},
$$

(2.14)

with $\check{g}(y) = g(-y)$. The convergence of this last integral follows by Hölder’s inequality and (2.12).

2.3. The distributional Dunkl transform

Let us first review some basic facts on tempered distributions with compact support, which can be found in the book [6, Sections 2.2–2.3]. Let $S$ and $C^\infty(\mathbb{R})$ denote the space of Schwartz functions, and the space of smooth functions on $\mathbb{R}$, respectively. Recall that $f_n \rightarrow f$ in $S$ if and only if $f_n, f \in S$ and $\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}} |\lambda^a(f_n)(x) - f^a(x)| = 0$ for all $a, b \in \mathbb{Z}_+ := N \cup \{0\}$, whereas $f_n \rightarrow f$ in $C^\infty$ if and only if $f_n, f \in C^\infty(\mathbb{R})$ and $\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}} |f_n^a(x) - f^a(x)| = 0$ for every $N \in \mathbb{N}$ and $b \in \mathbb{Z}_+$. We denote by $S'$ and $E'$ the dual spaces of the spaces $S$ and $C^\infty$, respectively. Each element in $S'$ is called a tempered distribution on $\mathbb{R}$. We shall use the notation $(u, f)$ to denote the action of $u \in S'$ on a test function $f$.

A support of a tempered distribution $u$ is the smallest closed set $K \subset \mathbb{R}$ with the property $(u, f) = 0$ whenever $f \in S$ and $\operatorname{supp} f \subset \mathbb{R} \setminus K$. A linear functional $u$ on $C^\infty(\mathbb{R})$ is an element in the space $E'$ if and only if $u$ is a tempered distribution with compact support, if and only if there exist $C > 0$ and $m, N \in \mathbb{N}$ such that

$$
|u(f)| \leq C \sum_{j=0}^{N} \sup_{|x| \leq m} |f^{(j)}(x)| \quad \text{for all } f \in C^\infty(\mathbb{R}).
$$

(2.15)
If \( u \in S' \) and \( h \in C^\infty(\mathbb{R}) \) satisfies \(|h^{(r)}(x)| \leq c_r(1 + |x|)^{k_r} \) for all \( r \in \mathbb{Z}_+ \) and some \( k_r > 0 \), then the product \( hu \) is a tempered distribution defined by

\[
\langle hu, f \rangle = \langle u, hf \rangle \quad \text{for all } f \in S. \tag{2.16}
\]

Given \( a > 0 \), the dilation \( \text{Dil}_a u \) of a tempered distribution \( u \) is a tempered distribution defined by

\[
\langle \text{Dil}_a u, f \rangle = a^{-1} \langle u, f(a^{-1}) \rangle, \quad f \in S. \tag{2.17}
\]

Every function \( g \in L^p(\mathbb{R}, d\mu_a) \) with \( 1 \leq p \leq \infty \) is a tempered distribution on \( \mathbb{R} \) in the following sense:

\[
\langle g, f \rangle := \int f(x) g(x) d\mu_a(x), \quad f \in S, \tag{2.18}
\]

which can be easily seen from the following simple inequality

\[
\|f\|_{p, a} \leq c \sum_{j=0}^{\lfloor \frac{2}{p} + 2a \rfloor + 3} \sup_{x \in \mathbb{R}} |x^j f(x)|, \tag{2.19}
\]

where \( \lfloor a \rfloor \) denotes the integer part of the real number \( a \).

The following definition of the distributional Dunkl transform was given in [1, Section 4]:

**Definition 2.4.** The distributional Dunkl transform \( \mathcal{F}_\alpha^d u \) of a tempered distribution \( u \) is defined by

\[
\langle \mathcal{F}_\alpha^d u, f \rangle = \langle u, \mathcal{F}_\alpha f \rangle, \quad f \in S.
\]

Since \( \mathcal{F}_\alpha : S \to S \) is a continuous linear isomorphism on \( S \), it follows that \( \mathcal{F}_\alpha^d \) defined above is a continuous bijection from \( S' \) to \( S' \) with \( (\mathcal{F}_\alpha^d)^{-1} : S' \to S' \) being given by

\[
\langle (\mathcal{F}_\alpha^d)^{-1} u, f \rangle = \langle u, \mathcal{F}_\alpha^{-1} f \rangle, \quad f \in S.
\]

Moreover, \( \mathcal{F}_\alpha^d g \) coincides with the function \( \mathcal{F}_\alpha g \in L^p(\mathbb{R}, d\mu_a) \) when \( g \in L^p(\mathbb{R}, d\mu_a) \) and \( 1 \leq p \leq 2 \), on account of (2.10). Thus, we shall use \( \mathcal{F}_\alpha \) to denote either the Dunkl transform defined in Section 2.1 or the distributional Dunkl transform defined above, whenever it causes no confusion.

The following proposition collects several useful facts on the distributional Dunkl transform:

**Proposition 2.5.**

(i) If \( u \in S' \) and \( a > 0 \) then

\[
\mathcal{F}_\alpha \text{Dil}_a u = a^{2d} \text{Dil}_a^{-1} (\mathcal{F}_\alpha u).
\]

If particular, if \( \text{supp} \mathcal{F}_\alpha u \subset [-R, R] \) for some \( R > 0 \) then \( \text{supp} \mathcal{F}_\alpha \text{Dil}_a u \subset [-aR, aR] \).

(ii) If \( f \in L^p(\mathbb{R}, d\mu_a) \), \( 1 \leq p \leq \infty \) and \( g \in S \) then

\[
\mathcal{F}_\alpha (f *_a g) = (\mathcal{F}_\alpha f)(\mathcal{F}_\alpha g).
\]

(iii) If \( u \) is a tempered distribution whose Dunkl transform \( \mathcal{F}_\alpha u \) is a tempered distribution supported in \([-R, R]\), then

\[
F(z) = [\mathcal{F}_\alpha u, E_\alpha(iz)], \quad z \in \mathbb{C}
\]

defines an entire function on \( \mathbb{C} \) that satisfies

\[
|F(z)| \leq C(1 + |z|)^N \exp(R|\text{Im} z|), \quad z \in \mathbb{C}, \tag{2.20}
\]

for some \( C, N > 0 \), and coincides with \( u \), when restricted to \( \mathbb{R} \), in the sense that

\[
\langle u, f \rangle = \int_{\mathbb{R}} F(x) f(x) d\mu_\alpha(x) \quad \text{for all } f \in S. \tag{2.21}
\]
Proof. (i) follows directly from the definitions of $\operatorname{Dil}_0$ and $\mathcal{F}_\alpha$, and Lemma 2.2(v).

(ii) Using (2.14) and Definition 2.4, we obtain, for any $\varphi \in \mathcal{S}$,

$$\langle \mathcal{F}_\alpha (f \ast_\alpha g), \varphi \rangle = \langle f \ast_\alpha g, \mathcal{F}_\alpha \varphi \rangle = \int f(y) T^{\ast_\alpha g}_\varphi (y) d\mu_\alpha (y) \mathcal{F}_\alpha \varphi (x) d\mu_\alpha (x).$$

Since $g \in \mathcal{S}$, $T^{\ast_\alpha g}_\varphi$ is in $\mathcal{S}$ as well. Thus, the Fubini theorem is applicable so that

$$\langle \mathcal{F}_\alpha (f \ast_\alpha g), \varphi \rangle = \int f(y) \left( \int T^{\ast_\alpha g}_\varphi (y) \mathcal{F}_\alpha \varphi (x) d\mu_\alpha (x) \right) d\mu_\alpha (y) = \int f(y) \mathcal{F}_\alpha \varphi (-y) d\mu_\alpha (y) = \int f(y) (\mathcal{F}_\alpha \varphi) \ast_\alpha g(-y) d\mu_\alpha (y),$$

where we used the identities $T^{\ast_\alpha g}_\varphi (x) = T^{-y} \varphi (-y)$, and $\mathcal{F}_\alpha \varphi (-x) = \mathcal{F}_\alpha \varphi (x)$ respectively in the last two steps. Since $\mathcal{F}_\alpha^2 \varphi (x) = \varphi (-x)$ for $\varphi \in \mathcal{S}$, the integral in (2.22) equals

$$\langle f, \mathcal{F}_\alpha^2 (\mathcal{F}_\alpha \varphi) \ast_\alpha g \rangle = \langle f, \mathcal{F}_\alpha (\varphi \ast_\alpha \mathcal{F}_\alpha g) \rangle = \langle \mathcal{F}_\alpha f, \varphi \ast_\alpha \mathcal{F}_\alpha g \rangle = \langle \mathcal{F}_\alpha f, \mathcal{F}_\alpha \varphi \ast_\alpha g \rangle,$$

where we used the identity $\mathcal{F}_\alpha (f \ast_\alpha g) = \mathcal{F}_\alpha f \ast_\alpha \mathcal{F}_\alpha g$ for $f, g \in \mathcal{S}$ in the first step (see [16]), and Definition 2.4 in the second step. This proves the desired identity in (ii).

(iii) First, observe that the function $F(z) := \langle \mathcal{F}_\alpha u, E_\alpha (iz \cdot) \rangle$ is well defined since $\mathcal{F}_\alpha u \in \mathcal{E}'$ is a continuous linear functional on $C^\infty$, and the function $\xi \mapsto E_\alpha (iz \xi)$ is in $C^\infty (\mathbb{R})$. Next, we show that $F$ is an entire function satisfying (2.20). Since $E_\alpha (z)$ is an entire function on $\mathbb{C}$, it has a power series expansion $E_\alpha (z) = \sum_{m=0}^{\infty} a_m z^m$ which converges uniformly in every compact subset of $\mathbb{C}$. Thus, for any fixed $z \in \mathbb{C}$, $E_\alpha (iz \cdot) = \sum_{m=0}^{\infty} (a_m i^m z) g_m(x), \ x \in \mathbb{R}$, where $g_m(x) = x^m$, and the last series converges in the topology of $C^\infty (\mathbb{R})$ (with respect to the variable $x$). Since $\mathcal{F}_\alpha u \in \mathcal{E}'$, it follows that

$$\langle \mathcal{F}_\alpha u, E_\alpha (iz \cdot) \rangle = \sum_{n=0}^{\infty} a_n i^n \langle \mathcal{F}_\alpha u, g_n \rangle z^n,$$

where the series converges for every $z \in \mathbb{C}$. This shows that $F(z) = \langle \mathcal{F}_\alpha u, E_\alpha (iz \cdot) \rangle$ is an entire function on $\mathbb{C}$. To show that $F$ satisfies (2.20), we use a result from [8, 14.2], which asserts that for any $\epsilon > 0$, there exists a $C^\infty$ function $\eta$ such that $\eta(\epsilon) = 1$ for $|x| \leq 1$, $\eta(\epsilon) = 0$ for $|x| \geq 1 + \epsilon$ and $|\eta^{(j)}(x)| \leq c_{\epsilon j}^{-1}$ for all $j \in \mathbb{N}$. Let $z$ be a fixed complex number, and set $f_z(\xi) = E_\alpha (iz \xi) \eta (\xi z^{-1})$ for some $R > 0$. Then using (2.15), we have

$$|F(z)| = |\langle \mathcal{F}_\alpha u, f_z \rangle| \leq C \sum_{j=0}^{N} \sup_{|x| \leq m} \left| f_z^{(j)}(x) \right| \text{ for some } m, N \in \mathbb{N},$$

which using (2.6), is dominated by

$$C R (N+1)^{2N} \sum_{j=0}^{N} \frac{|\eta|^{(j)}(|z|^{-1} z^{-1}) e^{R (|1+j| N)|\operatorname{Im} z|}. $$

Setting $\epsilon = R^{-1} (1 + |z|)^{-1}$, we deduce the desired estimate (2.20).

Finally, we prove (2.21). Given any $f \in \mathcal{S}$, we have

$$\langle u, f \rangle = \langle u, \mathcal{F}_\alpha^2 f \rangle = \langle \mathcal{F}_\alpha u, \mathcal{F}_\alpha \tilde{f} \rangle = \left\langle \mathcal{F}_\alpha u, \int \mathcal{F}_\alpha \varphi (x) d\mu_\alpha (y) \right\rangle,$$

Since $f \in \mathcal{S}$, it is easily seen that the Riemann sums of the integral $\int \mathcal{F}_\alpha (f(y) E_\alpha (ix \cdot) d\mu_\alpha (y)$ as well as their derivatives (with respect to $x$) converge uniformly on every compact subset of $\mathbb{R}$; that is, the Riemann sums converge to this integral in the topology of $C^\infty (\mathbb{R})$. Since $\mathcal{F}_\alpha u \in \mathcal{E}'$ is a continuous linear functional on $C^\infty (\mathbb{R})$, we obtain

$$\left\langle \mathcal{F}_\alpha u, \int f(y) E_\alpha (ix \cdot) d\mu_\alpha (y) \right\rangle = \int f(y) \left\langle \mathcal{F}_\alpha u, E_\alpha (ix \cdot) \right\rangle d\mu_\alpha (y) = \int f(y) F(y) d\mu_\alpha (y)$$

proving the desired Eq. (2.21). □
2.4. The Paley–Wiener theorem

Given \( 1 \leq p \leq \infty \), we denote by \( \mathcal{H}_p^{\alpha} \) the space of entire functions \( f : \mathbb{C} \to \mathbb{C} \) satisfying

\[
|f(z)| \leq C f e^{\alpha |\text{Im} z|}, \quad z \in \mathbb{C}
\]

(2.23)

for some \( C_f > 0 \), and belonging to \( L^p(\mathbb{R}, d\mu_\alpha) \) when restricted to the real line. It was proved by N.B. Anderson and M. de Jeu [1, Theorem 12] (see also [10]) that \( f \in \mathcal{H}_p^{\alpha} \) if and only if there exists a function \( \psi \in L^2(\mathbb{R}, d\mu_\alpha) \) supported in \( [-\sigma, \sigma] \) and such that

\[
f(z) = \int_{-\sigma}^{\sigma} \psi(y) E_\alpha(iyz) \, d\mu_\alpha(y), \quad z \in \mathbb{C}.
\]

N.B. Anderson and M. de Jeu [1, Remark 13(1)] also indicated that if \( f \in \mathcal{H}_p^{\alpha} \) and \( 1 \leq p \leq \infty \) then \( \mathcal{F}_a f \) is supported in \( [-\sigma, \sigma] \). We observe that this last fact can also be deduced directly from Theorem 12 of [1]. To see this, we assume, without loss of generality, that \( f \in \mathcal{H}_p^{\alpha} \) with \( p > 2 \), since \( \mathcal{H}_p^{\alpha} \subset \mathcal{H}_2^{\alpha} \) for \( p \leq 2 \). Hölder’s inequality and Lemma 2.1(ii) then imply that the function \( f(z)_{\mathcal{F}_a+\mathcal{E}^2} \) belongs to the space \( \mathcal{H}_p^{\alpha} \) for any given \( \varepsilon > 0 \). Thus, using Theorem 12 of [1], we conclude that \( \mathcal{F}_a f \) is supported in \( [-\sigma - \varepsilon, \sigma + \varepsilon] \). On the other hand, however, since \( \mathcal{F}_a f \) is in the topology of \( \mathcal{S}' \) as \( \varepsilon \to 0^+ \), and since \( \mathcal{F}_a \mathcal{S}' \to \mathcal{S}' \) is continuous, it follows that \( \mathcal{F}_a (f_{\mathcal{F}_a+\mathcal{E}^2}) \to \mathcal{F}_a f \) in \( \mathcal{S}' \) as \( \varepsilon \to 0^+ \), which in turn implies that \( \mathcal{F}_a f \) is supported in \( [-\sigma, \sigma] \). Clearly, the above argument has the advantage that it applies equally well to the case of higher-dimensional Euclidean spaces.

The result of [1, Remark 13(1)], together with Proposition 2.5(iii), implies the following.

**Proposition 2.6.** Let \( \sigma > 0 \), \( 1 \leq p \leq \infty \) and \( f \in C(\mathbb{R}) \). Then \( f \) can be extended to an entire function in \( \mathcal{P}^{\alpha} \) if and only if \( f \in L^p(\mathbb{R}, d\mu_\alpha) \) and \( \mathcal{F}_a f \) is supported in \( [-\sigma, \sigma] \).

**Proof.** The sufficient part of the proposition is an immediate consequence of Proposition 2.5(iii). Thus, it remains to show the necessary part; that is, \( f \in \mathcal{P}^{\alpha} \) implies \( \mathcal{F}_a f \subset [-\sigma, \sigma] \). Since \( \alpha > -\frac{1}{2} \), we have \( \mathcal{P}^{\alpha} \subset \mathcal{P}^{\alpha} \). However, by [14, Lemma 3] it follows that any function in \( \mathcal{P}^{\alpha} \) must be bounded on \( \mathbb{R} \). Thus, using the Phragmen–Lindelöf theorem (see [15, pp. 125–126]), we conclude that for \( f \in \mathcal{P}^{\alpha} \) and any \( \varepsilon > 0 \),

\[
|f(z)| \leq C_{\varepsilon, \alpha} e^{(\sigma + \varepsilon)|\text{Im} z|}, \quad z \in \mathbb{C},
\]

which means \( f \in \mathcal{H}_p^{\alpha} \). Thus, using the result of N.B. Anderson and M. de Jeu [1, Remark 13(1)], we deduce \( \mathcal{F}_a f \subset [-\sigma - \varepsilon, \sigma + \varepsilon] \). Letting \( \varepsilon \to 0^+ \), we deduce the desired result. \( \square \)

**Remark 2.7.** Since \( L^p(\mathbb{R}, d\mu_\alpha) \cap C(\mathbb{R}) \subset L^p(\mathbb{R}) \) for \( \alpha > -\frac{1}{2} \), Proposition 2.6, together with the classical Paley–Wiener theorem, implies that given \( 1 \leq p \leq \infty \), a continuous function \( f \) can be extended to an entire function in \( \mathcal{P}^{\alpha} \) if and only if \( f \in L^p(\mathbb{R}, d\mu_\alpha) \) and its distributional Fourier transform is supported in \( [-\sigma, \sigma] \).

3. Proof of Theorem 1.2

The proof of Theorem 1.2 relies on several lemmas.

**Lemma 3.** Let \( \varphi \) be an even Schwartz function on \( \mathbb{R} \) and let \( T^y \) be the generalized translation operator given in (2.11). Then for \( x, y \in \mathbb{R}, \varepsilon > 0 \) and any \( \ell > 0 \), we have

\[
(\varepsilon + |y|)^{2\alpha + 1} |T^y \varphi_\varepsilon(x)| \leq C \varepsilon^{-1} (1 + \varepsilon^{-1} |x| - |y|)^{-\ell},
\]

(3.1)

where \( \varphi_\varepsilon(x) = \varepsilon^{2 - 2\alpha} \varphi(x/\varepsilon) \), and the constant \( C \) depends only on \( \ell, \alpha \) and \( \varphi \).

**Proof.** We first show that for any \( s > 0 \),

\[
|T^y \varphi_\varepsilon(x)| \leq C_{s, \varphi} \varepsilon^{-1} \left( \frac{1}{|xy| + \varepsilon^2} \right)^{\alpha + \frac{1}{2}} (1 + \varepsilon^{-1} |x| - |y|)^{-s}.
\]

(3.2)

Since \( \varphi \) is even, (2.13) is applicable so that

\[
T^y \varphi_\varepsilon(x) = C \varepsilon^{-2\alpha - 2} \int_{-1}^{1} \varphi(\varepsilon^{-1} \sqrt{x^2 + y^2 - 2ytx})(1 - t^2)^{-\alpha - 1/2} (1 + t) \, dt = \varepsilon^{-2\alpha - 2} T^y \varphi(x/\varepsilon).
\]

(3.3)
Thus, for the proof of (3.2), it suffices to consider the case of $\varepsilon = 1$. Without loss of generality, we may assume $s > 2\alpha + 3$. Since $\varphi$ is a Schwartz function, using (3.3) with $\varepsilon = 1$, we have

$$|T^y \varphi(x)| \leq c \int_{-1}^1 \frac{1}{(1 + x^2 + y^2 - 2xyt)^{\alpha + \frac{1}{2}}(1 - t)^{\alpha - \frac{1}{2}}} dt.$$  \hfill (3.4)

Since $x^2 + y^2 - 2xyt \geq x^2 + y^2 - 2|x||y||t| \geq ||x| - |y||^2$ for all $-1 \leq t \leq 1$, this implies

$$|T^y \varphi(x)| \leq c \int_{-1}^1 (1 + ||x| - |y||)^2 s \int_0^{1-t} (1 - t)^{\alpha - \frac{1}{2}} dt \leq c (1 + ||x| - |y||)^{-2s},$$  \hfill (3.5)

which proves (3.2) for the case of $|xy| \leq 1 = \varepsilon$. On the other hand, if $|xy| \geq 1$, using (3.4), we have

$$|T^y \varphi(x)| \leq c \int_{-1}^1 \frac{1}{(1 + x^2 + y^2 - 2|xy||t|)^{\alpha + 1}}(1 - t)^{\alpha - \frac{1}{2}} dt$$

$$\leq c \int_0^{1} \frac{1}{(1 + x^2 + y^2 - 2|xy||t|)^{\alpha + \frac{1}{2}}} \alpha u^{\alpha - 1} \int_0^{\infty} \frac{u^{\alpha - \frac{1}{2}}}{(1 + u)^{\frac{3}{2}}} du$$

$$\leq c \int_0^{1} \frac{1}{(1 + |x| - |y||)^{\alpha}} \frac{1}{(1 + u)} u^{\alpha - \frac{1}{2}} \int_0^{\infty} \frac{u^{\alpha - \frac{1}{2}}}{(1 + u)^{\frac{3}{2}}} du$$

where we used a change of variable $u = 2|xy|(1 - t)$ in the third step, the estimate $(1 + (|x| - |y|)^2 + u)^{-\frac{1}{2}} \geq (1 + (|x| - |y||^2 + u)^{\frac{1}{2}}$ in the fourth step, and the fact that $\alpha > -\frac{1}{2}$ and $s > 2$ in the last step. This proves (3.2) for the remaining case $|xy| > 1 = \varepsilon$.

Now we turn to the proof of (3.1). If $|y| \leq 4||x| - |y||$ then using (3.2) with $s = \epsilon + 2\alpha + 1$, we have

$$(\varepsilon + |y|)^{2\alpha + 1} |T^y \varphi_\varepsilon(x)| \leq c (\varepsilon + |x| - |y||)^{2\alpha + 1} \varepsilon^{-2\alpha - 2} (1 + e^{-1}||x| - |y||)^{-s}$$

$$= c e^{-1} (1 + e^{-1}||x| - |y||)^{-\epsilon}.$$  \hfill (3.7)

On the other hand, if $|y| > 4||x| - |y||$ then $|x| \sim |y|$, and by (3.2) applied to $s = \epsilon$, we have

$$(\varepsilon + |y|)^{2\alpha + 1} |T^y \varphi_\epsilon(x)| \leq c (\varepsilon + |y|)^{2\alpha + 1} (|y|^2 + \varepsilon^2)^{-\alpha - \frac{1}{2}} e^{-1} (1 + e^{-1}||x| - |y||)^{-\epsilon}$$

$$\leq c e^{-1} (1 + e^{-1}||x| - |y||)^{-\epsilon}.$$  \hfill (3.8)

Thus, in either case, we obtain the desired inequality (3.1). \hfill \Box

A sequence $\{t_k; \ k \in \mathbb{Z}\}$ of real numbers is called $\beta$-separated for some $\beta > 0$ if $|t_k - t_j| \geq \beta$ for all $k \neq j \in \mathbb{Z}$.

**Lemma 3.2.** If $\{t_k; \ k \in \mathbb{Z}\} \subset \mathbb{R}$ is $\beta$-separated for some $\beta > 0$, then for any $f \in \text{PW}^{p, \alpha}_{\beta}$ with $\sigma > 0$ and $1 \leq p \leq \infty$, we have

$$\left(\sigma^{-2\alpha - 2} \sum_{k \in \mathbb{Z}} \left| t_k + 1 \right|^{2\alpha + 1} |f(t_k \sigma^{-1})|^p \right)^{1/p} \leq c \|f\|_{p, \alpha}$$  \hfill (3.9)

with the usual change when $p = \infty$, where the constant $c$ depends only on $\alpha$ and $\beta$.

**Proof.** Clearly, from the definition of $\text{PW}^{p, \alpha}_{\beta}$, it suffices to prove the lemma for the case of $\sigma = 1$. Let $\varphi$ be an even Schwartz function on $\mathbb{R}$ satisfying $J\varphi(x) = 1$ for $|x| \leq 1$, and $J\varphi(x) = 0$ for $|x| \geq 2$. The existence of such a function follows from the fact that $J\varphi$ is an isomorphism from $S$ to $S$ which maps even functions to even functions. Assume
that \( f \in PW_{1}^{p, \alpha} \). Invoking Proposition 2.6, we know that \( \text{supp} \mathcal{F}_{\alpha} f \subset [-1, 1] \), and hence by Proposition 2.5(ii), \( \mathcal{F}_{\alpha} f = (\mathcal{F}_{\alpha} f)(\mathcal{F}_{\alpha} \varphi) = \mathcal{F}_{\alpha} (f \ast \varphi) \). Since \( \mathcal{F}_{\alpha} \) is a bijection from \( S' \) to \( S' \), it follows that

\[
f(x) = (f \ast \varphi)(x) = \int f(y) (T_{x} \varphi(y)) \, d\mu_{\alpha}(y),
\]

where the equalities hold for all \( x \in \mathbb{R} \) because both \( f \) and \( f \ast \varphi \) are continuous functions on \( \mathbb{R} \). Thus, using Hölder's inequality, we have, for \( 1 \leq p < \infty \)

\[
|f(x)|^{p} \leq \left( \int |f(y)|^{p} |T_{x} \varphi(y)| \, d\mu_{\alpha}(y) \right) \left( \int |T_{x} \varphi(y)| \, d\mu_{\alpha}(y) \right)^{p-1}.
\]

which together with (2.12) implies that for all \( x \in \mathbb{R} \),

\[
|f(x)|^{p} \leq (\|f\|_{1, \alpha})^{p-1} \int |f(y)|^{p} |T_{x} \varphi(y)| \, d\mu_{\alpha}(y) \leq c \int |f(y)|^{p} |T_{x} \varphi(y)| \, d\mu_{\alpha}(y).
\]

Therefore,

\[
\sum_{k \in \mathbb{Z}} (1 + |tk|)^{2\alpha+1} |f(tk)|^{p} \leq c \sum_{k \in \mathbb{Z}} \left( (1 + |tk|)^{2\alpha+1} |T_{tk} \varphi(y)| \right) \, d\mu_{\alpha}(y).
\]

However, using Lemma 3.1 with \( \varepsilon = 1 \), we have, for \( \ell > 1 \),

\[
\sum_{k \in \mathbb{Z}} (1 + |tk|)^{2\alpha+1} |T_{tk} \varphi(y)| \leq c \sum_{k \in \mathbb{Z}} \left[ (1 + |tk| + |y|)^{-\ell} + (1 + |tk| - |y|)^{-\ell} \right]
\]

\[
\leq c_{\beta} \sum_{k \in \mathbb{Z}} \left[ \int_{tk}^{tk+\beta} (1 + |t| - |y|)^{-\ell} \, dt + \int_{tk}^{tk+\beta} (1 + |t| + |y|)^{-\ell} \, dt \right]
\]

\[
\leq c_{\beta} \int_{tk}^{tk+\beta} |t|^{-\ell} \, dt \leq c_{\beta, \ell}.
\]

where we used the fact that \( \{tk: k \in \mathbb{Z}\} \) is \( \beta \)-separated in the third step. Combining (3.9) with (3.8), we deduce the desired inequality (3.6) for the case of \( 1 \leq p < \infty \). Clearly, the above proof with a slight modification works equally well for the case \( p = \infty \). \( \square \)

As a consequence of the proof of Lemma 3.2, we have the following useful corollary.

**Corollary 3.3.** If \( f \in PW_{1}^{p, \alpha} \) and \( 1 \leq p < \infty \) then

\[
\sup_{x \in \mathbb{R}} |f(x)| \leq K \sigma^{(2\alpha+2)/p} \|f\|_{p, \mu},
\]

where \( K \) is a positive constant independent of \( f \) and \( \sigma \).

**Proof.** It suffices to prove the corollary for the case of \( \sigma = 1 \). Using (3.7), we obtain

\[
|f(x)| \leq c \sup_{y \in \mathbb{R}} |T_{x} \varphi(y)|^{\frac{1}{p}} \|f\|_{p, \alpha},
\]

which, using Lemma 3.1, is estimated by

\[
c (1 + |x|)^{-2\alpha+1} \|f\|_{p, \alpha} \leq K \|f\|_{p, \alpha}.
\]

Let \( \varphi \) be an even Schwartz function on \( \mathbb{R} \) such that \( \mathcal{F}_{\alpha} \varphi(x) = 1 \) for \( |x| \leq 1 \), and \( \mathcal{F}_{\alpha} \varphi(x) = 0 \) for \( |x| \geq \frac{1}{\delta} \) with \( \delta \in (0, 1) \) being a fixed constant. Note that the existence of such a function \( \varphi \) follows from the facts that \( \mathcal{F}_{\alpha} : S \to S \) is a linear isomorphism and \( \mathcal{F}_{\alpha}^{-1} : S \to S \) maps even functions to even functions. For a fixed \( \delta \in (0, 1) \), we define

\[
\Phi_{j}(x) = c_{\delta, j}^{2\alpha+2} \frac{T_{\delta j} \varphi(x)}{(j_{\alpha}(s_{j})^{2}), \quad j \in \mathbb{Z},}
\]

(3.10)
Lemma 3.4. Let \( \{a_k: k \in \mathbb{Z}\} \) be a sequence of complex numbers in \( \ell^p(\mathbb{Z}) \) for some \( 1 \leq p \leq \infty \). Then the series
\[
g(x) := \sum_{k \in \mathbb{Z}} a_k \Phi_k(x)
\]
converges uniformly on every compact subset of \( \mathbb{R} \), and unconditionally in the \( L^p(\mathbb{R}, d\mu^\alpha) \)-norm when \( p < \infty \). Furthermore, the function \( g \) is bounded, continuous and satisfies
\[
\|g\|_{p, \alpha} = \left\| \sum_{k \in \mathbb{Z}} a_k \Phi_k \right\|_{p, \alpha} \leq C_{\alpha, \varphi} \|a_k\|_{\ell^p, \alpha}.
\]  

Proof. First, we claim that for any \( N \in \mathbb{N} \),
\[
\sum_{k \in \mathbb{Z}, |k| \leq N} |\Phi_k(x)| \leq C_{\alpha, \varphi},
\]  
where \( C_{\alpha, \varphi} \) is independent of \( N \). To see this, we use (3.10), Lemma 2.1(ii), (iii) and Lemma 3.1 to obtain
\[
|\Phi_k(x)| \leq C (1 + |s_k|)^{2\alpha + 1} |T^{\delta k} \varphi(x)| \leq C (1 + |\delta| |s_k| - |x|)^{-2},
\]  
which implies
\[
\sum_{k \in \mathbb{Z}, |k| \leq N} |\Phi_k(x)| \leq C \sum_{k=0}^{\infty} \left[ \frac{1}{(1 + \delta s_k)^2} + \frac{1}{(1 + |\delta s_k| - N)^2} \right].
\]
Since, by Lemma 2.1(iii), \( s_{n+1} > s_n + \varepsilon_\alpha \) for some \( \varepsilon_\alpha > 0 \) and all \( n \in \mathbb{Z} \), the last infinite sum is estimated by
\[
C_{\alpha} \sum_{k=0}^{\infty} \frac{1}{(1 + \delta)^2} \left( \frac{1}{(1 + |\delta| - N)^2} \right) dt \leq C \int_{\mathbb{R}} \frac{1}{(1 + |t|)^2} dt \leq C < \infty.
\]
This proves the claim (3.13).

Now using (3.13), we have
\[
\sum_{k \in \mathbb{Z}, |k| \leq N} |a_k| |\Phi_k(x)| \leq \left\| |a_k| \right\|_{\ell^\infty} \sum_{k \in \mathbb{Z}, |k| \leq N} |\Phi_k(x)| \leq C \sum_{k \in \mathbb{Z}} |a_k| < \infty,
\]
which shows that the series \( \sum_{k \in \mathbb{Z}} a_k \Phi_k(x) \) converges uniformly on every compact subset of \( \mathbb{R} \) to a bounded continuous function \( g \).

To show the unconditional convergence of the series in the \( L^p(\mathbb{R}, d\mu^\alpha) \), we let \( A \) be an arbitrary finite subset of \( \mathbb{Z} \), and then use Hölder’s inequality to obtain
\[
\left( \sum_{k \in A} |a_k \Phi_k(x)| \right)^p \leq \left( \sum_{k \in A} |a_k|^p |\Phi_k(x)| \right) \left( \sum_{k \in \mathbb{Z}} |\Phi_k(x)| \right)^{p-1} \leq C \sum_{k \in \mathbb{Z}} |a_k|^p (1 + |k|)^{2\alpha + 1} |T^{\delta s_k} \varphi(x)|,
\]
where the last step uses (3.13) and the first inequality in (3.14). Integrating this last inequality over \( \mathbb{R} \), and using (2.12), we obtain
\[
\left\| \sum_{k \in A} a_k \Phi_k(x) \right\|_{p, \alpha} \leq C \sum_{k \in \mathbb{Z}} |a_k|^p (1 + |k|)^{2\alpha + 1} \|T^{\delta s_k} \varphi\|_{1, \alpha} \leq C \|a_k\|_{\ell^p, \alpha}.
\]
Since the space \( L^p(\mathbb{R}, d\mu^\alpha) \) is complete, this implies that the series \( \sum_{k \in \mathbb{Z}} a_k \Phi_k \) converges to the function \( g \) unconditionally in \( L^p(\mathbb{R}, d\mu^\alpha) \), and (3.12) holds. \( \square \)

Now we are in a position to prove Theorem 1.2.
Proof of Theorem 1.2. Again, it is sufficient to prove Theorem 1.2 for the case of $\sigma = 1$. Let $\{\Phi_j : j \in \mathbb{Z}\}$ be as defined in (3.10), and let $f \in PW_{1}^{p,\alpha}$. Using Lemma 2.1(iii) and Lemma 3.2, we have

$$\left\| \left\{ f(s_j \delta) \right\}_{s \in \mathbb{Z}} \right\|_{\ell^p,\alpha(\mathbb{Z})} \leq c \left\| f \right\|_{p,\alpha} < \infty.$$  \hfill (3.15)

Thus, applying Lemma 3.4 to the sequence $\{a_k\} = \{f(\delta s_k)\}$, we conclude that the series $\sum_{j \in \mathbb{Z}} f(s_j \delta) \Phi_j(x)$ converges uniformly on every compact subset of $\mathbb{R}$, and unconditionally in the space $L^p(\mathbb{R}, d\mu_{\alpha})$ when $p < \infty$. Moreover, by (3.12), we have

$$\left\| \sum_{j \in \mathbb{Z}} f(s_j \delta) \Phi_j(x) \right\|_{p,\alpha} \leq c \left\| \left\{ f(s_j \delta) \right\} \right\|_{\ell^p,\alpha(\mathbb{Z})}.$$  

To complete the proof of Theorem 1.2, it remains to show that for every $x \in \mathbb{R}$,

$$f(x) = \sum_{j \in \mathbb{Z}} f(\delta s_j) \Phi_j(x).$$  \hfill (3.16)

Let’s first prove (3.16) for the case of $1 \leq p \leq 2$. Since $PW_{1}^{p,\alpha} \subset PW_{2}^{p,\alpha}$ for $1 \leq p \leq 2$, it suffices to show (3.16) for $f \in PW_{1}^{2,\alpha}$ in this case.

Now consider the function $F(\xi) := F_{\alpha} f(\delta^{-1} \xi)$ for $\xi \in \mathbb{R}$. Clearly, $F \in C(\mathbb{R})$ (cf. (1.13)), and $\mathcal{F}_\alpha f \in L^2((-1, 1), d\mu_{\alpha})$ when restricted to $[-1, 1]$. Let $a_{\alpha,j} = \frac{\sqrt{2}}{n_{\alpha,j}^2} \int_{-1}^{1} F(t) E_{\alpha}(-is\delta t) d\mu_{\alpha}(t)$ for each $j \in \mathbb{Z}$, where $c_{\alpha,j}$ is given by (3.11). It is known that the sequence $\{a_{\alpha,j} : j \in \mathbb{Z}\}$ of functions (restricted on $[-1, 1]$) forms an orthonormal basis for the Hilbert space $L^2((-1, 1), d\mu_{\alpha})$ (see [4]). Thus, there exists a subsequence $\{n_{\alpha,j} : j \in \mathbb{N}\}$ of $\mathbb{N}$ such that for a.e. $\xi \in [-1, 1]$,

$$F(\xi) = \lim_{k \to \infty} \sum_{j=-n_{\alpha,k}}^{n_{\alpha,k}} \frac{c_{\alpha,j}}{(j_{\alpha}(s_j))^2} \int_{-1}^{1} F(t) E_{\alpha}(-is\delta t) d\mu_{\alpha}(t) E_{\alpha}(is\delta \xi)$$

$$= \lim_{k \to \infty} \sum_{j=-n_{\alpha,k}}^{n_{\alpha,k}} \frac{c_{\alpha,j}}{(j_{\alpha}(s_j))^2} \int_{-1}^{1} \mathcal{F}_\alpha f(\xi) E_{\alpha}(-is\delta \xi) d\mu_{\alpha}(x) E_{\alpha}(is\delta \xi)$$

$$= \lim_{k \to \infty} \sum_{j=-n_{\alpha,k}}^{n_{\alpha,k}} \frac{c_{\alpha,j} \delta^{2\alpha+2}}{(j_{\alpha}(s_j))^2} f(s_j \delta) E_{\alpha}(-is\delta \xi),$$

where we used the fact that $E_{\alpha}(it) = E_{\alpha}(-it)$ for $t \in \mathbb{R}$ in the first step, a change of variable $v = t \delta^{-1}$ in the second step, and Proposition 2.5(iii) and the fact that $f \in C(\mathbb{R})$ in the last step. Since $F$ is supported in $[-1, 1]$ and $\mathcal{F}_\alpha f(\delta^{-1} \xi) = 1$ for $|\xi| \leq \delta$, we have $F(\xi) = F(\xi) \mathcal{F}_\alpha \phi(\delta^{-1} \xi)$ for a.e. $\xi \in \mathbb{R}$. It then follows that for a.e. $\xi \in [-1/\delta, 1/\delta]$,

$$\mathcal{F}_\alpha f(\xi) = F(\delta \xi) = F(\delta \xi) \mathcal{F}_\alpha \phi(\xi) = \lim_{k \to \infty} \sum_{j=-n_{\alpha,k}}^{n_{\alpha,k}} c_{\alpha,j} \delta^{2\alpha+2} \int_{-1}^{1} f(s_j \delta) E_{\alpha}(-is\delta \xi) \mathcal{F}_\alpha \phi(\xi)$$

$$= \lim_{k \to \infty} \sum_{j=-n_{\alpha,k}}^{n_{\alpha,k}} f(s_j \delta) \mathcal{F}_\alpha \Phi_j(\xi),$$

where the last step uses (3.10) and (2.11). The above equation obviously holds for $\xi \neq [-1/\delta, 1/\delta]$ since both $\mathcal{F}_\alpha f$ and $\mathcal{F}_\alpha \Phi_j$ are supported in $[-1/\delta, 1/\delta]$. Thus, we have proven that

$$\mathcal{F}_\alpha f(\xi) = \lim_{k \to \infty} \mathcal{F}_\alpha \left( \sum_{j=-n_{\alpha,k}}^{n_{\alpha,k}} f(s_j \delta) \Phi_j(\xi) \right) \quad \text{for a.e. } \xi \in \mathbb{R}. \hfill (3.17)$$

On the other hand, however, by (3.15) and Lemma 3.4, the series $\sum_{j \in \mathbb{Z}} f(s_j \delta) \Phi_j(x)$ converges unconditionally to a continuous function $g \in L^2(\mathbb{R}, d\mu_{\alpha})$ in $L^2(\mathbb{R}, d\mu_{\alpha})$-norm, which, by the Plancherel theorem (Lemma 2.2(iii)), implies that $\mathcal{F}_\alpha \left( \sum_{j=-n_{\alpha,k}}^{n_{\alpha,k}} f(s_j \delta) \Phi_j \right)$ converges to $\mathcal{F}_\alpha g$ in $L^2(\mathbb{R}, d\mu_{\alpha})$-norm as $k \to \infty$. By (3.17), this in turn implies $\mathcal{F}_\alpha f = \mathcal{F}_\alpha g$. Since $\mathcal{F}_\alpha$ is a bijection on $L^2(\mathbb{R}, d\mu_{\alpha})$, we deduce (3.16) for a.e. $x \in \mathbb{R}$. That (3.16) holds for all $x \in \mathbb{R}$ follows from the fact that both $f$ and $g$ are continuous on $\mathbb{R}$.

Finally, we show (3.16) for the case of $2 < p \leq \infty$. For $\varepsilon > 0$, we define

$$f^\varepsilon(z) = f(z) j_{\alpha+1}(\varepsilon z).$$
By (1.1), (2.4) and Hölder’s inequality, it’s easily seen that \( f^\varepsilon \in PW^{2,\alpha}_{1+\varepsilon} \). Now invoking (1.10) for the already proven case of \( p = 2 \), we get
\[
f^\varepsilon (x) = \sum_{j \in \mathbb{Z}} f (\delta s_j (1 + \varepsilon)^{-1}) \Phi_j ((1 + \varepsilon)x) j_{\alpha + 1} (\varepsilon (1 + \varepsilon)^{-1} s_j).
\]
Letting \( \varepsilon \to 0^+ \) gives
\[
f (x) = \lim_{\varepsilon \to 0^+} f^\varepsilon (x)
= \sum_{j \in \mathbb{Z}} \lim_{\varepsilon \to 0^+} \left[ f (\delta s_j (1 + \varepsilon)^{-1}) \Phi_j ((1 + \varepsilon)x) j_{\alpha + 1} (\varepsilon (1 + \varepsilon)^{-1} s_j) \right]
= \sum_{j \in \mathbb{Z}} f (\delta s_j) \Phi_j (x),
\]
where the second step is justified since by (3.13),
\[
\sum_{|j| \geq N} |f (\delta s_j (1 + \varepsilon)^{-1}) j_{\alpha + 1} (1 + \varepsilon)^{-1} \delta s_j| \Phi_j ((1 + \varepsilon)x) | \leq C \|f\|_\infty \sum_{|j| \geq N} \sup_{|y| \leq 2|x|} | \Phi_j (y) | \to 0, \quad \text{as } N \to \infty.
\]
If \( 1 \leq p < \infty \), the uniform convergence on \( \mathbb{R} \) of the series in (3.16) follows from Corollary 3.3 and (3.12). This completes the proof of Theorem 1.2. \( \square \)

4. Proof of Theorem 1.1

Clearly, it suffices to prove Theorem 1.1 for \( \sigma = 1 \). Recall that (see [4])
\[
\mathcal{F}_\sigma U_k (\xi) = c_{\sigma, k} \chi_{[-1, 1]} (\xi) E_\sigma (\frac{i \xi s_k}{\| \|}), \quad k \in \mathbb{Z}
\]
with \( c_{\sigma, k} \) being given by (3.11). This implies that \( U_k (x) = C_{\sigma} T^k j_{\alpha + 1} (x) \) for \( x \in \mathbb{R} \) and \( k \in \mathbb{Z} \). Thus, by (2.12) and Lemma 2.1(ii), we have \( \|u_k\|_{p, \alpha} \leq 3 \|j_{\alpha + 1}\|_{p, \alpha} < \infty \) whenever \( p > p_1 (\alpha) \), which in turn implies that \( \{U_k : k \in \mathbb{Z}\} \subset PW^{p,\alpha}_{\sigma} \) for \( p > p_1 (\alpha) \).

(i) Firstly, we claim that if \( 1 \leq p < p_2 (\alpha) \) and \( \{a_k\} \in \ell^p (\mathbb{Z}) \) then the series \( f (z) = \sum_{k \in \mathbb{Z}} a_k U_k (z) \) converges uniformly in every compact subset of \( C \). Clearly, this combined with (1.3) will imply that \( f (s_k) = a_k \) for all \( k \in \mathbb{Z} \). To prove the claim, let \( C_1 \) be a positive constant such that \( |s_k| \geq C_1 |k| \) for all \( k \in \mathbb{Z} \). Given an arbitrary fixed positive number \( M \), let \( N \) be a positive integer \( > 2M/C_1 \). If \( |k| \geq N \) and \( |z| \leq M \), then \( |s_k| \geq C_1 N > 2|z| \), and hence by (1.2) and Lemma 2.1, we have
\[
|U_k (z)| \leq \left( \sum_{|k| \geq N} \frac{1}{|z - s_k|} \right)^{\frac{1}{2}} \leq C (1 + |z|)^{-\frac{1}{2}} e^{\frac{1}{2} \Pi (k |z|^{-1})} \leq C M |k|^{-\frac{1}{2}}.
\]
By Hölder’s inequality, this implies
\[
\sum_{|k| \geq N} |a_k| \sup_{|z| \leq M} |U_k (z)| \leq C M \sum_{|k| \geq N} |k|^{-\frac{1}{2}} |a_k| \leq C M \|a_k\|_{\ell^p (\mathbb{Z})} \left( \sum_{k \geq N} k^{\frac{p-1}{p}} \right)^{\frac{1}{p} - \frac{1}{2}} \leq C M \|a_k\|_{\ell^p (\mathbb{Z})},
\]
which converges to 0 as \( N \to \infty \) as \( (\alpha - \frac{1}{2} - \frac{2\alpha + 1}{p}) \leq -1 \) provided \( p < p_2 (\alpha) \). This proves the desired claim.

Secondly, we show that if \( p_1 (\alpha) < p < p_2 (\alpha) \) and \( \{a_k\} \in \ell^p (\mathbb{Z}) \), then the series (1.8) converges unconditionally to a function \( f \in PW^{p,\alpha}_{\sigma} \) in the \( L^p (\mathbb{R}, d\mu_{\sigma}) \)-norm. Let \( \Phi \) be given by (3.10), and let \( S_{\sigma} \) be as defined in Lemma 2.3. Since the function \( \mathcal{F}_\sigma \varphi (\xi) = 1 \) whenever \( \xi \in [-1, 1] \), it is easily seen that \( \mathcal{F}_\sigma U_k (\xi) = c_{\sigma, k} \chi_{[-1, 1]} (\xi) E_\sigma (\frac{i \xi s_k}{\| \|}) \chi_{[-1, 1]} \). This implies
\[
U_k (x) = S_{\sigma} (\Phi_k (x)), \quad x \in \mathbb{R}.
\]
Now, using (4.2), Lemmas 2.3 and 3.4, we obtain, for every finite subset \( \Lambda \) of \( \mathbb{Z} \),
\[
\left\| \sum_{k \in \Lambda} a_k U_k \right\|_{p, \alpha} \leq \sum_{k \in \Lambda} |a_k| \leq C \left( \sum_{k \in \Lambda} |a_k|^p (1 + |k|)^{2\alpha + 1} \right)^{\frac{1}{p}}.
\]
This proves unconditional convergence of the series (1.8) in \( L^p (\mathbb{R}, d\mu_{\sigma}) \) as well as the inequality
\[
\left\| \sum_{k \in \mathbb{Z}} a_k U_k \right\|_{p, \alpha} \leq C \left( \sum_{k \in \mathbb{Z}} |a_k|^p (1 + |k|)^{2\alpha + 1} \right)^{\frac{1}{p}}.
\]
That the limit function $f$ of the series (1.8) belongs to the space $PW^{p,\alpha}_1$ follows from Corollary 3.3 and the known fact that if a sequence $(f_n)$ of entire functions of exponential type $\leq \sigma$ converges to a function $f$ uniformly on $\mathbb{R}$, then $f$ can be extended to an entire function of exponential type $\leq \sigma$ (see [11]).

Thirdly, we show that if $p_1(\alpha) < p < p_2(\alpha)$ and $f \in PW^{p,\alpha}_1$ then

$$f(x) = \sum_{k \in \mathbb{Z}} f(s_k)U_k(x), \quad x \in \mathbb{R},$$

(4.4)

where the series converges in $L^p(\mathbb{R},d\mu_{\alpha})$-norm. Note that (4.4) already implies that $f \in PW^{p,\alpha}_1$ is uniquely determined by its values on the set of nodes $(s_k: k \in \mathbb{Z})$. For the proof of (4.4), we first claim that if $p_1(\alpha) < p < p_2(\alpha)$ and $f \in PW^{p,\alpha}_1$ then $S_0f = f$. To see this, we define $f'(z) = j_{\alpha+1}(z)f((1-\epsilon)z)$ for $\epsilon > 0$. By Lemma 2.1(ii) and Corollary 3.3, $f' \in PW^{p,\alpha}_1$. Hence, using the definition of $S_0$ (in Lemma 2.3) and Proposition 2.6, we have

$$S_0(f^{\epsilon})(x) = f^{\epsilon}(x)$$

(4.5)

for all $\epsilon > 0$ and a.e. $x \in \mathbb{R}$. On the other hand, however, using the dominated convergence theorem and the fact that $f \in C(\mathbb{R}) \cap L^p(\mathbb{R},d\mu_{\alpha})$, we obtain that $\lim_{\epsilon \to 0} \|f^{\epsilon} - f\|_{p,\alpha} = 0$. Thus, letting $\epsilon \to 0+$ on both sides of (4.5), and using Lemma 2.3, we prove the claim $S_0f = f$. Next, using Lemmas 2.1 and 3.2, we observe that

$$\|\{s_k\}\|_{L^p(\mathbb{Z})} \leq C\|f\|_{p,\alpha},$$

(4.6)

which in turn implies that the series on the right-hand side of (4.4) converges in $L^p(\mathbb{R},d\mu_{\alpha})$-norm. To show this series converges to $f$, we use Theorem 1.2 to obtain

$$f(x) = \sum_{j \in \mathbb{Z}} f(\delta s_j)\Phi_j(x),$$

with $\delta \in (0,1)$ being a fixed constant. Since this last series converges in $L^p(\mathbb{R},d\mu_{\alpha})$-norm, using Lemma 2.3 and the facts that $S_0f = f$ and $S_0\Phi_j = U_j$, we obtain

$$f(x) = \sum_{j \in \mathbb{Z}} f(\delta s_j)U_j(x),$$

(4.7)

where the series converges in $L^p(\mathbb{R},d\mu_{\alpha})$. Observe that for $|s_j| \geq C_1|j| \geq C_1N \geq 2|x|$,}

$$\sum_{|j| \geq N} |f(\delta s_j)U_j(x)| \leq C|x|^{j_{\alpha+1}(x)} \sum_{|j| \geq N} |f(\delta s_j)| |j|^{\alpha-\frac{1}{2}}$$

\[ \leq C \|f(\delta s_j)\|_{L^{p,\alpha}(\mathbb{Z})} \left( \sum_{|j| \geq N} |j|^{\alpha-\frac{1}{2} - \frac{2\alpha+1}{p}} \right)^{\frac{p-1}{p}} \]

\[ \leq C' \|f\|_{p,\alpha} \left( \sum_{|j| \geq N} |j|^{\alpha-\frac{1}{2} - \frac{2\alpha+1}{p}} \right)^{\frac{p-1}{p}}, \]

where the constant $C'_x$ is independent of $\delta$ as $\delta \to 1$ because of Lemma 3.2. The last series converges to $0$ as $N \to \infty$ because $p < p_2(\alpha)$ implies $(\alpha - \frac{1}{2} - \frac{2\alpha+1}{p}) \frac{p-1}{p} < -1$. Now letting $\delta \to 1$ on both sides of (4.7), we deduce the desired Eq. (4.4).

Finally, we point out that (1.9) follows by (4.6), (4.4) and (4.3). This completes the proof of (i).

(ii) Firstly, we show that a function $f \in PW^{p,\alpha}_1$ is uniquely determined by its values on $|s_j: j \in \mathbb{Z}|$ if and only if $p \leq p_2(\alpha)$. To see this, assume that $f \in PW^{p,\alpha}_1$ satisfies $f(s_k) = 0$ for all $k \in \mathbb{Z}$, and consider the function

$$g(z) = \begin{cases} \frac{f(z)}{z}, & \text{if } z \neq 0, \\ f'(0), & \text{if } z = 0. \end{cases}$$

Clearly, $g \in PW^{p,\alpha}_1$ for some $p_1(\alpha) < q < p_2(\alpha)$, and $g(s_j) = 0$ for $j \neq 0$. Thus, by the already proven result in part (i), we have $g(z) = f'(0)j_{\alpha+1}(z)$, which in turn implies that $f(z) = f'(0)j_{\alpha+1}(z)$. However, since $\|x_{\alpha+1}(x)\|_{p_2(\alpha)} = \infty$ and $f \in PW^{p_2(\alpha),\alpha}_1$, we must have $f'(0) = 0$, hence $f(z) = 0$, proving that each $f \in PW^{p_2(\alpha),\alpha}_1$ is uniquely determined by its values on $|s_j: j \in \mathbb{Z}|$. Since $PW^{p,\alpha}_1 \subset PW^{p,\alpha}_1$ for $p \leq q$, this also implies the uniqueness result for the case of $p \leq p_2(\alpha)$. On the other hand, for $p > p_2(\alpha)$, consider the function $F(z) := zj_{\alpha+1}(z)$. It is easily seen that $F \in PW^{p,\alpha}_1$ for all $p > p_2(\alpha)$, and $F(s_j) = 0$ for all $j \in \mathbb{Z}$, however, $F \neq 0$. Thus, $f \in PW^{p,\alpha}_1$ is not uniquely determined by its values on $|s_j: j \in \mathbb{Z}|$ when $p > p_2(\alpha)$. 
Next, note that it has already been proved in part (i) that the interpolation series (1.8) converges uniformly in each compact subset of \( \mathbb{C} \) to an entire function \( f \) which satisfies \( f(s_k) = a_k \), for all \( k \in \mathbb{Z} \) and any given sequence \( \{a_k: k \in \mathbb{Z}\} \in \ell^{p,\alpha}(\mathbb{Z}) \) with \( 1 \leq p \leq p_1(\alpha) \). To see that the function \( f \) may not be in \( PW_{1,\alpha}^p \), we consider the following interpolation problem:

\[
f \in \bigcup_{p \leq p_2(\alpha)} PW_{1,\alpha}^p,
\quad f(s_j) = 0 \quad \text{for } j \neq 0, \text{ and } f(0) = 1.
\]

By the uniqueness result we just proved, this interpolation problem has a unique solution \( f(z) = j_{\alpha+1}(z) \), however, \( \|j_{\alpha+1}\|_{p,\alpha} = \infty \) whenever \( p \leq p_1(\alpha) \).

Finally, we give a counterexample, showing that the interpolation series (1.8) may be divergent if \( \{a_k\} \in \ell^{p,\alpha}(\mathbb{Z}) \) and \( p \geq p_2(\alpha) \). Using (2.3) and (2.5), we can choose a positive integer \( n_0 \in \mathbb{N} \) sufficiently large so that

\[
j_{\alpha}(s_{2k}) \sim k^{-\alpha - \frac{1}{2}} \quad \text{whenever } k \geq n_0.
\]

Set

\[
a_k = \begin{cases} (k/2)^{-\alpha - \frac{1}{2}} (\log k)^{-1}, & \text{if } k \geq 2n_0 \text{ is even}, \\ 0, & \text{otherwise}. \end{cases}
\]

Clearly, \( \{a_k: k \in \mathbb{Z}\} \in \ell^{p,\alpha}(\mathbb{Z}) \) when \( p \geq p_2(\alpha) \). We shall show that the series

\[
f(x) = \sum_{k \in \mathbb{Z}} a_k U_k(x) = c \sum_{k=n_0}^{\infty} k^{-\alpha - \frac{1}{2}} (\log k)^{-1} \frac{j_{\alpha+1}(x)}{j_{\alpha}(s_{2k})} \frac{x}{x-s_{2k}}
\]

diverges on \([\varepsilon, \infty)\) for some \( \varepsilon > 0 \). In fact, since \( j_{\alpha+1}(0) = 1 \), there exists a constant \( \varepsilon \in (0,1) \) such that \( j_{\alpha+1}(x) \geq \frac{1}{2} \) whenever \( x \in [-\varepsilon, \varepsilon] \). It then follows by (4.9) and (4.8) that for \( \frac{\varepsilon}{2} \leq |x| \leq \varepsilon \),

\[
|f(x)| \geq c \sum_{k=n_0}^{\infty} k^{-1} (\log k)^{-1} = \infty
\]

as desired. This completes the proof of (ii).

(iii) Consider the function \( U_0(x) = j_{\alpha+1}(x) \). It’s easily seen from (2.3) that \( \|U_0\|_{p,\alpha} = \infty \) whenever \( p \leq p_1(\alpha) \). The claim in statement (iii) for the case of \( p \leq p_1(\alpha) \) then follows. To show the case \( p \geq p_2(\alpha) \), let \( n_0 \) be a positive integer large enough so that (4.8) holds, and define, for \( N \geq 2n_0 \),

\[
f_N(x) = \sum_{k=n_0}^{N} U_{2k}(x) = c \sum_{k=n_0}^{N} \frac{j_{\alpha+1}(x)}{j_{\alpha}(s_{2k})} \frac{x}{x-s_{2k}}.
\]

Since each \( U_k \) belongs to \( PW_{1,\alpha}^p \), it follows that \( f_N \in PW_{1,\alpha}^p \). Also, it is clear that

\[
f_N(s_k) = \begin{cases} 1, & \text{if } 2n_0 \leq k \leq 2N \text{ is even}, \\ 0, & \text{otherwise}. \end{cases}
\]

It follows that

\[
\left( \sum_{k \in \mathbb{Z}} |f_N(s_k)|^p \right)^{\frac{1}{p}} \left( 1 + |k| \right)^{2\alpha + 1} \sim N^{-\frac{2\alpha + 2}{p}}.
\]

On the other hand, using (4.8), and (4.10), we obtain that for \( x \in [-N, -1] \),

\[
|f_N(x)| = c |xj_{\alpha+1}(x)| \sum_{k=n_0}^{N} \frac{1}{j_{\alpha}(s_{2k})} \frac{1}{|x| + s_{2k}} \geq c |xj_{\alpha+1}(x)| \sum_{N/2 \leq k \leq N} \frac{1}{j_{\alpha}(s_{2k})} \frac{1}{s_{2k}} \sim |xj_{\alpha+1}(x)| N^{\alpha + \frac{1}{2}}.
\]

Thus, using (2.3), we have

\[
\|f_N\|_{p,\alpha} \geq CN^{\alpha + \frac{1}{2}} \left( \int_{-N}^{1} |xj_{\alpha+1}(x)|^p |x|^{2\alpha + 1} dx \right)^{\frac{1}{p}} \sim \begin{cases} N^{\alpha + \frac{1}{2}}, & \text{if } p > p_2(\alpha), \\ N^{\alpha + \frac{1}{2}} (\log N)^{\frac{1}{p}}, & \text{if } p = p_2(\alpha). \end{cases}
\]

This combined with (4.11) yields

\[
\|f_N\|_{p,\alpha} \geq \begin{cases} N^{\left(\frac{1}{p_2(\alpha)} - \frac{1}{2}\right)(2\alpha + 1)}, & \text{if } p > p_2(\alpha), \\ (\log N)^{\frac{1}{p}}, & \text{if } p = p_2(\alpha), \end{cases}
\]

which tends to \( \infty \) as \( N \to \infty \). This proves the claim for \( p \geq p_2(\alpha) \).
(iv) Clearly, statement (i) implies that the system \( \{U_k: k \in \mathbb{Z}\} \) forms an unconditional basis for \( PW^1_t\) when \( p_1(\alpha) < p < p_2(\alpha) \). On the other hand, \( \{U_k: k \in \mathbb{Z}\} \) cannot be a basis for \( PW^1_t \) in the case of \( p \leq p_1(\alpha) \) because of the fact that \( \|U_0\|_{p, \alpha} = \|f\|_{p, \alpha} = \infty \). Next, we claim that \( \{U_k: k \in \mathbb{Z}\} \) is not a basis for \( PW^1_t \) whenever \( p > p_2(\alpha) \). Assume otherwise, and consider the function \( f(z) = z^{1/2}(z) \). By (2.3), \( f \in PW^1_t \) for every \( p > p_2(\alpha) \). Thus, by the assumption, there exists a sequence of complex numbers \( a_k, k \in \mathbb{Z} \) such that \( \sum_{k \in \mathbb{Z}} a_k U_k \to f \) in \( L^p(\mathbb{R}, d\mu_\alpha) \)-norm for some \( p > p_2(\alpha) \), which, using Corollary 3.3, in turn implies that the series converges to \( f \) uniformly on \( \mathbb{R} \). Thus, by (1.3), we have \( a_k = f(\delta_k) = \delta_k j_{\alpha+1}(\delta_k) = 0 \) for all \( k \in \mathbb{Z} \). This leads to a contradiction that \( xj_{\alpha+1}(x) = \sum_{k \in \mathbb{Z}} a_k U_k(x) = 0 \) for every \( x \in \mathbb{R} \).

Finally, we prove the assertion for the case of \( p = p_2(\alpha) \). Let \( n_0 \) be a sufficiently positive supports such that (4.9) holds. We define, for \( N \geq n_0 \),

\[
f_N(x) := 2N \sum_{k=N}^{2N} c_k U_{2k}(x), \quad \text{and} \quad g_N(x) := 2N \sum_{k=N}^{2N} (-1)^k c_k U_{2k}(x),
\]

where \( c_k = N^{\alpha+\frac{1}{2}} j_\alpha(s_{2k}) \sim 1 \) for \( N \leq k \leq 2N \). It will be proven that

\[
\|f_N\|_{p_2(\alpha)} \geq c_1 N^{\alpha+\frac{1}{2}} (\log N)^{\frac{1}{2}} \quad \text{and} \quad \|g_N\|_{p_2(\alpha)} \leq c_2 N^{\alpha+\frac{1}{2}}
\]

(4.13) for some positive constants \( c_1, c_2 \) independent of \( N \), which, using Lemma 2.7 of [7, p. 213], will imply that \( \{U_k: k \in \mathbb{Z}\} \) is not an unconditional basis for \( PW^1_t \). The proof of the first inequality in (4.13) is almost identical to that of (4.12). To show the second inequality in (4.13), we observe that

\[
g_N(x) = N^{\alpha+\frac{1}{2}} j_\alpha(\delta_{2N}) \sum_{k=N}^{2N} \frac{(-1)^k}{x-s_{2k}}.
\]

(4.14)

Thus, if \( x \in (-\infty, s_{2N} - 1] \), we have

\[
|g_N(x)| = N^{\alpha+\frac{1}{2}} |j_\alpha(\delta_{2N})| \sum_{k=N}^{2N} \frac{(-1)^k}{s_{2k} - x} \leq N^{\alpha+\frac{1}{2}} |j_\alpha(\delta_{2N})| \frac{1}{s_{2N} - x} \leq C(1 + |x|)^{-\alpha+\frac{1}{2}} N^{\alpha+\frac{1}{2}} (s_{2N} - x)^{-1};
\]

(4.15) whereas if \( x \in [s_{4N} + 1, \infty) \), we have

\[
|g_N(x)| = N^{\alpha+\frac{1}{2}} |j_\alpha(\delta_{2N})| \sum_{k=N}^{2N} \frac{(-1)^k}{x-s_{2k}} \leq N^{\alpha+\frac{1}{2}} |j_\alpha(\delta_{2N})| \frac{1}{x-s_{4N}} \leq C(1 + |x|)^{-\alpha+\frac{1}{2}} N^{\alpha+\frac{1}{2}} (x-s_{4N})^{-1}.
\]

(4.16)

Next, since \( s_k \sim k \) and

\[
|j'_{\alpha+1}(x)| = \frac{|x j_{\alpha+2}(x)|}{2\alpha + 4} \leq C(1 + |x|)^{-\alpha-\frac{1}{2}},
\]

it follows that if \( |x-s_{2k}| \leq 1 \) for some \( N \leq k \leq 2N \),

\[
N^{\alpha+\frac{1}{2}} \frac{|x j_{\alpha+1}(x)|}{|x-s_{2k}|} = N^{\alpha+\frac{1}{2}} \frac{|x|}{|x-s_{2k}|} \leq C.
\]

Using (4.14), this implies that for \( x \in (s_{2N} - 1, s_{4N} + 1) \),

\[
|g_N(x)| \leq C + N^{\alpha+\frac{1}{2}} |j_{\alpha+1}(x)| \left| \sum_{N \leq k \leq 2N : |x-s_{2k}| \geq 1} \frac{(-1)^k}{x-s_{2k}} \right| \leq C + CN^{\alpha+\frac{1}{2}} |j_{\alpha+1}(x)| \leq C + CN^{\alpha+\frac{1}{2}} (1 + |x|)^{-\alpha-\frac{1}{2}}.
\]

(4.17) Now recalling \( p_2(\alpha) = \frac{2(\alpha+1)}{\alpha+\frac{1}{2}} \), we obtain from (4.15), (4.16) and (4.17) that

\[
\int_{\mathbb{R}} |g_N(x)| |x|^{2\alpha+1} \, dx \leq CN^{2\alpha+2} \int_{-\infty}^{s_{2N}-1} \frac{dx}{(1 + |x|)(s_{2N} - x)^{p_2(\alpha)}} + C N^{2\alpha+2} \int_{s_{4N}+1}^{\infty} \frac{dx}{(1 + |x|)(x-s_{4N})^{p_2(\alpha)}}
\]

\[
+ C \int_{s_{2N}+1}^{s_{4N}+1} |x|^{2\alpha+1} \, dx + CN^{2\alpha+2} \int_{s_{2N}-1}^{s_{4N}+1} \frac{1}{1 + |x|} \, dx,
\]

which, using (iii) of Lemma 2.1, is dominated by \( CN^{2\alpha+2} \). This proves the second inequality in (4.13).
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References