Modules Whose Proper Submodules Are Finitely Generated

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INTRODUCTION

Throughout the following, $R$ will denote a ring. The ring $R$ will always have a unit and will be commutative unless otherwise noted. In [1], Matlis developed a structure theory for Artinian modules over a commutative, Noetherian, one-dimensional Cohen–Macaulay ring $R$ and used the theory to study these rings. The building blocks in the structure theory are the simple divisible $R$-modules, which satisfy the following condition:

**Definition.** An $R$-module $M$ is *almost finitely generated* (a.f.g.) if $M$ is not finitely generated as an $R$-module but every proper $R$-submodule of $M$ is finitely generated.

This paper explores the structure and properties of a.f.g. modules and the rings connected with them. In Section 1 we establish the basic properties of a.f.g. modules over commutative rings. We show that an a.f.g. module is either isomorphic to the field of quotients of a domain or is Artinian (and simple divisible). The main theorem states that the endomorphism ring of an Artinian a.f.g. module is a commutative, complete, local Noetherian domain of Krull dimension one.

In Section 2 we use the ideas of quotient equivalence and Matlis duality to characterize the Artinian a.f.g. modules over a commutative Noetherian ring. Section 3 relates the properties of injectivity and quasi-injectivity to the a.f.g. property. We show that every quotient equivalence class of Artinian a.f.g. modules contains exactly one quasi-injective module, which is then used to describe the class.

In Section 4 we show that an a.f.g. module over a commutative ring has finite width. It then follows that a commutative Noetherian ring $R$ has Krull dimension one.

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dimension \( \leq 1 \) if and only if every Artinian \( R \)-module has finite width. In Section 5 we give an example of a one-dimensional non-Noetherian domain with a faithful Artinian a.f.g. module. It is shown that if we generalize the definition of a.f.g. module to left modules over noncommutative rings, some of the good properties (e.g., finite width) no longer hold.

1. Properties of A.F.G. Modules

**Notation.** Finitely generated will be abbreviated f.g. If \( R \) is an integral domain, then \( Q(R) \) will denote the field of fractions of \( R \).

**Example 1.** If \( R \) is a discrete valuation ring, then \( Q(R) \) is an a.f.g. \( R \)-module.

**Example 2.** For any integral prime \( p \), the abelian group \( \mathbb{Z}_{p^n} \) is an a.f.g. \( \mathbb{Z} \)-module since every proper subgroup of \( \mathbb{Z}_{p^n} \) is cyclic of order \( p^n \) for some integer \( n > 0 \).

**Example 3.** If \( R \) is any ring and \( N \) is an Artinian \( R \)-module which is not f.g., then the set \( S(N) = \{ L \subseteq N \mid L \) is an \( R \)-submodule and \( L \) is not f.g.\} \) is nonempty. Since \( N \) is Artinian, \( S(N) \) has elements minimal with respect to inclusion; these submodules of \( N \) will be a.f.g. \( R \)-modules.

**Notation.** Let \( M \) be an \( R \)-module. We will write \( \text{ann}_R(M) \) for the annihilator in \( R \) of \( M \).

**Definition.** For any subset \( L \) of \( R \), the annihilator in \( M \) of \( L \) (denoted \( \text{ann}_R(L) \)) is the set \( \{ x \in M \mid \text{for all } r \in L, rx = 0 \} \).

**Definition.** Let \( M \) be an \( R \)-module and \( t \) an element of \( R \). If \( tM = M \), then \( M \) is divisible by \( t \). If \( R \) is a domain and \( M \) is divisible by every nonzero element of \( R \), then \( M \) is a divisible \( R \)-module.

**Proposition 1.1.** Let \( R \) be a ring, not necessarily commutative. Let \( M \) be an a.f.g. \( R \)-module and \( T = \text{End}_R(M) \), the ring of \( R \)-endomorphisms of \( M \). Then:

1. For any proper \( R \)-submodule \( K \) of \( M \), \( M/K \) is an a.f.g. \( R \)-module.
2. \( M \) is indecomposable.
3. For any \( h \in T \), \( h \neq 0 \), we have \( hM = M \), and \( T \) is a (possibly
noncommutative) domain. If \( R \) is commutative, then \( \text{ann}_R(M) \) is a prime ideal of \( R \) and \( M \) is a divisible module over the domain \( R/\text{ann}_R(M) \);

(4) For any \( g \in T \) which is not a unit we have \( \bigcup_{i=1}^{\infty} \text{ann}_M(g^i) = M \).

Thus the (possibly noncommutative) domain \( T \) has a single maximal two-sided ideal.

Proof. (1) We have an exact sequence

\[ 0 \to K \to M \to M/K \to 0. \]

Since \( K \) is f.g. and \( M \) is not f.g., \( M/K \) cannot be f.g. However, every proper \( R \)-submodule of \( M/K \) is a quotient of a proper submodule of \( M \), hence is f.g., so \( M/K \) is a.f.g.

(2) Suppose that \( M = K \oplus L \) with \( K \) and \( L \) proper \( R \)-submodules of \( M \). Then \( K \) and \( L \) are f.g., which implies \( M \) is f.g., contradicting the hypothesis that \( M \) is an a.f.g. \( R \)-module.

(3) Since \( h \neq 0 \), \( \text{ann}_M(h) \) is a proper \( R \)-submodule of \( M \) and thus \( hM = M/\text{ann}_M(h) \) is a.f.g. by part (1). But \( hM \) is also an \( R \)-submodule of \( M \), hence must equal \( M \).

If \( h_1, h_2 \) are nonzero elements of \( T \), then \( h_1h_2M = h_1M = M \) so \( h_1h_2 \neq 0 \). Thus \( T \) is a (possibly noncommutative) domain; regarding \( R/\text{ann}_R(M) \) as a subring of \( T \) in the usual way, one sees that the rest of (3) holds.

(4) We may assume \( g \neq 0 \) and choose \( x \in \text{ann}_M(g), x \neq 0 \). If \( i \) is a positive integer, then by (3) we have \( g^iM = M \) so there exists \( y_i \in M \) with \( g^i(y_i) = x \). Then \( y_i \in \text{ann}_M(g^{i+1}) \) and \( y_i \notin \text{ann}_M(g^j) \) so the union \( \bigcup_{i=1}^{\infty} \text{ann}_M(g^i) \) is strictly ascending. Thus \( \bigcup_{i=1}^{\infty} \text{ann}_M(g^i) \) is an \( R \)-submodule of \( M \) which is not f.g., hence \( \bigcup_{i=1}^{\infty} \text{ann}_M(g^i) = M \).

Let \( \mathfrak{n} \) denote the set of nonunits of \( T \). It follows easily from (3) above that \( \mathfrak{n} \) is closed under multiplication from both sides. To establish that \( \mathfrak{n} \) is closed under addition we first show that for all \( g \in \mathfrak{n}, 1 - g \) is a unit in \( T \). Since \( \bigcup_{i=1}^{\infty} \text{ann}_M(g^i) = M \), for any \( x \in M \) the sum \( \sum_{j=0}^{\infty} g^j(x) = x + g(x) + g^2(x) + \cdots \) has only finitely many nonzero terms. Thus we can define an endormorphism \( f \) of \( M \) by \( f(x) = \sum_{j=0}^{\infty} g^j(x) \); \( f \) is a two-sided inverse for \( 1 - g \).

Now if \( \mathfrak{n} \) is not closed under addition, there exist \( h, k \in \mathfrak{n} \) such that \( h + k = u \) for some unit \( u \) of \( T \). Then we have \( u^{-1}h + u^{-1}k = 1 \) with \( u^{-1}h, u^{-1}k \subset u \). But by the last paragraph, \( u^{-1}h = 1 \) \( u^{-1}k \) is a unit of \( T \), which contradicts \( u^{-1}h \in \mathfrak{n} \). Thus \( \mathfrak{n} \) is closed under addition and is the unique maximal two-sided ideal of \( T \).

DEFINITION. Let \( M \) be an \( R \)-module. We say that \( r \in R \) is a zero-divisor of \( R \) on \( M \) if there exists \( x \in M, x \neq 0, \) such that \( rx = 0 \).
DEFINITION. Let $M$ be an $R$-module. The set of prime ideals of $R$ associated to $M$ (denoted $\text{Ass}_R(M)$) is the set $\{p | p$ is a prime ideal of $R$ and $p = \text{ann}_R(y)$ for some $y \in M\}$.

DEFINITION. Let $M$ be an $R$-module. We say that $M$ has ACC (ascending chain condition) if every ascending chain of submodules of $M$ terminates, and we say that $M$ has DCC (descending chain condition) if every descending chain of submodules of $M$ terminates. Often we will refer to modules with ACC as Noetherian $R$-modules and to modules with DCC as Artinian $R$-modules.

**Lemma 1.2.** Let $N$ be a nonzero Noetherian $R$-module. Then there is a finite series of submodules $0 = N_0 \subset N_1 \subset \cdots \subset N_k = N$ and a finite set $\{p_1, \ldots, p_k\}$ of prime ideals of $R$ such that for each $j, j = 1, \ldots, k, N_j/N_{j-1} \cong R/p_j$. $\text{Ass}_R(N)$ is a nonempty subset of $\{p_1, \ldots, p_k\}$ and these two sets have the same minimal elements under inclusion.

**Proof:** It is sufficient to consider the case where $N$ is faithful. By [3, Chap. 6, Exercise 4] $R$ is a Noetherian ring. The lemma then follows from [4, Corollaire 2 and Theorem 1, p. 5].

**Proposition 1.3.** Let $R$ be a commutative ring and $M$ an $afg. R$-module. Then $\text{Ass}_R(M)$ consists of a single prime ideal $p$ and $p$ is the set of zero-divisors of $R$ on $M$. One of two cases occurs:

1. $p = \text{ann}_R(M)$. Then $M$ is isomorphic to $Q(R/%ann_R(M))$, $p$ is not a maximal ideal and $M$ is not an Artinian $R$-module;

2. $p \supseteq \text{ann}_R(M)$. Then $p$ is a maximal ideal of $R$ and every proper submodule of $M$ has finite length, so $M$ is an Artinian $R$-module.

In either case $\text{ann}_R(M)$ is a prime ideal of $R$ which is not a maximal ideal.

**Proof:** To see that $\text{Ass}_R(M)$ is nonempty, choose a nonzero $x \in M$; since every submodule of $Rx$ is a proper submodule of $M$ and hence f.g., $Rx$ is a Noetherian $R$-module. By Lemma 1.2, $\text{Ass}_R(Rx)$ is nonempty and thus $\text{Ass}_R(M)$ is nonempty.

Suppose that $p \in \text{Ass}_R(M)$. Then there exists $y \in M$ with $\text{ann}_R(y) = p$. Let $t \in R$ be a zero-divisor on $M$. Then by Proposition 1.1(4), $\bigcup_{j=1}^{\infty} \text{ann}_M(t^j) = M$ so for some $j$, $t^jy = 0$ which implies $t^j \in p$ and finally $t \in p$. This shows that $p$ is the set of zero-divisors on $M$ and thus $p$ is the only element of $\text{Ass}_R(M)$.

If $p = \text{ann}_R(M)$ the preceding paragraph and Proposition 1.1(3) show that $M$ is torsion-free and divisible over the domain $R/%\text{ann}_R(M)$. Hence $M$ is naturally a $Q(R/%\text{ann}_R(M))$-vector space. By Proposition 1.1(2) $M$ is
indecomposable, which implies $M$ is isomorphic to $Q(R/\text{ann}_R(M))$. If $p$ were a maximal ideal, then since $p = \text{ann}_R(M)$ we would have $M = Q(R/\text{ann}_R(M)) = R/\text{ann}_R(M)$ and thus $M$ would be f.g. as an $R$-module. However, $M$ is an a.f.g. $R$-module, hence not f.g., and therefore $p$ is not a maximal ideal, so $R/p$ is not Artinian. Hence $M$ is not Artinian.

If $p \not\supseteq \text{ann}_R(M)$, then let $M_1 = \text{ann}_R(p)$ and choose $t \in R$, $t \notin p$. From Proposition 1.1(3) we have $tM = M$ so $M_1$ is divisible in $M$ by $t$. Suppose $x \in M$ with $tx \in M_1$; then $p(tx) = 0$. But since $t \notin p$, $\text{ann}_M(t) = 0$ so we must have $px = 0$ and thus $x \in M_1$. Combining the above we have $tM_1 = M_1$. Now since $p \not\supseteq \text{ann}_R(M)$ it follows that $M_1$ is a proper submodule of $M$, hence f.g., and we may apply Nakayama’s lemma to conclude that there is $s \in R$ with $(1 - st)M_1 = 0$. This implies that $st = 1$ in $R/p$; since $t$ was an arbitrary element of $R - p$ this shows that $R/p$ is a field, so $p$ is a maximal ideal.

Let $N$ be a proper submodule of $M$. Then $N$ is a Noetherian $R$-module so by Lemma 1.2 there is a series of submodules

$$0 = N_0 \subset N_1 \subset \cdots \subset N_k = N$$

and a set $\{p_1, \ldots, p_k\}$ of prime ideals of $R$ such that for each $j$, $j = 1, \ldots, k$, $N_j/N_{j-1} \cong R/p_j$.

For any $r \in p$ we have $\bigcup_{i=1}^{\infty} \text{ann}_M(r^i) = M$ by Proposition 1.1(4); since $N$ is f.g. there is some integer $l$ such that $r^lN = 0$. Then for each $j$, $r^lN_j \subset N_{j-1}$, which implies $r^l \in p_j$, hence $r \in p_j$. Thus $p \subseteq p_j$ for each $j$, and since $p$ is a maximal ideal this implies $p = p_j$. Thus the above series of submodules is a composition series for $N$ and $N$ has finite length. Since a module of finite length is Artinian, we have shown that every proper submodule of $M$ is Artinian, which implies that $M$ is Artinian.

The last statement of the proposition follows easily from an examination of the two cases.

**Definition.** Let $M$ be an Artinian a.f.g. $R$-module. By Proposition 1.3 there is a maximal ideal $p$ of $R$ such that $\text{Ass}_R(M) = \{p\}$. We say $p$ is the associated maximal ideal of $M$.

**Definition [1, p. 46].** An $R$-module $M$ is a simple divisible module if it is a nonzero torsion divisible module having no proper nonzero divisible submodules.

**Remark.** If $M$ is an Artinian a.f.g. $R$-module it is clear that $M$ is simple divisible over $R/\text{ann}_R(M)$.

We now examine the first case of Proposition 1.3. The next proposition is essentially [1, Theorem 7.1]; it gives necessary and sufficient conditions on a domain $R$ for $Q(R)$ to be a.f.g. over $R$. 


**Definition.** The *Krull dimension of R* (denoted $K. \dim(R)$) is defined to be the supremum of lengths of chains of prime ideals of $R$.

**Notation.** We write DVR for discrete valuation ring.

**Definition.** A ring $R$ is *almost DVR* if $R$ is a local Noetherian domain of Krull dimension 1 and the integral closure $\bar{R}$ of $R$ in $Q(R)$ is a f.g. $R$-module and is a DVR.

**Proposition 1.4.** Let $R$ be a domain, $Q(R)$ the field of fractions of $R$. The following are equivalent:

1. $Q(R)$ is a.f.g. over $R$;
2. $Q(R)/R$ is Artinian a.f.g. over $R$;
3. $R$ is an almost DVR.

**Proof.** (1) $\Rightarrow$ (2) As an a.f.g. $R$-module $Q(R)$ cannot equal $R$ and thus by Proposition 1.1(1) $Q(R)/R$ is a.f.g. over $R$. Since $Q(R)/R$ is a torsion $R$-module, it must be of the second type described in Proposition 1.3, hence is Artinian.

(2) $\Rightarrow$ (1) We need to show that proper $R$-submodules of $Q(R)$ are finitely generated. Let $N$ be a nonzero proper submodule of $Q(R)$ and choose $s \in N$, $s \neq 0$. Then $Q(R)/Rs$ is isomorphic to $Q(R)/R$, hence is a.f.g. There is an exact sequence of $R$-modules

$$0 \to Rs \to N \to N/Rs \to 0.$$ 

The module $N/Rs$ is isomorphic to a proper submodule of $Q(R)/Rs$, hence is f.g. Since $Rs$ is also f.g., it follows that $N$ is f.g. over $R$, and (1) is established.

(1) $\Rightarrow$ (3) Since $R$ does not equal $Q(R)$, $R$ has at least one nonzero prime ideal. However, if $P$ is a nonzero prime ideal of $R$ and $R_p$ is the localization of $R$ at $P$, then we have $R \subseteq R_p \subseteq Q(R)$ which shows that $R_p$ is a finitely generated $R$-module. This can only happen if $R = R_p$, which implies that the set of nonunits of $R$ is the only nonzero prime ideal of $R$.

Every ideal of $R$ is a proper $R$-submodule of $Q(R)$, hence f.g. Thus $R$ is a Noetherian ring.

Let $\bar{R}$ denote the integral closure of $R$ in $Q(R)$. By [3, Corollary 5.22] $\bar{R}$ can be expressed as an intersection of valuation domains, so we may choose a valuation domain $V$ such that $R \subseteq \bar{R} \subseteq V \subseteq Q(R)$. Then $V$ is a f.g. $R$-module, hence is a Noetherian ring integral over $R$. Thus $\bar{R} = V$ and $\bar{R}$ is a discrete valuation ring.

(3) $\Rightarrow$ (1) Let $\bar{R}g$ be the maximal ideal of $\bar{R}$. Then for any $q \in Q(R)$, there is a unit $u$ of $\bar{R}$ and an integer $n$ (possibly negative) such that $q = ug^n$. 
It follows that the proper $\bar{R}$-submodules of $Q(R)$ are of the form $g^n\bar{R}$ for $n \in \mathbb{Z}$, and thus $Q(R)$ is an a.f.g. $\bar{R}$-module.

Now suppose that $N$ is an $R$-submodule of $Q(R)$ which is not f.g. as an $R$-module. Then $\bar{R}N$ is an $\bar{R}$-submodule of $Q(R)$; if $\bar{R}N$ were f.g. over $\bar{R}$, then $\bar{R}N$ would also be f.g. over $R$ and (since $\bar{R}$ is Noetherian) we could conclude that $N$ was f.g. over $R$, a contradiction. So $RN$ is not f.g. as an $\bar{R}$-module, hence $\bar{R}N = Q(R)$. Since $\bar{R}$ is a f.g. $R$-module we may choose $s \in R$, $s \neq 0$, such that $s\bar{R} \subseteq R$. Then

$$Q(R) = sQ(R) = s\bar{R}N \subseteq RN = N \subseteq Q(R),$$

so $N = Q(R)$. This shows that $Q(R)$ is the only $R$-submodule of $Q(R)$ which is not f.g., so $Q(R)$ is an a.f.g. $R$-module.

**Proposition 1.5.** Let $M$ be an a.f.g. $R$-module. Suppose that for any $t, v \in R$ the following conditions are equivalent:

1. $\text{ann}_M(t) \subseteq \text{ann}_M(v);$
2. $Rt \supseteq Rv.$

Then $R$ is a domain, and for some $x \in M$ and $R$-homomorphism $h$ the following sequence is exact:

$$0 \to \text{ann}_R(x) \xrightarrow{\subseteq} Q(R) \to M \to 0.$$

Therefore $R$ is almost DVR.

**Proof.** First note that if $r \in \text{ann}_R(M)$, then $\text{ann}_M(r) = M = \text{ann}_M(0)$, so by (1) $\Rightarrow$ (2) above $r$ and 0 generate the same ideal, hence $r = 0$. Thus $\text{ann}_R(M) = 0$, and since $\text{ann}_R(M)$ is a prime ideal by Proposition 1.1(3), $R$ is a domain.

If $M$ is not Artinian over $R$, then by the first case of Proposition 1.3 there is an isomorphism from $Q(R)$ to $M$, and by choosing $x$ to be any nonzero element of $Q(R)$ we obtain the exact sequence in the statement of Proposition 1.5. By Proposition 1.4, $R$ is almost DVR.

Thus we may assume that $M$ is Artinian; denote the associated maximal ideal of $M$ by $\mathfrak{n}$. By Proposition 1.3, $\mathfrak{n}$ is strictly larger than $\text{ann}_R(M) = 0$, so we may choose a nonzero element $t$ of $\mathfrak{n}$. We claim that $R[1/t] = Q(R)$; to prove this it suffices to show that $t$ is contained in every nonzero prime ideal of $R$. Let $q$ be a nonzero prime ideal of $R$ and choose a nonzero element $z$ of $q$. Then $\text{ann}_M(z) \subseteq M$ and thus $\text{ann}_M(z)$ is f.g. Since $t \in \mathfrak{n}$ necessarily $\text{ann}_M(t) \neq 0$ so by Proposition 1.1(4) we have $\bigcup_{i=1}^{\infty} \text{ann}_M(t^i) = M$. Thus there is an integer $j$ such that $\text{ann}_M(t^j) \supseteq \text{ann}_M(z)$. By (1) $\Rightarrow$ (2) above this implies that $t^j \in Rz$ so $t^j \in q$ and finally $t \in q.$
Choose $x_0 \in M$, $x_0 \neq 0$. Since $M$ is divisible by $t$, we may inductively construct a sequence $(x_i)$ in $M$ such that $tx_i = x_{i-1}$ for $i \geq 1$. Then $\bigcup_{i=1}^{\infty} Rx_i = M$ is a nonzero submodule of $M$ and divisible by $t$, hence $\bigcup_{i=1}^{\infty} Rx_i = M$.

For each $i \geq 0$, let $Rt^{-i}$ denote the $R$-submodule of $Q(R)$ generated by $t^{-i}$ and define a map $f_i: Rt^{-i} \to Rx_i$ by setting $f_i(t^{-i}) = x_i$. These maps are consistent and thus have as their limit a map $f \in \text{Hom}_R(Q(R), M)$ which is surjective since $\bigcup_{i=0}^{\infty} Rx_i = M$.

Making some identifications we have $\text{Ker}(f) = \bigcup_{i=1}^{\infty} \text{Ker}(f_i) = \bigcup_{i=1}^{\infty} t^{-i} \text{Ann}_R(x_i)$. We wish to show that this union is not strictly ascending, so that $\text{Ker}(f)$ is isomorphic to an ideal of $R$.

Since $\text{Ann}_M(t)$ is f.g. and $\bigcup_{i=0}^{\infty} Rx_i = M$, there is an integer $k$ such that for all $j \geq k$, $Rx_j \supseteq \text{Ann}_M(t)$. Taking double annihilators gives $\text{Ann}_M(\text{Ann}_M(x_j)) \supseteq \text{Ann}_M(\text{Ann}_M(\text{Ann}_M(t))) = \text{Ann}_M(t)$. Now, since $\text{Ann}_M(\text{Ann}_M(x_j)) = \bigcap_{r \in \text{Ann}_M(x_j)} \text{Ann}_M(r)$ we see that for any $r \in \text{Ann}_M(x_j)$ the containment $\text{Ann}_M(r) \supseteq \text{Ann}_M(\text{Ann}_M(x_j))$ holds, and combining this with the last statement gives

$$\text{Ann}_M(r) \supseteq \text{Ann}_M(t) \quad \text{for } r \in \text{Ann}_M(x_j).$$

Another application of $(1) \Rightarrow (2)$ shows that $r \in Rt$ and thus $\text{Ann}_M(x_j) \subseteq Rt$ for $j \geq k$. Then

$$[\text{Ann}_M(x_j)] = [\text{Ann}_M(tx_{j+1})] = \bigcap_{i=0}^{j} \text{Ann}_M(x_i) = \text{Ann}_M(x_{j+1}),$$

where the second equality is easily verified by checking containments. It follows that for $j \geq k$ we have the following equality of submodules of $Q(R)$:

$$t^{-j} \text{Ann}_M(x_j) = t^{-k} \text{Ann}_M(x_k)$$

and thus $\text{Ker}(f) = \bigcup_{i=0}^{\infty} t^{-i} \text{Ann}_M(x_i) = t^{-k} \text{Ann}_M(x_k)$. Then setting $x = x_k$ and $h = ft^{-k}$ gives the exact sequence in the statement of Proposition 1.5.

Since $Q(R)/\text{Ann}_M(x)$ is a.f.g. over $R$, by Proposition 1.1(1) we see that $Q(R)/R$ is a.f.g. over $R$. Then by Proposition 1.4, $R$ is almost DVR.

**Definition.** Let $M$ be an $R$-module and $S$ a subring of $\text{End}_R(M)$. $S$ is dense in $\text{End}_R(M)$ if for any $g \in \text{End}_R(M)$ and finite family $x_1, \ldots, x_k$ of elements of $M$ there exists $f$ in $S$ such that $f(x_i) = g(x_i)$ for $i = 1, \ldots, k$. (The term dense will be used only in this sense, and not in connection with ideal-adic ring topologies.)

**Lemma 1.6.** Let $R$ be a commutative domain, not a field, and $I$ a nonzero ideal of $R$. There is a natural monomorphism of rings

$$H: \text{End}_R(I) \to \text{End}_R(Q(R)/I)$$
and the image of $H$ is dense in $\text{End}_R(Q(R)/I)$. Thus $\text{End}_R(Q(R)/I)$ is a commutative ring.

Proof. $\text{End}_R(I)$ can be identified with $\{q \in Q(R) | qI \subseteq I\}$. For any $q \in \text{End}_R(I)$, the map $f_q: Q(R)/I \to Q(R)/I$ given by setting $f_q(u) = qu$ for each $u \in Q$ is well defined and thus is an $R$-module homomorphism. Define $H$ by setting $H(q) = f_q$ for each $q \in \text{End}_R(I)$. Note that if $f_v = 0$ for some $v \in Q(R)$, then $vQ(R) \subseteq I$. Since $I \in R \subseteq Q(R)$ this implies $v = 0$ so $H$ is a monomorphism.

Given a finite family $\tilde{y}_1, ..., \tilde{y}_k$ of elements of $Q(R)/I$ and $g \in \text{End}_R(Q(R)/I)$ we need to show that there is $q \in \text{End}_R(I)$ such that $f_q(\tilde{y}_j) = g(\tilde{y}_j)$ for each $j$. For each $\tilde{y}_j$ choose a coset representative $y_j$ in $Q(R)$, and pick $d \in R$ such that for each $j$, $y_j \in Rd^{-1}$. Denote $g(d^{-1})$ by $\tilde{z}$ and pick a coset representative $z$ for $\tilde{z}$. Now, for $r \in I$

$$dzr = dr\tilde{z} = drg(d^{-1}) = g(r) = g(0) = 0$$

so $dzI \subseteq I$. It is then clear that $H(dz) = f_{dz}$ agrees with $g$ on $d^{-1}$, hence they agree on the $\tilde{y}_j$'s.

To establish the commutativity of $\text{End}_R(Q(R)/I)$ we need to show that $g_1g_2(x) = g_2g_1(x)$ for any $x \in Q(R)/I$ and $g_1, g_2 \in \text{End}_R(Q(R)/I)$. By the last paragraph we may choose $q_1, q_2 \in \text{End}_R(I)$ such that $H[q_1](x) = g_1(x)$ and $H[q_2](g_1(x)) = g_2(x)$ for $i = 1, 2$. Then we have $g_1g_2(x) = H[q_1]g_2(x) = g_2(x)$, $g_2g_1(x) = H[q_2]g_1(x) = g_1(x)$, $H[q_1]g_2(x) = g_2g_1(x)$, and $H[q_2]g_1(x) = g_2g_1(x)$.

The next lemma collects some well-known facts about Artinian modules which will be needed later.

**Definition.** Let $n$ be an ideal of $R$, $M$ an $R$-module. Define $T_n(M) = \{x \in M | n^kx = 0 \text{ for some integer } k\}$. $T_n(M)$ is the $n$-torsion submodule of $M$.

*Notation.* Let $n$ be an ideal of $R$. $\hat{R}_n$ will denote the completion of $R$ in the $n$-adic topology.

**Lemma 1.7.** Let $M$ be a nonzero Artinian $R$-module. Then we have the following:

1. $\text{Ass}_R(M) = \{n_1, ..., n_k\}$ is a finite set of maximal ideals.
2. $M = \sum_{i=1}^k T_{n_i}(M)$. Thus if $M$ is an indecomposable $R$-module, $\text{Ass}_R(M)$ consists of a single maximal ideal.
3. Let $n = \cap_{i=1}^k n_i$. $M$ is naturally an $\hat{R}_n$-module, and for any $t \in \hat{R}_n$ and finite family $x_1, ..., x_h$ of elements of $M$, there is $r \in R$ such that $rx_i = tx_i$.
for $i = 1, \ldots, h$. $M$ is an Artinian $\hat{R}_g$-module, and the $\hat{R}_g$-submodules of $M$ are exactly the $R$-submodules of $M$.

(4) $\text{End}_{\hat{R}_g}(M)$ may be identified with $\text{End}_g(M)$.

Proof. We will establish (3) and (4). For (3) we need to define an $\hat{R}_g$-module structure on $M$. Let $y \in M$ and $(t_i)$ a sequence in $R$ which is Cauchy in the $n$-adic topology. By (2) of this proposition there is an integer $h$ such that $n^hy = 0$. Since $(t_i)$ is Cauchy there is an integer $N_h$ such that for $i, j \geq N_h$, $t_i - t_j \subseteq n^n$, which implies $t_iy = t_jy$. Thus the sequence $(t_i, y)$ of elements of $M$ is eventually constant, and it is easy to verify that setting $(t_i)y = t_{N_h}y$ makes $M$ an $\hat{R}_g$-module.

Given $t = (t_i) \in \hat{R}_g$ and a finite family $x_1, \ldots, x_j$ of elements of $M$, we may choose an integer $N$ large enough so that $(t_i)x_j = t_Nx_j$ for each $1 \leq i \leq j$.

To prove (4) it suffices to show that every $R$-endomorphism of $M$ is an $\hat{R}_g$-endomorphism of $M$. Let $f \in \text{End}_g(M)$, $t \in \hat{R}_g$, and $x \in M$. By (3) we may choose $r \in R$ such that $rx = tx$ and $tf(x) = tf(x)$. Then $tf(x) = tf(x) = f(tx) - f(tx)$; since $t$ and $x$ were arbitrary this shows $f$ is an $\hat{R}_g$-endomorphism of $M$.

**Theorem 1.8.** Let $R$ be a domain, $M$ a faithful Artinian a.f.g. $R$-module. Then $M$ is naturally a faithful Artinian a.f.g. module over a domain $S$ which satisfies the following conditions:

1. $R \subseteq S \subseteq Q(R)$;
2. $S$ is almost DVR with maximal ideal $n$;
3. $M$ is isomorphic to $Q(R)/I$ for some ideal $I$ of $S$;
4. $\text{End}_R(M)$ is isomorphic to $\hat{S}_g$. Thus $\text{End}_R(M)$ is almost DVR and complete and $M$ is a faithful Artinian a.f.g. module over $\text{End}_R(M)$.

Proof. Let $C$ denote the center of the ring $\text{End}_R(M)$. By Proposition 1.1(3) $C$ is a domain so (identifying $R$ with a subring of $C$ in the usual way) we may consider $C$ and $Q(R)$ as subrings of $Q(C)$ and define $S = C \cap Q(R)$. Then $M$ is naturally an $S$-module and since $R \subseteq S$, every proper $S$-submodule is also an $R$-submodule, hence has finite length over $R$ and over $S$. It is then clear that $M$ is a faithful Artinian $S$-module; to show that $M$ is a.f.g. as an $S$-module it suffices to show that $M$ does not have finite length as an $S$-module. Since $M$ is an Artinian a.f.g. $R$-module, we see by the second case of Proposition 1.3 that there exists $r \in R$ such that $r$ is a zero-divisor on $M$ but $r \notin \text{ann}_R(M) = 0$. Then $\text{ann}_M(r) \neq 0$ and $rM = M$; we have an exact sequence of $S$-modules and homomorphisms

$$0 \rightarrow \text{ann}_M(r) \rightarrow M \rightarrow M \rightarrow 0.$$
Since \( \text{ann}_M(r) \neq 0 \) and length is additive, \( M \) cannot have finite length as an \( S \)-module and thus \( M \) is a faithful Artinian a.f.g. module over \( S \).

We wish to establish (2) by using Proposition 1.5; to this end, suppose that \( t, v \in S \) with \( \text{ann}_M(t) \subseteq \text{ann}_M(v) \). If \( t = 0 \), then certainly \( \text{ann}_M(t) = M \) and \( \text{ann}_M(v) = M \) so \( v = 0 \). Otherwise \( tM = M \) and we may define \( g: M \to M \) as follows: given \( x \in M \) choose \( y \in M \) such that \( tx = y \) and define \( g(x) = ty \).

Then the annihilator condition implies that \( g \) is a well-defined \( S \)-module homomorphism, so \( g \in \text{End}_S(M) \). We now show that \( g \in C \cap Q(R) = S \).

Let \( f \in \text{End}_S(M) \), \( x, y \in M \) with \( tx = x \). Since \( t \in C \) we have \( f(x) = f(tx) = tf(x) \); then using the definition of \( g \) and the fact that \( v \in C \) gives \( g(f(x)) = uf(x) = f(vy) = f(g(x)) \). Thus \( g \in C \).

Since \( t, u \in S \subseteq Q(R) \) there exist \( a_1, a_2, b_1, b_2 \in R \) such that \( ub_1(z) = a_1(z) \) and \( tb_2(z) = a_2(z) \) for all \( z \in M \). Then \( a_2b_1(z) = tb_2b_1(z) \) so \( g(a_2b_1(z)) = gb_2b_1(z) = gb_2b_1(z) = a_1b_2(z) \) for \( z \in M \), which shows that \( g \in Q(R) \). Thus \( g \in C \cap Q(R) = S \).

It is clear from the definition of \( g \) that \( v = gI \) so \( v \in St \) as needed for (1) \( \Rightarrow \) (2) of Proposition 1.5. Since (2) \( \Rightarrow \) (1) of Proposition 1.5 is always satisfied, we may conclude that \( S \) is almost DVR and that \( M \) is isomorphic to \( Q(S)/I \), where \( I = \text{ann}_S(x) \) for some \( x \in M \). Clearly \( Q(R) = Q(S) \) so (3) has been established. Since \( R \subseteq S \subseteq C \) we may identify \( \text{End}_S(M) \) and \( \text{End}_S(M) \), and compute the latter using the isomorphism \( M = Q(S)/I \).

Let \( S_1 = \text{End}_S(I) \). By Lemma 1.6 we may identify \( S_1 \) with a dense subring of \( \text{End}_S(M) \) and conclude that \( \text{End}_S(M) \) is commutative, so \( S = \text{End}_S(M) \cap Q(R) \). But also \( S_1 \subseteq \text{End}_S(M) \cap Q(R) \) so in fact \( S = S_1 \) and thus \( S \) is dense in \( \text{End}_S(M) \).

We must show that \( \mathcal{S}_g \) is isomorphic to \( \text{End}_S(M) \). By Lemma 1.7(3) \( M \) is naturally an \( \mathcal{S}_g \)-module, and it is easily checked that \( \text{ann}_{\mathcal{S}_g}(M) = \bigcap_{i=1}^{\infty} (S) = 0 \). Thus by Lemma 1.7(4) we may identify \( \mathcal{S}_g \) with a subring of \( \text{End}_S(M) \).

Using the divisibility of \( M \) over \( S \) we may choose a sequence \( (x_i) \) of elements of \( M \) such that the union \( \bigcup_{i=1}^{\infty} Sx_i \) is strictly ascending, hence equal to \( M \). Let \( f \in \text{End}_S(M) \); since \( S \) is dense in \( \text{End}_S(M) \) we may choose a sequence \( (s_i) \) of elements of \( S \) such that for \( i \geq 1, f(x_i) = s_i x_i \). We now show that \( (s_i) \) is a Cauchy sequence in the \( n \)-adic topology of \( S \).

For \( k \) a positive integer choose a nonzero element \( v_k \) of \( n^k \). Then \( \text{ann}_S(v_k) \) is f.g. so there is an integer \( h \) (depending on \( k \)) such that \( \text{ann}_M(v_k) \subseteq Sx_h \).

Now, for \( t, f \geq h \) we have \( s_ix_h = f(x_h) = s_jx_h \) so \( s_i - s_j \in \text{ann}_S(x_h) \) and thus \( \text{ann}_M(v_k) \subseteq \text{ann}_M(s_i - s_j) \). Again using the fact that \( S \) satisfies the hypotheses of Proposition 1.5 we have \( s_i - s_j \in Sv_k \) and thus \( s_i - s_j \in n^k \) for \( i, j \geq h \). Since \( k \) was arbitrary we have shown that \( (s_i) \) is a Cauchy sequence in \( S \), hence \( (s_i) \) may be identified with an element \( t \) of \( \mathcal{S}_g \), and since \( \bigcup_{i=1}^{\infty} Sx_i = M \) the endomorphisms \( f \) and \( t \) are equal. Thus \( \text{End}_R(M) \) is isomorphic to \( \mathcal{S}_g \).
By Lemma 1.7(3) the $\text{End}_R(M)$-submodules of $M$ are exactly the $S$-submodules of $M$, which suffices to show that $M$ does not have finite length as an $\text{End}_R(M)$-module although its proper submodules do. This implies that $M$ is faithful, Artinian, and a.f.g. as a module over $\text{End}_R(M)$.

It is known (see [5, Exercise 1, p. 122]) that the completion of an almost DVR is almost DVR. An alternative way of seeing that $\text{End}_R(M)$ is almost DVR is the following: apply the first part of this theorem (with $\text{End}_R(M) \cong \hat{S}_n$ in the role of $R$) to obtain an almost DVR $S_2$ such that $\hat{S}_n \subseteq S_2 \subseteq \text{End}_R(M)$. Since $\hat{S}_n$ is commutative and $\hat{S}_n \cong \text{End}_R(M)$ we have $\text{End}_{\hat{S}_n}(M) = \hat{S}_n$, which implies that $\hat{S}_n = S_2$ and thus $\hat{S}_n$ is almost DVR.

**COROLLARY 1.9.** Let $M$ be an Artinian a.f.g. module over a commutative ring $R$. Then $\text{End}_R(M)$ is almost DVR and complete.

**Proof.** By Proposition 1.1(3), $\text{ann}_R(M)$ is a prime ideal. The result then follows from Theorem 1.8.

## 2. PROPERTIES OF RINGS HAVING A.F.G. MODULES

**DEFINITION.** Two $R$-modules $M$ and $N$ are quotient equivalent if each is a homomorphic image of the other. This will be denoted $M \sim_e N$. The equivalence class of $M$ will be denoted by $[M]_e$.

**Remark.** This equivalence relation was introduced in [1, p. 50]. The next lemma generalizes [1, Lemma 5.8].

**LEMMA 2.1.** Let $M$ be an Artinian a.f.g. $R$-module. Then for any proper $R$-submodule $K$ of $M$ we have $(M/K) \sim_e M$.

**Proof.** We need a homomorphism from $M/K$ onto $M$. Since we are in case 2 of Proposition 1.3, $\text{ann}_R(M) \subseteq n$, where $n$ is the associated maximal ideal of $M$. Choose $t \in n$, $t \notin \text{ann}_R(M)$; then by Proposition 1.1 we have $t^i M = M$ for $i \geq 1$ and $\bigcup_{j=1}^{\infty} \text{ann}_M(t^j) = M$. Since $K$ is a proper submodule and thus f.g., there is an integer $h$ such that $K \subseteq \text{ann}_M(t^h)$. Letting $\pi$ denote the natural surjection from $M/K$ to $M/\text{ann}_M(t^h)$, we have $M/K \twoheadrightarrow M/\text{ann}_M(t^h)$ which gives the desired surjection.

The next proposition is a partial generalization of [1, Corollary 8.4].

**PROPOSITION 2.2.** Let $M$ be an Artinian a.f.g. $R$-module and $q = \text{ann}_R(M)$. Then there is a discrete valuation ring $V$ between $R/q$ and $Q(R/q)$ such that $M \sim_e Q(R/q)/V$.

**Proof.** We may regard $M$ as a faithful Artinian a.f.g. module over the
domain $R/q$. By Proposition 1.8 there is an almost DVR $S$ between $R/q$ and $Q(R/q)$ such that $M$ is isomorphic to $Q(R/q)/I$ for some ideal $I$ of $S$. By the definition of almost DVR the integral closure $\overline{S}$ of $S$ in $Q(R/q)$ is a DVR, and clearly $I \subseteq \overline{S} \supseteq Q(R/q)$ so $Q(R/q)/\overline{S}$ is a nonzero quotient of $Q(R/q)/I$. Then take $V = \overline{S}$ and Lemma 2.1 gives the desired equivalence.

PROPOSITION 2.3. Let $R$ be a domain, $V$ a discrete valuation ring between $R$ and $Q(R)$, $V_g$ the maximal ideal of $V$ and $m = R \cap V_g$. The following conditions are equivalent:

1. $Q(R)/V$ is an Artinian a.f.g. $R$-module.
2. (a) $V/V_g$ has finite length over $R$.
   (b) For each $R$-module $L$ with $V \subseteq L \subseteq Q(R)$ there is an integer $k$ (depending on $L$) such that $g^kL \subseteq V$.

If these conditions are satisfied, then $m$ is a maximal ideal of $R$ and is the unique associated prime of $Q(R)/V$ over $R$.

Proof. (1) $\Rightarrow$ (2) Note that $Q(R)/V$ has a proper $R$-submodule isomorphic to $V_g^{-1}/V$, hence $V_g^{-1}/V$ must have finite length over $R$. Since $V/V_g$ is $R$-isomorphic to $V_g^{-1}/V$, (2)(a) is established. If $L$ is a proper $R$-submodule of $Q(R)$, then $L/V$ is isomorphic to a proper submodule of $Q(R)/V$ so $L/V$ is f.g. as an $R$-module. Since $\bigcup_{i=1}^{\infty} (V_g^{-1}/V)$ is a strictly ascending union in $Q/V$, there is a $k$ such that $L/V \subseteq g^{-k}V/V$ and then $g^kL \subseteq V$.

(2) $\Rightarrow$ (1) Since there is a DVR between $R$ and $Q(R)$, $R$ is not a field, hence the nonzero divisible $R$-module $Q(R)/V$ cannot be f.g. over $R$.

A proper $R$-submodule of $Q(R)/V$ is of the form $L/V$ for some proper submodule $L$ of $Q(R)$. By (2)(b), there is an integer $k$ such that $V \subseteq L \subseteq V_g^{-k}$. By (2)(a), $V_g^{-k}/V$ has finite length over $R$, so the submodule $L/V$ has finite length over $R$. Thus $Q(R)/V$ is Artinian and a.f.g. over $R$.

Since $m = R \cap V_g$, $m$ is a prime ideal. We may regard $R/m$ as a submodule of $V/V_g$, and thus as a submodule of $Q(R)/V$; hence $m \in \text{Ass}_R(Q(R)/V)$ so by Proposition 1.3, $m$ is the lone maximal ideal of $R$ associated to $Q(R)/V$.

Remarks. Let $R$ be a Noetherian domain of Krull dimension 1.

1. If $V$ is a valuation domain between $R$ and $Q(R)$, then by the theorem of Krull–Akizuki [5, Theorem 33.2] $V$ is a discrete valuation ring and condition (2)(a) of Proposition 2.3 holds.

2. In [1, Corollary 8.4] Matlis shows that if $V_1, \ldots, V_n$ are the valuation rings between $R$ and $Q(R)$, then the modules $Q(R)/V_1, \ldots, Q(R)/V_n$ are a complete set of representatives of the equivalence classes (under ~) of Artinian a.f.g. modules.
Notation. Let $M$ be an $R$-module, then $E_R(M)$ denotes the injective envelope of $M$ as an $R$-module.

The ideas of completion and Matlis duality are useful in studying Artinian a.f.g. modules over Noetherian rings. For convenience we recall the duality theorem [6, Corollary 4.3] in part.

**Proposition 2.4.** Let $R$ be a Noetherian complete local ring, $\mathfrak{n}$ its maximal ideal, and $E = E_R(R/\mathfrak{n})$, the injective envelope of $R/\mathfrak{n}$. Then:

1. An $R$-module has ACC if and only if it is a homomorphic image of $R^k$ for some $k$.
2. An $R$-module has DCC if and only if it is a submodule of $E^k$ for some $k$.
3. If $X, Y$ are the categories of $R$-modules with ACC and DCC, respectively, then the contravariant, exact functor $\text{Hom}_R(\cdot, E)$ establishes a one-to-one correspondence $X \leftrightarrow Y$. In particular, $M \cong \text{Hom}_R(M, E)$ for $M$ in either category.
4. If $I$ is an ideal of $R$, then $\text{ann}_R(\text{ann}_R(I)) = I$. If $K$ is a submodule of $E$, then $\text{ann}_R(\text{ann}_R(K)) = K$.

**Definition and Lemma 2.5.** For $R$ a ring let $L = \oplus R/\mathfrak{n}$, the sum being taken over the set of maximal ideals $\mathfrak{n}$ of $R$. Then $E_R(L)$ is the universal injective of $R$ and has the following property: if $M$ is an $R$-module and $x$ a nonzero element of $M$, then there is a homomorphism $f: M \to E_R(L)$ such that $f(x) \neq 0$.

Proof. Since $x \neq 0$, $\text{ann}_R(x)$ is a proper ideal, hence $\text{ann}_R(x) \subset \mathfrak{n}$ for some $\mathfrak{n} \subset \text{maxspec}(R)$. There is then a nonzero map $f_0: Rx \to R/\mathfrak{n} \subseteq L$ and since $E_R(L)$ is injective $f_0$ may be extended to give $f: M \to E_R(L)$.

**Definition.** For $I$ an ideal of $R$, define the dimension of $I$ (denoted $\dim_R(I)$) by $\dim_R(I) = K. \dim(R/I)$.

The next proposition generalizes [1, Corollary 8.3] in part.

**Proposition 2.6.** Let $R$ be a complete local Noetherian ring, $\mathfrak{n}$ its maximal ideal and $E = E_R(R/\mathfrak{n})$. Then there is a one-to-one correspondence between the set of prime ideals $p$ of dimension one of $R$ and the set of equivalence classes $[M]_e$ of Artinian a.f.g. $R$-modules given by $p \mapsto [\text{Hom}_R(R/p, E)]_e$ and $[M]_e \mapsto \text{ann}_R(M)$.

Proof. To shorten statements we define $N^* = \text{Hom}_R(N, E)$ for each $R$-module $N$. The following are useful consequences of Proposition 2.4:
(a) If $N$ satisfies ACC or DCC, then $(N^*)^* \simeq N$. Thus $\text{ann}_R(N) = \text{ann}_R(N^*)$. Also note that $R \simeq (R^*)^* \simeq E^*$.

(b) If $N$ satisfies ACC (resp. DCC), then $N^*$ satisfies DCC (resp. ACC). In particular, if $N$ has finite length, then so does $N^*$.

Now, if $M$ is a nonzero $R$-module, then since $E$ is the universal injective for $R$ there is a nonzero homomorphic image $M_1$ of $M$ inside $E$. If $M$ is an Artinian a.f.g. $R$-module, then by Propositions 1.1 and 2.1, $M_1$ is Artinian a.f.g. and is quotient equivalent to $M$. There is an exact sequence

$$0 \to M_1 \to E \to E/M_1 \to 0.$$  

Applying $\text{Hom}_R(\ , E)$ and replacing $E^*$ by $R$ gives

$$0 \to (E/M_1)^* \to R \to M_1^* \to 0.$$  

Thus $(E/M_1)^*$ can be identified with an ideal $I$ of $R$, and applying $\text{Hom}_R(\ , E)$ again gives

$$M_1 \simeq (R/I)^*.$$  

Now, if $\dim_R(I) = 0$, then $R/I$ has finite length and by (b) this would imply that $M_1$ has finite length, contradicting the fact that $M_1$ is a.f.g. over $R$. Thus $\dim_R(I) \geq 1$, so we may choose a prime $p$ such that $I \subseteq p$ and $\dim_R(p) = 1$. Notice that since $R/p$ satisfies ACC but not DCC, $(R/p)^*$ satisfies DCC but not ACC, hence is not f.g. We have an exact sequence

$$0 \to p/I \to R/I \to R/p \to 0.$$  

Applying $\text{Hom}_R(\ , E)$ gives

$$0 \to (R/p)^* \to (R/I)^* \to (p/I)^* \to 0.$$  

Since $(R/p)^*$ is not f.g. and $(R/I)^*$ is a.f.g. the map $(R/p)^* \to (R/I)^*$ is an isomorphism and thus $(p/I)^* = 0$. Since $E$ is the universal injective for $R$, this implies $p/I = 0$ and $p = I$ as desired. We have shown $M \simeq (R/p)^*$ for $p$ a prime of dimension 1.

Finally, note that in order to prove that every prime $p$ of dimension 1 gives rise to an Artinian a.f.g. module in this way, it remains to be shown that proper submodules of $(R/p)^*$ have finite length. However, if $L \subseteq (R/p)^*$ we have an exact sequence

$$0 \to L \to (R/p)^* \to K \to 0,$$

where $K \neq 0$. Applying $\text{Hom}_R(\ , E)$ gives

$$0 \to K^* \to R/p \to L^* \to 0.$$
and since $K^*$ is nonzero, there is an ideal $I$ such that $I \not\supseteq p$ and $L^* = R/I$. Since $\dim_R(p) = 1$, $\dim_R(I) = 0$ and thus $R/I$ has finite length. Then by (b), $(R/I)^*$ has finite length, and since $L$ is isomorphic to $(R/I)^*$, $(R/p)^*$ is an Artinian a.f.g. $R$-module.

It is easily verified that if $M, N$ are $R$-modules with $M \sim N$, then $\text{ann}_R(M) = \text{ann}_R(N)$; thus the set map given by $[M]_c \rightarrow \text{ann}_R(M)$ is well defined. If $M$ is an Artinian a.f.g. $R$-module we have seen that $M \sim (R/p)^*$ for some prime $p$ of dimension one, so $\text{ann}_R(M) = \text{ann}_R((R/p)^*) = \text{ann}(R/p) = p$ is dimension one as desired.

**COROLLARY 2.1.** Let $R$ be a commutative Noetherian ring, $\mathfrak{n}$ a maximal ideal of $R$, $p$ a prime ideal of $R$ which is contained in $\mathfrak{n}$. Let $\hat{R}_\mathfrak{n}$ denote the completion of $R$ in the $\mathfrak{n}$-adic topology and $f : R \rightarrow \hat{R}_\mathfrak{n}$ the canonical ring homomorphism. The following conditions are equivalent:

1. There exists an Artinian a.f.g. $R$-module $M$ such that $\text{Ass}_R(M) = \{\mathfrak{n}\}$ and $\text{ann}_R(M) = p$.

2. There is a prime ideal $\mathfrak{p}$ of $\hat{R}_\mathfrak{n}$ such that $\dim_{\hat{R}_\mathfrak{n}}(\mathfrak{p}) = 1$ and $f^{-1}(\mathfrak{p}) = p$.

In particular, if there are no prime ideals of $R$ strictly between $p$ and $\mathfrak{n}$ (but $p \neq \mathfrak{n}$) then these conditions are satisfied for $p$.

**Proof.** Before we begin, recall that if $M$ is an Artinian $R$-module with $\text{Ass}_R(M) = \mathfrak{n}$ then by Lemma 1.7(3) $M$ is naturally an $\hat{R}_\mathfrak{n}$-module and the $\hat{R}_\mathfrak{n}$-submodules of $M$ are exactly the $R$-submodules of $M$. Thus $M$ will be Artinian a.f.g. over $R$ if and only if $M$ is Artinian a.f.g. over $\hat{R}_\mathfrak{n}$.

By [3, Proposition 10.15(iii) and (iv)] $\hat{R}_\mathfrak{n}$ is a complete local Noetherian ring with maximal ideal $\mathfrak{p}_\mathfrak{n}$.

1. $\Rightarrow$ (2) $M$ is an Artinian a.f.g. $\hat{R}_\mathfrak{n}$-module. Letting $P$ denote $\text{ann}_{\hat{R}_\mathfrak{n}}(M)$, we see by Proposition 2.6 that $P$ is a prime ideal of dimension one, and we have $p = \text{ann}_R(M) = f^{-1}(\text{ann}_{\hat{R}_\mathfrak{n}}(M)) = f^{-1}(P)$ as desired.

2. $\Rightarrow$ (1) By Proposition 2.6 there exists an Artinian a.f.g. $\hat{R}_\mathfrak{n}$-module $M$ having associated maximal ideal $\mathfrak{g}$ and annihilator $P$. $M$ will be Artinian a.f.g. over $R$ and since $f^{-1}(\mathfrak{g}) = \mathfrak{n}$ and $f^{-1}(P) = p$ we have (1).

For the last part of the proposition, note that there are no prime ideals of $R$ strictly between $p$ and $\mathfrak{n}$ if and only if $K$. $\dim(R/p_\mathfrak{n}) = 1$, where $( \cdot )_\mathfrak{n}$ indicates localization at $\mathfrak{n}$. Let $S$ be the $(n_\mathfrak{n}/p_\mathfrak{n})$-adic completion of $R_\mathfrak{n}/p_\mathfrak{n}$. Then by [3, Corollary 11.19] $S$ is a complete local Noetherian ring and $K$. $\dim(S) = 1$; hence by Proposition 2.6 $S$ has at least one Artinian a.f.g. module $M$. As before, $M$ will be Artinian a.f.g. over $R_\mathfrak{n}/P_\mathfrak{n}$ and since $0$ is the only nonmaximal prime ideal of $R_\mathfrak{n}/p_\mathfrak{n}$, we have $\text{ann}_R((R_\mathfrak{n}/p_\mathfrak{n})(M)) = 0$. Thus $R_\mathfrak{n}$ has an Artinian a.f.g. module with associated maximal ideal $\mathfrak{n}_\mathfrak{n}$ and annihilator $p_\mathfrak{n}$; since $R_\mathfrak{n}$-modules of finite length also have finite length over...
$R, M$ will be Artinian a.f.g. over $R$ with $\text{Ass}_R(M) = \{n\}$ and $\text{ann}_R(M) = p$, so by the previous part of the proof both conditions are satisfied for $p$.

**Corollary 2.8.** Let $R$ be a commutative Noetherian ring, $\mathfrak{n}$ a maximal ideal of $R$, $R_\mathfrak{n}$ the localization of $R$ at $\mathfrak{n}$. Let $\#(R, \mathfrak{n})$ denote the number of distinct equivalence classes of Artinian a.f.g. $R$-modules with associated maximal ideal $\mathfrak{n}$. There are then the following relationships between $\#(R, \mathfrak{n})$ and the Krull dimension of $R_\mathfrak{n}$:

$$\#(R, \mathfrak{n}) = 0 \iff K. \text{dim}(R_\mathfrak{n}) = 0,$$

$$0 < \#(R, \mathfrak{n}) < \infty \iff K. \text{dim}(R_\mathfrak{n}) = 1.$$

**Proof.** Let $\hat{R}_\mathfrak{n}$ denote the $\mathfrak{n}$-adic completion of $R_\mathfrak{n}$. By [3, Corollary 11.19] we have $K. \text{dim}(\hat{R}_\mathfrak{n}) = K. \text{dim}(R_\mathfrak{n})$ and by the proof of Proposition 2.7 the Artinian a.f.g. $R_\mathfrak{n}$-modules can be viewed as Artinian a.f.g. over $\hat{R}_\mathfrak{n}$ and vice versa. Thus it suffices to prove the statements for a complete local ring $R_\mathfrak{n}$. By Proposition 2.6, $\#(R, \mathfrak{n})$ is then equal to the number of distinct prime ideals of dimension one of $R_\mathfrak{n}$. The first statement is then clear, and the second follows from the fact that $R_\mathfrak{n}$ has only finitely many prime ideals of dimension one if and only if $K. \text{dim}(R_\mathfrak{n}) = 1$ ([7, Theorem 144]).

**Remarks.** From Proposition 1.8, and from Proposition 2.6 and its corollaries, one sees that for a Noetherian ring $R$ there is a strong connection between the one-dimensional prime ideals of $R$ and the Artinian a.f.g. $R$-modules. However, the hypothesis of completeness in Proposition 2.6 is necessary; the following examples show that one-dimensional prime ideals of $\hat{R}_\mathfrak{n}$ can contract badly in $R_\mathfrak{n}$.

**Example 2.9.** For $i > 0$ there exists a regular local Noetherian domain $T_i$ of Krull dimension $i$ which has a faithful Artinian a.f.g. module.

**Proof.** Let $k$ be a finite or countably infinite field, $x$ an indeterminate, and $k[[x]]$ the ring of formal power series in $x$. Then $k[[x]]$ has the cardinality of the continuum, which is higher than that of $k$. Thus, starting with $y_1 = x$ we may select an infinite sequence $(y_i)$ of elements of $k[[x]]$ which are algebraically independent over $k$. For $i > 0$, define $T_i = k[y_1, \ldots, y_i]_{(y_1, \ldots, y_i)}$; that is, the localization at the ideal generated by $y_1, \ldots, y_i$ of the ring generated by $k$ and $y_1, \ldots, y_i$. Thus $T_i$ is isomorphic to the polynomial ring in $i$ variables localized at the origin.

Now, by Proposition 1.4 the discrete valuation ring $R = k[[x]]_{(x)}$ has a torsion-free a.f.g. module, namely, its field of fractions $k(x)$. Let $M$ denote $k(x)/k[[x]]_{(x)}$. By Proposition 1.1(1), $M$ is a.f.g. and by Proposition 1.3, $M$ is Artinian with $\text{ann}_R(M) \not\subseteq xk[[x]]_{(x)}$. Since $M$ is a.f.g., $\text{ann}_R(M)$ is prime, which implies that $\text{ann}_R(M) = 0$. Thus $M$ is faithful, Artinian and a.f.g. over $R$. 

Applying Proposition 1.8, we see that \( \text{End}_R(M) \) is the \((\chi)\)-adic completion of \( R \), namely, \( k[[\chi]] \). Since for each \( i > 0 \) we have \( k[[\chi]] \subseteq T_i \), every proper \( T_i \)-submodule of \( M \) has finite length over \( T_i \). Since \( M \) is faithful and not finitely generated over \( k[[\chi]] \) and \( T_i \sim k[[\chi]] \), \( M \) is faithful and not finitely generated over \( T_i \). Thus \( M \) is a faithful Artinian a.f.g. \( T_i \)-module.

**Remarks.** In each case the \( i \)-dimensional prime \( 0 \) of \( T_i \) is (by Proposition 2.7) the contraction of a one-dimensional prime of the completion of \( T_i \).

If we define \( T_\infty = \bigcup_{i=1}^\infty T_i \cong k[y_1, y_2, \ldots]_{(y_1, y_2, \ldots)} \), then since \( k[[\chi]] \subseteq T_\infty \subseteq k[[\chi]] \), we have an example of a non-Noetherian quasi-local domain with a faithful a.f.g. module. The same can be said for the completion of \( T_\infty \) in the topology given by its maximal ideal.

3. **Injectivity, Quasi-injectivity and the A.F.G. Property**

**Definition.** Let \( M \) be an \( R \)-module. We say that \( M \) is **simply embedded** if \( M \) has a simple submodule \( L \) which is essential in \( M \). (Since \( L \) is simple, this means every nonzero submodule of \( M \) contains \( L \).)

**Definition.** Let \( M \) be an \( R \)-module. If for any submodule \( N \) of \( M \) and \( R \)-homomorphism \( g: N \to M \) we can find an endomorphism \( g': M \to M \) which extends \( g \), \( M \) is said to be **quasi-injective**.

**Definition.** Let \( M \) be an \( R \)-module. The **socle** of \( M \) (denoted \( \text{socle} M \)) is the sum of all the simple submodules of \( M \).

**Proposition 3.1.** Let \( R \) be a commutative ring, \( M \) an Artinian a.f.g. \( R \)-module. The following conditions are equivalent:

1. \( M \) is simply embedded;
2. \( M \) is quasi-injective.

If these conditions hold for \( M \), then \( R/\text{ann}_R(M) \) is dense in \( \text{End}_R(M) \).

**Proof.** (1) \( \Rightarrow \) (2) Let \( \pi \) denote the maximal ideal of \( R \) associated to \( M \). We may choose \( t \in \pi, t \notin \text{ann}_R(M) \). Since \( M \) is divisible by \( t \), we may begin with a nonzero element \( x_0 \) of \( M \) and construct a sequence \( (x_i) \) in \( M \) with \( tx_i = x_{i-1} \) for \( i > 0 \). Then \( \bigcup_{i=1}^\infty Rx_i \) is a strictly ascending union, so must equal \( M \). Let \( N \) be a proper \( R \)-submodule of \( M \) and \( f: N \to M \) an \( R \)-homomorphism. We shall show that \( f \) can be extended to \( M \). Since \( N \) is proper the submodule \( N + f(N) \) is f.g. and so there is an integer \( j \) such that \( N + f(N) \subseteq Rx_j \). Now, \( R/\text{ann}_R(x_j) \cong Rx_j \subseteq M \). Hence \( R/\text{ann}_R(x_j) \) is simply embedded and Artinian. Let \( \pi \) denote the maximal ideal of \( R \) associated to
there is an integer $k$ such that $n^k \subset \text{ann}_R(x_j)$, which implies that $n/\text{ann}_R(x_j)$ is the unique maximal ideal of $R/\text{ann}_R(x_j)$. Thus $R/\text{ann}_R(x_j)$ is a simply embedded, Artinian local ring, and therefore by [8, Exercise 6.1] $R/\text{ann}_R(x_j)$ is a self-injective ring. Thus $Rx_j$ is quasi-injective as an $R$-module and we may extend $f$ to an $R$-endomorphism of $Rx_j$. However, $\text{End}_R(Rx_j) = R/\text{ann}_R(x_j)$ so actually there is an $s \in R$ such that $f(x) = sx$ for all $x \in N$, and clearly we can then extend $f$ to all of $M$.

This establishes both (2) and the last statement of the proposition.

(2) $\Rightarrow$ (1) Assume that $M$ is not simply embedded. Then since $M$ is Artinian with a single associated maximal ideal $n$, the socle $S$ of $M$ is an $(R/n)$-vector space of dimension greater than one. Thus we may choose $f, g \in \text{End}_R(S)$ with $fg \neq gf$. Since by Proposition 1.8 $\text{End}_R(M)$ is commutative, it will not be possible to extend both $f$ and $g$ to endomorphisms of $M$ and thus $M$ is not quasi-injective.

**Corollary 3.2.** Let $M$ be an Artinian $a.f.g.$ $R$-module, $n$ the associated maximal ideal of $M$, and $E = E_R(R/n)$. Then there is a unique submodule $M_0$ of $E$ satisfying $M \sim^e M_0$. $M_0$ is the unique simply embedded representative of $[M]_e$.

**Proof.** Since $M$ contains a copy of $R/n$ there is a nonzero homomorphism $f: M \to E$. Let $M_0 = f(M)$; then $M_0$ is simply embedded and by Lemma 2.1 we have $M \sim^e M_0$.

Suppose that $M_1$ is a simply embedded representative of $[M]_e$. Since $M_1$ is isomorphic to a quotient of $M$, $n$ is the associated maximal ideal of $M_1$ and then since $M_1$ is simply embedded we may regard $M_1$ as a submodule of $E$. Since $M_1 \sim^e M_0$ there is an epimorphism $g: M_0 \to M_1$. Using the injectivity of $E$ we may extend $g$ to an endomorphism $g_1$ of $E$. However, $M_0$ is quasi-injective by Proposition 3.1 and by [9, Chapt. 3, Proposition 1] a quasi-injective module is a fully invariant submodule of its injective envelope. Thus $M_0 = g_1(M_0) = g(M_0) = M_1$, giving the desired uniqueness.

**Proposition 3.3.** Let $M$ be an Artinian $a.f.g.$ $R$-module and $n$ the associated maximal ideal of $M$. If $M$ is injective over $R$, then $M$ is isomorphic to $E_R(R/n)$. The following conditions are equivalent:

1. $E_R(R/n)$ is Artinian $a.f.g.$
2. The localization $R_n$ is almost DVR.

If these conditions hold, then $E_R(R/n)$ is the unique simply embedded representative of $[M]_e$.

**Proof.** If $M$ is injective, then certainly $M$ is quasi-injective so by Proposition 3.1 we see that $M$ is isomorphic to a submodule of $E_R(R/n)$. Since $R/n$ is simple, $E_R(R/n)$ is indecomposable and hence $M \cong E_R(R/n)$.
It is not difficult to see that $E_R(R/\mathfrak{n})$ is naturally an $R^n$-module and is isomorphic to the $R^n$-injective envelope of $R^n/\mathfrak{n}_n$. Also, $E_R(R/\mathfrak{n})$ will be Artinian a.f.g. over $R$ if and only if it is so over $R_n$. Thus in order to prove the equivalence of (1) and (2) we may assume that $R = R_n$ is quasi-local.

(1) $\Rightarrow$ (2) We are going to use Proposition 1.5. Let $E = E_R(R/\mathfrak{n})$ and suppose that $t, u \in S$ with $\text{ann}_E(t) \subseteq \text{ann}_E(v)$. There is an exact sequence

$$0 \to (t, v)/Rt \to R/Rt \to R/(t, v) \to 0.$$ Applying the exact functor $\text{Hom}_R(\cdot, E)$ gives

$$0 \to \text{Hom}_R(R/(t, v), E) \to \text{Hom}_R(R/Rt, E) \to \text{Hom}_R((t, v)/Rt, E) \to 0.$$ Making some identifications, we have

$$0 \to \text{ann}_E((t, v)) \to \text{ann}_E(Rt) \to \text{ann}_E((t, v)/Rt, E) \to 0,$$

where the first map is the inclusion. However, $\text{ann}_E((t, v)) = \text{ann}_E(t) \cap \text{ann}_E(v) = \text{ann}_E(t)$; thus by exactness $\text{Hom}_R((t, v)/Rt, E) = 0$. Since $R$ is quasi-local, $E$ is the universal injective of $R$ and thus $(t, v)/Rt$ is zero, which implies $v \in Rt$. Therefore, the hypotheses of Proposition 1.5 are satisfied, and $R$ is almost DVR.

(2) $\Rightarrow$ (1) This is (6) $\Rightarrow$ (5) of [1, Theorem 7.1]. An alternative proof follows.

Assuming that $R$ is almost DVR, Proposition 1.4 shows that $Q(R)/R$ is Artinian a.f.g. Let $M_0$ denote the simply embedded representative of $[Q(R)/R]_e$. Regarding $M_0$ as a submodule of $E$, we wish to show $M_0 = E$. Given any $x \in E$ there is a map $f_x : R \to E$ with $f_x(1) = x$. Using the injectivity of $E$ we may extend $f_x$ to a map $g_x : Q(R) \to E$. Since $E$ is torsion, $\text{Ker}(g_x)$ is nonzero; choosing a nonzero $y \in \text{Ker}(g_x)$ we have a natural quotient map $\pi_y : Q(R)/Ry \to Q(R)/\text{Ker}(g_x)$. However, $Q(R)/Ry$ is isomorphic to $Q(R)/R$, hence there is a surjection $h_x : M_0 \to Q(R)/Ry$. Letting $\bar{g}_x$ denote the map induced by $g_x$ we have a composition

$$M_0 \xrightarrow{h_x} Q(R)/Ry \xrightarrow{\pi_y} Q(R)/\text{Ker}(g_x) \xrightarrow{\bar{g}_x} E.$$ Since $h_x$ and $\pi_y$ are surjections, $x$ is contained in $\bar{g}_x \circ \pi_y \circ h_x(M_0)$. Using the injectivity of $E$ we may extend $\bar{g}_x \circ \pi_y \circ h_x$ to $F_x : E \to E$ with $x \in F_x(M_0)$. But by Proposition 3.1, $M_0$ is quasi-injective and thus by [9, Chapt. 3, Proposition 1] is a fully invariant submodule of $E$, so $x \in M_0$. Since $x$ was arbitrary in $E$, we have established $M_0 = E$ and thus $E$ is an Artinian a.f.g. module.
COROLLARY 3.4. Let $M$ be an Artinian $a.f.g.$ $R$-module, $M_0$ the simply embedded representative of $[M]_e$ and $T = \text{End}_R(M_0)$. Then every member of $[M]_e$ is naturally a $T$-module, and there is a one-to-one correspondence between the set of elements $N$ of $[M]_e$ and the set of isomorphism classes $[I]$ of nonzero ideals of $T$, given by

$$N \mapsto \left[\text{Hom}_T(N, M_0)\right]_0 \quad \text{and} \quad [I] \mapsto \text{Hom}_T(I, M_0).$$

Proof. A typical member $N$ of $[M]_e$ is of the form $M_0/K$ for some proper $R$-submodule $K$ of $M_0$. By Proposition 3.1, $R/\text{ann}_R(M_0)$ is dense in $T$ and thus $K$ is a $T$-submodule of $M_0$, so $N \cong M_0/K$ is naturally a $T$-module.

By Corollary 1.9 $T$ is a complete almost DVR; let $n$ denote the maximal ideal of $T$. Since $M_0$ is simply embedded over $R$, it is likewise over $T$. Hence by Proposition 3.3 $M_0$ is isomorphic to $E_\tau(T/n)$. The rest of this proposition follows from the duality theorem quoted in Proposition 2.4—nonzero quotients of $M_0 \cong E_\tau(T/n)$ correspond to nonzero submodules of $T$.

Remark. Let $m$ denote the maximal ideal of $R$ associated to $M_0$. From the fact that $M_0$ is simply embedded over $R$ and over $T$ it follows that $R/m \cong T/n$ as $R$-modules.

4. Finite Width and the A.F.G. Property

The idea of width of a module is due to Brameret in [10, p. 36051. The definitions and elementary results used here will be drawn from Wichman's thesis [11].

DEFINITION. Let $R$ be a commutative ring. An $R$-module $M$ is said to have width $n$ if $n$ is the smallest integer such that for any set of $n + 1$ elements of $M$, at least one of the elements is in the submodule of $M$ generated by the remaining $n$ elements. We denote this by $W(R, M) = n$; if there is no such integer $n$ we will say $W(R, M) = \infty$.

EXAMPLES. For any prime $p \in \mathbb{Z}$, $W(\mathbb{Z}, \mathbb{Z}_{p\mathbb{Z}}) = 1, W(\mathbb{Z}, \mathbb{Z}) = \infty$. If $V$ is an $n$-dimensional vector space over a field $k$, $W(k, V) = n$.

The next result is from Propositions 1.1 and 1.2 of [11]. The proof is direct and will be omitted.

PROPOSITION 4.1. Let $0 \rightarrow K \rightarrow M \rightarrow M/K \rightarrow 0$ an exact sequence of $R$-modules. Then

$$\max\{W(R, K), W(R, M/K)\} \leq W(R, M) \leq W(R, K) + W(R, M/K).$$
PROPOSITION 4.2. Let $R$ be an almost DVR. $Q(R)/R$ is an Artinian a.f.g. $R$-module and for every Artinian a.f.g. $R$-module $M$ we have $M \sim Q(R)/R$. Then

$$W(R, R) = W(R, Q(R)) = W(R, M) < \infty.$$  

Proof. By Proposition 1.4 $Q(R)/R$ is an Artinian a.f.g. $R$-module. If $M$ is an Artinian a.f.g. $R$-module, then from Proposition 1.8 we see that $M$ is a nonzero proper quotient of $Q(R)$ which implies that $M$ is a quotient of $Q(R)/R$ and hence by Lemma 2.1 we have $M \sim Q(R)/R$. It is then clear from Proposition 4.1 and the definition of quotient equivalence that $W(R, M) = W(R, Q(R)/R)$, so it suffices to prove the last statement of this proposition in the case $M \equiv Q(R)/R$. This statement may be deduced from Corollary 1.5, Proposition 1.17 and Theorem 1.12 of [11]; a direct proof follows.

Since $R$ is a submodule of $Q(R)$, $W(R, R) \leq W(R, Q(R))$ by Proposition 4.1. If $L$ is a f.g. $R$-submodule of $Q(R)$, then by choosing a common denominator for a generating set of $L$ we see that $L$ is isomorphic to an ideal of $R$, which implies $W(R, R) \geq W(R, Q(R))$. Thus $W(R, R) = W(R, Q(R))$.

Since $Q(R)/R$ is a quotient of $Q(R)$, $W(R, Q(R)/R) \leq W(R, Q(R))$ by Proposition 4.1. Let $q_1, \ldots, q_m$ a set of elements of $Q(R)$, no one of which is in the span of the others. Then none of the $q_i$'s are zero so we may choose $q \in \bigcap_{i=1}^n Rq_i, q \neq 0$. Let $\tilde{q}_i$ denote the image of $q_i$ in $Q(R)/Rq$ for $i = 1, \ldots, m$. From the choice of $q$ it follows quickly that no one of the $\tilde{q}_i$'s is in the span of the others, and since $Q(R)/Rq \cong Q(R)/R$ this implies $W(R, Q(R)/R) \geq W(R, Q(R))$. Thus $W(R, Q(R)/R) = W(R, Q(R))$.

Recall that $R$ is a local Noetherian domain with maximal ideal $n$ and there is a discrete valuation domain $V, R \subseteq V \subseteq Q(R)$, which is f.g. as an $R$-module. Hence if we pick $g \in V$ with $Vg$ the maximal ideal of $V$, there is an integer $j > 0$ such that $Vg^j \subset n$.

Now suppose that $\{x_1, \ldots, x_b\} \subset R$ is a finite set generating an ideal $L$ of $R$. Then $L$ is f.g. over $R$ so $VL$ is f.g. over $V$ and thus for some integer $h$, $VL = Vg^h$. Then

$$Vg^{j+h} = Vg^jVg^h = Vg^jVL = Vg^jL \subseteq nL \subset L \subset Vg^h.$$  

If $A$ is an $R$-module of finite length, let $l_R(A)$ denote its length. Then $l_R(L/nL) \leq l_R(V/Vg^{j+h}) = l_R(Vg^j/Vg^h) = jl_R(V/Vg)$ which is finite and does not depend on $L$. By the Nakayama lemma we may extract a subset $S$ of $\{x_1, \ldots, x_b\}$ which is a generating set for $L$ and which has no more than $jl_R(V/Vg)$ elements. This implies that $W(R, R) \leq jl_R(V/Vg)$, which finishes the proof.
Proposition 4.3. Let $R$ be a commutative ring and $M$ an a.f.g. $R$-module. Then $W(R, M) < \infty$.

Proof. If $M$ is not Artinian, then by Proposition 1.4 $R$ is almost DVR and by Proposition 4.2, $W(R, M) < \infty$.

If $M$ is an Artinian a.f.g. $R$-module, then by Corollary 3.2 there is a simply embedded Artinian a.f.g. $R$-module $M_0$ such that $M \sim e M_0$. Then by Proposition 4.1, $W(R, M) = W(R, M_0)$ so it suffices to prove that $W(R, M_0) < \infty$.

By Proposition 3.1, $R/\text{ann}_R(M)$ is dense in $\text{End}_R(M)$; thus the $R$-submodules of $M_0$ are the same as the $\text{End}_R(M_0)$-submodules of $M_0$. It follows that $W(R, M_0) = W(\text{End}_R(M_0), M_0)$. By Theorem 1.8, $\text{End}_R(M_0)$ is almost DVR and $M_0$ is an Artinian a.f.g. module over $\text{End}_R(M_0)$. Then by Proposition 4.2, $W(\text{End}_R(M_0), M_0) < \infty$ and thus $W(R, M_0) < \infty$.

Proposition 4.4. Let $R$ be a commutative ring, $M$ an Artinian a.f.g. $R$-module, $M_0$ the simply embedded representative of $[M]_e$. If $W(R, M) = k$, then $k$ is the least integer such that $M_0$ contains an isomorphic copy of every member of $[M]_e$.

Proof. Let $T$ denote $\text{End}_R(M_0)$ and $\mathfrak{n}$ the maximal ideal of $T$. In Corollary 3.4 we showed that every member of $[M]_e$ is naturally a $T$-module. In the proof of Proposition 4.3 we showed $W(R, M_0) = W(T, M_0)$ and from the definition of quotient equivalence and Proposition 4.1 we have $W(R, M_0) = W(R, M)$. Thus it suffices to prove the proposition in the case where $R = T$.

Let $N$ be a member of $[M]_e$. Since $N$ is an Artinian module, the socle of $N$ is a finite-dimensional $T/\mathfrak{n}$-vector space. Since $W(T, N) = W(T, M) = k$, socle $(N)$ has dimension $j \leq k$ over $T/\mathfrak{n}$. Then using the fact that an Artinian module is an essential extension of its socle and that $E_T(T/\mathfrak{n}) \simeq M_0$ (by Proposition 3.3)

$$E_T(N) \simeq (E_T(T/\mathfrak{n}))^l \simeq M_0^l$$

and thus $N$ is isomorphic to a submodule of $M_0^l$.

To see that $k$ is least for this property, note that by the definition of width there must exist a set $x_1, ..., x_k$ of elements of $M$, no one of which is in the span of the others. Let $L$ be the submodule generated by the $x_i$'s. It is clear from the way in which the $x_i$'s were chosen that no proper subset of $\{x_1, ..., x_k\}$ is a generating set for $L$. Then by the Nakayama lemma $L/\mathfrak{n}L$ has dimension $k$ as a vector space over $T/\mathfrak{n}$. From this and $W(T, M) = k$ it follows that the socle of $M/\mathfrak{n}L$ is generated by exactly $k$ elements so $k$ is least.
DEFINITIONS. Let $M$ be an $R$-module. A DCC series for $M$ is a series of submodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_k = M,$$

where for each $i$, $1 \leq i \leq k$, $M_i/M_{i-1}$ is either Artinian a.f.g. or simple. The modules $M_i/M_{i-1}$ are the factors of the DCC series. If $M$ has a DCC series we define the DCC dimension of $M$ (denoted $\text{DCC dim}_R(M)$) as follows:

(a) If every factor of every DCC series of $M$ is simple, then $\text{DCC dim}_R(M) = 0$;

(b) If $M$ has a DCC series in which at least one of the factors is an Artinian a.f.g. module, then $\text{DCC dim}_R(M) = 1$.

It is convenient to make the convention that $\text{DCC dim}_R(M) = -1$ if and only if $M = 0$.

Remarks. (1) If $M$ has a DCC series, then $\text{DCC dim}_R(M)$ agrees with the codeviation of the set of submodules of $M$ (ordered by inclusion) as defined by Lemonnier in [2]. The ideas of [2] can be used to extend the definitions of DCC series and DCC dimension so as to apply to any Artinian module over any ring.

(2) There is a generalized Jordan–Holder theorem (similar to [1, Theorem 5.10]) for DCC series.

The next proposition is due to Matlis.

PROPOSITION 4.5. Let $R$ be a commutative Noetherian ring, $\mathfrak{n}$ a maximal ideal of $R$. The following are equivalent:

(1) $R_{\mathfrak{n}}$ has Krull dimension 1;

(2) $\text{DCC dim}_R(E_{R}(R/\mathfrak{n})) = 1$.

Proof. Since $\text{K.dim}(R_{\mathfrak{n}}) = K_{\mathfrak{n}}$, since $E = E_{R}(R/\mathfrak{n})$ may be regarded in the same role over $R_{\mathfrak{n}}$ and $\hat{R}_{\mathfrak{n}}$, and since the a.f.g. $R$-modules with associated ideal $\mathfrak{n}$ are a.f.g. over $\hat{R}_{\mathfrak{n}}$, it suffices to prove the equivalence in the case where $R = \hat{R}_{\mathfrak{n}}$ is a complete local Noetherian ring.

(1) $\Rightarrow$ (2) Suppose that $I \subset J$ are ideals of $R$ such that $J/I \cong R/P$, where $P$ is a prime ideal of $R$. Since $\text{K.dim}(R) = 1$, either $P = \mathfrak{n}$ or $\dim(P) = 1$. We have an exact sequence

$$0 \to R/P \to R/I \to R/J \to 0$$

and applying $\text{Hom}_R(\cdot, E)$ gives an exact sequence

$$0 \to \text{Hom}_R(R/J, E) \to \text{Hom}_R(R/I, E) \to \text{Hom}_R(R/P, E) \to 0.$$
Thus we have an exact sequence

\[ 0 \to \text{ann}_E(J) \to \text{ann}_E(I) \to \text{Hom}_R(R/P, E) \to 0. \]

Therefore if \( P = n \), \( \text{ann}_E(I)/\text{ann}_E(J) \cong R/n \) is simple, and if \( \dim(P) = 1 \), then by Proposition 2.5 we see that \( \text{ann}_E(I)/\text{ann}_E(J) \) is an Artinian a.f.g. \( R \)-module.

Now by Lemma 1.2 there exists a finite set of ideals of \( R \), \( \{I_j\} \), \( 0 \leq j \leq k \) such that \( 0 = I_0 \subset I_1 \subset \cdots \subset I_k = R \) and such that \( I_j/I_{j-1} \cong R/P_j \), where \( P_j \) is a prime ideal of \( R \). Hence we have a chain of \( R \)-submodules of \( E \)

\[ E = \text{ann}_E(I_0) \supset \text{ann}_E(I_1) \supset \cdots \supset \text{ann}_E(R) = 0. \]

By the preceding paragraph the factors of this series are either simple \( R \)-modules or Artinian a.f.g. \( R \)-modules. If all the factors were simple, then we would have \( I_j/I_{j-1} \cong R/n \) for each \( j \), which would imply \( n^k = 0 \) and then \( n \) is the only prime ideal of \( R \), contradicting \( K \dim(R) = 1 \). Thus at least one of the factors is an Artinian a.f.g. \( R \)-module, hence \( \text{DCC dim}_R(E) = 1 \).

(2) \( \Rightarrow \) (1) By definition we have a chain of \( R \)-submodules of \( E \)

\[ E = E_0 \supset E_1 \supset \cdots \supset E_k = 0 \]

such that the factors of this series are either simple \( R \)-modules or Artinian a.f.g. modules. At least one of the factors is an Artinian a.f.g. \( R \)-module because \( \text{DCC dim}_R(E) = 1 \).

Define \( I_j = \text{ann}_R(E_j) \), giving a chain of ideals of \( R \)

\[ 0 = I_0 \subset I_1 \subset \cdots \subset I_k = R. \]

By the duality presented in Proposition 2.4, we have

\[ \text{Hom}_R(R/I_j, E) \cong \text{ann}_E(I_j) = \text{ann}_E(\text{ann}_R(E_j)) = E_j. \]

Because of the exact sequence

\[ 0 \to \text{Hom}_R(R/I_j, E) \to \text{Hom}_R(R/I_{j-1}, E) \to \text{Hom}_R(I_j/I_{j-1}, E) \to 0 \]

we see that \( \text{Hom}_R(I_j/I_{j-1}, E) \cong E_{j-1}/E_j \). Hence by Proposition 2.4, \( I_j/I_{j-1} \cong \text{Hom}_R(E_{j-1}/E_j, E) \). Since \( E_{j-1}/E_j \) is either a simple \( R \)-module or an Artinian a.f.g. \( R \)-module, it follows from Proposition 2.5 that \( \text{Hom}_R(E_{j-1}/E_j, E) \cong R/P_j \), where either \( P_j = n \) or \( P_j \) is a prime ideal of \( R \) of dimension 1. Because at least one of the factors \( E_{j-1}/E_j \) is not simple, \( \dim(P_j) = 1 \) for at least one of the \( P_j \)'s. Looking at the chain of ideals \( \{I_j\} \) we see that \( \prod_{j=1}^k P_j \) annihilates \( R \), hence \( \prod_{j=1}^k P_j = 0. \)
Let \( P \) be any prime ideal of \( R \). Then \( \prod_{j=1}^{k} P_j \subseteq P \) and hence there exists \( j \) such that \( P_j \subseteq P \). Thus \( \dim(P) \leq 1 \) and \( K\dim(R) = 1 \).

**Remark.** In the situation of Proposition 4.5, Matlis has shown that \( R_n \) is Cohen–Macaulay if and only if every factor in a DCC series for \( E_n(R/n) \) is a.f.g.

**Definition.** Let \( R \) be a ring (not necessarily commutative) and let \( M \) be a left \( R \)-module. Define a series of submodules \( \{L_i(M)\}^\infty_{i=0} \) as follows:

\[
L_0(M) = 0 \quad \text{and for each } i > 0, \quad L_i(M)/L_{i-1}(M) = \socle(M/L_{i-1}(M)).
\]

\( \{L_i(M)\}^\infty_{i=0} \) is the ascending Loewy chain of \( M \).

**Corollary 4.6.** Let \( R \) be a commutative Noetherian ring. The following conditions are equivalent:

1. \( K\dim(R) \leq 1 \).
2. Every Artinian \( R \)-module has finite width.

**Proof:**  
(1) \( \Rightarrow \) (2) We first establish that for each maximal ideal \( \mathfrak{n} \) of \( R \), \( E_n(R/n) \) has finite width. If \( K\dim(R_n) = 0 \), then \( E_n(R/n) \) has finite length, and an easy induction on length using Proposition 4.1 shows that \( W(R, E_n(R/n)) \leq l_R(E_n(R/n)) \) and thus is finite. If \( K\dim(R_n) = 1 \), then by Proposition 5.5 \( E_n(R/n) \) has a DCC series in which each factor is either Artinian a.f.g. or simple. From Proposition 4.3 we see that this means all factors have finite width, and another induction using Proposition 4.1 shows \( W(R, E_n(R/n)) < \infty \).

Let \( M \) be an Artinian \( R \)-module. Then \( M \) is an essential extension of its socle, which is a finite direct sum of simple \( R \)-modules. Then we have an isomorphism \( E_n(M) \simeq \bigoplus_{i=1}^{k} E_n(R/n_i) \) (where the \( n_i \)'s are not necessarily distinct) and thus \( M \) is isomorphic to a submodule of a finite direct sum of \( R \)-modules of finite width, so by Proposition 4.1 again we have \( W(R, M) < \infty \).

(2) \( \Rightarrow \) (1) Suppose that \( K\dim(R) > 1 \). Then we may choose a maximal ideal \( \mathfrak{n} \) such that \( K\dim(R_\mathfrak{n}) = d \) for \( d > 1 \). By Proposition 2.4, \( E = E_n(R/n) \) is an Artinian \( R \)-module; we shall show \( W(R, E) = \infty \).

It is not hard to see that for \( i \geq 1 \), \( L_i(E) = \text{ann}_E(n^i) \). Note that \( \mathfrak{n}[L_i(E)] = L_{i-1}(E) \) so by Nakayama’s lemma the minimal size of a generating set for \( L_i(E) \) is \( \dim_{R/n}(L_i(E)/L_{i-1}(E)) \). The situation is the same over \( R_\mathfrak{n} \) so we may assume \( R \) is complete.

Applying \( \text{Hom}_R(\mathfrak{n}, E) \) to the exact sequence

\[
0 \to n^{i-1}/n^i \to R/n^i \to R/n^{i-1} \to 0
\]
gives

\[
0 \to \text{Hom}_R(R/n^{i-1}, E) \to \text{Hom}_R(R/n^i, E) \to \text{Hom}(n^{i-1}/n^i, E) \to 0.
\]
Making some identifications gives

\[ 0 \to L_{i-1}(E) \xrightarrow{\epsilon} L_i(E) \to \text{Hom}_R(n^{i-1}/n^i, E) \to 0. \]

The functor \( \text{Hom}_R(\cdot, E) \) preserves finite length so we have

\[ 0 \to L_{i-1}(E) \xrightarrow{\epsilon} L_i(E) \to \text{Hom}_R(n^{i-1}/n^i, E) \to 0. \]

However, by the Hilbert–Samuel polynomial theorem [3, Theorem 11.14] \( l_R(n^{i-1}/n^i) \) is (for large \( i \)) a polynomial in \( i \) of degree \( d - 1 \), which is at least one. So although \( L_i(E) \) is f.g. for each \( i \), as \( i \) grows the number of generators needed for \( L_i(E) \) grows without bound and thus \( W(R, E) = \infty \).

5. Examples

**Example 5.1.** For any integer \( h > 0 \), there exists an almost DVR \( R_h \) such that \( W(R_h, M) = h \) for every Artinian a.f.g. \( R \)-module \( M \).

**Proof.** By Proposition 4.2 it suffices to produce an almost DVR \( R_h \) with \( W(R_h, R_h) = h \). For \( h > 0 \), define \( R_h = k[[x^h, x^{h+1}, \ldots, x^{2h-1}]] \), where \( k \) is any field. Then for each \( h \), \( k[[x]] \) is generated by \( \{1, x, \ldots, x^{h-1}\} \) as an \( R_h \)-module, so \( R_h \) is almost DVR. A minimal generating set for the maximal ideal \( n_h \) of \( R_h \) is \( \{x^h, x^{h+1}, \ldots, x^{2h-1}\} \) so \( W(R_h, R_h) \geq h \). To show equality, it suffices to show \( W(R_h, R_h) \leq h \).

Let \( I \) be an ideal of \( R_h \). There is \( z \in I \) such that \( k[[x]] I = k[[x]] z \). Then since \( n_h = x^h k[[x]] \) we have

\[ x^h k[[x]] z = x^h k[[x]] I = n_h I \subset I \subset k[[x]] I = k[[x]] z. \]

We can then relate lengths as follows:

\[ l_{R_h}(I/n_h I) \leq l_{R_h}(k[[x]] z/x^h k[[x]] z) = l_{R_h}(k[[x]]/x^h k[[x]]) = h. \]

Since \( I \) is f.g. and \( R_h \) is local, by the Nakayama lemma we have shown that from any generating set of \( I \) we may extract a generating set having not more than \( h \) elements, and we are done.

**Remark.** Let \( k\{x\} \) denote the field of fractions of \( k[[x]] \). Then by Proposition 1.4, \( M = k\{x\}/k[[x]] \) is an Artinian a.f.g. \( R_h \)-module for each \( h > 0 \). The socle of \( M \) as an \( R_h \)-module is \( h \)-dimensional and is generated by the images of \( x^{-1}, \ldots, x^{-h} \).
I wish to thank Jon Johnson of Elmhurst College for communicating to me the following:

**Example 5.2.** There exists a non-Noetherian local domain \((R, m)\) of Krull dimension 1 which has a faithful Artinian a.f.g. module \(N\). Further, \(N\) is isomorphic to the \(m\)-torsion submodule of \(E_R(R/m)\).

**Proof.** Let \(k\) be a field. We may choose a power series \(y\) in \(k[[x]]\) which is transcendental over \(k[x]\) and has leading term \(x\); then \(F=k(x,y)\) is isomorphic to the field of rational functions in two variables over \(k\).

Let \(t = \sum_{j=0}^{\infty} \left(\frac{1}{10}\right)^{2^j} = 0.110100010\ldots\) We define a valuation \(v_t\) on \(F\) taking values in the real numbers as follows: set \(v_t(x) = 1\), \(v_t(y) = t\) and \(v_t(c) = 0\) for \(c \in k\). Since \(t\) is irrational, there is exactly one way to extend \(v_t\) to a valuation of \(F\). Let \((V_t, m_t)\) denote the valuation ring of \(F\) defined by \(v_t\). Let \((V_1, m_1)\) denote the valuation ring \(F \cap k[[x]]\) and \(v_1\) the corresponding valuation of \(F\); note that \(v_1\) is discrete and that for any \(q \in k[x,y]\), \(v_1(q)\) is the degree of the lowest nonzero term in the power series \(q\).

Define \(m = m_1 \cap m_t\), so \(m = \{z \in F | v_1(z) \geq 1\} \text{ and } v_t(z) > 0\}. Define \(R = k + m\). It is apparent that \(R\) is a ring and \(m\) is an ideal of \(R\). If \(c \in k\), \(c \neq 0\), and \(z \in m\) then \(1/(c + z) = (1/c) - (z/c(c + z))\) and checking valuations shows \(z/c(c + z) \in m\), so \(1/(c + z) \in R\) and thus \(m\) is the set of nonunits of \(R\).

If \(w, z\) are nonzero elements of \(m\), then for some integer \(i\), \(v_t(w^i) > v_t(z)\) and \(v_t(w^i) > v_t(z)\) which implies \(w^i/z \in m\) and \(w^i \in Rz\). It follows that \(m\) is the only nonzero prime ideal of \(R\).

We next define a set of elements of \(m\) useful in describing the powers of \(m\). For \(j \leq 0\) define

\[z_j = (y^{(10^{2j})}/x^{10^{2j+1}}) + x\]

(where \([\ ]\) denotes the greatest integer function). Then \(v_t(z_j) = \min\{10^{2j} - [10^{2j}t], 1\} = 1\), and

\[v_t(z_j) = \min\{10^{2j} - [10^{2j}t], 1\} = \sum_{i=j}^{\infty} \left(\frac{1}{10}\right)^{2^i} < \left(\frac{1}{10}\right)^{2^{j-1}}.\]

We claim that for \(i \geq 1\), \(m^i = \{w \in R | v_t(w) \geq i\} \text{ and } v_t(w) > 0\}. For \(i = 1\), this is the definition of \(m\). For \(i > 1\) and \(w\) such that \(v_t(x) \geq i\), \(v_t(w) > 0\), note that for some \(j\), \(v_t(z_j) < (1/(i-1)) v_t(w)\). Then \(v_t(w/z_j^{i-1}) \geq 1\) and \(v_t(w/z_j^{i-1}) > 0\) so \(w \in m z_j^{i-1} \subset m^i\). Since the opposite containment follows from the definition of \(m\), the claim is established.

We can then see that none of the powers of \(m\) are f.g.; if (say) \(m^i\) were generated by \(u_1, \ldots, u_n\) and \(d = \min\{v_t(u_h) | h = 1, \ldots, n\}\), then by valuation properties we would have \(v_t(w) > d > 0\) for any \(w \in m^i\). However, for some \(j\)
we have \( v_i(z'_i) < d \) and since \( z'_i \in m^i \) this contradicts the assumption of finite generation. Thus \( R \) is not Noetherian.

In order to provide \( R \) with an Artinian a.f.g. module, we show the containments \( k[x]_{(x)} \subset R \subset k[[x]] \). It is clear that \( k[x] \subset R \): if \( w \in k[x] \) and \( w \not\in xk[x] \), then \( v_i(x) = 0 \) so \( w \in m \) and thus \( 1/w \in R \), so the localization \( k[x]_{(x)} \) is contained in \( R \). From the definition of \( R \), \( R \subset V_1 = k[[x]] \cap F \) so \( R \subset k[[x]] \).

Since \( k[x]_{(x)} \) is a DVR, \( N = k(x)/k[x]_{(x)} \) is Artinian a.f.g. over \( k[x]_{(x)} \) by Proposition 1.4. By Proposition 1.8 there is an almost DVR \( S \) such that:

1. \( k[x]_{(x)} \subseteq S \subset k(x) \), and
2. \( \text{End}_{k[x]_{(x)}}(N) \cong \hat{S}_2 \).

Since \( k[x]_{(x)} \) is a discrete valuation ring, (a) implies \( k[x]_{(x)} = S \) and then (b) implies \( \text{End}_{k[x]_{(x)}}(N) \cong k[[x]] \).

Every proper \( R \)-submodule of \( N \) has finite length over \( k[x]_{(x)} \), hence has finite length over \( R \). By Proposition 1.8 \( N \) is an Artinian a.f.g. \( k[[x]] \)-module, hence not f.g. over \( k[[x]] \) and thus not f.g. over \( R \). We have shown that \( N \) is an Artinian a.f.g. \( R \)-module.

Note that \( x'k[[x]] \cap R = \{ w \in R | v_i(w) \geq i \} \). However, if \( w \in R \) and \( v_i(w) > 0 \), then \( w \in m \), so also \( v_i(w) > 0 \), so \( x'k[[x]] \cap R = \{ w \in R | v_i(w) \geq i \} \) and \( v_i(w) > 0 \) = \( m^i \). (Thus the topology of \( R \) induced by the \((x)\)-adic topology of \( k[[x]] \) is the \( m \)-adic topology, and since \( k[x]_{(x)} \subset R \) we see that \( \hat{R} \cong k[[x]] \).) Since \( x'k[[x]] \cap k[x]_{(x)} = x'k[x]_{(x)} \), for each \( i \) we have

\[
k[x]_{(x)}/x'k[x]_{(x)} \subseteq R/m^i \subseteq k[[x]]/x'k[[x]].
\]

Since the composition is an isomorphism it follows that

\[
R/m^i \cong k[[x]]/x'k[[x]]
\]
as \( R \)-modules. Using this isomorphism we may represent \( N \) as a union of the \( R \)-modules \( R/m^i \).

Since \( N \) is simply embedded as an \( k[x]_{(x)} \)-module, \( N \) is simply embedded over \( R \), and thus by Proposition 3.1 \( N \) may be regarded as a quasi-injective submodule of \( E = E_2(R/m^i) \). Let \( T_{m^i}(E) \) denote the \( m \)-torsion submodule of \( E \); then \( N \subset T_{m^i}(E) \) is clear. If \( s \in T_{m^i}(E) \), then for some \( i \), \( m^i s = 0 \) so there is a natural surjection \( f_0 : R/m^i \to Rs \). Using the injectivity of \( E \) and the fact that \( N \) contains a copy of \( R/m^i \), we may extend \( f_0 \) to an endomorphism \( f \) of \( E \) with \( s \in f(N) \). But since \( N \) is quasi-injective this implies \( s \in N \), and thus \( T_{m^i}(E) = N \).
Remark 1. Since $R$ is not almost DVR, Proposition 3.3 shows the inclusion $N \subset E$ is strict.

Remark 2. The technique of Example 2.9 may be adapted to add variables to $R$, thereby producing non-Noetherian domains of larger finite Krull dimension which have a faithful Artinian a.f.g. module.

We may extend the definition of a.f.g. module to left modules over noncommutative rings in an obvious way. Some of the properties of a.f.g. modules over commutative rings hold more generally; however, the most interesting properties are lost. We now show that an Artinian a.f.g. left module over a noncommutative ring may have infinite width, and may have nonzero quotients which are not quotient equivalent to it.

**Notation.** Let $\mathbb{N} = \{1, 2, \ldots\}$. Recall that if $R$ is a (possibly noncommutative) ring and $M$ is a left $R$-module, then $L_i(M)$ denotes the $i$th module in the ascending Loewy chain of $M$ as previously defined.

**Example 5.3.** Given $F: \mathbb{N} \to \mathbb{N}$ there is a noncommutative ring $R$ and an Artinian a.f.g. left $R$-module $M$ such that $l_i(L_i(M)) = \sum_{j=1}^{F(j)} F(j)$ for $i > 0$.

**Proof.** Let $k$ be a field and $W$ a $k$-vector space with basis $B = \{e_{h,j} | h, j \in \mathbb{N} \text{ and } h \leq F(j)\}$. For each basis element $e_{h,j}$ let $P_{h,j}: W \to W$ be the projection map on $e_{h,j}$. For $j > 1$, $h \leq F(j)$, and $1 \leq n \leq F(j-1)$, define $D_{h,j,n} \in \text{End}_k(W)$ by $D_{h,j,n}(e_{h,j}) = e_{n,h-1}$ and $D_{h,j,n}(e_{a,b}) = 0$ if $a \neq h$ or $b \neq j$.

Let $R$ be the subring of $\text{End}_k(W)$ generated by $k$, the $P_{h,j}$'s and the $D_{h,j,n}$'s. Let $M$ be $W$ with the natural $R$-module structure.

There is a natural way to identify $B$ with a set of lattice points in the first quadrant of the Cartesian plane. The $j$th row of the set has length $F(j)$; from the definitions of the maps $D_{h,j,n}$ we see that for $e_{h,j} \in B$, the $R$-submodule $Re_{h,j}$ contains every $e_{a,b} \text{ "below" } e_{h,j}$, that is, every $e_{a,b} \in B$ with $b < j$.

It is not hard to see that the socle of $M$ is $\sum_{h=1}^{F(1)} Re_{h,1}$ and is $F(1)$ dimensional as a $k$-vector space. It follows that for $i > 0$, $L_i(M) = \sum_{h=1}^{F(i)} Re_{h,i}$ and $l_i(L_i(M)) = \sum_{j=1}^{F(i)} F(j)$.

If $N$ is an $R$-submodule of $M$, then since the $P_{h,j}$'s are in $R$, $N$ has a basis (as a $k$-vector space) which is a subset of $B$. Thus if $N$ is a proper submodule of $M$, there is some $e_{h,j} \in B$ such that $e_{h,j} \notin N$. By a preceding remark, this implies $N \subset L_j(M)$, hence $N$ has finite length. Since $L_i(M) \subsetneq L_{i+1}(M)$ for each $i > 0$, $M$ is not of finite length and thus we have shown that $M$ is an Artinian a.f.g. left $R$-module.

**Example 5.4.** There is a noncommutative ring $R$ and an Artinian a.f.g. left $R$-module $M$ with a submodule $J$ such that $M$ is not quotient equivalent to $M/J$. 

Proof. Define $F: \mathbb{N} \to \mathbb{N}$ by $F(1) = 2$ and $F(i) = 1$ for $i > 1$. Let $R, M$ as in Example 5.3. Let $J = L_1(M)$; then $M/J$ is uniserial, which implies that any quotient of $M/J$ is uniserial. Since $l_R(L_1(M)) = 2$, $M$ cannot be isomorphic to a quotient of $M/J$.

Example 5.5. There is a noncommutative ring $R$ and an Artinian a.f.g. left $R$-module $M$ such that $W(R, M) = 0$.

Proof. Take any unbounded function $F: \mathbb{N} \to \mathbb{N}$ and $R, M$ as in Example 5.3.

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References