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NOTE

On Mathieu's Inequality

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The inequalities

$$\frac{1}{x^2 + 1/(2\zeta(3))} < \sum_{n=1}^{\infty} \frac{2n}{(n^2+x^2)^2} < \frac{1}{x^2 + 1/6}$$

hold for all real numbers $x \neq 0$. The constants $1/(2\zeta(3))$ and $1/6$ are best possible. © 1998 Academic Press

1. INTRODUCTION

In 1890, Mathieu [11] conjectured (in connection with work on elasticity of solid bodies) that the inequality

$$S(x) := \sum_{n=1}^{\infty} \frac{2n}{(n^2+x^2)^2} < \frac{1}{x^2} \tag{1.1}$$

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is valid for all real numbers $x \neq 0$. In 1949, Schröder [15] established a weaker form of (1.1) and used his result to solve the problem of the rectangular plate. The first proof of Mathieu's inequality was published in 1952 by Berg [2]. An interesting new proof was given by van der Corput and Heflinger [3] in 1956; they established a general integral inequality and showed that (1.1) can be deduced from their theorem. Moreover, they corrected an error in an approach presented by Emersleben [7]. All these proofs are quite intricate. A very short and elementary proof of inequality (1.1) was published by Makai [10] in 1957, who also established that the following converse of (1.1) holds for all real x:

$$\frac{1}{x^2 + 1/2} < S(x). \tag{1.2}$$

Several interesting extensions, refinements, and results related to (1.1) and (1.2) can be found in [2–10, 14, 16, 17]; see also the monographs [12, pp. 360–362] and [13, pp. 629–634]. Recently, Jakimovski and Russell [9] showed that an extended form of $S(x)$ plays a role in examining Mercerian theorems for Cesàro summability.

From (1.1) and (1.2) we conclude that the double inequality

$$\frac{1}{x^2 + a} < S(x) < \frac{1}{x^2 + b} \tag{1.3}$$

holds for all $x \neq 0$, if $a = 1/2$ and $b = 0$. It is natural to look for an improvement of this result. More precisely, we ask for the smallest constant a and the largest constant b such that (1.3) is valid for all $x \neq 0$. In this note this problem is solved. According to numerical results, Elbert [6] conjectured in 1982 that the left-hand inequality of (1.3) holds with $a = 1/(2\zeta(3)) = 0.415\ldots$. In the next section we establish Elbert's conjecture and we show that the right-hand inequality holds with $b = 1/6 = 0.166\ldots$.

2. THE MAIN RESULT

The principal tools we need to prove our theorem are an inequality for infinite series and an expansion of $S(x)$ in powers of $1/x$.

LEMMA 1. *If $y > 0$ is a real number, then*

$$\sum_{n=1}^{\infty} \frac{n}{(n^2 + y)^3} < \left(\sum_{n=1}^{\infty} \frac{n}{(n^2 + y)^2} \right)^2. \tag{2.1}$$

The validity of this inequality was conjectured by the first two authors [1]. Very recently, Wilkins [18] found a remarkable proof; he used series and integral representations for the trigamma function to establish (2.1).

LEMMA 2. *For all sufficiently large* x, *the asymptotic estimate*

$$S(x) = \frac{1}{x^2} - \frac{1}{6x^4} + O\left(\frac{1}{x^6}\right) \tag{2.2}$$

is valid.

The representation (2.2) was proved by Wang and Wang [17] in 1981; see also [13, p. 630]. A generalization of this result was given by Elbert [6] and Russell [14].

Using the lemmas just stated, we can prove the following theorem, which provides the best possible values of a and b in (1.3).

THEOREM. *For all real numbers* $x \neq 0$, *we have*

$$\frac{1}{x^2 + 1/(2\zeta(3))} < \sum_{n=1}^{\infty} \frac{2n}{(n^2 + x^2)^2} < \frac{1}{x^2 + 1/6}. \tag{2.3}$$

The constants $1/(2\zeta(3))$ *and* $1/6$ *are best possible*.

Proof. A simple calculation reveals that (2.3) is equivalent to

$$1/6 < f(y) < 1/(2\zeta(3)), \qquad y > 0, \tag{2.4}$$

where

$$f(y) = \left(\sum_{n=1}^{\infty} \frac{2n}{(n^2 + y)^2}\right)^{-1} - y.$$

First, we show that f is strictly decreasing on $[0, \infty)$. For all $y > 0$, straightforward differentiation yields

$$\left(\sum_{n=1}^{\infty} \frac{n}{(n^2 + y)^2}\right)^2 f'(y) = \sum_{n=1}^{\infty} \frac{n}{(n^2 + y)^3} - \left(\sum_{n=1}^{\infty} \frac{n}{(n^2 + y)^2}\right)^2,$$

so that Lemma 1 implies

$$f'(y) < 0 \qquad \text{for } y > 0.$$

Since $f(0) = 1/(2\zeta(3))$, the second inequality of (2.4) holds for all $y > 0$.

Now, we prove the first inequality of (2.4). For this it suffices to show that

$$\lim_{y \to \infty} f(y) = 1/6. \tag{2.5}$$

Lemma 2 implies

$$S(\sqrt{y}) = \frac{1}{y} - \frac{1}{6y^2} + O\left(\frac{1}{y^3}\right).$$

Therefore,

$$f(y) = \frac{1}{S(\sqrt{y})} - y = \frac{\frac{1}{6} - y^2 O\left(\frac{1}{y^3}\right)}{1 - \frac{1}{6y} + yO\left(\frac{1}{y^3}\right)},$$

from which (2.5) follows. Since f is monotonic, the bounds given in (2.4) cannot be improved. This completes the proof of the theorem.

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