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Characterization of subdifferentials of a singular convex functional in Sobolev spaces of order minus one

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Abstract

Subdifferentials of a singular convex functional representing the surface free energy of a crystal under the roughening temperature are characterized. The energy functional is defined on Sobolev spaces of order -1, so the subdifferential mathematically formulates the energy's gradient which formally involves 4th order spacial derivatives of the surface's height. The subdifferentials are analyzed in the negative Sobolev spaces of arbitrary spacial dimension on which both a periodic boundary condition and a Dirichlet boundary condition are separately imposed. Based on the characterization theorem of subdifferentials, the smallest element contained in the subdifferential of the energy for a spherically symmetric surface is calculated under the Dirichlet boundary condition.

Keywords: Subdifferential; Negative Sobolev space; Singular functional; 4th order PDE

1. Introduction

When a time evolution problem has a structure of gradient flow and its governing energy functional has good properties such as convexity and lower semi-continuity, the evolution problem can be formulated into a well-posed initial value problem whose right-hand side is given by subdifferential of the energy functional. An advantage of the subdifferential formulation is that smoothness of the energy functional is not required, enabling us to handle a large class of physical models, which only have a formal meaning at most, within mathematical context. However, this

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mathematical formulation might look too abstract to extract physical insights which the model is initially expected to present. The abstract appearance is mainly due to the multi-valued nature of subdifferential. In this formulation the time derivative of unknown is not described by an equality, but is only contained in a set of possible gradients of the energy at the time. This ambiguity motivates us to characterize the subdifferential of the singular functional explicitly so that one can interpret the abstract evolution problem involving subdifferential as a natural formulation of the original singular model.

Our intention is especially to give an interpretation to the subdifferential formulation of the following 4th order equation.

$$\frac{\partial}{\partial t} f = -\Delta \operatorname{div}\bigl(|\nabla f|^{-1}\nabla f + \mu|\nabla f|^{p-2}\nabla f\bigr) \quad \bigl(\mu > 0,\ p \in (1,\infty)\bigr), \tag{1.1}$$

where f is a time-dependent, real-valued function defined on a bounded domain Ω of $\mathbb{R}^d$ obeying an appropriate boundary condition. Apparently Eq. (1.1) loses a mathematical meaning when $\nabla f = \mathbf{0}$. However, if we put the mathematical rigor aside temporarily, we can go on to rewrite Eq. (1.1) symbolically into a gradient flow equation

$$\frac{\partial}{\partial t} f = -\frac{\delta F(f)}{\delta f} \tag{1.2}$$

governed by the energy functional

$$F(f) = \int_{\Omega} \left(\bigl|\nabla f(\mathbf{x})\bigr| + \frac{\mu}{p}\bigl|\nabla f(\mathbf{x})\bigr|^p \right) d\mathbf{x}. \tag{1.3}$$

Here the functional derivative of F is taken with respect to the metric of the space $H^{-1}(\Omega)$ so that

$$\frac{\delta F(f)}{\delta f} = \Delta \operatorname{div}\bigl(|\nabla f|^{-1}\nabla f + \mu|\nabla f|^{p-2}\nabla f\bigr).$$

Recall that if we choose a Dirichlet boundary condition for instance, $H^{-1}(\Omega)$ is defined as the dual space of $H_0^1(\Omega)$. Using the isometry $-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$, we can formally regard $H^{-1}(\Omega)$ as a Hilbert space having the inner product $\int_\Omega (-\Delta)^{-1} f(\mathbf{x}) \cdot g(\mathbf{x})\, d\mathbf{x}$ ($f, g \in H^{-1}(\Omega)$). The function spaces will be defined later in this section in more rigorous context.

The idea of the subdifferential formulation is simply to replace the formal functional derivative by the subdifferential of F. The formulation of (1.2) is

$$\frac{d}{dt} f \in -\partial F(f). \tag{1.4}$$

We wish to postpone the mathematical definition of subdifferential until the following subsections. Here let us only note that subdifferential is an extended concept of derivative since its value is no other than the usual derivative if the functional is differentiable. The strength of the abstract theory guarantees the unique solvability of the initial value problem of (1.4). In this paper we characterize the value of $\partial F(f)$ so that we can regain a visible expression like (1.1) from (1.4).

Physically the solution f to Eq. (1.1) models the height of a crystalline surface driven by surface diffusion under the roughening temperature. Spohn [13] systematically derived Eq. (1.1) and formulated it into a free boundary value problem with evolving facets. Kashima [10] proposed the subdifferential formulation (1.4) of the singular problem (1.1) under the Dirichlet boundary condition and characterized the subdifferential of the energy by revising the characterization theorem of subdifferentials for 2nd order equations by Attouch and Damlamian [3]. Odisharia [12, Chapter 3] derived a free boundary value problem, which is consistent with Spohn's free boundary formulation [13], from the subdifferential formulation by Kashima [10]. Odisharia's derivation excludes a speculation by Kashima in [10] that the subdifferential formulation of (1.1) is inconsistent with the free boundary value problem with facets. Developments on the subject have been continuing until today. Recently Giga and Kohn [8] proved that the solution to the initial value problem of (1.4) under the periodic boundary condition becomes uniformly zero in finite time and obtained an upper bound on the extinction time independently of the volume of the domain. Kohn and Versieux [11] proposed a finite element approximation of (1.4) and established an error estimate between the solution to (1.4) and the fully discrete finite element solution. More topics on singular diffusion equations including (1.1) are found in the article [7].

This paper improves the previous results in [10]. The article [10] tried to characterize H^{-1}-subdifferentials of a class of convex functionals including (1.3) under the Dirichlet boundary condition in a way parallel to the general L^2-theory [3]. In this paper by restricting the argument to the functional (1.3) we construct our proofs in a self-contained manner using only a few basic facts from convex analysis and characterize its H^{-1}-subdifferentials under both the periodic boundary condition and the Dirichlet boundary condition separately. The characterization is carried out in arbitrary spacial dimension, improving the results in [10], where the dimension is assumed to be less than equal to 4. In addition to the removal of the dimensional constraint, the characterized value of the subdifferential seems more natural especially in the periodic setting as a formulation of (1.1). The main task in our proof is to characterize the conjugate functional of the energy functional and a technical difference from the argument [10, Subsection 3.3] lies in this part, too. Though it was also aimed to simplify the proof of the characterization of the conjugate functional of (1.3) in [10, Subsection 3.3], its argument needed the Sobolev embedding theorem and consequently characterized the conjugate functional under a restrictive assumption on the exponent p. In this paper we complete the characterization of the conjugate functional for all $p > 1$. Remark that this approach is different from the method used to characterize L^2-subdifferential of total variation in [1, Chapter 1], which is based on a fact that the functional of total variation is positive homogeneous of degree 1. By applying the characterization theorem we calculate the smallest element in the subdifferential of the energy functional under the Dirichlet boundary condition for a spherically symmetric surface in any spacial dimension. The smallest element is called canonical restriction. Our calculation of the canonical restriction is seen as an extension of that of 1-dimensional case presented in [10, Section 4] for the Dirichlet problem, [12, Chapter 3] for the periodic problem. The canonical restriction is relevant to the study of the crystalline motion since the general theory (see e.g. [5]) suggests that it actually represents the speed of the surface during the time evolution. From the canonical restriction we can, therefore, predict how the surface behaves in the next moment, which was in fact the strategy of Odisharia [12, Chapter 3] to derive the free boundary value problem.

In the rest of this section we prepare notations, introduce function spaces, and state the main results concerning the characterization of subdifferentials. In Section 2 we give proofs of the characterization theorems first for the periodic problem, then for the Dirichlet problem.

In Section 3 we calculate the canonical restriction under the Dirichlet boundary by assuming a spherical symmetry of the surface.

1.1. Function spaces with a periodic boundary condition

Here we introduce notations and function spaces to formulate the periodic problem. Throughout the paper the number d ($\in \mathbb{N}$) denotes the spacial dimension and p ($\in (1,\infty)$) is used to define the exponent of the spaces of integrable functions. The notation $\mathbb{T}^d$ stands for a d-dimensional flat torus; $\mathbb{T}^d := \prod_{i=1}^d (\mathbb{R}/\omega_i\mathbb{Z})$ with $\omega_i > 0$ ($i = 1, 2, \dots, d$). Set $\Omega_{\mathrm{per}} := \prod_{i=1}^d (0, \omega_i)$ ($\subset \mathbb{R}^d$).

We consider the following real Banach space of periodic integrable functions.

$$L^p(\mathbb{T}^d; \mathbb{R}^m) := \left\{ \mathbf{f} \in L^p_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^m) \;\middle|\; \begin{array}{l} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x} + (m_1\omega_1, \dots, m_d\omega_d)) \\ \text{a.e. } \mathbf{x} \in \mathbb{R}^d,\ \forall (m_1, \dots, m_d) \in \mathbb{Z}^d \end{array} \right\},$$

where $m \in \mathbb{N}$ and the notation $\mathbf{f} \in L^p_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^m)$ means that for any open bounded set O ($\subset \mathbb{R}^d$), $\mathbf{f}|_O \in L^p(O; \mathbb{R}^m)$. The norm of $L^p(\mathbb{T}^d; \mathbb{R}^m)$ is defined by

$$\|\mathbf{f}\|_{L^p(\mathbb{T}^d;\mathbb{R}^m)} := \left(\int_{\Omega_{\mathrm{per}}} \left|\mathbf{f}(\mathbf{x})\right|^p d\mathbf{x} \right)^{1/p}.$$

Among these spaces $L^2(\mathbb{T}^d; \mathbb{R}^m)$ is a Hilbert space having the inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle_{L^2(\mathbb{T}^d;\mathbb{R}^m)} := \int_{\Omega_{\mathrm{per}}} \langle \mathbf{f}(\mathbf{x}), \mathbf{g}(\mathbf{x}) \rangle_{\mathbb{R}^m}\, d\mathbf{x}.$$

When $m = 1$, let us simply write $L^p(\mathbb{T}^d)$ instead of $L^p(\mathbb{T}^d; \mathbb{R})$.

The space $L^p_{\mathrm{ave}}(\mathbb{T}^d)$ is a subspace of $L^p(\mathbb{T}^d)$ defined by

$$L^p_{\mathrm{ave}}(\mathbb{T}^d) := \left\{ f \in L^p(\mathbb{T}^d) \;\middle|\; \int_{\Omega_{\mathrm{per}}} f(\mathbf{x})\, d\mathbf{x} = 0 \right\}.$$

The Sobolev spaces $W^{1,p}(\mathbb{T}^d)$, $W^{1,p}_{\mathrm{ave}}(\mathbb{T}^d)$ are defined by

$$W^{1,p}(\mathbb{T}^d) := \left\{ f \in L^p(\mathbb{T}^d) \mid \nabla f \left(\in \mathcal{D}'(\mathbb{R}^d; \mathbb{R}^d)\right) \text{ satisfies } \nabla f \in L^p(\mathbb{T}^d; \mathbb{R}^d) \right\},$$
$$W^{1,p}_{\mathrm{ave}}(\mathbb{T}^d) := W^{1,p}(\mathbb{T}^d) \cap L^p_{\mathrm{ave}}(\mathbb{T}^d).$$

We use the notation $H^1_{\mathrm{ave}}(\mathbb{T}^d)$ in place of $W^{1,2}_{\mathrm{ave}}(\mathbb{T}^d)$.

Poincaré's inequality states that there exists a constant C (> 0) such that

$$\|f\|_{L^p(\mathbb{T}^d)} \leqslant C \|\nabla f\|_{L^p(\mathbb{T}^d;\mathbb{R}^d)}, \quad \forall f \in W^{1,p}_{\mathrm{ave}}(\mathbb{T}^d).$$

This inequality enables us to adapt $\|\nabla \cdot\|_{L^p(\mathbb{T}^d;\mathbb{R}^d)}$ as the norm of $W^{1,p}_{\mathrm{ave}}(\mathbb{T}^d)$ and $\langle \nabla \cdot, \nabla \cdot \rangle_{L^2(\mathbb{T}^d;\mathbb{R}^d)}$ as the inner product of the Hilbert space $H^1_{\mathrm{ave}}(\mathbb{T}^d)$.

Throughout the paper we use the notation $\langle\cdot,\cdot\rangle$ to denote the scalar product of duality between a real Banach space and its topological dual space. We do not specify which duality is being described by $\langle\cdot,\cdot\rangle$ if it is clear from the context.

Let $H_{\mathrm{ave}}^{-1}(\mathbb{T}^d)$ denote the topological dual space of $H_{\mathrm{ave}}^{1}(\mathbb{T}^d)$. We define a linear operator $-\Delta_{\mathrm{per}}: H_{\mathrm{ave}}^{1}(\mathbb{T}^d) \to H_{\mathrm{ave}}^{-1}(\mathbb{T}^d)$ by

$$\langle -\Delta_{\mathrm{per}} f, \cdot\rangle := \langle \nabla f, \nabla \cdot\rangle_{L^2(\mathbb{T}^d;\mathbb{R}^d)}, \quad \forall f \in H_{\mathrm{ave}}^{1}(\mathbb{T}^d).$$

Because of our choice of the inner product of $H_{\mathrm{ave}}^{1}(\mathbb{T}^d)$ and Riesz' representation theorem, the operator $-\Delta_{\mathrm{per}}: H_{\mathrm{ave}}^{1}(\mathbb{T}^d) \to H_{\mathrm{ave}}^{-1}(\mathbb{T}^d)$ is an isometry. The dual space $H_{\mathrm{ave}}^{-1}(\mathbb{T}^d)$ can be considered as a Hilbert space equipped with the inner product $\langle\cdot,\cdot\rangle_{H_{\mathrm{ave}}^{-1}(\mathbb{T}^d)}$ defined by

$$\langle f, g\rangle_{H_{\mathrm{ave}}^{-1}(\mathbb{T}^d)} := \langle (-\Delta_{\mathrm{per}})^{-1} f, g\rangle, \quad \forall f, g \in H_{\mathrm{ave}}^{-1}(\mathbb{T}^d).$$

Introduce the space of smooth periodic functions by

$$C^\infty(\mathbb{T}^d;\mathbb{R}^m) := \left\{ \mathbf{f} \in C^\infty(\mathbb{R}^d;\mathbb{R}^m) \;\middle|\; \begin{array}{l} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x} + (m_1\omega_1, \dots, m_d\omega_d)), \\ \forall \mathbf{x} \in \mathbb{R}^d,\ \forall (m_1, \dots, m_d) \in \mathbb{Z}^d \end{array} \right\}.$$

Again let us simply write $C^\infty(\mathbb{T}^d)$ instead of $C^\infty(\mathbb{T}^d;\mathbb{R})$. We define a subspace of $C^\infty(\mathbb{T}^d)$ by

$$C_{\mathrm{ave}}^\infty(\mathbb{T}^d) := \left\{ f \in C^\infty(\mathbb{T}^d) \;\middle|\; \int_{\Omega_{\mathrm{per}}} f(\mathbf{x})\, d\mathbf{x} = 0 \right\}.$$

We will make use of the following density property.

Lemma 1.1. *The set $C_{\mathrm{ave}}^\infty(\mathbb{T}^d)$ is dense in $W_{\mathrm{ave}}^{1,p}(\mathbb{T}^d)$.*

Proof. Let $\rho \in C_0^\infty(\mathbb{R}^d)$ be such that

$$\rho(\mathbf{x}) \geqslant 0 \quad (\forall \mathbf{x} \in \mathbb{R}^d), \qquad \rho(\mathbf{x}) = 0 \quad \text{if } |\mathbf{x}| \geqslant 1, \qquad \int_{\mathbb{R}^d} \rho(\mathbf{x})\, d\mathbf{x} = 1. \tag{1.5}$$

For any $f \in W_{\mathrm{ave}}^{1,p}(\mathbb{T}^d)$ and $\delta > 0$ define a function $f_\delta : \mathbb{R}^d \to \mathbb{R}$ by

$$f_\delta(\mathbf{x}) := \int_{\mathbb{R}^d} \delta^{-d} \rho\left(\frac{\mathbf{x} - \mathbf{y}}{\delta}\right) f(\mathbf{y})\, d\mathbf{y}.$$

By using standard properties of the mollifier and the periodicity of f one can check that $f_\delta \in C_{\mathrm{ave}}^\infty(\mathbb{T}^d)$ and f_δ converges to f in $W_{\mathrm{ave}}^{1,p}(\mathbb{T}^d)$ as $\delta \searrow 0$. □

Remark that these spaces of periodic functions are equivalent to those axiomatically defined on the compact Riemannian manifold $\mathbb{T}^d$, the flat torus. See e.g. [9] for the construction of $\mathbb{T}^d$ as a Riemannian manifold and [4] for Sobolev spaces on Riemannian manifolds in general.

We define a subset X_{per} of $H^{-1}_{\mathrm{ave}}(\mathbb{T}^d)$ as follows. An f $(\in H^{-1}_{\mathrm{ave}}(\mathbb{T}^d))$ belongs to X_{per} if there exists $\tilde{f} \in W^{1,p}_{\mathrm{ave}}(\mathbb{T}^d)$ such that

$$\langle f, \phi\rangle = \lim_{n\to\infty} \int_{\Omega_{\mathrm{per}}} \tilde{f}(\mathbf{x})\phi_n(\mathbf{x})\,d\mathbf{x}, \quad \forall\phi \in H^1_{\mathrm{ave}}\big(\mathbb{T}^d\big),$$

where $\{\phi_n\}_{n=1}^{\infty}$ $(\subset C^{\infty}_{\mathrm{ave}}(\mathbb{T}^d))$ is any sequence converging to ϕ in $H^1_{\mathrm{ave}}(\mathbb{T}^d)$ as $n \to \infty$.

Note that for any $f \in X_{\mathrm{per}}$ such $\tilde{f}$ $(\in W^{1,p}_{\mathrm{ave}}(\mathbb{T}^d))$ uniquely exists. From now we use the notation "$\tilde{\cdot}$" to indicate the corresponding function of $W^{1,p}_{\mathrm{ave}}(\mathbb{T}^d)$ to a given element of X_{per}. It follows that X_{per} is a real linear space and the map $f \mapsto \tilde{f} : X_{\mathrm{per}} \to W^{1,p}_{\mathrm{ave}}(\mathbb{T}^d)$ is linear.

By using these notions we now define the functional $F_{\mathrm{per}} : H^{-1}_{\mathrm{ave}}(\mathbb{T}^d) \to \mathbb{R} \cup \{\infty\}$ by

$$F_{\mathrm{per}}(f) := \begin{cases} \int_{\Omega_{\mathrm{per}}} \sigma(\nabla \tilde{f}(\mathbf{x}))\,d\mathbf{x} & \text{if } f \in X_{\mathrm{per}}, \\ \infty & \text{otherwise}, \end{cases}$$

where $\sigma : \mathbb{R}^d \to \mathbb{R}$ is defined by

$$\sigma(\mathbf{y}) := |\mathbf{y}| + \frac{\mu}{p}|\mathbf{y}|^p \quad \big(\mu > 0,\ p \in (1,\infty)\big).$$

Lemma 1.2. *The functional $F_{\mathrm{per}} : H^{-1}_{\mathrm{ave}}(\mathbb{T}^d) \to \mathbb{R} \cup \{\infty\}$ is convex, lower semi-continuous and not identically ∞.*

Proof. Being convex and not identically ∞ can be seen from the definition. To show the lower semi-continuity of F_{per}, assume that $\{f_n\}_{n=1}^{\infty}$ $(\subset H^{-1}_{\mathrm{ave}}(\mathbb{T}^d))$ converges to f in $H^{-1}_{\mathrm{ave}}(\mathbb{T}^d)$ as $n \to \infty$ and $F_{\mathrm{per}}(f_n) \leqslant \lambda$ $(\forall n \in \mathbb{N})$, where $\lambda \geqslant 0$.

Since $\{\tilde{f}_n\}_{n=1}^{\infty}$ is bounded in $W^{1,p}_{\mathrm{ave}}(\mathbb{T}^d)$, there are $g \in W^{1,p}_{\mathrm{ave}}(\mathbb{T}^d)$ and a subsequence $\{\tilde{f}_{n(j)}\}_{j=1}^{\infty}$ of $\{\tilde{f}_n\}_{n=1}^{\infty}$ such that $\tilde{f}_{n(j)}$ weakly converges to g in $W^{1,p}_{\mathrm{ave}}(\mathbb{T}^d)$ as $j \to \infty$. Mazur's theorem for $H^{-1}_{\mathrm{ave}}(\mathbb{T}^d) \times W^{1,p}_{\mathrm{ave}}(\mathbb{T}^d)$ guarantees that for any $k \in \mathbb{N}$ there exist $j_k \in \mathbb{N}$ and $\alpha_l^k \in [0,1]$ $(l = 1,\dots,j_k)$ satisfying $\sum_{l=1}^{j_k} \alpha_l^k = 1$ such that as $k \to \infty$

$$\sum_{l=1}^{j_k} \alpha_l^k f_{n(l)} \to f \quad \text{in } H^{-1}_{\mathrm{ave}}\big(\mathbb{T}^d\big), \qquad \sum_{l=1}^{j_k} \alpha_l^k \tilde{f}_{n(l)} \to g \quad \text{in } W^{1,p}_{\mathrm{ave}}\big(\mathbb{T}^d\big).$$

Moreover, for any $\psi \in C^{\infty}_{\mathrm{ave}}(\mathbb{T}^d)$

$$\langle f, \psi\rangle = \lim_{k\to\infty}\left\langle \sum_{l=1}^{j_k} \alpha_l^k f_{n(l)}, \psi\right\rangle = \lim_{k\to\infty} \int_{\Omega_{\mathrm{per}}} \sum_{l=1}^{j_k} \alpha_l^k \tilde{f}_{n(l)}(\mathbf{x})\psi(\mathbf{x})\,d\mathbf{x}$$

$$= \int_{\Omega_{\mathrm{per}}} g(\mathbf{x})\psi(\mathbf{x})\,d\mathbf{x}.$$

Hence, for any $\phi \in H^1_{\mathrm{ave}}(\mathbb{T}^d)$ and $\{\phi_n\}_{n=1}^{\infty}$ $(\subset C^{\infty}_{\mathrm{ave}}(\mathbb{T}^d))$ converging to ϕ in $H^1_{\mathrm{ave}}(\mathbb{T}^d)$

$$\langle f, \phi\rangle = \lim_{n\to\infty}\langle f, \phi_n\rangle = \lim_{n\to\infty}\int_{\Omega_{\mathrm{per}}} g(\mathbf{x})\phi_n(\mathbf{x})\,d\mathbf{x},$$

which means that $f \in X_{\mathrm{per}}$ and $g = \tilde{f}$.

Then by the convexity and the continuity of $\int_{\Omega_{\mathrm{per}}}\sigma(\cdot)\,d\mathbf{x}$ in $L^p(\mathbb{T}^d;\mathbb{R}^d)$

$$\begin{aligned} F_{\mathrm{per}}(f) &= \int_{\Omega_{\mathrm{per}}} \sigma\big(\nabla \tilde{f}(\mathbf{x})\big)\,d\mathbf{x} = \lim_{k\to\infty}\int_{\Omega_{\mathrm{per}}} \sigma\left(\sum_{l=1}^{j_k}\alpha_l^k \nabla \tilde{f}_{n(l)}(\mathbf{x})\right)d\mathbf{x} \\ &\leqslant \limsup_{k\to\infty}\sum_{l=1}^{j_k}\alpha_l^k F_{\mathrm{per}}(f_{n(l)}) \leqslant \lambda, \end{aligned}$$

which concludes that F_{per} is lower semi-continuous in $H^{-1}_{\mathrm{ave}}(\mathbb{T}^d)$. □

1.2. Function spaces with a Dirichlet boundary condition

Here we prepare some notions necessary to formulate the Dirichlet problem. Let Ω be an open bounded subset of $\mathbb{R}^d$. By Poincaré's inequality we may choose $\|\nabla\cdot\|_{L^p(\Omega;\mathbb{R}^d)}$ as the norm of $W_0^{1,p}(\Omega)$ and $\langle\nabla\cdot,\nabla\cdot\rangle_{L^2(\Omega;\mathbb{R}^d)}$ as the inner product of $H_0^1(\Omega)$. Let $H^{-1}(\Omega)$ denote the topological dual space of the Hilbert space $H_0^1(\Omega)$. We define a linear map $-\Delta_{\mathrm{D}} : H_0^1(\Omega) \to H^{-1}(\Omega)$ by

$$\langle -\Delta_{\mathrm{D}} f, \cdot\rangle := \langle \nabla f, \nabla\cdot\rangle_{L^2(\Omega;\mathbb{R}^d)}, \quad \forall f \in H_0^1(\Omega).$$

By using Riesz' representation theorem we can prove that the linear map $-\Delta_{\mathrm{D}} : H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isometry. The dual space $H^{-1}(\Omega)$ is a Hilbert space having the inner product $\langle\cdot,\cdot\rangle_{H^{-1}(\Omega)}$ defined by

$$\langle f, g\rangle_{H^{-1}(\Omega)} := \big\langle(-\Delta_{\mathrm{D}})^{-1} f, g\big\rangle, \quad \forall f, g \in H^{-1}(\Omega).$$

Let X_{D} denote a subset of $H^{-1}(\Omega)$ consisting of any $f \in H^{-1}(\Omega)$ for which there exists $\tilde{f} \in W_0^{1,p}(\Omega)$ such that

$$\langle f, \phi\rangle = \lim_{n\to\infty}\int_{\Omega} \tilde{f}(\mathbf{x})\phi_n(\mathbf{x})\,d\mathbf{x}, \quad \forall \phi \in H_0^1(\Omega),$$

where $\{\phi_n\}_{n=1}^{\infty}$ $(\subset C_0^{\infty}(\Omega))$ is any sequence converging to ϕ in $H_0^1(\Omega)$ as $n \to \infty$. For given $f \in X_{\mathrm{D}}$ such $\tilde{f}$ $(\in W_0^{1,p}(\Omega))$ uniquely exists. As in the periodic case we use the notation "$\tilde{\cdot}$" to represent the function of $W_0^{1,p}(\Omega)$ associated with a given element of X_{D}.

We define the functional $F_{\mathrm{D}} : H^{-1}(\Omega) \to \mathbb{R} \cup \{\infty\}$ by

$$F_{\mathrm{D}}(f) := \begin{cases} \int_\Omega \sigma(\nabla \tilde{f}(\mathbf{x}))\, d\mathbf{x} & \text{if } f \in X_{\mathrm{D}}, \\ \infty & \text{otherwise.} \end{cases}$$

The following lemma can be proved in the same way as in Lemma 1.2.

Lemma 1.3. *The functional $F_{\mathrm{D}} : H^{-1}(\Omega) \to \mathbb{R} \cup \{\infty\}$ is convex, lower semi-continuous and not identically ∞.*

1.3. Subdifferentials

Subdifferential is an extended concept of differential. Subdifferential of a functional becomes a multi-valued operator if the functional is not differentiable in the normal sense. Let us see this by calculating the subdifferential of the energy density σ. The subdifferential $\partial\sigma(\cdot) : \mathbb{R}^d \to 2^{\mathbb{R}^d}$ is defined by

$$\partial\sigma(\mathbf{x}) := \left\{\mathbf{y} \in \mathbb{R}^d \;\middle|\; \langle \mathbf{y}, \mathbf{z}\rangle_{\mathbb{R}^d} + \sigma(\mathbf{x}) \leqslant \sigma(\mathbf{x} + \mathbf{z}),\ \forall \mathbf{z} \in \mathbb{R}^d\right\}, \quad \forall \mathbf{x} \in \mathbb{R}^d.$$

It follows directly from the definition that

$$\partial\sigma(\mathbf{x}) = \begin{cases} \{|\mathbf{x}|^{-1}\mathbf{x} + \mu|\mathbf{x}|^{p-2}\mathbf{x}\} & \text{if } \mathbf{x} \neq \mathbf{0}, \\ \{\mathbf{y} \in \mathbb{R}^d \mid |\mathbf{y}| \leqslant 1\} & \text{if } \mathbf{x} = \mathbf{0}. \end{cases}$$

From this characterization we see that if $\mathbf{x} \neq \mathbf{0}$ the only element of $\partial\sigma(\mathbf{x})$ is nothing but the gradient of $\sigma(\cdot)$ at $\mathbf{x}$. However, at $\mathbf{x} = \mathbf{0}$, where $\sigma(\cdot)$ is not differentiable, $\partial\sigma(\mathbf{x})$ becomes multi-valued.

We define the subdifferential $\partial F_{\mathrm{per}}(\cdot) : H^{-1}_{\mathrm{ave}}(\mathbb{T}^d) \to 2^{H^{-1}_{\mathrm{ave}}(\mathbb{T}^d)}$ of F_{per} by

$$\partial F_{\mathrm{per}}(f) := \left\{g \in H^{-1}_{\mathrm{ave}}(\mathbb{T}^d) \;\middle|\; \langle g, h\rangle_{H^{-1}_{\mathrm{ave}}(\mathbb{T}^d)} + F_{\mathrm{per}}(f) \leqslant F_{\mathrm{per}}(f + h),\ \forall h \in H^{-1}_{\mathrm{ave}}(\mathbb{T}^d)\right\}$$

and the subdifferential $\partial F_{\mathrm{D}}(\cdot) : H^{-1}(\Omega) \to 2^{H^{-1}(\Omega)}$ of F_{D} by

$$\partial F_{\mathrm{D}}(f) := \left\{g \in H^{-1}(\Omega) \;\middle|\; \langle g, h\rangle_{H^{-1}(\Omega)} + F_{\mathrm{D}}(f) \leqslant F_{\mathrm{D}}(f + h),\ \forall h \in H^{-1}(\Omega)\right\}.$$

Our main purpose is to characterize $\partial F_{\mathrm{per}}(\cdot)$ and $\partial F_{\mathrm{D}}(\cdot)$. The results are the following.

Theorem 1.4. *If $\partial F_{\mathrm{per}}(f) \neq \emptyset$,*

$$\partial F_{\mathrm{per}}(f) = \left\{ -(-\Delta_{\mathrm{per}})\,\mathrm{div}\,\mathbf{g} \;\middle|\; \begin{array}{l} \mathbf{g} \in L^{p/(p-1)}(\mathbb{T}^d; \mathbb{R}^d) \textit{ satisfying that } \mathrm{div}\,\mathbf{g} \in H^1_{\mathrm{ave}}(\mathbb{T}^d), \\ \mathbf{g}(\mathbf{x}) \in \partial\sigma(\nabla \tilde{f}(\mathbf{x})) \textit{ a.e. } \mathbf{x} \in \mathbb{R}^d \end{array} \right\}.$$

Theorem 1.5. *If $\partial F_{\mathrm{D}}(f) \neq \emptyset$,*

$$\partial F_{\mathrm{D}}(f) = \left\{ -(-\Delta_{\mathrm{D}})\,\mathrm{div}\,\mathbf{g} \;\middle|\; \begin{array}{l} \mathbf{g} \in L^{p/(p-1)}(\Omega; \mathbb{R}^d) \textit{ satisfying that } \mathrm{div}\,\mathbf{g} \in H^1_0(\Omega), \\ \mathbf{g}(\mathbf{x}) \in \partial\sigma(\nabla \tilde{f}(\mathbf{x})) \textit{ a.e. } \mathbf{x} \in \Omega, \\ \langle h, \mathrm{div}\,\mathbf{g}\rangle + \int_\Omega \langle \nabla \tilde{h}(\mathbf{x}), \mathbf{g}(\mathbf{x})\rangle_{\mathbb{R}^d}\, d\mathbf{x} = 0,\ \forall h \in X_{\mathrm{D}} \end{array} \right\}.$$

By assuming an additional condition on p ($\in (1,\infty)$) we can simplify the characterization of Theorem 1.5 as follows.

Corollary 1.6. *Assume that*

$$p \in (1,\infty) \quad \text{if } d \leqslant 4,$$
$$p \in \left[\frac{2d}{d+4}, \infty\right) \quad \text{if } d \geqslant 5. \tag{1.6}$$

If $\partial F_{\mathrm{D}}(f) \neq \emptyset$,

$$\partial F_{\mathrm{D}}(f) = \left\{ -(-\Delta_{\mathrm{D}}) \operatorname{div} \mathbf{g} \;\middle|\; \begin{array}{l} \mathbf{g} \in L^{p/(p-1)}(\Omega;\mathbb{R}^d) \text{ satisfying that } \operatorname{div}\mathbf{g} \in H_0^1(\Omega), \\ \mathbf{g}(\mathbf{x}) \in \partial\sigma(\nabla \tilde{f}(\mathbf{x})) \text{ a.e. } \mathbf{x} \in \Omega \end{array} \right\}.$$

Remark 1.7. In Lemmas 1.2 and 1.3 we have seen that both $F_{\mathrm{per}} : H_{\mathrm{ave}}^{-1}(\mathbb{T}^d) \to \mathbb{R} \cup \{\infty\}$ and $F_{\mathrm{D}} : H^{-1}(\Omega) \to \mathbb{R} \cup \{\infty\}$ are convex, lower semi-continuous, and not identically ∞. These properties are sufficient to ensure the unique solvability of the initial value problems to find $f_{\mathrm{per}} \in C([0,\infty); H_{\mathrm{ave}}^{-1}(\mathbb{T}^d))$ and $f_{\mathrm{D}} \in C([0,\infty); H^{-1}(\Omega))$ such that

$$\begin{cases} \dfrac{d}{dt} f_{\mathrm{per}}(t) \in -\partial F_{\mathrm{per}}\big(f_{\mathrm{per}}(t)\big) \text{ a.e. } t>0, \\ f_{\mathrm{per}}(0) = f_{\mathrm{per},0} \ (\in \overline{X_{\mathrm{per}}}), \end{cases} \qquad \begin{cases} \dfrac{d}{dt} f_{\mathrm{D}}(t) \in -\partial F_{\mathrm{D}}\big(f_{\mathrm{D}}(t)\big) \text{ a.e. } t>0, \\ f_{\mathrm{D}}(0) = f_{\mathrm{D},0} \ (\in \overline{X_{\mathrm{D}}}) \end{cases}$$

(see e.g. [5]). Theorems above characterize the right-hand sides of these evolution systems and provide us with explicit representations comparable to the right-hand side of the original model (1.1).

2. Proof of the characterization of subdifferentials

In this section we prove Theorems 1.4, 1.5 and Corollary 1.6. Let us fix some notational conventions and recall a few basic facts from convex analysis beforehand. For a real Banach space B let B^* denote its topological dual space. For a functional $E : B \to \mathbb{R} \cup \{\infty\}$ being not identically ∞ its conjugate functional $E^* : B^* \to \mathbb{R} \cup \{\infty\}$ is defined by

$$E^*(v) := \sup_{u \in B} \big\{ \langle v, u \rangle - E(u) \big\}, \quad \forall v \in B^*.$$

Lemma 2.1. *Assume that* $E : B \to \mathbb{R} \cup \{\infty\}$ *is convex, lower semi-continuous and not identically* ∞. *The following hold true.*

(1) $E^* : B^* \to \mathbb{R} \cup \{\infty\}$ *is convex, lower semi-continuous and not identically* ∞.
(2) $(E^*)^*(v) = E(v)$, $\forall v \in B$.

For a functional defined on a real Hilbert space H we adapt the inner product $\langle \cdot,\cdot \rangle_H$ to define its conjugate functional. To distinguish from Banach spaces' case, let us change a notation.

For a functional $F : H \to \mathbb{R} \cup \{\infty\}$ being not identically ∞ we define its conjugate functional $F^{\#} : H \to \mathbb{R} \cup \{\infty\}$ by

$$F^{\#}(v) := \sup_{u \in H} \big\{ \langle v, u \rangle_H - F(u) \big\}, \quad \forall v \in H.$$

Moreover, we define its subdifferential $\partial F : H \to 2^H$ by

$$\partial F(u) := \big\{ v \in H \,\big|\, \langle v, w \rangle_H + F(u) \leqslant F(u + w),\ \forall w \in H \big\}.$$

Lemma 2.2. *Assume that a functional $F : H \to \mathbb{R} \cup \{\infty\}$ is convex, lower semi-continuous and not identically ∞. The following statements are equivalent to each other.*

(i) $v \in \partial F(u)$.
(ii) $u \in \partial F^{\#}(v)$.
(iii) $F(u) + F^{\#}(v) = \langle u, v \rangle_H$.

We use Lemmas 2.1 and 2.2 without providing the proofs. See e.g. [6] to verify them.

The conjugate functional $\sigma^{\#} : \mathbb{R}^d \to \mathbb{R}$ of σ and its subdifferential $\partial\sigma^{\#}(\cdot) : \mathbb{R}^d \to 2^{\mathbb{R}^d}$ can be calculated from the definitions.

Lemma 2.3. *For any $\mathbf{y} \in \mathbb{R}^d$*

$$\sigma^{\#}(\mathbf{y}) = \begin{cases} 0 & \text{if } |\mathbf{y}| \leqslant 1, \\ (1 - \frac{1}{p})\mu^{-1/(p-1)}(|\mathbf{y}| - 1)^{p/(p-1)} & \text{if } |\mathbf{y}| > 1, \end{cases} \tag{2.1}$$

$$\partial\sigma^{\#}(\mathbf{y}) = \begin{cases} \{\mathbf{0}\} & \text{if } |\mathbf{y}| \leqslant 1, \\ \{\mu^{-1/(p-1)}(|\mathbf{y}| - 1)^{1/(p-1)}|\mathbf{y}|^{-1}\mathbf{y}\} & \text{if } |\mathbf{y}| > 1. \end{cases} \tag{2.2}$$

2.1. Proof for the periodic problem

We are going to characterize the subdifferential of the periodic energy F_{per}. We introduce the real Banach space $H^{-1}_{\text{ave}}(\mathbb{T}^d) \times L^p(\mathbb{T}^d; \mathbb{R}^d)$ having the norm defined by $\|(f, \mathbf{g})\| := \|f\|_{H^{-1}_{\text{ave}}(\mathbb{T}^d)} + \|\mathbf{g}\|_{L^p(\mathbb{T}^d;\mathbb{R}^d)}$. Define functionals $Q, R : H^{-1}_{\text{ave}}(\mathbb{T}^d) \times L^p(\mathbb{T}^d; \mathbb{R}^d) \to \mathbb{R} \cup \{\infty\}$ by

$$Q((f, \mathbf{g})) := \int_{\Omega_{\text{per}}} \sigma\big(\mathbf{g}(\mathbf{x})\big)\, d\mathbf{x},$$

$$R((f, \mathbf{g})) := \begin{cases} 0 & \text{if } f \in X_{\text{per}} \text{ and } \mathbf{g} = \nabla \tilde{f}, \\ \infty & \text{otherwise.} \end{cases}$$

One can check that Q, R are convex, lower semi-continuous, and not identically ∞.

We define a linear map $\Phi_{p/(p-1)} : H^{-1}_{\text{ave}}(\mathbb{T}^d) \times L^{p/(p-1)}(\mathbb{T}^d; \mathbb{R}^d) \to (H^{-1}_{\text{ave}}(\mathbb{T}^d) \times L^p(\mathbb{T}^d; \mathbb{R}^d))^*$ by

$$\left\langle \Phi_{p/(p-1)}((u,\mathbf{v})),(f,\mathbf{g})\right\rangle := \langle u,f\rangle_{H^{-1}_{\mathrm{ave}}(\mathbb{T}^d)} + \int_{\Omega_{\mathrm{per}}} \left\langle \mathbf{v}(\mathbf{x}),\mathbf{g}(\mathbf{x})\right\rangle_{\mathbb{R}^d} d\mathbf{x},$$

$$\forall (u,\mathbf{v}) \in H^{-1}_{\mathrm{ave}}\left(\mathbb{T}^d\right) \times L^{p/(p-1)}\left(\mathbb{T}^d;\mathbb{R}^d\right),\ \forall (f,\mathbf{g}) \in H^{-1}_{\mathrm{ave}}\left(\mathbb{T}^d\right) \times L^{p}\left(\mathbb{T}^d;\mathbb{R}^d\right).$$

The map $\Phi_{p/(p-1)}$ is an isomorphism between these Banach spaces.

In our proof characterizing the conjugate functional $F^{\#}_{\mathrm{per}}$ $(: H^{-1}_{\mathrm{ave}}(\mathbb{T}^d) \to \mathbb{R} \cup \{\infty\})$ is crucial to characterize $\partial F_{\mathrm{per}}$. The first step is the following.

Lemma 2.4. *For any $u \in H^{-1}_{\mathrm{ave}}(\mathbb{T}^d)$*

$$F^{\#}_{\mathrm{per}}(u) = (Q+R)^*\left(\Phi_{p/(p-1)}((u,\mathbf{0}))\right). \tag{2.3}$$

Proof. Take any $(u,\mathbf{v}) \in H^{-1}_{\mathrm{ave}}(\mathbb{T}^d) \times L^{p/(p-1)}(\mathbb{T}^d;\mathbb{R}^d)$.

$$\begin{aligned}
&(Q+R)^*\left(\Phi_{p/(p-1)}((u,\mathbf{v}))\right) \\
&\quad = \sup_{(f,\mathbf{g})\in H^{-1}_{\mathrm{ave}}(\mathbb{T}^d)\times L^p(\mathbb{T}^d;\mathbb{R}^d)} \left\{\left\langle \Phi_{p/(p-1)}((u,\mathbf{v})),(f,\mathbf{g})\right\rangle - (Q+R)((f,\mathbf{g}))\right\} \\
&\quad = \sup_{f\in X_{\mathrm{per}}} \left\{ \langle u,f\rangle_{H^{-1}_{\mathrm{ave}}(\mathbb{T}^d)} + \int_{\Omega_{\mathrm{per}}} \left\langle \mathbf{v}(\mathbf{x}),\nabla\tilde{f}(\mathbf{x})\right\rangle_{\mathbb{R}^d} d\mathbf{x} - \int_{\Omega_{\mathrm{per}}} \sigma\left(\nabla\tilde{f}(\mathbf{x})\right) d\mathbf{x}\right\},
\end{aligned}$$

from which the claimed equality follows. □

We will characterize the right-hand side of (2.3) after characterizing Q^* and R^*.

Lemma 2.5. *For any $(u,\mathbf{v}) \in H^{-1}_{\mathrm{ave}}(\mathbb{T}^d) \times L^{p/(p-1)}(\mathbb{T}^d;\mathbb{R}^d)$*

$$Q^*\left(\Phi_{p/(p-1)}((u,\mathbf{v}))\right) = \begin{cases} \int_{\Omega_{\mathrm{per}}} \sigma^{\#}(\mathbf{v}(\mathbf{x}))\,d\mathbf{x} & \textit{if } u=0, \\ \infty & \textit{otherwise.}\end{cases}$$

Proof. Take any $(u,\mathbf{v}) \in H^{-1}_{\mathrm{ave}}(\mathbb{T}^d) \times L^{p/(p-1)}(\mathbb{T}^d;\mathbb{R}^d)$.

$$\begin{aligned}
&Q^*\left(\Phi_{p/(p-1)}((u,\mathbf{v}))\right) \\
&\quad = \sup_{(f,\mathbf{g})\in H^{-1}_{\mathrm{ave}}(\mathbb{T}^d)\times L^p(\mathbb{T}^d;\mathbb{R}^d)} \left\{\left\langle \Phi_{p/(p-1)}((u,\mathbf{v})),(f,\mathbf{g})\right\rangle - Q((f,\mathbf{g}))\right\} \\
&\quad = \sup_{(f,\mathbf{g})\in H^{-1}_{\mathrm{ave}}(\mathbb{T}^d)\times L^p(\mathbb{T}^d;\mathbb{R}^d)} \left\{ \langle u,f\rangle_{H^{-1}_{\mathrm{ave}}(\mathbb{T}^d)} + \int_{\Omega_{\mathrm{per}}} \left\langle \mathbf{v}(\mathbf{x}),\mathbf{g}(\mathbf{x})\right\rangle_{\mathbb{R}^d} d\mathbf{x} - \int_{\Omega_{\mathrm{per}}} \sigma\left(\mathbf{g}(\mathbf{x})\right) d\mathbf{x}\right\} \\
&\quad = \begin{cases} \sup_{\mathbf{g}\in L^p(\mathbb{T}^d;\mathbb{R}^d)} \int_{\Omega_{\mathrm{per}}} (\langle \mathbf{v}(\mathbf{x}),\mathbf{g}(\mathbf{x})\rangle_{\mathbb{R}^d} - \sigma(\mathbf{g}(\mathbf{x})))\,d\mathbf{x} & \text{if } u=0, \\ \infty & \text{otherwise.}\end{cases}
\end{aligned} \tag{2.4}$$

On one hand, it follows from the definition of $\sigma^{\#}$ that

$$\sup_{\mathbf{g}\in L^p(\mathbb{T}^d;\mathbb{R}^d)} \int_{\Omega_{\mathrm{per}}} \big(\langle \mathbf{v}(\mathbf{x}), \mathbf{g}(\mathbf{x})\rangle_{\mathbb{R}^d} - \sigma\big(\mathbf{g}(\mathbf{x})\big)\big)\, d\mathbf{x} \leqslant \int_{\Omega_{\mathrm{per}}} \sigma^{\#}\big(\mathbf{v}(\mathbf{x})\big)\, d\mathbf{x}. \tag{2.5}$$

On the other hand, let us define $\mathbf{h} \in L^p(\mathbb{T}^d;\mathbb{R}^d)$ by

$$\mathbf{h}(\mathbf{x}) := \begin{cases} \mathbf{0} & \text{if } |\mathbf{v}(\mathbf{x})| \leqslant 1, \\ \mu^{-1/(p-1)}(|\mathbf{v}(\mathbf{x})|-1)^{1/(p-1)}|\mathbf{v}(\mathbf{x})|^{-1}\mathbf{v}(\mathbf{x}) & \text{if } |\mathbf{v}(\mathbf{x})| > 1. \end{cases}$$

By (2.2)

$$\mathbf{h}(\mathbf{x}) \in \partial\sigma^{\#}\big(\mathbf{v}(\mathbf{x})\big) \quad \text{a.e. } \mathbf{x} \in \mathbb{R}^d. \tag{2.6}$$

By Lemma 2.2 the inclusion (2.6) implies that

$$\sigma^{\#}\big(\mathbf{v}(\mathbf{x})\big) = \langle \mathbf{v}(\mathbf{x}), \mathbf{h}(\mathbf{x})\rangle_{\mathbb{R}^d} - \sigma\big(\mathbf{h}(\mathbf{x})\big) \quad \text{a.e. } \mathbf{x} \in \mathbb{R}^d,$$

which leads to

$$\int_{\Omega_{\mathrm{per}}} \sigma^{\#}\big(\mathbf{v}(\mathbf{x})\big)\, d\mathbf{x} \leqslant \sup_{\mathbf{g}\in L^p(\mathbb{T}^d;\mathbb{R}^d)} \int_{\Omega_{\mathrm{per}}} \big(\langle \mathbf{v}(\mathbf{x}), \mathbf{g}(\mathbf{x})\rangle_{\mathbb{R}^d} - \sigma\big(\mathbf{g}(\mathbf{x})\big)\big)\, d\mathbf{x}. \tag{2.7}$$

By putting (2.4), (2.5) and (2.7) together, we obtain the result. □

To characterize R^* we need a couple of lemmas based on density properties of smooth functions in the periodic Sobolev spaces.

Lemma 2.6. *For any $f \in W^{1,p}_{\mathrm{ave}}(\mathbb{T}^d)$ and $\boldsymbol{\phi} \in C^\infty(\mathbb{T}^d;\mathbb{R}^d)$*

$$\int_{\Omega_{\mathrm{per}}} f(\mathbf{x})\, \operatorname{div}\boldsymbol{\phi}(\mathbf{x})\, d\mathbf{x} + \int_{\Omega_{\mathrm{per}}} \langle \nabla f(\mathbf{x}), \boldsymbol{\phi}(\mathbf{x})\rangle_{\mathbb{R}^d}\, d\mathbf{x} = 0.$$

Proof. Lemma 1.1 justifies the equality. □

Lemma 2.7. *For any $\mathbf{v} \in L^{p/(p-1)}(\mathbb{T}^d;\mathbb{R}^d)$ satisfying $\operatorname{div}\mathbf{v} \in H^1_{\mathrm{ave}}(\mathbb{T}^d)$ there exists $\{\mathbf{v}_n\}_{n=1}^\infty \subset C^\infty(\mathbb{T}^d;\mathbb{R}^d)$ such that as $n \to \infty$*

$$\mathbf{v}_n \to \mathbf{v} \quad \text{in } L^{p/(p-1)}\big(\mathbb{T}^d;\mathbb{R}^d\big),$$
$$\operatorname{div}\mathbf{v}_n \to \operatorname{div}\mathbf{v} \quad \text{in } H^1_{\mathrm{ave}}\big(\mathbb{T}^d\big).$$

Proof. As in Lemma 1.1 let us define a function $\mathbf{v}_\delta : \mathbb{R}^d \to \mathbb{R}^d$ by

$$\mathbf{v}_\delta(\mathbf{x}) := \int_{\mathbb{R}^d} \delta^{-d} \rho\left(\frac{\mathbf{x}-\mathbf{y}}{\delta}\right) \mathbf{v}(\mathbf{y})\, d\mathbf{y}$$

by choosing a function $\rho \in C_0^\infty(\mathbb{R}^d)$ having the properties (1.5) and $\delta > 0$. The function $\mathbf{v}_\delta$ is contained in $C^\infty(\mathbb{T}^d; \mathbb{R}^d)$ and converges to $\mathbf{v}$ in the way claimed above as $\delta \searrow 0$. □

Then we have

Lemma 2.8. *For any* $(u, \mathbf{v}) \in H_{\mathrm{ave}}^{-1}(\mathbb{T}^d) \times L^{p/(p-1)}(\mathbb{T}^d; \mathbb{R}^d)$

$$R^*\big(\Phi_{p/(p-1)}((u, \mathbf{v}))\big) = \begin{cases} 0 & \textit{if } \operatorname{div} \mathbf{v}\, (\in \mathcal{D}'(\mathbb{R}^d)) \textit{ satisfies } \operatorname{div} \mathbf{v} = (-\Delta_{\mathrm{per}})^{-1} u, \\ \infty & \textit{otherwise.} \end{cases}$$

Proof. Take any $(u, \mathbf{v}) \in H_{\mathrm{ave}}^{-1}(\mathbb{T}^d) \times L^{p/(p-1)}(\mathbb{T}^d; \mathbb{R}^d)$.

$$\begin{aligned}
&R^*\big(\Phi_{p/(p-1)}((u, \mathbf{v}))\big) \\
&\quad = \sup_{(f,\mathbf{g}) \in H_{\mathrm{ave}}^{-1}(\mathbb{T}^d) \times L^p(\mathbb{T}^d; \mathbb{R}^d)} \big\{ \big\langle \Phi_{p/(p-1)}((u, \mathbf{v})), (f, \mathbf{g}) \big\rangle - R((f, \mathbf{g})) \big\} \\
&\quad = \sup_{f \in X_{\mathrm{per}}} \left\{ \langle u, f \rangle_{H_{\mathrm{ave}}^{-1}(\mathbb{T}^d)} + \int_{\Omega_{\mathrm{per}}} \big\langle \mathbf{v}(\mathbf{x}), \nabla \tilde{f}(\mathbf{x}) \big\rangle_{\mathbb{R}^d}\, d\mathbf{x} \right\} \\
&\quad \geqslant \sup_{\phi \in C_{\mathrm{ave}}^\infty(\mathbb{T}^d)} \left\{ \int_{\Omega_{\mathrm{per}}} (-\Delta_{\mathrm{per}})^{-1} u(\mathbf{x}) \cdot \phi(\mathbf{x})\, d\mathbf{x} + \int_{\Omega_{\mathrm{per}}} \big\langle \mathbf{v}(\mathbf{x}), \nabla \phi(\mathbf{x}) \big\rangle_{\mathbb{R}^d}\, d\mathbf{x} \right\} \\
&\quad = \sup_{\mathbf{y} \in \mathbb{R}^d} \sup_{\phi \in C_{\mathrm{ave}}^\infty(\mathbb{T}^d)} \int_{\Omega_{\mathrm{per}} + \mathbf{y}} \Big((-\Delta_{\mathrm{per}})^{-1} u(\mathbf{x}) \cdot \phi(\mathbf{x}) + \big\langle \mathbf{v}(\mathbf{x}), \nabla \phi(\mathbf{x}) \big\rangle_{\mathbb{R}^d} \Big)\, d\mathbf{x} \\
&\quad \geqslant \sup_{\mathbf{y} \in \mathbb{R}^d} \sup_{\phi \in C_0^\infty(\Omega_{\mathrm{per}} + \mathbf{y})} \int_{\Omega_{\mathrm{per}} + \mathbf{y}} \Big((-\Delta_{\mathrm{per}})^{-1} u(\mathbf{x}) \cdot \phi(\mathbf{x}) + \big\langle \mathbf{v}(\mathbf{x}), \nabla \phi(\mathbf{x}) \big\rangle_{\mathbb{R}^d} \Big)\, d\mathbf{x} \\
&\quad = \begin{cases} 0 & \text{if } \operatorname{div} \mathbf{v}\, (\in \mathcal{D}'(\Omega_{\mathrm{per}} + \mathbf{y})) \text{ satisfies } \operatorname{div} \mathbf{v} = (-\Delta_{\mathrm{per}})^{-1} u|_{\Omega_{\mathrm{per}} + \mathbf{y}}\ (\forall \mathbf{y} \in \mathbb{R}^d), \\ \infty & \text{otherwise,} \end{cases}
\end{aligned} \tag{2.8}$$

where we have used the fact that $\int_{\Omega_{\mathrm{per}} + \mathbf{y}} (-\Delta_{\mathrm{per}})^{-1} u(\mathbf{x})\, d\mathbf{x} = 0$. From the inequality (2.8) we can deduce that

$$R^*\big(\Phi_{p/(p-1)}((u, \mathbf{v}))\big) \geqslant \begin{cases} 0 & \text{if } \operatorname{div} \mathbf{v}\, (\in \mathcal{D}'(\mathbb{R}^d)) \text{ satisfies } \operatorname{div} \mathbf{v} = (-\Delta_{\mathrm{per}})^{-1} u, \\ \infty & \text{otherwise.} \end{cases} \tag{2.9}$$

To confirm this, assume that $\operatorname{div}\mathbf{v}\,(\in \mathcal{D}'(\Omega_{\text{per}}+\mathbf{y}))$ satisfies $\operatorname{div}\mathbf{v} = (-\Delta_{\text{per}})^{-1}u|_{\Omega_{\text{per}}+\mathbf{y}}$ ($\forall \mathbf{y}\in \mathbb{R}^d$). For any proposition P let $1_{\text{P}}\,(\in\{0,1\})$ be defined by

$$1_{\text{P}} := \begin{cases} 1 & \text{if P is true,} \\ 0 & \text{otherwise.} \end{cases}$$

Take a function $\eta\in C_0^\infty(\mathbb{R})$ such that $0\leqslant \eta(x)\leqslant 1$ ($\forall x\in\mathbb{R}$), $\eta(x)=0$ if $|x|\geqslant 1$ and $\int_{\mathbb{R}}\eta(x)\,dx=1$. By using η we define functions $f_{i,n}\in C_0^\infty(\mathbb{R})$ ($i\in\{1,\dots,d\}$, $n\in\mathbb{Z}$) by

$$f_{i,n}(x) := \int_{\mathbb{R}} \left(\frac{\omega_i}{8}\right)^{-1} \eta\left(\frac{x-y}{\omega_i/8}\right) 1_{y\in[0,\omega_i/2)+\omega_i n/2}\,dy.$$

Remark that $\operatorname{supp} f_{i,n}\subset (0,\omega_i)+\omega_i n/2-\omega_i/4$ and $\sum_{n\in\mathbb{Z}} f_{i,n}(x)=1$ ($\forall x\in\mathbb{R}$). For any $\mathbf{n}\,(=(n_1,\dots,n_d))\in\mathbb{Z}^d$ set $f_{\mathbf{n}}(\mathbf{x}) := \prod_{i=1}^d f_{i,n_i}(x_i)$. We see that $f_{\mathbf{n}}\in C_0^\infty(\Omega_{\text{per}}+\mathbf{y}_{\mathbf{n}})$ and $\sum_{\mathbf{n}\in\mathbb{Z}^d} f_{\mathbf{n}}(\mathbf{x})=1$ ($\forall \mathbf{x}\in\mathbb{R}^d$), where $\mathbf{y}_{\mathbf{n}} := (\omega_1 n_1/2-\omega_1/4, \omega_2 n_2/2-\omega_2/4,\dots,\omega_d n_d/2-\omega_d/4)\,(\in\mathbb{R}^d)$. For any $\phi\in C_0^\infty(\mathbb{R}^d)$ there exists $N\in\mathbb{N}$ such that

$$\operatorname{supp}\phi \subset \bigcup_{\substack{\mathbf{n}\in\mathbb{Z}^d\\ |n_i|\leqslant N\ (i=1,\dots,d)}} (\Omega_{\text{per}}+\mathbf{y}_{\mathbf{n}}).$$

Then for any $\mathbf{x}\in\operatorname{supp}\phi$

$$\sum_{\substack{\mathbf{n}\in\mathbb{Z}^d\\ |n_i|\leqslant N+1\ (i=1,\dots,d)}} f_{\mathbf{n}}(\mathbf{x}) = 1.$$

Thus, by assumption

$$\begin{aligned}
\int_{\mathbb{R}^d} \big\langle \mathbf{v}(\mathbf{x}), -\nabla\phi(\mathbf{x})\big\rangle_{\mathbb{R}^d}\,d\mathbf{x} &= \sum_{\substack{\mathbf{n}\in\mathbb{Z}^d\\ |n_i|\leqslant N+1\ (i=1,\dots,d)}} \int_{\Omega_{\text{per}}+\mathbf{y}_{\mathbf{n}}} \big\langle \mathbf{v}(\mathbf{x}), -\nabla\big(f_{\mathbf{n}}(\mathbf{x})\phi(\mathbf{x})\big)\big\rangle_{\mathbb{R}^d}\,d\mathbf{x} \\
&= \sum_{\substack{\mathbf{n}\in\mathbb{Z}^d\\ |n_i|\leqslant N+1\ (i=1,\dots,d)}} \int_{\Omega_{\text{per}}+\mathbf{y}_{\mathbf{n}}} (-\Delta_{\text{per}})^{-1}u(\mathbf{x})\cdot f_{\mathbf{n}}(\mathbf{x})\phi(\mathbf{x})\,d\mathbf{x} \\
&= \int_{\mathbb{R}^d} (-\Delta_{\text{per}})^{-1}u(\mathbf{x})\cdot\phi(\mathbf{x})\,d\mathbf{x}.
\end{aligned}$$

Hence, $\operatorname{div}\mathbf{v}\,(\in\mathcal{D}'(\mathbb{R}^d))$ satisfies $\operatorname{div}\mathbf{v}=(-\Delta_{\text{per}})^{-1}u$, which means that the right-hand side of (2.8) is larger than equal to that of (2.9), resulting in the inequality (2.9).

To show that the inequality (2.9) is actually the equality, let us assume that $\operatorname{div}\mathbf{v} = (-\Delta_{\text{per}})^{-1}u$. By Lemma 2.7 we can take a sequence $\{\mathbf{v}_n\}_{n=1}^\infty\,(\subset C^\infty(\mathbb{T}^d;\mathbb{R}^d))$ such that $\mathbf{v}_n\to\mathbf{v}$ in $L^{p/(p-1)}(\mathbb{T}^d;\mathbb{R}^d)$, $\operatorname{div}\mathbf{v}_n\to\operatorname{div}\mathbf{v}$ in $H^1_{\text{ave}}(\mathbb{T}^d)$ as $n\to\infty$. Applying Lemma 2.6, we observe that

$$\begin{aligned}R^*\big(\Phi_{p/(p-1)}((u,\mathbf{v}))\big) &= \sup_{f\in X_{\mathrm{per}}}\left\{\langle \operatorname{div}\mathbf{v}, f\rangle + \int_{\Omega_{\mathrm{per}}} \big\langle \mathbf{v}(\mathbf{x}), \nabla \tilde f(\mathbf{x})\big\rangle_{\mathbb{R}^d}\, d\mathbf{x}\right\}\\ &= \sup_{f\in X_{\mathrm{per}}}\lim_{n\to\infty}\left\{\int_{\Omega_{\mathrm{per}}} \big(\operatorname{div}\mathbf{v}_n(\mathbf{x})\tilde f(\mathbf{x}) + \big\langle \mathbf{v}_n(\mathbf{x}), \nabla \tilde f(\mathbf{x})\big\rangle_{\mathbb{R}^d}\big)\, d\mathbf{x}\right\}\\ &= 0,\end{aligned}$$

which concludes the proof. □

For any $u\in H^{-1}_{\mathrm{ave}}(\mathbb{T}^d)$ let $Y_{\mathrm{per}}(u)$ ($\subset L^{p/(p-1)}(\mathbb{T}^d;\mathbb{R}^d)$) be defined by

$$Y_{\mathrm{per}}(u) := \big\{\mathbf{s}\in L^{p/(p-1)}\big(\mathbb{T}^d;\mathbb{R}^d\big)\ \big|\ \operatorname{div}\mathbf{s} = (-\Delta_{\mathrm{per}})^{-1}u\big\}.$$

Using Lemmas 2.5 and 2.8, we show the following.

Lemma 2.9. *For any* $(u,\mathbf{v})\in H^{-1}_{\mathrm{ave}}(\mathbb{T}^d)\times L^{p/(p-1)}(\mathbb{T}^d;\mathbb{R}^d)$

$$(Q+R)^*\big(\Phi_{p/(p-1)}((u,\mathbf{v}))\big) = \begin{cases}\min_{\mathbf{s}\in Y_{\mathrm{per}}(u)}\int_{\Omega_{\mathrm{per}}}\sigma^{\#}(\mathbf{v}(\mathbf{x})-\mathbf{s}(\mathbf{x}))\,d\mathbf{x} & \text{if } Y_{\mathrm{per}}(u)\neq\emptyset,\\ \infty & \text{otherwise.}\end{cases}$$

Remark 2.10. A direct application of the general theorem [2, Proposition 3.4] on inf-convolution can shorten the proof of Lemma 2.9 below. However, we prove the lemma by referring only to the basic facts Lemma 2.1 and Lemma 2.2 for self-containedness of the paper.

Proof of Lemma 2.9. Define a functional $S: H^{-1}_{\mathrm{ave}}(\mathbb{T}^d)\times L^{p/(p-1)}(\mathbb{T}^d;\mathbb{R}^d)\to\mathbb{R}\cup\{\infty\}$ by

$$\begin{aligned}S((u,\mathbf{v})) := \inf_{(r,\mathbf{s})\in H^{-1}_{\mathrm{ave}}(\mathbb{T}^d)\times L^{p/(p-1)}(\mathbb{T}^d;\mathbb{R}^d)} &\big\{Q^*\big(\Phi_{p/(p-1)}\big((u,\mathbf{v})-(r,\mathbf{s})\big)\big)\\ &+ R^*\big(\Phi_{p/(p-1)}((r,\mathbf{s}))\big)\big\},\quad \forall (u,\mathbf{v})\in H^{-1}_{\mathrm{ave}}\big(\mathbb{T}^d\big)\times L^{p/(p-1)}\big(\mathbb{T}^d;\mathbb{R}^d\big).\end{aligned}\tag{2.10}$$

Lemmas 2.5 and 2.8 imply that

$$S((u,\mathbf{v})) = \begin{cases}\inf_{\mathbf{s}\in Y_{\mathrm{per}}(u)}\int_{\Omega_{\mathrm{per}}}\sigma^{\#}(\mathbf{v}(\mathbf{x})-\mathbf{s}(\mathbf{x}))\,d\mathbf{x} & \text{if } Y_{\mathrm{per}}(u)\neq\emptyset,\\ \infty & \text{otherwise.}\end{cases}\tag{2.11}$$

We need to show that $S((u,\mathbf{v})) = (Q+R)^*(\Phi_{p/(p-1)}((u,\mathbf{v})))$.

By using the convexity of Q^* and R^* in (2.10) we can prove that S is convex as well. Moreover, from (2.11) and (2.1) we see that S is not identically ∞. To show the lower semi-continuity of S, let us assume that $(u_n,\mathbf{v}_n)$ converges to $(u,\mathbf{v})$ in $H^{-1}_{\mathrm{ave}}(\mathbb{T}^d)\times L^{p/(p-1)}(\mathbb{T}^d;\mathbb{R}^d)$ as $n\to\infty$ and there is $\lambda\geqslant 0$ such that $S((u_n,\mathbf{v}_n))\leqslant\lambda$ ($\forall n\in\mathbb{N}$). The equality (2.11) ensures that there exists $\{\mathbf{s}_i^n\}_{i=1}^{\infty}$ ($\subset L^{p/(p-1)}(\mathbb{T}^d;\mathbb{R}^d)$) such that $\operatorname{div}\mathbf{s}_i^n = (-\Delta_{\mathrm{per}})^{-1}u_n$ ($\forall i\in\mathbb{N}$) and

$$\lim_{i\to\infty}\int_{\Omega_{\mathrm{per}}}\sigma^{\#}\big(\mathbf{v}_n(\mathbf{x})-\mathbf{s}_i^n(\mathbf{x})\big)\,d\mathbf{x} = \inf_{\mathbf{s}\in Y_{\mathrm{per}}(u_n)}\int_{\Omega_{\mathrm{per}}}\sigma^{\#}\big(\mathbf{v}_n(\mathbf{x})-\mathbf{s}(\mathbf{x})\big)\,d\mathbf{x}.$$

There exists $\lambda' \geqslant 0$ such that

$$\int_{\Omega_{\mathrm{per}}} \sigma^{\#}\big(\mathbf{v}_n(\mathbf{x}) - \mathbf{s}_i^n(\mathbf{x})\big)\,d\mathbf{x} \leqslant \lambda + \lambda', \quad \forall i \in \mathbb{N}.$$

By this inequality and (2.1) $\{\mathbf{s}_i^n\}_{i=1}^{\infty}$ is bounded in $L^{p/(p-1)}(\mathbb{T}^d;\mathbb{R}^d)$. Thus, we can extract a subsequence $\{\mathbf{s}_{i(l)}^n\}_{l=1}^{\infty}$ from $\{\mathbf{s}_i^n\}_{i=1}^{\infty}$ so that $\mathbf{s}_{i(l)}^n$ weakly converges to some $\mathbf{s}_n$ in $L^{p/(p-1)}(\mathbb{T}^d;\mathbb{R}^d)$ as $l \to \infty$. Moreover, Mazur's theorem for the space $L^{p/(p-1)}(\mathbb{T}^d;\mathbb{R}^d) \times \mathbb{R}$ guarantees that for any $k \in \mathbb{N}$ there exist $l_k \in \mathbb{N}$ and $\beta_j^k \in [0,1]$ $(j = 1, \ldots, l_k)$ satisfying $\sum_{j=1}^{l_k} \beta_j^k = 1$ such that as $k \to \infty$

$$\sum_{j=1}^{l_k} \beta_j^k \mathbf{s}_{i(j)}^n \to \mathbf{s}_n \quad \text{in } L^{p/(p-1)}\big(\mathbb{T}^d;\mathbb{R}^d\big),$$

$$\sum_{j=1}^{l_k} \beta_j^k \int_{\Omega_{\mathrm{per}}} \sigma^{\#}\big(\mathbf{v}_n(\mathbf{x}) - \mathbf{s}_{i(j)}^n(\mathbf{x})\big)\,d\mathbf{x} \to \inf_{\mathbf{s}\in Y_{\mathrm{per}}(u_n)} \int_{\Omega_{\mathrm{per}}} \sigma^{\#}\big(\mathbf{v}_n(\mathbf{x}) - \mathbf{s}(\mathbf{x})\big)\,d\mathbf{x}.$$

Furthermore, by extracting a subsequence from $\{\sum_{j=1}^{l_k} \beta_j^k \mathbf{s}_{i(j)}^n\}_{k=1}^{\infty}$ we may assume that as $k \to \infty$

$$\sum_{j=1}^{l_k} \beta_j^k \mathbf{s}_{i(j)}^n(\mathbf{x}) \to \mathbf{s}_n(\mathbf{x}) \quad \text{a.e. } \mathbf{x} \in \mathbb{R}^d,$$

where we used the same notation for simplicity. Then, by Fatou's lemma and the convexity of $\sigma^{\#}$ we have that

$$\begin{aligned}\int_{\Omega_{\mathrm{per}}} \sigma^{\#}\big(\mathbf{v}_n(\mathbf{x}) - \mathbf{s}_n(\mathbf{x})\big)\,d\mathbf{x} &\leqslant \liminf_{k\to\infty} \sum_{j=1}^{l_k} \beta_j^k \int_{\Omega_{\mathrm{per}}} \sigma^{\#}\big(\mathbf{v}_n(\mathbf{x}) - \mathbf{s}_{i(j)}^n(\mathbf{x})\big)\,d\mathbf{x} \\ &= \inf_{\mathbf{s}\in Y_{\mathrm{per}}(u_n)} \int_{\Omega_{\mathrm{per}}} \sigma^{\#}\big(\mathbf{v}_n(\mathbf{x}) - \mathbf{s}(\mathbf{x})\big)\,d\mathbf{x}. \end{aligned} \tag{2.12}$$

Since the set $Y_{\mathrm{per}}(w)$ is a convex, closed subset of $L^{p/(p-1)}(\mathbb{T}^d;\mathbb{R}^d)$ for any $w \in H_{\mathrm{ave}}^{-1}(\mathbb{T}^d)$ with $Y_{\mathrm{per}}(w) \neq \emptyset$, we obtain

$$\operatorname{div} \mathbf{s}_n = (-\Delta_{\mathrm{per}})^{-1} u_n. \tag{2.13}$$

By (2.12), (2.13) we have that $\mathbf{s}_n \in Y_{\mathrm{per}}(u_n)$ and

$$\int_{\Omega_{\mathrm{per}}} \sigma^{\#}\big(\mathbf{v}_n(\mathbf{x}) - \mathbf{s}_n(\mathbf{x})\big)\,d\mathbf{x} = \min_{\mathbf{s}\in Y_{\mathrm{per}}(u_n)} \int_{\Omega_{\mathrm{per}}} \sigma^{\#}\big(\mathbf{v}_n(\mathbf{x}) - \mathbf{s}(\mathbf{x})\big)\,d\mathbf{x} \leqslant \lambda. \tag{2.14}$$

It follows from (2.14) that $\{\mathbf{s}_n\}_{n=1}^{\infty}$ is bounded in $L^{p/(p-1)}(\mathbb{T}^d;\mathbb{R}^d)$. By using Mazur's theorem for the space $H_{\mathrm{ave}}^{-1}(\mathbb{T}^d)\times L^{p/(p-1)}(\mathbb{T}^d;\mathbb{R}^d)\times L^{p/(p-1)}(\mathbb{T}^d;\mathbb{R}^d)$ one can show that there are sequences $\{n(j)\}_{j=1}^{\infty}$, $\{m_k\}_{k=1}^{\infty}$ $(\subset\mathbb{N})$, $\gamma_j^k\in[0,1]$ $(j=1,\dots,m_k)$ with $\sum_{j=1}^{m_k}\gamma_j^k=1$ $(\forall k\in\mathbb{N})$ and $\mathbf{s}\in L^{p/(p-1)}(\mathbb{T}^d;\mathbb{R}^d)$ such that as $k\to\infty$

$$\sum_{j=1}^{m_k}\gamma_j^k(u_{n(j)},\mathbf{v}_{n(j)})\to(u,\mathbf{v})\quad\text{in } H_{\mathrm{ave}}^{-1}\big(\mathbb{T}^d\big)\times L^{p/(p-1)}\big(\mathbb{T}^d;\mathbb{R}^d\big),$$

$$\sum_{j=1}^{m_k}\gamma_j^k\mathbf{s}_{n(j)}\to\mathbf{s}\quad\text{in } L^{p/(p-1)}\big(\mathbb{T}^d;\mathbb{R}^d\big).$$

Moreover, by taking a subsequence if necessary we may claim that as $k\to\infty$

$$\sum_{j=1}^{m_k}\gamma_j^k\mathbf{v}_{n(j)}(\mathbf{x})\to\mathbf{v}(\mathbf{x}),\qquad\sum_{j=1}^{m_k}\gamma_j^k\mathbf{s}_{n(j)}(\mathbf{x})\to\mathbf{s}(\mathbf{x})\quad\text{a.e. }\mathbf{x}\in\mathbb{R}^d.$$

Then, Fatou's lemma, the convexity of $\sigma^{\#}$ and (2.14) prove that

$$\int_{\Omega_{\mathrm{per}}}\sigma^{\#}\big(\mathbf{v}(\mathbf{x})-\mathbf{s}(\mathbf{x})\big)\,d\mathbf{x}\leqslant\liminf_{k\to\infty}\sum_{j=1}^{m_k}\gamma_j^k\int_{\Omega_{\mathrm{per}}}\sigma^{\#}\big(\mathbf{v}_{n(j)}(\mathbf{x})-\mathbf{s}_{n(j)}(\mathbf{x})\big)\,d\mathbf{x}\leqslant\lambda.\tag{2.15}$$

Note that for any $\phi\in C_0^{\infty}(\mathbb{R}^d)$

$$\begin{aligned}\int_{\mathbb{R}^d}\big\langle\mathbf{s}(\mathbf{x}),-\nabla\phi(\mathbf{x})\big\rangle_{\mathbb{R}^d}\,d\mathbf{x}&=\lim_{k\to\infty}\sum_{j=1}^{m_k}\gamma_j^k\int_{\mathbb{R}^d}\big\langle\mathbf{s}_{n(j)}(\mathbf{x}),-\nabla\phi(\mathbf{x})\big\rangle_{\mathbb{R}^d}\,d\mathbf{x}\\&=\lim_{k\to\infty}\sum_{j=1}^{m_k}\gamma_j^k\int_{\mathbb{R}^d}(-\Delta_{\mathrm{per}})^{-1}u_{n(j)}(\mathbf{x})\cdot\phi(\mathbf{x})\,d\mathbf{x}\\&=\int_{\mathbb{R}^d}(-\Delta_{\mathrm{per}})^{-1}u(\mathbf{x})\cdot\phi(\mathbf{x})\,d\mathbf{x},\end{aligned}$$

which means that

$$\operatorname{div}\mathbf{s}=(-\Delta_{\mathrm{per}})^{-1}u.\tag{2.16}$$

By combining (2.15), (2.16) with (2.11) we arrive at $S((u,\mathbf{v}))\leqslant\lambda$, which concludes that S is lower semi-continuous.

Since $S:H_{\mathrm{ave}}^{-1}(\mathbb{T}^d)\times L^{p/(p-1)}(\mathbb{T}^d;\mathbb{R}^d)\to\mathbb{R}\cup\{\infty\}$ is convex, lower semi-continuous and not identically ∞, we can apply Lemma 2.1(2) to deduce that

$$\big(S^*\big)^*((u,\mathbf{v}))=S((u,\mathbf{v})),\quad\forall(u,\mathbf{v})\in H_{\mathrm{ave}}^{-1}\big(\mathbb{T}^d\big)\times L^{p/(p-1)}\big(\mathbb{T}^d;\mathbb{R}^d\big).\tag{2.17}$$

In order to characterize S^* ($: (H^{-1}_{\mathrm{ave}}(\mathbb{T}^d) \times L^{p/(p-1)}(\mathbb{T}^d;\mathbb{R}^d))^* \to \mathbb{R} \cup \{\infty\}$), take any $(f, \mathbf{g}) \in H^{-1}_{\mathrm{ave}}(\mathbb{T}^d) \times L^p(\mathbb{T}^d;\mathbb{R}^d)$. Recalling (2.10), we observe that

$$\begin{aligned}
&S^*\big(\Phi_p((f,\mathbf{g}))\big)\\
&\quad= \sup_{(u,\mathbf{v})\in H^{-1}_{\mathrm{ave}}(\mathbb{T}^d)\times L^{p/(p-1)}(\mathbb{T}^d;\mathbb{R}^d)} \big\{\big\langle \Phi_p((f,\mathbf{g})),(u,\mathbf{v})\big\rangle - S((u,\mathbf{v}))\big\}\\
&\quad= \sup_{(u,\mathbf{v})\in H^{-1}_{\mathrm{ave}}(\mathbb{T}^d)\times L^{p/(p-1)}(\mathbb{T}^d;\mathbb{R}^d)} \ \sup_{(r,\mathbf{s})\in H^{-1}_{\mathrm{ave}}(\mathbb{T}^d)\times L^{p/(p-1)}(\mathbb{T}^d;\mathbb{R}^d)}\\
&\qquad \cdot \big\{\big\langle \Phi_p((f,\mathbf{g})),(u,\mathbf{v})\big\rangle - Q^*\big(\Phi_{p/(p-1)}\big((u,\mathbf{v})-(r,\mathbf{s})\big)\big) - R^*\big(\Phi_{p/(p-1)}((r,\mathbf{s}))\big)\big\}\\
&\quad= \sup_{(r,\mathbf{s})\in H^{-1}_{\mathrm{ave}}(\mathbb{T}^d)\times L^{p/(p-1)}(\mathbb{T}^d;\mathbb{R}^d)} \ \sup_{(u,\mathbf{v})\in H^{-1}_{\mathrm{ave}}(\mathbb{T}^d)\times L^{p/(p-1)}(\mathbb{T}^d;\mathbb{R}^d)}\\
&\qquad \cdot \big\{\big\langle \Phi_p((f,\mathbf{g})),(u,\mathbf{v})-(r,\mathbf{s})\big\rangle - Q^*\big(\Phi_{p/(p-1)}\big((u,\mathbf{v})-(r,\mathbf{s})\big)\big)\\
&\qquad + \big\langle \Phi_p((f,\mathbf{g})),(r,\mathbf{s})\big\rangle - R^*\big(\Phi_{p/(p-1)}((r,\mathbf{s}))\big)\big\}\\
&\quad= \big(Q^*\big)^*((f,\mathbf{g})) + \big(R^*\big)^*((f,\mathbf{g}))\\
&\quad= Q((f,\mathbf{g})) + R((f,\mathbf{g})).
\end{aligned} \tag{2.18}$$

To derive the last equality of (2.18) we applied Lemma 2.1(2) to Q, R. Moreover, by using (2.18) one can verify that for $(u,\mathbf{v}) \in H^{-1}_{\mathrm{ave}}(\mathbb{T}^d) \times L^{p/(p-1)}(\mathbb{T}^d;\mathbb{R}^d)$

$$\begin{aligned}
\big(S^*\big)^*((u,\mathbf{v})) &= \sup_{(f,\mathbf{g})\in H^{-1}_{\mathrm{ave}}(\mathbb{T}^d)\times L^p(\mathbb{T}^d;\mathbb{R}^d)} \big\{\big\langle (u,\mathbf{v}),\Phi_p((f,\mathbf{g}))\big\rangle - S^*\big(\Phi_p((f,\mathbf{g}))\big)\big\}\\
&= \sup_{(f,\mathbf{g})\in H^{-1}_{\mathrm{ave}}(\mathbb{T}^d)\times L^p(\mathbb{T}^d;\mathbb{R}^d)} \big\{\big\langle \Phi_{p/(p-1)}((u,\mathbf{v})),(f,\mathbf{g})\big\rangle - (Q+R)((f,\mathbf{g}))\big\}\\
&= (Q+R)^*\big(\Phi_{p/(p-1)}((u,\mathbf{v}))\big).
\end{aligned} \tag{2.19}$$

Combining (2.19) with (2.17) yields

$$S((u,\mathbf{v})) = (Q+R)^*\big(\Phi_{p/(p-1)}(u,\mathbf{v})\big), \quad \forall (u,\mathbf{v}) \in H^{-1}_{\mathrm{ave}}\big(\mathbb{T}^d\big) \times L^{p/(p-1)}\big(\mathbb{T}^d;\mathbb{R}^d\big).$$

Finally remark that the argument leading to (2.14) essentially showed that ‘inf’ in (2.11) can be replaced by ‘min’, which results in the desired equality. □

Lemmas 2.4 and 2.9 complete the characterization of $F^{\#}_{\mathrm{per}}$.

Lemma 2.11. *For any $u \in H^{-1}_{\mathrm{ave}}(\mathbb{T}^d)$*

$$F^{\#}_{\mathrm{per}}(u) = \begin{cases} \min_{\mathbf{s}\in Y_{\mathrm{per}}(-u)} \int_{\Omega_{\mathrm{per}}} \sigma^{\#}(\mathbf{s}(\mathbf{x}))\,d\mathbf{x} & \text{if } Y_{\mathrm{per}}(-u) \neq \emptyset,\\ \infty & \text{otherwise}. \end{cases}$$

All the preparations have been done to prove Theorem 1.4.

Proof of Theorem 1.4. Assume that $\partial F_{\mathrm{per}}(f) \neq \emptyset$ throughout the proof. If $u \in \partial F_{\mathrm{per}}(f)$, according to Lemma 2.2 we equivalently have that

$$F_{\mathrm{per}}(f) + F_{\mathrm{per}}^{\#}(u) = \langle f, u \rangle_{H_{\mathrm{ave}}^{-1}(\mathbb{T}^d)},$$

or by Lemma 2.11 that

$$F_{\mathrm{per}}(f) + \min_{\mathbf{s} \in Y_{\mathrm{per}}(-u)} \int_{\Omega_{\mathrm{per}}} \sigma^{\#}\bigl(\mathbf{s}(\mathbf{x})\bigr)\, d\mathbf{x} = \langle f, u \rangle_{H_{\mathrm{ave}}^{-1}(\mathbb{T}^d)}.$$

Let $\mathbf{g} \in Y_{\mathrm{per}}(-u)$ be a minimizer. We have $-\operatorname{div} \mathbf{g} = (-\Delta_{\mathrm{per}})^{-1} u$ and

$$F_{\mathrm{per}}(f) + \int_{\Omega_{\mathrm{per}}} \sigma^{\#}\bigl(\mathbf{g}(\mathbf{x})\bigr)\, d\mathbf{x} = \langle f, u \rangle_{H_{\mathrm{ave}}^{-1}(\mathbb{T}^d)},$$

which lead to

$$\int_{\Omega_{\mathrm{per}}} \bigl(\sigma\bigl(\nabla \tilde{f}(\mathbf{x})\bigr) + \sigma^{\#}\bigl(\mathbf{g}(\mathbf{x})\bigr)\bigr)\, d\mathbf{x} = \langle f, -\operatorname{div} \mathbf{g} \rangle. \tag{2.20}$$

By Lemma 2.7 we can choose a sequence $\{\mathbf{g}_n\}_{n=1}^{\infty}$ $(\subset C^{\infty}(\mathbb{T}^d; \mathbb{R}^d))$ so that as $n \to \infty$

$$\begin{aligned} \mathbf{g}_n &\to \mathbf{g} \quad \text{in } L^{p/(p-1)}\bigl(\mathbb{T}^d; \mathbb{R}^d\bigr), \\ \operatorname{div} \mathbf{g}_n &\to \operatorname{div} \mathbf{g} \quad \text{in } H_{\mathrm{ave}}^1\bigl(\mathbb{T}^d\bigr). \end{aligned}$$

Then by using Lemma 2.6 we see that

$$\begin{aligned} \langle f, -\operatorname{div} \mathbf{g} \rangle &= -\lim_{n\to\infty} \int_{\Omega_{\mathrm{per}}} \tilde{f}(\mathbf{x}) \operatorname{div} \mathbf{g}_n(\mathbf{x})\, d\mathbf{x} = \lim_{n\to\infty} \int_{\Omega_{\mathrm{per}}} \bigl\langle \nabla \tilde{f}(\mathbf{x}), \mathbf{g}_n(\mathbf{x}) \bigr\rangle_{\mathbb{R}^d}\, d\mathbf{x} \\ &= \int_{\Omega_{\mathrm{per}}} \bigl\langle \nabla \tilde{f}(\mathbf{x}), \mathbf{g}(\mathbf{x}) \bigr\rangle_{\mathbb{R}^d}\, d\mathbf{x}. \end{aligned}$$

Therefore, we can deduce from (2.20) that

$$\int_{\Omega_{\mathrm{per}}} \bigl(\sigma\bigl(\nabla \tilde{f}(\mathbf{x})\bigr) + \sigma^{\#}\bigl(\mathbf{g}(\mathbf{x})\bigr) - \bigl\langle \nabla \tilde{f}(\mathbf{x}), \mathbf{g}(\mathbf{x}) \bigr\rangle_{\mathbb{R}^d}\bigr)\, d\mathbf{x} = 0.$$

Since the integrand of the integral above is non-negative, we obtain

$$\sigma\bigl(\nabla \tilde{f}(\mathbf{x})\bigr) + \sigma^{\#}\bigl(\mathbf{g}(\mathbf{x})\bigr) - \bigl\langle \nabla \tilde{f}(\mathbf{x}), \mathbf{g}(\mathbf{x}) \bigr\rangle_{\mathbb{R}^d} = 0 \quad \text{a.e. } \mathbf{x} \in \mathbb{R}^d,$$

or equivalently $\mathbf{g}(\mathbf{x}) \in \partial\sigma(\nabla \tilde{f}(\mathbf{x}))$ a.e. $\mathbf{x} \in \mathbb{R}^d$ by Lemma 2.2. Since $u = -(-\Delta_{\mathrm{per}}) \cdot \operatorname{div} \mathbf{g}$, we have proved the inclusion '$\subset$' of the claim of Theorem 1.4.

To show the opposite inclusion '⊃', take any $u \in H^{-1}_{\mathrm{ave}}(\mathbb{T}^d)$ for which there is $\mathbf{g} \in L^{p/(p-1)}(\mathbb{T}^d;\mathbb{R}^d)$ such that $\operatorname{div}\mathbf{g} \in H^1_{\mathrm{ave}}(\mathbb{T}^d)$, $u = -(-\Delta_{\mathrm{per}})\operatorname{div}\mathbf{g}$ and $\mathbf{g}(\mathbf{x}) \in \partial\sigma(\nabla\tilde{f}(\mathbf{x}))$ a.e. $\mathbf{x} \in \mathbb{R}^d$. Then by exactly following the argument above the other way round we can reach

$$F_{\mathrm{per}}(f) + \int_{\Omega_{\mathrm{per}}} \sigma^{\#}\big(\mathbf{g}(\mathbf{x})\big)\,d\mathbf{x} = \langle f, u\rangle_{H^{-1}_{\mathrm{ave}}(\mathbb{T}^d)}.$$

By taking infimum over such $\mathbf{g}$s and by Lemma 2.11 one has

$$F_{\mathrm{per}}(f) + F^{\#}_{\mathrm{per}}(u) \leqslant \langle f, u\rangle_{H^{-1}_{\mathrm{ave}}(\mathbb{T}^d)},$$

which is equivalent to the inclusion $u \in \partial F_{\mathrm{per}}(f)$ by the definition of $F^{\#}_{\mathrm{per}}$ and Lemma 2.2. We have proved the inclusion '⊃' as well. □

2.2. *Proof for the Dirichlet problem*

The major part of the proof for Theorem 1.5 can be constructed by straightforwardly translating the proof for Theorem 1.4 into the context with the Dirichlet boundary condition. Let us, therefore, explain only different parts from the periodic problem and be brief about the parallel parts.

To characterize the conjugate functional $F^{\#}_{\mathrm{D}}$ $(: H^{-1}(\Omega) \to \mathbb{R}\cup\{\infty\})$ we introduce functionals $Q_{\mathrm{D}}, R_{\mathrm{D}} : H^{-1}(\Omega) \times L^p(\Omega;\mathbb{R}^d) \to \mathbb{R}\cup\{\infty\}$ by

$$Q_{\mathrm{D}}((f,\mathbf{g})) := \int_{\Omega} \sigma\big(\mathbf{g}(\mathbf{x})\big)\,d\mathbf{x},$$

$$R_{\mathrm{D}}((f,\mathbf{g})) := \begin{cases} 0 & \text{if } f \in X_{\mathrm{D}} \text{ and } \mathbf{g} = \nabla\tilde{f}, \\ \infty & \text{otherwise,} \end{cases}$$

where $H^{-1}(\Omega) \times L^p(\Omega;\mathbb{R}^d)$ is the real Banach space with the norm $\|(f,\mathbf{g})\|_{\mathrm{D}} := \|f\|_{H^{-1}(\Omega)} + \|\mathbf{g}\|_{L^p(\Omega;\mathbb{R}^d)}$. The functionals Q_{D}, R_{D} are convex, lower semi-continuous and not identically ∞.

The difference from the periodic problem mainly lies in a lack of a density property like Lemma 2.7, which worked conveniently in the periodic case. Consequently in the Dirichlet problem the characterization of R^*_{D}, $F^{\#}_{\mathrm{D}}$ and ∂F_{D} inherits an additional constraint, which is to require a function $\mathbf{w}$ $(\in L^{p/(p-1)}(\Omega;\mathbb{R}^d))$ satisfying $\operatorname{div}\mathbf{w} \in H^1_0(\Omega)$ to obey

$$\langle \operatorname{div}\mathbf{w}, h\rangle + \int_{\Omega} \big\langle \mathbf{w}(\mathbf{x}), \nabla\tilde{h}(\mathbf{x})\big\rangle_{\mathbb{R}^d}\,d\mathbf{x} = 0, \quad \forall h \in X_{\mathrm{D}}. \tag{2.21}$$

The first difference appears in the characterization of R^*_{D} $(: (H^{-1}(\Omega) \times L^p(\Omega;\mathbb{R}^d))^* \to \mathbb{R}\cup\{\infty\})$, while the characterization of Q^*_{D} can be carried out in the same way as in Lemma 2.5. Using the isomorphism $\Psi_{p/(p-1)} : H^{-1}(\Omega) \times L^{p/(p-1)}(\Omega;\mathbb{R}^d) \to (H^{-1}(\Omega) \times L^p(\Omega;\mathbb{R}^d))^*$ defined by

$$\left\langle \Psi_{p/(p-1)}((u,\mathbf{v})),(f,\mathbf{g})\right\rangle := \langle u,f\rangle_{H^{-1}(\Omega)} + \int_{\Omega} \left\langle \mathbf{v}(\mathbf{x}),\mathbf{g}(\mathbf{x})\right\rangle_{\mathbb{R}^d} d\mathbf{x},$$
$$\forall (u,\mathbf{v}) \in H^{-1}(\Omega)\times L^{p/(p-1)}\left(\Omega;\mathbb{R}^d\right),\ \forall (f,\mathbf{g}) \in H^{-1}(\Omega)\times L^{p}\left(\Omega;\mathbb{R}^d\right),$$

we have

Lemma 2.12. *For any* $(u,\mathbf{v}) \in H^{-1}(\Omega)\times L^{p/(p-1)}(\Omega;\mathbb{R}^d)$

$$R_{\mathrm{D}}^{*}\left(\Psi_{p/(p-1)}((u,\mathbf{v}))\right) = \begin{cases} 0 & \text{if } \mathbf{v} \text{ satisfies (2.21) and } \operatorname{div}\mathbf{v} = (-\Delta_{\mathrm{D}})^{-1}u, \\ \infty & \text{otherwise.} \end{cases}$$

Proof. Take any $(u,\mathbf{v}) \in H^{-1}(\Omega)\times L^{p/(p-1)}(\Omega;\mathbb{R}^d)$.

$$\begin{aligned} R_{\mathrm{D}}^{*}\left(\Psi_{p/(p-1)}((u,\mathbf{v}))\right) &= \sup_{f\in X_{\mathrm{D}}} \left\{ \langle u,f\rangle_{H^{-1}(\Omega)} + \int_{\Omega} \left\langle \mathbf{v}(\mathbf{x}),\nabla \tilde{f}(\mathbf{x})\right\rangle_{\mathbb{R}^d} d\mathbf{x} \right\} \\ &\geqslant \sup_{\phi\in C_0^{\infty}(\Omega)} \int_{\Omega} \left((-\Delta_{\mathrm{D}})^{-1}u(\mathbf{x})\cdot\phi(\mathbf{x}) + \left\langle \mathbf{v}(\mathbf{x}),\nabla\phi(\mathbf{x})\right\rangle_{\mathbb{R}^d}\right) d\mathbf{x} \\ &= \begin{cases} 0 & \text{if } \operatorname{div}\mathbf{v} = (-\Delta_{\mathrm{D}})^{-1}u, \\ \infty & \text{otherwise.} \end{cases} \end{aligned}$$

On the assumption that $\operatorname{div}\mathbf{v} = (-\Delta_{\mathrm{D}})^{-1}u$ we have that

$$\begin{aligned} R_{\mathrm{D}}^{*}\left(\Psi_{p/(p-1)}((u,\mathbf{v}))\right) &= \sup_{f\in X_{\mathrm{D}}} \left\{ \langle \operatorname{div}\mathbf{v},f\rangle + \int_{\Omega} \left\langle \mathbf{v}(\mathbf{x}),\nabla \tilde{f}(\mathbf{x})\right\rangle_{\mathbb{R}^d} d\mathbf{x} \right\} \\ &= \begin{cases} 0 & \text{if } \mathbf{v} \text{ satisfies (2.21)}, \\ \infty & \text{otherwise.} \end{cases} \qquad \square \end{aligned}$$

For any $u \in H^{-1}(\Omega)$ let us define a subset $Y_{\mathrm{D}}(u)$ of $L^{p/(p-1)}(\Omega;\mathbb{R}^d)$ by

$$Y_{\mathrm{D}}(u) := \left\{ \mathbf{s} \in L^{p/(p-1)}\left(\Omega;\mathbb{R}^d\right) \,\middle|\, \operatorname{div}\mathbf{s} = (-\Delta_{\mathrm{D}})^{-1}u,\ \mathbf{s} \text{ satisfies (2.21)} \right\}.$$

By noting that $Y_{\mathrm{D}}(u)$ is convex and closed in $L^{p/(p-1)}(\Omega;\mathbb{R}^d)$ for any $u \in H^{-1}(\Omega)$ with $Y_{\mathrm{D}}(u) \neq \emptyset$, we can straightforwardly modify the proof of Lemma 2.9 to conclude the following.

Lemma 2.13. *For any* $(u,\mathbf{v}) \in H^{-1}(\Omega)\times L^{p/(p-1)}(\Omega;\mathbb{R}^d)$

$$(Q_{\mathrm{D}}+R_{\mathrm{D}})^{*}\left(\Psi_{p/(p-1)}((u,\mathbf{v}))\right) = \begin{cases} \min_{\mathbf{s}\in Y_{\mathrm{D}}(u)} \int_{\Omega} \sigma^{\#}(\mathbf{v}(\mathbf{x})-\mathbf{s}(\mathbf{x}))\,d\mathbf{x} & \text{if } Y_{\mathrm{D}}(u) \neq \emptyset, \\ \infty & \text{otherwise.} \end{cases}$$

Since the Dirichlet analogue of Lemma 2.4 holds naturally, we obtain from Lemma 2.13 that

Lemma 2.14. *For any* $u \in H^{-1}(\Omega)$

$$F_{\mathrm{D}}^{\#}(u) = \begin{cases} \min_{\mathbf{s}\in Y_{\mathrm{D}}(-u)} \int_\Omega \sigma^{\#}(\mathbf{s}(\mathbf{x}))\,d\mathbf{x} & \text{if } Y_{\mathrm{D}}(-u) \neq \emptyset, \\ \infty & \text{otherwise}. \end{cases}$$

On these preparations we can prove Theorem 1.5.

Proof of Theorem 1.5. Assume that $\partial F_{\mathrm{D}}(f) \neq \emptyset$. By Lemmas 2.2 and 2.14 the inclusion $u \in \partial F_{\mathrm{D}}(f)$ is equivalent to the equality

$$F_{\mathrm{D}}(f) + \min_{\mathbf{s}\in Y_{\mathrm{D}}(-u)} \int_\Omega \sigma^{\#}\big(\mathbf{s}(\mathbf{x})\big)\,d\mathbf{x} = \langle f, u\rangle_{H^{-1}(\Omega)}. \tag{2.22}$$

If $\mathbf{g}\,(\in Y_{\mathrm{D}}(-u))$ is a minimizer, $u = -(-\Delta_{\mathrm{D}})\operatorname{div}\mathbf{g}$ and the equality (2.22) coupled with (2.21) leads to

$$\int_\Omega \sigma\big(\nabla \tilde f(\mathbf{x})\big)\,d\mathbf{x} + \int_\Omega \sigma^{\#}\big(\mathbf{g}(\mathbf{x})\big)\,d\mathbf{x} = \int_\Omega \big\langle \mathbf{g}(\mathbf{x}), \nabla \tilde f(\mathbf{x})\big\rangle_{\mathbb{R}^d}\,d\mathbf{x},$$

which is equivalent to the inclusion that $\mathbf{g}(\mathbf{x}) \in \partial\sigma(\nabla \tilde f(\mathbf{x}))$ a.e. $\mathbf{x} \in \Omega$ by Lemma 2.2. We have proved the inclusion '$\subset$' of Theorem 1.5. The opposite inclusion '$\supset$' can be shown by arguing the other way around. □

Proof of Corollary 1.6. We show that

- the bilinear form $(f, g) \mapsto \int_\Omega f(\mathbf{x})g(\mathbf{x})\,d\mathbf{x}$ is well defined on $W_0^{1,p}(\Omega) \times H_0^1(\Omega)$,
- $f \mapsto \int_\Omega f(\mathbf{x})g(\mathbf{x})\,d\mathbf{x}$ is continuous in $W_0^{1,p}(\Omega)$ $\big(\forall g \in H_0^1(\Omega)\big)$,
- $g \mapsto \int_\Omega f(\mathbf{x})g(\mathbf{x})\,d\mathbf{x}$ is continuous in $H_0^1(\Omega)$ $\big(\forall f \in W_0^{1,p}(\Omega)\big)$, (2.23)

in the assumed circumstance by means of the Sobolev embedding theorem. Note that if (2.23) holds, the constraint (2.21) is trivial by the density property of $C_0^\infty(\Omega)$ in $H_0^1(\Omega)$ and $W_0^{1,p}(\Omega)$.

If $d \leqslant 2$, $H_0^1(\Omega) \subset L^{p/(p-1)}(\Omega)$, thus (2.23) is true.

If $p \geqslant d$, $W_0^{1,p}(\Omega) \subset L^2(\Omega)$. Therefore (2.23) holds.

If $d \geqslant 3$ and $1 < p < d$, $H_0^1(\Omega) \subset L^{2d/(d-2)}(\Omega)$ and $W_0^{1,p}(\Omega) \subset L^{dp/(d-p)}(\Omega)$. From this we see that the inequality

$$\frac{2d}{d-2} \geqslant \frac{dp/(d-p)}{dp/(d-p)-1}$$

is sufficient to guarantee (2.23). This inequality is equivalent to $p \geqslant 2d/(d+4)$.

By summing up, the condition (1.6) is seen to be sufficient for (2.23) to be true. □

3. Canonical restriction for a spherically symmetric surface

In this section we will find the smallest element in $\partial F_{\mathrm{D}}(f)$ with respect to the norm $\|\cdot\|_{H^{-1}(\Omega)}$ by giving a spherically symmetric surface $\tilde{f}$. Let us write the smallest element called canonical restriction as $\partial F_{\mathrm{D}}^{c}(f)$. It is known (see e.g. [5]) that the solution to the initial value problem

$$\begin{cases} \dfrac{d}{dt} f_{\mathrm{D}}(t) \in -\partial F_{\mathrm{D}}\big(f_{\mathrm{D}}(t)\big) & \text{a.e.} t > 0, \\ f_{\mathrm{D}}(0) = f_{\mathrm{D},0}\ (\in \overline{X_{\mathrm{D}}}) \end{cases}$$

satisfies

$$\frac{d^+}{dt} f_{\mathrm{D}}(t) = -\partial F_{\mathrm{D}}^{c}\big(f_{\mathrm{D}}(t)\big) \quad \text{all } t > 0,$$

where d^+/dt means the right derivative. Hence, the canonical restriction provides useful information on the time evolution of the crystalline surface as already discussed for 1-dimensional problems in [10, Section 4], [12, Chapter 3]. Here we argue a general dimensional problem under the constraint (1.6).

We fix $f \in X_{\mathrm{D}}$ whose $\tilde{f}\ (\in W_0^{1,p}(\Omega))$ satisfies that $\nabla \tilde{f}(\mathbf{x}) = 0$ a.e. $\mathbf{x} \in \Omega_0$ and $\nabla \tilde{f}(\mathbf{x}) \neq 0$ a.e. $\mathbf{x} \in \Omega \backslash \overline{\Omega_0}$ with an open set Ω_0 satisfying $\overline{\Omega_0} \subset \Omega$.

Using this $\tilde{f}$, we define a function $\mathbf{u}_{\tilde{f}} : \Omega \backslash \overline{\Omega_0} \to \mathbb{R}^d$ by

$$\mathbf{u}_{\tilde{f}}(\mathbf{x}) := \big|\nabla \tilde{f}(\mathbf{x})\big|^{-1} \nabla \tilde{f}(\mathbf{x}) + \mu \big|\nabla \tilde{f}(\mathbf{x})\big|^{p-2} \nabla \tilde{f}(\mathbf{x}).$$

For any functions $\mathbf{g} : \Omega_0 \to \mathbb{R}^m$, $\mathbf{h} : \Omega \backslash \overline{\Omega_0} \to \mathbb{R}^m$ $(m \in \mathbb{N})$, let $(\mathbf{g}|\mathbf{h}) : \Omega \to \mathbb{R}^m$ be defined by

$$(\mathbf{g}|\mathbf{h})(\mathbf{x}) := \begin{cases} \mathbf{g}(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega_0, \\ \mathbf{h}(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega \backslash \overline{\Omega_0}. \end{cases}$$

The following lemma tells us a way to find $\partial F_{\mathrm{D}}^{c}(f)$.

Lemma 3.1. *Assume that the condition* (1.6) *holds and that* $\mathbf{g}\ (\in L^{p/(p-1)}(\Omega_0; \mathbb{R}^d))$ *and* $\mathbf{u}_{\tilde{f}}\ (\in L^{p/(p-1)}(\Omega \backslash \overline{\Omega_0}; \mathbb{R}^d))$ *satisfy the following conditions.*

(i) *there exists* $\boldsymbol{\psi} \in C_0^\infty(\Omega; \mathbb{R}^d)$ *such that* $\boldsymbol{\psi}|_{\Omega_0} = \mathbf{g}$.
(ii) $\nabla \Delta \operatorname{div} \mathbf{g}(\mathbf{x}) = \mathbf{0}$, $\forall \mathbf{x} \in \Omega_0$.
(iii) $|\mathbf{g}(\mathbf{x})| \leqslant 1$, $\forall \mathbf{x} \in \Omega_0$.
(iv) $\operatorname{div}(\mathbf{g}|\mathbf{u}_{\tilde{f}}) \in H_0^1(\Omega)$.

Then, $\partial F_{\mathrm{D}}^{c}(f) = -(-\Delta_{\mathrm{D}}) \operatorname{div}(\mathbf{g}|\mathbf{u}_{\tilde{f}})$.

Proof. By the conditions (iii), (iv) and Corollary 1.6, $-(-\Delta_{\mathrm{D}}) \operatorname{div}(\mathbf{g}|\mathbf{u}_{\tilde{f}}) \in \partial F_{\mathrm{D}}(f)$. Since $\partial F_{\mathrm{D}}(f)$ is a non-empty, closed convex set in $H^{-1}(\Omega)$, the canonical restriction $\partial F_{\mathrm{D}}^{c}(f)$ uniquely exists. By Corollary 1.6 we may write $\partial F_{\mathrm{D}}^{c}(f) = -(-\Delta_{\mathrm{D}}) \operatorname{div} \mathbf{G}$ with some $\mathbf{G} \in L^{p/(p-1)}(\Omega; \mathbb{R}^d)$ satisfying $\mathbf{G}|_{\Omega \setminus \overline{\Omega_0}} = \mathbf{u}_{\tilde{f}}$. By convexity of $\partial F_{\mathrm{D}}(f)$ and minimality of $\|-\Delta_{\mathrm{D}} \operatorname{div} \mathbf{G}\|_{H^{-1}(\Omega)}$ we have that

$$\lim_{\varepsilon \searrow 0} \frac{d}{d\varepsilon} \big\| (1-\varepsilon)(-\Delta_{\mathrm{D}}) \operatorname{div} \mathbf{G} + \varepsilon(-\Delta_{\mathrm{D}}) \operatorname{div}(\mathbf{g}|\mathbf{u}_{\tilde{f}}) \big\|_{H^{-1}(\Omega)}^2$$
$$= 2 \int_{\Omega_0} \big\langle \nabla \operatorname{div} \mathbf{g}(\mathbf{x}) - \nabla \operatorname{div} \mathbf{G}(\mathbf{x}), \nabla \operatorname{div} \mathbf{G}(\mathbf{x}) \big\rangle_{\mathbb{R}^d} \, d\mathbf{x} \geqslant 0. \tag{3.1}$$

On the other hand, we can derive the equality that for any $\boldsymbol{\psi} \in C_0^\infty(\Omega; \mathbb{R}^d)$

$$\int_{\Omega_0} \big\langle \nabla \operatorname{div} \mathbf{g}(\mathbf{x}) - \nabla \operatorname{div} \mathbf{G}(\mathbf{x}), \boldsymbol{\psi}(\mathbf{x}) \big\rangle_{\mathbb{R}^d} \, d\mathbf{x} = \int_{\Omega_0} \big\langle \mathbf{g}(\mathbf{x}) - \mathbf{G}(\mathbf{x}), \nabla \operatorname{div} \boldsymbol{\psi}(\mathbf{x}) \big\rangle_{\mathbb{R}^d} \, d\mathbf{x}.$$

Then by the assumptions (i), (ii)

$$\int_{\Omega_0} \big\langle \nabla \operatorname{div} \mathbf{g}(\mathbf{x}) - \nabla \operatorname{div} \mathbf{G}(\mathbf{x}), \nabla \operatorname{div} \mathbf{g}(\mathbf{x}) \big\rangle_{\mathbb{R}^d} \, d\mathbf{x} = 0. \tag{3.2}$$

Combining (3.1) with (3.2) gives

$$\int_{\Omega_0} \big| \nabla \operatorname{div} \mathbf{g}(\mathbf{x}) - \nabla \operatorname{div} \mathbf{G}(\mathbf{x}) \big|^2 \, d\mathbf{x} \leqslant 0,$$

or $-\Delta_{\mathrm{D}} \operatorname{div}(\mathbf{g}|\mathbf{u}_{\tilde{f}}) = -\Delta_{\mathrm{D}} \operatorname{div} \mathbf{G}$. □

3.1. A spherically symmetric surface

Let us apply Lemma 3.1 to find the canonical restriction $\partial F_{\mathrm{D}}^{c}(f)$ under assumptions that both Ω_0 and Ω are spherical domains and $\tilde{f} : \Omega \to \mathbb{R}$ is spherically symmetric. More precisely we assume that

$$\Omega_0 = \big\{ \mathbf{x} \in \mathbb{R}^d \ \big| \ |\mathbf{x}| < r_0 \big\}, \qquad \Omega = \big\{ \mathbf{x} \in \mathbb{R}^d \ \big| \ |\mathbf{x}| < r \big\}$$

with $0 < r_0 < r$ and $\tilde{f}(\mathbf{x}) := h(|\mathbf{x}|)$ with $h \in C^1([0, r])$ satisfying

$$h(r) = 0, \qquad h^{(1)}(s) = 0 \quad \big(\forall s \in [0, r_0]\big) \quad \text{and} \quad h^{(1)}(s) < 0 \quad \big(\forall s \in (r_0, r)\big).$$

Here and below let the notation $u \in C^l([a, b])$ ($l \in \mathbb{N} \cup \{0\}$, $a < b$) mean that $u \in C^l((a, b))$ and $u^{(k)} \in C([a, b])$ ($k \in \{0, 1, \ldots, l\}$). The corresponding f ($\in H^{-1}(\Omega)$) to this $\tilde{f}$ ($\in W_0^{1,p}(\Omega)$) is characterized by

$$\langle f, \phi \rangle = \int_{\Omega} \tilde{f}(\mathbf{x}) \phi(\mathbf{x}) \, d\mathbf{x}, \quad \forall \phi \in H_0^1(\Omega).$$

To organize the calculation of the canonical restriction $\partial F_{\mathrm{D}}^{c}(f)$ below, we define a function $H : [r_0, r] \to \mathbb{R}$ by

$$H(s) := -1 + \mu\left|h^{(1)}(s)\right|^{p-2} h^{(1)}(s), \quad \forall s \in [r_0, r].$$

Theorem 3.2. *Assume that $H \in C^3([r_0, r])$,*

$$H^{(1)}(r) + \frac{d-1}{r} H(r) = 0, \tag{3.3}$$

$$H^{(1)}(r_0) \in [-9/r_0, 0]. \tag{3.4}$$

Then for all $\phi \in H_0^1(\Omega)$

$$\begin{aligned}\left\langle \partial F_{\mathrm{D}}^{c}(f), \phi\right\rangle = {} & \frac{d(d+2)}{r_0^2}\left(H^{(1)}(r_0) + \frac{1}{r_0}\right)\int_{\Omega_0} \phi(\mathbf{x})\, d\mathbf{x} \\ & + \int_{\Omega \setminus \overline{\Omega_0}} \left(H^{(3)}\left(|\mathbf{x}|\right) + \frac{2(d-1)}{|\mathbf{x}|} H^{(2)}\left(|\mathbf{x}|\right) + \frac{(d-1)(d-3)}{|\mathbf{x}|^2} H^{(1)}\left(|\mathbf{x}|\right)\right. \\ & \left. - \frac{(d-1)(d-3)}{|\mathbf{x}|^3} H\left(|\mathbf{x}|\right)\right)\phi(\mathbf{x})\, d\mathbf{x} \\ & + \left(H^{(2)}(r_0) - \frac{3}{r_0} H^{(1)}(r_0) - \frac{3}{r_0^2}\right)\int_{\partial\Omega_0} \phi(\mathbf{x})\, dS,\end{aligned} \tag{3.5}$$

where dS denotes the surface measure.

Remark 3.3. The surface integral over $\partial\Omega_0$ in (3.5) corresponds to the appearance of delta functions in 1-dimensional case [10, Theorem 4.1]. The surface integral disappears and the canonical restriction can be identified with a function being constant on the facet Ω_0 if $H^{(2)}(r_0) - 3/r_0 H^{(1)}(r_0) - 3/r_0^2 = 0$. This remark was missed in the conclusion of [10] and was properly taken into account in [12, Chapter 3] during its derivation of the free boundary value problem.

Proof of Theorem 3.2. First note that $\mathbf{u}_{\tilde{f}}(\mathbf{x}) = H(|\mathbf{x}|)\mathbf{x}/|\mathbf{x}|$ $(\forall \mathbf{x} \in \Omega \setminus \overline{\Omega_0})$ and by the assumption (3.3)

$$\operatorname{div} \mathbf{u}_{\tilde{f}}(\mathbf{x}) = 0, \quad \forall \mathbf{x} \in \partial\Omega. \tag{3.6}$$

Next let us find $\mathbf{g} : \Omega_0 \to \mathbb{R}^d$ satisfying (i), (ii), (iv) of Lemma 3.1. Postulate that $\mathbf{g}(\mathbf{x}) = \eta(|\mathbf{x}|)\mathbf{x}/|\mathbf{x}|$ with a function $\eta : [0, r_0] \to \mathbb{R}$. Then we have that

$$\begin{aligned}\nabla \Delta \operatorname{div} \mathbf{g}(\mathbf{x}) = {} & \left(|\mathbf{x}|^4 \eta^{(4)}\left(|\mathbf{x}|\right) + 2(d-1)|\mathbf{x}|^3 \eta^{(3)}\left(|\mathbf{x}|\right) + (d-1)(d-5)|\mathbf{x}|^2 \eta^{(2)}\left(|\mathbf{x}|\right)\right. \\ & \left. - 3(d-1)(d-3)|\mathbf{x}|\eta^{(1)}\left(|\mathbf{x}|\right) + 3(d-1)(d-3)\eta\left(|\mathbf{x}|\right)\right)\frac{\mathbf{x}}{|\mathbf{x}|^5}.\end{aligned}$$

The general solution to the ODE

$$s^4\eta^{(4)}(s) + 2(d-1)s^3\eta^{(3)}(s) + (d-1)(d-5)s^2\eta^{(2)}(s)$$
$$- 3(d-1)(d-3)s\eta^{(1)}(s) + 3(d-1)(d-3)\eta(s) = 0 \quad (s > 0)$$

is given by

$$\eta(s) = C_1 s + C_2 s^3 + C_3 s^{-(d-1)} + C_4 \begin{cases} s\log s & \text{if } d = 2, \\ s^{-(d-3)} & \text{if } d \neq 2, \end{cases} \quad \forall C_i \in \mathbb{R}\ (i = 1, 2, 3, 4).$$

Since we are looking for $\mathbf{g} \in C^\infty(\Omega_0; \mathbb{R}^d)$, $C_3 = C_4 = 0$. Therefore,

$$\mathbf{g}(\mathbf{x}) = \big(C_1|\mathbf{x}| + C_2|\mathbf{x}|^3\big)\frac{\mathbf{x}}{|\mathbf{x}|}, \quad \forall \mathbf{x} \in \Omega_0.$$

To determine C_1, C_2 we use the continuity conditions on $\partial\Omega_0$. Since $\operatorname{div}(\mathbf{g}|\mathbf{u}_{\tilde f}) \in L^2(\Omega)$, $\langle \mathbf{g}(\mathbf{x}), \mathbf{x}/|\mathbf{x}|\rangle_{\mathbb{R}^d} = \langle \mathbf{u}_{\tilde f}(\mathbf{x}), \mathbf{x}/|\mathbf{x}|\rangle_{\mathbb{R}^d}$ ($\forall \mathbf{x} \in \partial\Omega_0$), or $\eta(r_0) = H(r_0)$. Coupling this with the fact $h^{(1)}(r_0) = 0$ yields

$$C_1 r_0 + C_2 r_0^3 = -1. \tag{3.7}$$

Moreover, since $(\operatorname{div}\mathbf{g}|\operatorname{div}\mathbf{u}_{\tilde f}) \in H_0^1(\Omega)$, $\operatorname{div}\mathbf{g}(\mathbf{x}) = \operatorname{div}\mathbf{u}_{\tilde f}(\mathbf{x})$ ($\forall \mathbf{x} \in \partial\Omega_0$), which implies that $\eta^{(1)}(r_0) + (d-1)\eta(r_0)/r_0 = H^{(1)}(r_0) + (d-1)H(r_0)/r_0$, or by using the equality $\eta(r_0) = H(r_0)$,

$$C_1 + 3C_2 r_0^2 = H^{(1)}(r_0). \tag{3.8}$$

By solving (3.7)–(3.8) we have

$$\mathbf{g}(\mathbf{x}) = \left(\frac{1}{2r_0^2}\left(H^{(1)}(r_0) + \frac{1}{r_0}\right)|\mathbf{x}|^3 - \frac{1}{2}\left(H^{(1)}(r_0) + \frac{3}{r_0}\right)|\mathbf{x}|\right)\frac{\mathbf{x}}{|\mathbf{x}|},$$

which is seen to satisfy (i), (ii), (iv) of Lemma 3.1 by its construction and (3.6).

An elementary argument shows that this $\mathbf{g}$ obeys (iii) of Lemma 3.1 if and only if (3.4) holds.

We have checked that all the requirements of Lemma 3.1 are fulfilled, and thus obtain $\partial F_{\mathrm{D}}^c(f) = -(-\Delta_{\mathrm{D}})\operatorname{div}(\mathbf{g}|\mathbf{u}_{\tilde f})$. Then by direct calculation we can deduce (3.5). □

Example 3.4. Assume that $r = 2r_0$, $p = 2$ and $\mu = 1$. In this setting let us give a surface $\tilde f$ realizing all the assumptions of Theorem 3.2 plus

$$H^{(2)}(r_0) - \frac{3}{r_0}H^{(1)}(r_0) - \frac{3}{r_0^2} = 0, \tag{3.9}$$

so not having the surface integral over $\partial\Omega_0$ in (3.5). Note that now the condition (1.6) holds and

$$H(s) = -1 + h^{(1)}(s), \quad \forall s \in [r_0, 2r_0].$$

We can summarize the assumptions of Theorem 3.2 and (3.9) in terms of h as follows.

$$h \in C^1([0, 2r_0]) \cap C^4([r_0, 2r_0]),$$
$$h(2r_0) = 0,$$
$$h^{(1)}(s) = 0, \quad \forall s \in [0, r_0],$$
$$h^{(1)}(s) < 0, \quad \forall s \in (r_0, 2r_0),$$
$$h^{(2)}(2r_0) + \frac{d-1}{2r_0}\big(-1 + h^{(1)}(2r_0)\big) = 0,$$
$$h^{(2)}(r_0) \in [-9/r_0, 0],$$
$$h^{(3)}(r_0) - \frac{3}{r_0}h^{(2)}(r_0) - \frac{3}{r_0^2} = 0.$$

Define $h : [0, 2r_0] \to \mathbb{R}$ by

$$h(s) := \begin{cases} \int_{2r_0}^{r_0} (-\frac{3}{5r_0^3}(t-r_0)(t-2r_0)^2 + \frac{d-1}{2r_0^4}(t-r_0)^3(t-2r_0))\,dt & \text{if } s \in [0, r_0], \\ \int_{2r_0}^{s} (-\frac{3}{5r_0^3}(t-r_0)(t-2r_0)^2 + \frac{d-1}{2r_0^4}(t-r_0)^3(t-2r_0))\,dt & \text{if } s \in (r_0, 2r_0]. \end{cases}$$

Then, h obeys all the constraints listed above. With this h, define $\tilde{f}(\mathbf{x}) := h(|\mathbf{x}|)$ ($\forall \mathbf{x} \in \Omega$). By Theorem 3.2, $\partial F_{\mathrm{D}}^c(f) \in L^\infty(\Omega)$ and

$$\begin{aligned} \partial F_{\mathrm{D}}^c(f)(\mathbf{x}) = {} & \frac{2d(d+2)}{5r_0^3} 1_{\mathbf{x}\in\Omega_0} \\ & + \bigg(h^{(4)}\big(|\mathbf{x}|\big) + \frac{2(d-1)}{|\mathbf{x}|}h^{(3)}\big(|\mathbf{x}|\big) + \frac{(d-1)(d-3)}{|\mathbf{x}|^2}h^{(2)}\big(|\mathbf{x}|\big) \\ & - \frac{(d-1)(d-3)}{|\mathbf{x}|^3}h^{(1)}\big(|\mathbf{x}|\big) + \frac{(d-1)(d-3)}{|\mathbf{x}|^3}\bigg) 1_{\mathbf{x}\in\Omega\setminus\overline{\Omega_0}}. \end{aligned}$$

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