Available at WWW.MATHEMATICSWEB.ORG POWERED BY SCIENCE DIRECT

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

ELSEVIER

Journal of Computational and Applied Mathematics 159 (2003) 25–34

www.elsevier.com/locate/cam

Uniform pointwise convergence for a singularly perturbed problem using arc-length equidistribution☆

Yanping Chen*

Department of Mathematics, Xiangtan University, Xiangtan 411105, Hunan, China

Received 30 September 2002; received in revised form 20 January 2003

Abstract

A singularly perturbed two-point boundary value problem with an exponential boundary layer is solved numerically by using an adaptive grid method. The mesh is constructed adaptively by equidistributing a monitor function based on the arc-length of the exact solution. The error analysis for this approach was carried out by Qiu et al. (J. Comput. Appl. Math. 101 (1999) 1–25). In this work, their error bound will be improved to the optimal order which is independent of the perturbation parameter. The main ingredient used to obtain the improved result is the theory of the discrete Green's function.

MSC: 65L10; 65L12

Keywords: Singular perturbation; Adaptive mesh; Equidistribution principle; Uniform convergence

1. Introduction

We consider the numerical approximation of the singularly perturbed two-point boundary value problem:

$$Tu(x) := -\varepsilon u''(x) - p(x)u'(x) = 0 \quad \text{for } x \in (0,1), \quad u(0) = 0, \ u(1) = 1, \tag{1.1}$$

where $0 < \varepsilon \ll 1$ is a small positive parameter. It is also assumed that $p \in C^1[0,1]$, and there exist constants β and $\bar{\beta}$ such that

$$0 < \beta \leqslant p(x) \leqslant \bar{\beta} \quad \text{and} \quad |p'(x)| \leqslant \bar{\beta}, \quad \forall x \in [0,1]. \tag{1.2}$$

☆ This work is supported by National Science Foundation of China, the Backbone Teachers Foundation of China State Education Commission and the Special Funds for Major State Basic Research Projects.

* Tel.: +86-732-8293614; fax: +86-732-8293000.

E-mail address: ypchen@xtu.edu.cn (Y. Chen).

0377-0427/03/$ - see front matter
doi:10.1016/S0377-0427(03)00563-6

For $\varepsilon \ll 1$ the solution has a boundary layer of thickness $\mathcal{O}(\varepsilon)$ near the boundary $x = 0$ and it is well known that a central or upwind difference scheme on an even mesh will not give a satisfactory numerical solution in this case. To obtain a reliable numerical solution for (1.1) when $\varepsilon \ll 1$, it is advantageous to use a mesh that concentrates nodes in the boundary layer. One approach is the use of highly nonuniform layer-adapted meshes, see, e.g. [9,13–15]. Another approach is the use of adaptive mesh generated by equidistributing a monitor function over the domain of the problem. There has been a great deal of work done recently on the use of the adaptive methods. Of these two approaches, convergence results for the first approach is more satisfactory, see, e.g. [8,13,17,9]. However, the analysis for the second approach seems very difficult, in particular for problems in multi-dimensions and/or with interior layers.

There have been some theoretical results for one-dimensional adaptive mesh approach to the solution of the singularly perturbed problem (1.1), see, e.g. [2,4,5,7,10–12]. It seems that the best convergence result so far is the one obtained by Kopteva and Stynes [5] who investigated a quasi-linear convection-diffusion equation in conservative form. The mesh is generated by using the arc-length equidistribution principle. A practical algorithm, based on an iterative procedure to generate the mesh and to compute the arc-length, is proposed. It is noted that the equation of conservative form can be easily reduce to a first-order equation. By using this fact and the Green's function, a first-order error bound which is independent of the small perturbation parameter is obtained. For a fully discretized scheme with the moving mesh strategy, their result seems the first among such efforts. It should be pointed out that similar iterative idea was successfully implemented in some multi-dimensional moving mesh algorithms (see, e.g. [6]).

For the convection-diffusion problem of form (1.1), there are also several results on adaptive mesh arising from the equidistribution of a monitor function, see, e.g. [2,10–12]. Qiu et al. [12] studied the rate of convergence based on a semi-discretization approach which implies that the exact solution is used in the monitor function. This simplifies the analysis, and also can give a clear structure of grid distribution in the solution interval. They proved that for any given $\gamma \in (0,1)$ there exists a positive constant $C(\gamma)$ independent of ε and N, such that

$$\max_{0 \leqslant i \leqslant N} |u(x_i) - u_i^N| \leqslant C(\gamma) N^{-\gamma} \tag{1.3}$$

provided that N is sufficiently large, where N is the total number of grid points, and $\{u_i^N\}_{i=0}^N$ is the numerical approximation. The main purpose of this paper is to improve the result of (1.3) to the uniform order of convergence, namely, by replacing the right-hand side of (1.3) with $\mathcal{O}(N^{-1} \ln N)$. The main ingredients used to obtain the improved result are the discrete Green's function [1] and the theory of M-matrices [16]. The idea of using discrete Green's function was also employed in [4,5,7,8].

We will close this section by introducing the numerical method and the main result. Let

$$\Omega_N = \{x_j \mid 0 = x_0 < x_1 < \cdots < x_N = 1\}$$

be an arbitrary non-uniform mesh on $[0,1]$. On Ω_N we discretize (1.1) as follows:

$$T^N u_i^N := -\varepsilon D D^- u_i^N - p_i D^+ u_i^N = 0, \quad \text{for } 1 \leqslant i \leqslant N-1, \tag{1.4}$$

$$u_0^N = 0, \quad u_N^N = 1, \tag{1.5}$$

where the operators used are given by

$$D^- v_i = \frac{v_i - v_{i-1}}{h_i}, \quad D^+ v_i = \frac{v_{i+1} - v_i}{h_{i+1}}, \quad D v_i = \frac{v_{i+1} - v_i}{\hbar_i},$$
$$h_i = x_i - x_{i-1}, \quad \hbar_i = \frac{h_i + h_{i+1}}{2}.$$

The numerical mesh is constructed by equidistributing the arc-length function

$$M(x) = \sqrt{1 + (u'(x))^2} \tag{1.6}$$

over the domain $[0, 1]$. This gives rise to a mapping $x = x(\xi)$:

$$\frac{\mathrm{d}x}{\mathrm{d}\xi} = \frac{L}{\sqrt{1 + (u'(x))^2}}, \quad \xi \in (0, 1), \tag{1.7}$$

where L is the arc length of u over $(0, 1)$. More precisely

$$x_i = \int_0^{\xi_i} \frac{L}{\sqrt{1 + (u'(x))^2}} \, \mathrm{d}\xi, \quad \xi_i = \frac{i}{N}, \quad 0 \leqslant i \leqslant N.$$

The mesh size is given by

$$h_i = x_i - x_{i-1} = \int_{\xi_{i-1}}^{\xi_i} \frac{L}{\sqrt{1 + (u'(x))^2}} \, \mathrm{d}\xi, \quad 0 \leqslant i \leqslant N. \tag{1.8}$$

The problem (1.1) will be solved numerically by (1.4) and (1.5) and (1.8). This approach is called semi-discretization [12] since the mesh equation (1.8) involves the exact solution u. The fully discretized scheme will be investigated in a separate work.

Throughout the paper, C denotes a generic positive constant that is independent of ε, of the mesh, and can take different values in different places. The main result of this work is given below.

Theorem 1. *Let $u(x)$ be the exact solution to (1.1) and let $\{u_i^N\}_{i=0}^N$ be obtained by finite difference scheme (1.4) and (1.5) on the grid defined by (1.8). Then there exists a positive constant C independent of ε and N such that*

$$|u(x_i) - u_i^N| \leqslant CN^{-1} \ln N, \quad 0 \leqslant i \leqslant N. \tag{1.9}$$

2. Mesh structure

In this section, we will follow [12] to divide the domain [0,1] into three regions: a boundary layer region, a transition region and a regular solution region. In the regular solution region, the solution is smooth and its derivatives can be bounded by a constant which is independent of both ε and N; while within the boundary layer the exact solution is very steep and the derivatives are very large. Since we are mainly interested in very small perturbation parameter, we may assume that

$$\varepsilon \ln N \leqslant N^{-1}. \tag{2.1}$$

In the solution interval $[0, 1]$, we choose a point

$$x^* = \frac{2\varepsilon}{\beta} |\ln \varepsilon| \tag{2.2}$$

and let $\bar{x}$ denote a mesh transition parameter defined by

$$\bar{x} = \frac{\varepsilon}{p(0)} \ln N. \tag{2.3}$$

Let J be a positive integer satisfying

$$x_J \geqslant x^* \quad \text{and} \quad x_{J-1} < x^* \tag{2.4}$$

and let K be a positive integer satisfying

$$1 - \frac{LK}{DN} \geqslant \frac{2}{N} \quad \text{and} \quad 1 - \frac{L(K+1)}{DN} < \frac{2}{N}, \tag{2.5}$$

where $D = \varepsilon u'(0)/p(0) = \mathcal{O}(1)$, L is the arc-length of u. It can be verified that, see [12],

$$x_K < \bar{x} \quad \text{and} \quad x_K \geqslant \bar{x} - C_0\varepsilon, \tag{2.6}$$

where $C_0 = \ln((L/D) + (1/Dp(0)) + 3)/p(0)$. As shown in [12], the interval $(0,1)$ can be divided into three subregions: the boundary layer region $(0, x_K)$; the transition region (x_K, x_J); and the regular solution region $(x_J, 1)$.

Lemma 2.1 (Qiu et al. [12]). *If the mesh Ω_N is generated by (1.8), then*

- *There are $\mathcal{O}(N)$ grid points inside the boundary layer $(0, x_K)$. Moreover,*

$$h_i \leqslant C_1\varepsilon \quad \text{for } i \leqslant K; \tag{2.7}$$

- *There are $\mathcal{O}(1)$ grid points inside the transition region (x_K, x_J), where $\mathcal{O}(1)$ indicates a number independent of ε and N;*
- *There are $\mathcal{O}(N)$ grid points inside the regular solution region $(x_J, 1)$. Moreover, for $j \geqslant J+1$, we have $h_j \leqslant CN^{-1}$.*

3. Truncation error analysis

The exact solution of the problem (1.1) is

$$u(x) = \frac{G(x)}{G(1)}, \quad G(x) = \int_0^x \exp\left[-\frac{1}{\varepsilon}\int_0^t p(s)\,\mathrm{d}s\right] \mathrm{d}t. \tag{3.1}$$

As proved in [12] that

$$\frac{\beta}{\varepsilon}\mathrm{e}^{-\bar{\beta}x/\varepsilon} < u'(x) < \frac{2\bar{\beta}}{\varepsilon}\mathrm{e}^{-\beta x/\varepsilon} \quad \text{for } x \in [0,1], \tag{3.2}$$

provided $\frac{1}{2} \leqslant 1 - \mathrm{e}^{-\beta/\varepsilon}$, where β and $\bar{\beta}$ are defined in (1.2), and the above solution u can be splitted into two parts

$$u(x) = A(x) + Z(x), \tag{3.3}$$

where

$$A(x) = D[1 - \mathrm{e}^{-p(0)x/\varepsilon}], \quad D = \frac{\varepsilon u'(0)}{p(0)} = \mathcal{O}(1),$$

$$|Z(x)| \leqslant C\varepsilon \quad \text{for } x \in [0,1]. \tag{3.4}$$

The local truncation error of (1.4) at node x_i is defined by

$$\tau_i = T^N u_i - Tu(x_i), \quad i = 1,\ldots,N-1,$$

where u is the exact solution of (1.1) and $u_i = u(x_i)$. It is shown in [3,12] that

$$|\tau_i| \leqslant c \int_{x_{i-1}}^{x_{i+1}} |u''(s)|\,\mathrm{d}s, \quad i = 1,\ldots,N-1, \tag{3.5}$$

where c is a constant that is dependent of ε and N. The following results on the truncation error are useful in our error analysis.

Lemma 3.1. *For* $1 \leqslant i \leqslant N-1$,

$$\hbar_i|\tau_i| \leqslant C(\varepsilon + \mathrm{e}^{-p(0)x_{i-1}/\varepsilon}), \tag{3.6}$$

$$\hbar_i|\tau_i| \leqslant C\varepsilon^{-2}\hbar_i^2 \mathrm{e}^{-p(0)x_{i-1}/\varepsilon}. \tag{3.7}$$

Proof. From (3.2), it is clear that

$$u'(x) < 2\bar{\beta} \quad \text{for } x \geqslant x^*/2. \tag{3.8}$$

Furthermore, it is shown in [12] that

$$\frac{\beta}{2\varepsilon}\mathrm{e}^{-p(0)x/\varepsilon} < u'(x) < \frac{3\bar{\beta}}{\varepsilon}\mathrm{e}^{-p(0)x/\varepsilon} \quad \text{for } x \leqslant x^*. \tag{3.9}$$

It follows from (1.1) that

$$\tau_i = T^N u_i - Tu(x_i) = T^N u_i.$$

Using (1.4), (3.3), (3.4) and (3.8), (3.9) gives

$$\begin{aligned}
\hbar_i|\tau_i| &= \hbar_i|T^N u_i| \\
&= \varepsilon|D^+u_i - D^-u_i| + p_i\hbar_i h_{i+1}^{-1}|u(x_{i+1}) - u(x_i)| \\
&\leqslant 2\varepsilon \max_{x_{i-1}\leqslant x\leqslant x_{i+1}} |u'(x)| + C|A(x_{i+1}) - A(x_i)| + C|Z(x_{i+1})| + C|Z(x_i)| \\
&\leqslant C(\varepsilon + \mathrm{e}^{-p(0)x_{i-1}/\varepsilon}),
\end{aligned}$$

where we have used the fact $\hbar_i h_{i+1}^{-1} \leqslant 1$. Thus, (3.6) was proved. It can be shown, by using Eq. (1.1), the exact solution (3.1) and (3.8), (3.9), that

$$|u''(x)| = \varepsilon^{-1}p(x)u'(x) \leqslant 3\bar{\beta}^2\varepsilon^{-2}\mathrm{e}^{-p(0)x_{i-1}/\varepsilon} \quad \text{for } x \in (x_{i-1}, x_{i+1}).$$

This result, together with (3.5), leads to (3.7). □

We now define the error mesh function

$$e_i = u(x_i) - u_i^N.$$

It can be verified that the error mesh function satisfies

$$T^N e_i = T^N u_i - T^N u_i^N = T^N u_i - Tu(x_i) = \tau_i, \quad 1 \leqslant i \leqslant N-1 \tag{3.10}$$

with $e_0 = e_N = 0$. It is clear that T^N is an M-matrix.

Lemma 3.2. *For* $1 \leqslant i \leqslant N-1$,

$$|e_i| \leqslant \beta_0^{-1} \sum_{j=1}^{N-1} \hbar_j |\tau_j|, \tag{3.11}$$

where β_0 *is a constant.*

Proof. The proof can be obtained by using the discrete Green's function. For $j = 1, \dots, N-1$, the discrete Green's function $G(x_i, x_j)$ associated with the grid point x_j is defined by

$$T^N G(x_i, x_j) = \frac{\delta_{ij}}{\hbar_i} \quad \text{for } i = 1, \dots, N-1, \quad G(0, x_j) = G(1, x_j) = 0, \tag{3.12}$$

where the Kronecker function δ_{ij} is 1 if $i = j$ and 0 otherwise. For simplicity, denote $G_j^i = G(x_i, x_j)$. Then for each $1 \leqslant i \leqslant N-1$ we have

$$e_i = \sum_{j=1}^{N-1} \hbar_j G_j^i \tau_j \tag{3.13}$$

which can be verified directly by using (3.10) and (3.12). Let

$$B_j^i = \begin{cases} \beta_0^{-1}, & 0 \leqslant i \leqslant j, \\ \beta_0^{-1} \prod_{k=j+1}^{i} \left(1 + \frac{\beta_0 h_k}{\varepsilon}\right)^{-1}, & j+1 \leqslant i \leqslant N, \end{cases} \tag{3.14}$$

where $\beta_0 = \beta/2$, with β being defined by (1.2). Clearly G_j^i are nonnegative for all $0 \leqslant i, j \leqslant N$. We now show that B_j^i is an upper bound for G_j^i by using direct calculation:

- for $i > j$:

$$D^- B_j^i = h_i^{-1}(B_j^i - B_j^{i-1}) = h_i^{-1} B_j^i \left[1 - \left(1 + \frac{\beta_0 h_i}{\varepsilon}\right)\right] = -\frac{\beta_0 B_j^i}{\varepsilon},$$

$$D^+ B_j^i = h_{i+1}^{-1}(B_j^{i+1} - B_j^i) = h_{i+1}^{-1} B_j^i \left[\left(1 + \frac{\beta_0 h_{i+1}}{\varepsilon}\right)^{-1} - 1\right] = -\frac{\beta_0 B_j^i}{\varepsilon + \beta_0 h_{i+1}},$$

$$T^N B_j^i = -\varepsilon \frac{D^+ B_j^i - D^- B_j^i}{\hbar_i} - p_i D_+ B_j^i = \frac{\beta_0 B_j^i}{\hbar_i} \left(\frac{\varepsilon + p_i \hbar_i}{\varepsilon + \beta_0 h_{i+1}} - 1\right);$$

- for $i = j$:

$$D^- B_j^i = h_i^{-1}(B_j^i - B_j^{i-1}) = h_i^{-1}(\beta_0^{-1} - \beta_0^{-1}) = 0,$$

$$D^+ B_j^i = h_{i+1}^{-1}(B_j^{i+1} - B_j^i) = h_{i+1}^{-1} \beta_0^{-1} \left[\left(1 + \frac{\beta_0 h_{i+1}}{\varepsilon}\right)^{-1} - 1\right] = -\frac{1}{\varepsilon + \beta_0 h_{i+1}},$$

$$T^N B_j^i = -\varepsilon \frac{D^+ B_j^i - D^- B_j^i}{\hbar_i} - p_i D^+ B_j^i = \frac{1}{\hbar_i} \frac{\varepsilon + p_i \hbar_i}{\varepsilon + \beta_0 h_{i+1}};$$

- for $i < j$:

$$D^- B_j^i = 0, \quad D^+ B_j^i = 0, \quad T^N B_j^i = 0.$$

The above calculations lead to

$$T^N B_j^i = \begin{cases} 0, & i = 0, \dots, j-1, \\ \dfrac{1}{\hbar_i} \dfrac{\varepsilon + p_i \hbar_i}{\varepsilon + \beta_0 h_{i+1}}, & i = j, \\ \dfrac{\beta_0 B_j^i}{\hbar_i} \left(\dfrac{\varepsilon + p_i \hbar_i}{\varepsilon + \beta_0 h_{i+1}} - 1 \right), & i = j+1, \dots, N. \end{cases} \tag{3.15}$$

Since $p_i = p(x_i) \geqslant \beta$ and $\hbar_i \geqslant h_{i+1}/2$, for $i = 0, \dots, N$, we have $p_i \hbar_i \geqslant \beta_0 h_{i+1}$. These observations, together with (3.15), yield

$$T^N B_j^i \geqslant \frac{\delta_{ij}}{\hbar_i} = T^N G_j^i. \tag{3.16}$$

Also note that $B_j^0 \geqslant G_j^0$ and $B_j^N \geqslant G_j^N$. We can apply the M-Matrix theory to conclude that

$$0 \leqslant G_j^i \leqslant B_j^i \leqslant \beta_0^{-1} \quad \text{for } i, j = 0, \dots, N. \tag{3.17}$$

It follows from (3.12) to (3.17) that

$$|e_i| \leqslant \sum_{j=1}^{N-1} \hbar_j G_j^i |\tau_j| \leqslant \sum_{j=1}^{N-1} \hbar_j B_j^i |\tau_j| \leqslant \beta_0^{-1} \sum_{j=1}^{N-1} \hbar_j |\tau_j| \tag{3.18}$$

for $0 \leqslant i \leqslant N$. □

4. The proof of Theorem 1

We will apply Lemma 3.1 to bound the term $\hbar_j |\tau_j|$ in (3.11). It follows from (3.18) and Lemma 3.2 that, for $i = 0, \dots, N$,

$$\begin{aligned} |u(x_i) - u_i^N| &\leqslant \sum_{j=1}^{N-1} \hbar_j B_j^i |\tau_j| \\ &\leqslant C \sum_{j=1}^{K-1} \varepsilon^{-2} \hbar_j^2 B_j^i \mathrm{e}^{-p(0) x_{j-1}/\varepsilon} + C \sum_{j=K}^{J} (\varepsilon + \mathrm{e}^{-p(0) x_{j-1}/\varepsilon}) + C \sum_{j=J+1}^{N} \varepsilon^{-2} \hbar_j^2 \mathrm{e}^{-p(0) x_{j-1}/\varepsilon} \\ &=: I_1 + I_2 + I_3, \end{aligned} \tag{4.1}$$

where J and K are defined by (2.4) and (2.5), respectively. We now derive an auxiliary bound for B_j^i for $0 < j < i \leqslant K$. Observe that

$$\ln\left[\prod_{k=j+1}^{i}\left(1+\frac{\beta_0 h_k}{\varepsilon}\right)\right] \geqslant \sum_{k=j+1}^{i}\left[\frac{\beta_0 h_k}{\varepsilon}-\frac{1}{2}\left(\frac{\beta_0 h_k}{\varepsilon}\right)^2\right]$$

$$\geqslant \frac{\beta_0}{\varepsilon}(x_i - x_j) - \frac{1}{2}\sum_{k=1}^{K}\left(\frac{\beta_0 h_k}{\varepsilon}\right)^2 \tag{4.2}$$

for $0 < j < i \leqslant K$. Using (1.8) and (3.9), we have

$$h_k = \int_{\xi_{k-1}}^{\xi_k} \frac{L}{\sqrt{1+(u'(x))^2}}\,\mathrm{d}\xi \leqslant \int_{\xi_{k-1}}^{\xi_k} \frac{L}{u'(x)}\,\mathrm{d}\xi$$

$$\leqslant \frac{L}{N}\frac{1}{u'(x_k)} \leqslant \frac{2L}{\beta}\frac{\varepsilon}{N}\mathrm{e}^{p(0)x_k/\varepsilon}, \quad k = 1,\ldots,K. \tag{4.3}$$

Hence

$$\sum_{k=1}^{K}\left(\frac{\beta_0 h_k}{\varepsilon}\right)^2 \leqslant C\varepsilon^{-1}N^{-1}\sum_{k=1}^{K}\mathrm{e}^{p(0)x_k/\varepsilon}h_k \leqslant C\varepsilon^{-1}N^{-1}\int_0^{\bar{x}}\mathrm{e}^{p(0)x/\varepsilon}\,\mathrm{d}x$$

$$\leqslant CN^{-1}\mathrm{e}^{p(0)x/\varepsilon}|_0^{\bar{x}} = CN^{-1}[\mathrm{e}^{p(0)\bar{x}/\varepsilon}-1] \leqslant C.$$

Then, by the definition (3.14) and (4.2), we obtain

$$B_j^i \leqslant C\mathrm{e}^{-\beta_0(x_i-x_j)/\varepsilon} \quad \text{for } 0 < j < i \leqslant K. \tag{4.4}$$

Below we will prove that

$$|u(x_i) - u_i^N| \leqslant CN^{-1}\ln N, \quad 0 \leqslant i < K, \tag{4.5}$$

$$|u(x_i) - u_i^N| \leqslant CN^{-1}, \quad K \leqslant i \leqslant N. \tag{4.6}$$

4.1. Error in the regular solution region and the transition region

We will prove (4.6) first. It follows from Lemma 2.1 that $J - K = \mathcal{O}(1)$. This result, the second inequality in (2.6), (2.7) and (2.1) yield

$$I_2 = C\sum_{j=K}^{J}(\varepsilon + \mathrm{e}^{-p(0)x_{j-1}/\varepsilon})$$

$$\leqslant C(\varepsilon + \mathrm{e}^{-p(0)x_{K-1}/\varepsilon})$$

$$\leqslant C(\varepsilon + \mathrm{e}^{-p(0)(\bar{x}-(C_0+C_1)\varepsilon)/\varepsilon})$$

$$= C(\varepsilon + \mathrm{e}^{p(0)(C_0+C_1)}\mathrm{e}^{-p(0)\bar{x}/\varepsilon}) \leqslant CN^{-1}. \tag{4.7}$$

It follows from Lemma 2.1 that $h_j \leqslant CN^{-1}$ for $j \geqslant J+1$. Using (2.4) gives

$$e^{-p(0)x_{j-1}/\varepsilon} \leqslant e^{-p(0)x^*/\varepsilon} \leqslant e^{-\beta x^*/\varepsilon} = \varepsilon^2 \quad \text{for } j \geqslant J+1.$$

Using these facts we obtain

$$I_3 = C \sum_{j=J+1}^{N} \varepsilon^{-2} \hbar_j^2 e^{-p(0)x_{j-1}/\varepsilon}$$

$$\leqslant C \sum_{j=J+1}^{N} \varepsilon^{-2} N^{-2} e^{-p(0)x^*/\varepsilon} \leqslant CN^{-1}. \tag{4.8}$$

It follows from the definition (3.14) and (4.4) that, for $K \leqslant i \leqslant N$,

$$B_j^i \leqslant B_j^K \leqslant C e^{-\beta_0(x_K - x_j)/\varepsilon} \quad \text{for } i \geqslant K,\ j \leqslant K-1.$$

Moreover, it follows from (2.7) that

$$e^{p(0)h_j/\varepsilon} \leqslant C \quad \text{for } j \leqslant K.$$

Using the above two results and (4.3) gives that, for $i \geqslant K$,

$$I_1 \leqslant C \sum_{j=1}^{K-1} \varepsilon^{-2} \hbar_j^2 B_j^K e^{-p(0)x_{j-1}/\varepsilon}$$

$$\leqslant C\varepsilon^{-1} N^{-1} \sum_{j=1}^{K-1} e^{-\beta_0(x_K - x_j)/\varepsilon} \hbar_j$$

$$\leqslant C\varepsilon^{-1} N^{-1} e^{-\beta_0 x_K/\varepsilon} \int_0^{x_K} e^{\beta_0 x/\varepsilon}\, dx$$

$$= CN^{-1} e^{-\beta_0 x_K/\varepsilon} [e^{\beta_0 x_K/\varepsilon} - 1] \leqslant CN^{-1}. \tag{4.9}$$

Combining (4.1) and (4.7)–(4.9) gives the (4.6).

4.2. Error in boundary layer region

Now we need to bound the error for $0 \leqslant i < K$. It follows from (4.1) that only I_1 needs to be re-considered for $0 \leqslant i < K$; the estimates for I_2 and I_3 obtained in the last subsection remains valid for this range of index i. For $0 \leqslant i < K$,

$$I_1 \leqslant C \sum_{j=1}^{K-1} \varepsilon^{-2} \hbar_j^2 e^{-p(0)x_{j-1}/\varepsilon} \leqslant C\varepsilon^{-1} N^{-1} \sum_{j=1}^{K-1} \hbar_j$$

$$\leqslant C\varepsilon^{-1} N^{-1} x_K \leqslant C\varepsilon^{-1} N^{-1} \frac{\varepsilon}{p(0)} \ln N \leqslant CN^{-1} \ln N. \tag{4.10}$$

This result, together with (4.7) and (4.8), gives (4.5). This also completes the proof of Theorem 1.

Acknowledgements

The author is grateful to Professor Tao Tang for numerous discussions and to the referee for many helpful suggestions.

References

[1] V.B. Andreev, I.A. Savin, On the convergence, uniform with respect to the small parameter, of A.A. Samarskii's monotone scheme and its modifications, Comput. Math. Math. Phys. 35 (1995) 581–591.
[2] G.M. Beckett, J.A. Mackenzie, Convergence analysis of finite difference approximations on equidistributed grids to a singularly perturbed boundary value problem, Appl. Numer. Math. 35 (2000) 87–109.
[3] R.B. Kellogg, A. Tsan, Analysis of some difference approximations for a singular perturbation problem without turning points, Math. Comput. 32 (1978) 1025–1039.
[4] N. Kopteva, Maximum norm a posteriori error estimates for a one-dimensional convection-diffusion problem, SIAM J. Numer. Anal. 39 (2001) 423–441.
[5] N. Kopteva, M. Stynes, A robust adaptive method for quasi-linear one-dimensional convection-diffusion problem, SIAM J. Numer. Anal. 39 (2001) 1446–1467.
[6] R. Li, T. Tang, P. Zhang, Moving mesh methods in multiple dimensions based on harmonic maps, J. Comput. Phys. 170 (2001) 562–588.
[7] T. Linss, Uniform pointwise convergence of finite difference schemes using grid equidistribution, Computing 66 (2001) 27–39.
[8] T. Linss, H.-G. Roos, R. Vulanovic, Uniform pointwise convergence on Shishkin-type meshes for quasi-linear convection-diffusion problems, SIAM J. Numer. Anal. 38 (2000) 897–912.
[9] W.B. Liu, T. Tang, Error analysis for a Galerkin-spectral method with coordinate transformation for solving singularly perturbed problems, Appl. Numer. Math. 38 (2001) 315–345.
[10] J.A. Mackenzie, Uniform convergence analysis of an upwind finite-difference approximation of a convection-diffusion boundary value problem on an adaptive grid, IMA J. Numer. Anal. 19 (1999) 233–249.
[11] Y. Qiu, D.M. Sloan, Analysis of difference approximations to a singularly perturbed two-point boundary value problem on an adaptively generated grid, J. Comput. Appl. Math. 101 (1999) 1–25.
[12] Y. Qiu, D.M. Sloan, T. Tang, Numerical solution of a singularly perturbed two-point boundary value problem using equidistribution: analysis of convergence, J. Comput. Appl. Math. 116 (2000) 121–143.
[13] H.-G. Roos, M. Stynes, L. Tobiska, Numerical Methods for Singularly Perturbed Differential Equations, Springer, Berlin, 1996.
[14] G.I. Shishkin, Grid approximation of singularly perturbed elliptic and parabolic equations, Second Doctoral Thesis, Keldysh Institute of Applied Mathematics, USSR Academy of Sciences, Moscow, 1990.
[15] T. Tang, M.R. Trummer, Boundary layer resolving pseudospectral methods for singular perturbation problems, SIAM J. Sci. Comput. 17 (1996) 430–438.
[16] R.S. Varga, Matrix Iterative Analysis, Prentice-Hall, Englewood Cliffs, NJ, 1962.
[17] R. Vulanovic, A priori meshes for singularly perturbed quasilinear two-point boundary value problems, IMA J. Numer. Anal. 21 (2001) 349–366.