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On Finite Groups Satisfying the Permutizer Condition

James C. Beidleman

Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506

Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, Illinois 61801

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A group G satisfies the *permutizer condition* **P** if each proper subgroup H of G permutes with some cyclic subgroup not contained in H. The structure of finite groups with **P** is studied, the main result being that such groups are soluble with chief factors of order 4 or a prime. The classification of finite simple groups is used, as is detailed information about maximal factorizations of almost simple groups. © 1997 Academic Press

1. INTRODUCTION

A very well known property of groups is the *normalizer condition*, according to which every proper subgroup of a group is smaller than its normalizer. This is a relatively strong condition: for finite groups it is equivalent to nilpotence, while for groups in general it at least implies local nilpotence by a theorem of Plotkin (see [8, p. 364]).

A natural way to weaken the normalizer condition is to replace the normalizer of a subgroup H of a group G by its *permutizer*

$$P_G(H)$$

(see [9, p. 26]; the term is used in a different sense in [5, p. 26]). This is defined to be the subgroup generated by all cyclic subgroups of G that permute with H. Thus $H \leq P_G(H)$ and $H \neq P_G(H)$ if and only if $H\langle g\rangle$

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$= \langle g \rangle H$ for some $g \in G \setminus H$. A group G such that $H \neq P_G(H)$ for every proper subgroup H is said to satisfy the *permutizer condition* **P**, or to be a **P**-*group*.

Obviously a group satisfying the normalizer condition is a **P**-group. As examples of **P**-groups that do not satisfy the normalizer condition we cite nonnilpotent supersoluble groups (see (3.1) below) and the symmetric group S_4.

The main object of the present article is to study the structure of finite **P**-groups and to address the question: How close do such groups come to being supersoluble? It turns out that the answer is quite close. In the sequel *it is understood that all groups are finite*.

Our first task is to prove that **P**-groups are soluble, which is accomplished in Sect. 2. As one would expect, the proof depends on the classification of finite simple groups. But in addition it is vital to have detailed information about the factorizations of almost simple groups, (i.e., groups that lie between a simple group and its automorphism group). Fortunately the recent monograph [6] of Liebeck, Praeger, and Saxl provides the information needed.

Our main result on the structure of **P**-groups shows how near these groups are to being supersoluble.

THEOREM A. *Let G be a finite group satisfying the permutizer condition* **P**. *Then G is soluble and each chief factor of G has order* 4 *or a prime*. *In addition, if F is a chief factor of order* 4, *then G induces the full group of automorphisms in F, i.e., $G/C_G(F) \simeq S_3$.*

It seems difficult to give a really satisfactory characterization of **P**-groups, despite the wealth of information about their structure provided by Theorem A. In particular we do not know whether the converse of Theorem A is true.

However, there is a characterization of **P**-groups which involves a useful subgroup introduced in 1967 by O. H. Kegel [3]. If $\lambda\colon G \to S_4$ is a surjective homomorphism from a group G to the symmetric group S_4, let

$$D_\lambda(G)$$

denote the preimage of the normal subgroup of order 4 under λ. Then we define Kegel's D-subgroup to be

$$D(G) = \bigcap_{\lambda} D_\lambda(G),$$

where the intersection is over all surjective homomorphisms $\lambda\colon G \to S_4$, with the stipulation that $D(G) = G$ if no such λ's exist. Thus $G/D(G)$ is a subdirect product of copies of S_3.

Our second main result is

THEOREM B. *Let G be a finite group. Then G satisfies the permutizer condition* **P** *if and only if the following conditions hold*:

(i) *each chief factor of G has order 4 or a prime*;

(ii) *if L is a proper self-normalizing subgroup of G containing $O_{2'}(G)$, then L is maximal in some subgroup X such that $D(X/L_X)$ is supersoluble*.

Here L_X is the core of L in X. For example, one can read off that S_4 is a **P**-group, while A_4 is not, since $|D(S_4)| = 4$ but $D(A_4) = A_4$. This distinction between A_4 and S_4 is crucial in the study of **P**-groups.

We mention that in his paper Kegel studies a related property which he calls *: here a group G has * if and only if every maximal subgroup has a cyclic supplement of prime power order. It is easy to see from Theorem A that every **P**-group has *; however, Kegel's property is weaker than **P** (see [3, p. 213, Example 1]).

In the same paper Kegel raises an interesting question, which is apparently still open. If every maximal subgroup of a (finite) group has an abelian supplement, is the group necessarily soluble? In particular one can ask if solubility follows when every proper subgroup permutes with some abelian subgroup not contained in it.

2. THE SOLUBILITY OF FINITE **P**-GROUPS

The object of this section is to accomplish the critical step in the study of the permutizer condition by proving.

THEOREM 2.1. *A group with the permutizer condition* **P** *is soluble*.

The method of proof, in outline, is to take a counterexample G of smallest order and show that G may be assumed semisimple with simple completely reducible radical R. Thus $R \leq G \leq \mathrm{Aut}(R)$, i.e., G is almost simple. It is then shown that for each maximal subgroup of R there is a subgroup of G which possesses maximal factorizations. Now a description of the maximal factorizations of almost simple groups is available [6]. Using this and the classification of finite simple groups, we proceed to eliminate each possibility for R.

It will now be assumed that *G is an insoluble* **P**-*group of minimal order*. The first step is almost obvious.

(2.2) *G is a semisimple group*.

For, if S, the maximum soluble normal subgroup of G, is nontrivial, then G/S is soluble by minimality of $|G|$, and hence G is soluble. Therefore $S = 1$ and G is semisimple.

Next let R denote the completely reducible radical of G. Then $R \neq 1$ and G/R is soluble by minimality of $|G|$. Write

$$R = S_1 \times S_2 \times \cdots \times S_k,$$

where each S_i is a (nonabelian) simple group, and suppose that $k > 1$. Put

$$N_1 = N_G(S_1) \quad \text{and} \quad C_1 = C_G(S_1).$$

Then $R \leq N_1$ and $R \cap C_1 = S_2 \times \cdots \times S_k$. We are interested in the group

$$\overline{G} = N_1/C_1,$$

concerning which we shall prove

(2.3) *The group $\overline{G}$ is semisimple with simple completely reducible radical $\overline{R} = RC_1/C_1$. If $\overline{H}$ is any subgroup of $\overline{G}$ not containing $\overline{R}$, then $\overline{H}\langle \bar{x} \rangle = \langle \bar{x} \rangle \overline{H}$ for some $\bar{x} \in \overline{G} \setminus \overline{H}$.*

Proof. Suppose that A/C_1 is an abelian normal subgroup of $\overline{G}$. Since RC_1/C_1 is simple, $[R, A] \leq R \cap C_1 = S_2 \times \cdots \times S_k$. But A normalizes S_1, so $[S_1, A] = 1$ and $A = C_1$. Thus $\overline{G}$ is semisimple; its completely reducible radical is $\overline{R}$ since G/R is soluble.

Next let $\overline{H} = H/C_1$, where $R \nleq H$. Then $S_1 \nleq H$ since $S_2 \times \cdots \times S_k \leq C_1 \leq H$. Suppose that H does not permute with any cyclic subgroup of N_1 that is not contained in H. Since G is a **P**-group, there exists $g \in G \setminus H$ such that $\langle g \rangle H = H \langle g \rangle$. Therefore $(N_1 \cap \langle g \rangle)H = H(N_1 \cap \langle g \rangle)$, from which it follows that

$$N_1 \cap \langle g \rangle \leq H.$$

Hence $g \notin N_1$.

Now g permutes the direct factors S_i of R by conjugation; here we can assume that $S_1^g = S_2$. Let $s \in S_2$; then $s \in H$ and $gs \in \langle g \rangle H = H \langle g \rangle$, so that $gs = hg^i$ for some $h \in H$ and i. This shows that $gsg^{-1} = hg^{i-1}$, whence $g^{i-1} \in HS_2^{g^{-1}} = HS_1 \leq N_1$. But then $g^{i-1} \in N_1 \cap \langle g \rangle \leq H$ and so $gsg^{-1} = hg^{i-1} \in H$. This gives the contradiction $S_1 \leq H$.

We shall now replace G by the group $\overline{G}$ and R by $\overline{R}$. Thus *we can assume that G has the properties listed in* (2.3)—however, we no longer know that G has the property **P**.

With this notation we begin to utilize the maximal subgroups of R by proving

(2.4) *Let M be a maximal subgroup of the simple group R and write $L = N_G(M)$ and $G_1 = LR$. Then L is maximal in G_1 and $G_1 = \langle x \rangle L = L \langle x \rangle$ for some x in G_1. Further $L \cap \langle x \rangle = 1$.*

Proof. Clearly $L \cap R = M$, so $L \neq G_1$. Suppose that $L \leq H \leq G_1$; then $H = L(H \cap R)$ and $M \leq H \cap R \leq R$, so that either $H = L$ or $H = G_1$. Thus L is maximal in G_1.

Next assume that L has no cyclic supplements in G_1. Since $R \nleq L$, there exists $g \in G \setminus L$ such that $\langle g \rangle L = L\langle g \rangle$. Therefore $(G_1 \cap \langle g \rangle)L = L(G_1 \cap \langle g \rangle)$ and in consequence

$$G_1 \cap \langle g \rangle \leq L.$$

Now let $m \in M$. Then $gm \in \langle g \rangle L = L\langle g \rangle$, so we may write $gm = l g^i$ with l in L. Thus $gmg^{-1} = l g^{i-1}$ and $g^{i-1} \in G_1 \cap \langle g \rangle \leq L$. It follows that $gmg^{-1} \in L$ and $M^{g^{-1}} \leq L$. But $M^{g^{-1}} \leq R$, so we obtain a contradiction, $M^{g^{-1}} = M$ and $g \in L$. Therefore $G_1 = \langle x \rangle L$ for some x.

Finally, suppose that $L \cap \langle x \rangle \neq 1$. Then, since $G_1 = \langle x \rangle L$, the core of L in G_1 is nontrivial and hence contains R. But this would mean that $R \leq L$. The proof is now complete.

If H is an almost simple group with completely reducible radical R, then an expression $H = AB$, with A and B maximal subgroups of H not containing R, will be called a *maximal factorization* of H (cf. [6, p. 1]). The next result shows that for each maximal subgroup of R there is a subgroup of G with a maximal factorization. We continue with the notation of (2.4).

(2.5) *Let M be a maximal subgroup of the simple group R, and write $L = N_G(M)$. Then there is a subgroup $G(M)$ of $G_1 = LR$ containing R with a maximal factorization $G(M) = A(M)B(M)$ such that*

(a) $A(M) \cap R = M$;

(b) $B(M)$ contains an element of order $|R : M|$.

Proof. Choose $G(M)$ to be a subgroup which is minimal subject to $R \leq G(M) \leq G_1$ and the existence of factorizations $G(M) = A(M)R = A(M)\langle x \rangle$, where $A(M)$ is maximal in $G(M)$, $A(M) \cap R = M$, and $A(M) \cap \langle x \rangle = 1$. Such a choice is possible by (2.4). Let $B(M)$ be a maximal subgroup of $G(M)$ containing x. Then $G(M) = A(M)B(M)$; we show that this is a maximal factorization of $G(M)$.

If this is not the case, then $R \leq B(M)$ and it follows that $B(M) = (A(M) \cap B(M))R$ and $B(M) = (A(M) \cap B(M))\langle x \rangle$. Further $(A(M) \cap B(M)) \cap R = A(M) \cap R = M$. Next we claim that $A(M) \cap B(M)$ is maximal in $B(M)$. Indeed assume that $A(M) \cap B(M) \leq T \leq B(M)$. Now $M \leq T \cap R \leq R$, so that either $T \cap R = M$ or $R \leq T$. In the first case $T = T \cap (A(M) \cap B(M))R = (A(M) \cap B(M))M = A(M) \cap B(M)$. In the second case $T = B(M)$. Thus $A(M) \cap B(M)$ is maximal in $B(M)$ as claimed. However, $B(M) < G(M)$, so the minimality of $G(M)$ is contradicted. Therefore $G(M) = A(M)B(M)$ is a maximal factorization. Finally $|x| = |G(M) : A(M)| = |R : M|$ since $A(M) \cap \langle x \rangle = 1$.

Strategy

To complete the proof of Theorem 2.1 we will show that for each simple group R there is a maximal subgroup M for which there is no maximal factorization of the form $G(M) = A(M)B(M)$ as described in (2.5).

A list of all maximal factorizations of almost simple groups is displayed in [6] in a series of tables. These show the possibilities for $A \cap R$ and $B \cap R$, where $G = AB$ is a maximal factorization of an almost simple group G with simple completely reducible radical R. In the case where R is a classical group and A and B are geometric subgroups (i.e., stabilizers of certain subspaces), the tables give normal subgroups X_A and X_B of A and B, respectively; these can be slightly smaller than $A \cap R$ and $B \cap R$, but are often identical with them. In any case the composition factors (composition factors will always be nonabelian) of X_A and X_B are among those of $A \cap R$ and $B \cap R$, respectively. Notice that if $A = A(M)$ for some maximal subgroup M of R, then the composition factors of X_A are among those of M.

Our general method of excluding a particular simple group R is as follows. We try to choose a maximal subgroup M of R such that the composition factors of each X_A and X_B in the relevant table in [6] do not all appear as composition factors of M. If such a choice is possible, the simple group R can be excluded from consideration.

Usually M will be chosen to be a geometric subgroup when R is a classical group. A good reference for these geometric subgroups is the monograph of Kleidman and Liebeck [4].

The approach just described is generally successful, but it fails for certain classical groups of low degree. In these and some other cases it is invaluable to have an alternative and quite elementary argument with orders of elements (subsequently referred to as the *order argument*). This is embodied in the next result, which continues the notation of (2.4).

(2.6) *The simple group R has an element of order $|R:M|/d$, where d is the greatest common divisor of $|R:M|$ and the order of some element of* $\mathrm{Out}(R)$.

Proof. We have $G_1 = L\langle x \rangle = LR$, and also $L \cap \langle x \rangle = 1$. Hence $|x| = |R:M|$. If xR has order r, then $x^r \in R$ and the order of x^r is $|R:M|/d$, where $d = \gcd(|R:M|, r)$.

To exploit (2.6) it is generally best to choose M to be a maximal subgroup of R with small order. Since $\mathrm{Out}(R)$ tends to be small, the effect is to produce an element of R with large order. If the order is too large, then R may be excluded. The method is especially convenient if R appears in the ATLAS [1] since the maximum order of an element can usually be determined by inspecting the maximal subgroups of R. This procedure will often be adopted to deal with groups occurring in the ATLAS.

Many sporadic simple groups are complete. There is another simple method that can be applied when R is complete (and so $G = R$). Using the notation of (2.4), we observe

(2.7) *If the simple group R is complete, then every maximal subgroup of R has odd index.*

Proof. We have $R = M\langle x \rangle$ and $M \cap \langle x \rangle = 1$ since $L = M$ here. Consider the natural permutation representation of R on the set of right cosets of M. Since R is simple, each element is represented by an even permutation. But x is represented by an $|R : M|$–cycle, so $|R : M|$ is odd.

Actually A_7 is the only simple group with all its maximal subgroups of odd index. However, we shall not require this characterization, but merely employ (2.7) to exclude certain sporadic groups.

The Case-by-Case Analysis

We now proceed to examine the finite simple groups, showing that for each group R there is a choice of maximal subgroup M which leads to an impossible factorization. When R is a classical group, we typically appeal first to the tables in [6] and then use the order argument and the ATLAS to dispose of exceptional cases.

I. The Alternating Groups

Let $R = A_n$ where $n \geq 5$. Denote the setwise stabilizer of $\{n-1, n\}$ in S_n by S and let $M = S \cap R$. Then S and M are maximal in S_n and R, respectively. Also $|S_n : S| = |R : M| = \binom{n}{2}$. If $n = 6$, then (2.6) predicts the existence in R of an element of order 15, which is incorrect. Thus $n \neq 6$, and $\mathrm{Aut}(R) \simeq S_n$. Hence $G = R$ or $G = S_n$. In the first case (2.4) shows that $G = \langle x \rangle M$ and $\langle x \rangle \cap M = 1$. Here $x \notin S$, so $S_n = \langle x \rangle S$. Hence in any event $S_n = \langle x \rangle S$ and $\langle x \rangle \cap S = 1$. Modifying x by an element of S, we may assume that x is of the form $(n-1, \dots, n, \dots)$ or $(n-1, \dots)(n, \dots)$. Therefore $|x| \leq \frac{1}{4}n^2$. But $|x| = |R : M| = \binom{n}{2}$, a contradiction.

We now commence the discussion of the classical groups.

II. The Projective Groups

Let $R = PSL_n(q)$, where $q = p^e$ with p a prime and $q \geq 5$ if $n = 2$.

Case $n = 2$. Recall that the subgroups of $PSL_2(q)$ are known (see [2, p. 213]). Since $PSL_2(5) \simeq A_5$ and $PSL_2(9) \simeq A_6$, we can assume $q \neq 5$ or 9. Suppose that $q = 7$, so that $G = R = PSL_2(7)$ or $G \simeq PGL_2(7)$. The first case may be excluded since $PSL_2(7)$ has a maximal subgroup of index 8 but no elements of order 8 (see (2.5)). If $G \simeq PGL_2(7)$, let M be a Sylow 2-subgroup. Then M is maximal in G and so $G = \langle x \rangle M$ for some x by (2.3). But $\langle x \rangle \cap M = 1$ since M has trivial core in G. Hence x has order 21, which is impossible. Thus we can assume that $q = 8$ or $q \geq 11$.

Now R has a maximal subgroup M which is dihedral of order $2(q+1)/d$, where $d = \gcd(2, q-1)$ (see [2, p. 213]) and $|R:M| = q(q-1)/2$. Also $|\mathrm{Out}(R)| = de$ (see, for example, [6, p. 18]). Since the order of an element of R cannot exceed $q+1$, we obtain, using (2.6),

$$\frac{q(q-1)}{2de} \leq q + 1,$$

which can only hold if $q = 5$ or 9. Hence this case is complete.

Case $n = 3$. Suppose first that $e > 1$ and let r be a prime dividing e. Then R has a maximal subgroup $M \simeq PSL_3(q^{1/r})$ (see [4, p. 70]). Refer now to the list of possible factorizations in [6, p. 10]. The only (nonabelian) composition factor of X_A or X_B is $PSL_2(q)$, and this is never isomorphic with $PSL_3(q^{1/r})$, so we can exclude $PSL_3(q)$. When $q = 4$, there is an exceptional factorization ([6, p. 13]); however, $PSL_3(4)$ is easily excluded by choosing a maximal subgroup of order 72 and using the order argument and the ATLAS.

Now let $e = 1$, so that $q = p$. We can assume that $p \neq 2$ since $PSL_3(2) \simeq PSL_2(7)$. Now R has a maximal subgroup $M \simeq O_3(p)$ (see [4, p. 70]). Hence $|M| = p(p^2 - 1)$ and

$$|R:M| = \frac{p^2(p^3-1)}{d},$$

where $d = \gcd(3, p-1)$. Thus G has an element x of order $p^2(p^3-1)/d$.

Applying (2.5), we can assume that G has a maximal factorization $G = A(M)B(M)$ such that $A(M) \cap R = M$ and $x \in B(M)$. From the table in [6, p. 10] we conclude that X_A is soluble, so that in this case $A = B(M)$ and $B = A(M)$. Further $|X_A| = 3(p^3 - 1)$ (see the first line of the table). Now $x^l \in B(M) \cap R = A \cap R = X_A$ (see [6, p. 26]) for some l dividing $|\mathrm{Out}(R)| = 2d$. Therefore

$$\frac{p^2(p^3-1)}{dl} \leq |x^l| \leq 3(p^3-1)$$

and $p^2 \leq 3dl \leq 6d^2$. This yields $p = 7$. But $PSL_3(7)$ may be excluded by choosing a maximal subgroup of order 57 and using the order argument and the ATLAS.

Case $n > 3$. Choose M to be a geometric subgroup of type P_2, i.e., the stabilizer of a line. Them M has composition factors $PSL_{n-2}(q)$ if $(n, q) \neq (4, 2)$ or $(4, 3)$, and $PSL_2(q)$ if $q \geq 4$.

Now refer to the table on p. 10 of [6]. To exclude each possible factorization we need to check that both X_A and X_B have composition factors that are different from those of M. This is clear except in the first line of the table when n is prime and $A = A(M)$. But then $X_A = A \cap R = M$ is soluble, which is only true if $(n, q) = (4, 2)$ or $(4, 3)$. Finally, the groups $PSL_4(2)$ and $PSL_4(3)$ are excluded by the order argument.

There is an exceptional factorization for $PSL_5(2)$; this group is also dealt with using the order argument.

III. The Symplectic Groups

Let $R = PSp_{2n}(q)$, $q = p^e$ with p a prime and $(n, p) \neq (2, 2)$. Here we may assume $n \geq 2$.

Case $n = 2$. Suppose first that $e > 1$ and let r be a prime dividing e. Then R has a maximal subgroup $M \simeq PSp_4(q^{1/r})$ (see [4, p. 72]), which is simple unless $q = 4$, when A_6 is the only composition factor. Check the tables on pp. 10 and 12 of [6], comparing the composition factors of X_A and X_B with the above. As a result all cases can be excluded except for $q = 4$ and 16. The group $PSp_4(4)$ is dealt with by using the order argument and the ATLAS. However, $PSp_4(16)$ is not in the ATLAS and needs to examined.

Let $q = 16$. As before let $M \simeq PSp_4(4)$. Then $|R : M| = 2^8(4^4 + 1)(4^2 + 1) = l$, say. From the tables in [6, p. 10] we recognize that there is just one possible factorization with $X_A \simeq O_4^-(16)$ and $X_B \simeq Sp_4(4)$. Thus $B = A(M)$, $A = B(M)$, and $X_B = B \cap R = M$ (see [6, p. 50]). Also $x \in A$ by (2.5). Since $|\mathrm{Out}(R)| = 8$, it follows that $x^8 \in A \cap R = X_A$. But the largest possible order of an element of $O_4^-(16)$ is $16^2 + 1$, which is less than $l/8$. This disposes of $PSp_4(16)$.

Now consider the case $e = 1$, when $q = p$, so $R = PSp_4(p)$, $p \geq 3$. There is a maximal subgroup M of type P_2, i.e., the stabilizer of an isotropic line. Now $|R| = \frac{1}{2}p^4(p^4 - 1)(p^2 - 1)$ and by an easy computation $|M| = \frac{1}{2}p^4(p^2 - 1)(p - 1)$, so that

$$|R : M| = \frac{p^4 - 1}{p - 1}.$$

By [6, p. 10] the only possible factorization for R has $X_A \simeq PSp_2(p^2) \cdot 2$ and X_B of type P_1. Since $p > 2$, comparison of composition factors shows that $B = A(M)$ and $A = B(M)$. Hence A contains an element x of order $(p^4 - 1)/(p - 1)$. Since $|\mathrm{Out}(R)| = 2$, we have $x^2 \in A \cap R = X_A$. But the largest possible order of an element of X_A is $p^2 + 1$, which is less than $(p^4 - 1)/(p - 1)$, a contradiction.

Finally, an exceptional factorization occurs for $PSp_4(3)$, but this group is handled by the order argument and the ATLAS.

Case $n > 2$. Let M be a maximal subgroup of R of type P_2. A check of the possible factorizations in [6, pp. 10, 12, 13] reveals that no composition factors match those of M. This completes the discussion of the case.

IV. The Unitary Groups

Let $R = U_n(q)$, where $q = p^e$, p prime. We can assume that $n \geq 3$.

Case n odd. In this case there are only exceptional factorizations, occurring for the groups $U_3(3), U_3(5), U_3(8), U_9(2)$. The first three of these are quickly disposed of since they are in the ATLAS. To exclude the group $U_9(2)$, choose M to be a maximal subgroup of type P_2.

Case n even. Let $R = U_{2m}(q)$ where $m \geq 2$. We can assume that $(m, q) \neq (2, 2)$ since $U_4(2) \simeq PSp_4(3)$. Let M be a maximal subgroup of type P_1, with only one composition factor $U_{2m-2}(q)$. Now check the composition factors of X_A and X_B in the tables in [6, p. 11]; no coincidences occur.

There are a number of exceptional factorizations for $U_4(2), U_4(3), U_6(2), U_{12}(2)$. However, $U_4(2) \simeq PSp_4(3)$, while $U_4(3)$ and $U_6(2)$ are in the ATLAS and so may be handled using the order argument. Finally, $U_{12}(2)$ is excluded by choosing M of type P_1 as above, and observing that the composition factors of $A \cap R$ and $B \cap R$ in the factorization are *Suz* and $U_{11}(2)$ [6, p. 13].

V. The Orthogonal Groups $P\Omega_{2m+1}(q)$

Since $P\Omega_3(q) \simeq PSL_2(q)$ and $P\Omega_5(q) \simeq PSp_4(q)$, we can assume that $m \geq 3$. Also $P\Omega_{2m+1}(q) \simeq PSp_2m(q)$ if q is even, so we may also assume that q is odd. To exclude these groups choose M to be a maximal subgroup of type P_2.

There are in addition infinite families of factorizations for the groups $P\Omega_7(q), P\Omega_{13}(3^e), P\Omega_{25}(3^e)$ (see [6, p. 12]). These can be eliminated by the same choice of M as above. Finally $P\Omega_7(3)$ has numerous exceptional factorizations. Since this group appears in the ATLAS, it is best to exclude it using the order argument.

VI. The Orthogonal Groups $P\Omega_{2m}^+(q)$, $m \neq 4$

Since $P\Omega_4^+(q)$ is not simple and $P\Omega_6^+(q) \simeq PSL_4(q)$, we can assume that $m \geq 5$. If $m > 5$, choose a maximal subgroup M of type P_2. For $P\Omega_{16}^+(q)$

there is an additional infinite class of factorizations (see [6, p. 12]), and $P\Omega_{24}^{+}(2)$ has an exceptional factorization [6, p. 13]. These are also excluded by choosing M of type P_2.

A different choice of M is called for when $m = 5$; this is because of the isomorphism $P\Omega_6^{+}(q) \simeq PSL_4(q)$. Take M of type P_1. A check of the table in [6, p. 11] excludes this case.

VII. The Orthogonal Groups $P\Omega_8^{+}(q)$

These groups have a different set of maximal factorizations (see [6, p. 14]), but they can also be excluded by choosing a maximal subgroup M of type P_2.

VIII. The Orthogonal Groups $P\Omega_{2m}^{-}(q)$

Let $R = P\Omega_{2m}^{-}(q)$; we can assume that $m \geq 4$ because of low-degree isomorphisms. Choose M to be a maximal subgroup of type P_2.

IX. The Exceptional Simple Groups

Let R be an exceptional simple group of Lie type. Here our task is made easier by the fact that *all* factorizations of subgroups of $\mathrm{Aut}(R)$ containing R are known. This is work of Hering, Liebeck, and Saxl (see [6, p. 8]). In no case is there a cyclic factor, so all these groups may be excluded.

X. The Sporadic Groups

The sporadic groups for which there is a maximal factorization are M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_2, HS, He, Ru, Suz, Fi_{22}, and Co_1 (see [6, pp. 15–16]). The complete groups among these are M_{11}, M_{23}, M_{24}, Ru, and CO_1, each of which has a maximal subgroup of even index and so may be excluded by (2.7).

The remaining groups have outer automorphism group of order 2. They can all be excluded by using the order argument and the ATLAS. The proof of Theorem 2.1 is now complete.

An Alternative Approach

The above proof has the disadvantage of requiring in the case of the classical groups a detailed examination of many cases. We indicate briefly an alternative approach which avoids some of the case-by-case analysis and does not use the tables in [6]. This rests on the concept of a *primitive prime divisor* (ppd) of $b^f - 1$, where $b, f > 1$ are integers: this is a prime which divides $b^f - 1$ but does not divide $b^i - 1$ if $1 \leq i < f$. The importance of ppd's in the study of linear groups is underscored by a recent work of Niemeyer and Praeger [7].

Let R be a classical simple group with parameters n and $q = p^m$, where p is a prime. Choose a maximal subgroup M of R and let G_1 denote the group referred to in (2.4). Thus $G_1 = LR$, $L \cap R = M$, and L is maximal in G_1. Now it is straightforward to show that the order of an element of $GL_n(q)$ cannot be divisible by two ppd's p_{em}, p_{fm} of $p^{em} - 1$, $p^{fm} - 1$, respectively, where $n/2 < e < f$. Using this observation and the fact that p_{em} and p_{fm} do not divide $|\mathrm{Out}(R)|$, we may deduce that at least one of p_{em} and p_{fm} must divided $|M|$. A contradiction ensues if we are able to choose e, f such that $n/2 < e < f$ and $|M|$ is divisible by neither p_{em} nor p_{fm}.

This method is successful provided n is not too small, usually $n \geq 4$; otherwise the ppd's will not exist. As a result one is left with a number of low-degree-cases—actually 13—which must be handled by other means. (We are grateful to Alice Niemeyer and Cheryl Praeger for several helpful observations that led to the alternative approach, and for making their paper [7] available to us.)

3. THE STRUCTURE OF FINITE **P**-GROUPS

Having established that groups with **P** are soluble, we proceed to obtain detailed structural information about these groups.

We begin with an elementary observation which relates supersolubility to the property **P**. In any (finite) group G there is a unique maximum normal supersolubly embedded subgroup, denoted here by

$$\sigma(G).$$

So there is a G-invariant series in $\sigma(G)$ with cyclic factors, while $G/\sigma(G)$ has no nontrivial cyclic normal subgroups.

(3.1) *A group G is a* **P***-group if and only if $G/\sigma(G)$ is a* **P***-group.*

Proof. Assume that $G/\sigma(G)$ is a **P**-group and suppose that H is a proper subgroup of G which does not permute with any cyclic subgroup not contained in H. Then $\sigma(G) \nleq H$. Let $1 = S_0 < S_1 < \cdots < S_m = \sigma(G)$ be a G-invariant series with cyclic factors. There is an $i < m$ such that $S_i \leq H$ but $S_{i+1} \nleq H$. Writing $S_{i+1}/S_i = \langle xS_i \rangle$, we have $\langle x \rangle H = H\langle x \rangle$. Hence G is a **P**-group. The converse is obvious.

A Lemma on p-Groups

The following result plays a key role in the analysis of the chief factors of a **P**-group. The case where $p > 2$ may be found in [9, p. 27]; when $p = 2$ the result is in [10] (see also [11]). For the reader's convenience we include a proof which is shorter than those just cited.

(3.2) *Let Q be a p-group and let N be a nontrivial, elementary abelian normal subgroup of Q which has a complement X in Q. If $Q = \langle y \rangle X$ for some element y, then $|N| = p$ if p is odd and $|N| \le 4$ if $p = 2$.*

Proof. Assume that Q is a counterexample of minimal order.
Case $p > 2$. Let $1 \ne t \in N \cap Z(Q)$. The result is true for the group $Q/\langle t \rangle$, so $|N| = p^2$. Then we can write $N = \langle u, t \rangle$ for some u. Since $Q = XN$, we have $y = xu^a t^b$, where $x \in X$. Then

$$y^p = (xu^a)^p t^{bp} = x^p u^{ap} [u^a, x]^{\binom{p}{2}} = x^p$$

since $[N, Q] \le \langle t \rangle$. It follows that $y^p \in X$ and hence that $|N| = |Q : X| = |\langle y \rangle : \langle y \rangle \cap X| = p$.
Case $p = 2$. Again let $1 \ne t \in N \cap Z(Q)$. Then $|N : \langle t \rangle|$ divides 4 and $|N| = 8$. Let $1 \ne s\langle t \rangle \in N/\langle t \rangle \cap Z(Q/\langle t \rangle)$. Then $N = \langle s, t, u \rangle$ for some u, and $\langle s, t \rangle \lhd G$. Note that $[N, Q] \le \langle s, t \rangle$ and $[s, Q] \le \langle t \rangle$.

Now write $y = xu^a t^b s^c$, where $x \in X$. Since $s^{xu^a t^b} = s^x = s$ or st, we obtain

$$s^{1 + xu^a t^b + (xu^a t^b)^2 + (xu^a t^b)^3} = 1.$$

Consequently $y^4 = (xu^a t^b s^c)^4 = (xu^a)^4$. Since $s^x = s$ or st, we have $[x^2, \langle s, t \rangle] = 1$. Also $(xu^a)^2 = x^2 (u^a)^{1+x} \in x^2 \langle s, t \rangle$. Hence $y^4 = (xu^a)^4 = x^4$. Finally, $|N| = |Q : X| = |\langle y \rangle : \langle y \rangle \cup X| \le 4$, a contradiction.

We can now tackle the chief factors of **P**-groups.

THEOREM 3.3. *Let G be a group satisfying* **P**. *Then every chief factor of G has order* 4 *or a prime.*

Proof. Let G be a counterexample of minimal order and choose a minimal normal subgroup A of G. Of course G is soluble and A is elementary abelian. Each chief factor of G/A has order 4 or a prime. Therefore A must be an elementary abelian p-group, with $|A| > p$ and $|A| > 4$ if $p = 2$. Clearly A is the unique minimal normal subgroup of G. Put $N = \mathrm{Fit}(G)$, the Fitting subgroup of G. Then N is a p-group by uniqueness of A. Also $A \le Z(N)$ since $A \cap Z(N) \ne 1$. Next define F/N to be $O_{p'}(G/N)$. Then F splits over N, say with $F = VN$ and $V \cap N = 1$. The Frattini argument yields $G = N_G(V)F = LN$, where $L = N_G(V)$. Since $L \cap A \lhd LN = G$, either $L \cap A = 1$ or $A \le L$.

Suppose that $A \le L$. Then $[A, V] = 1$, and so $A \le Z(F)$ because $F = VN$. Now consider $O_{p'}(G/A) = T/A$. In the first place $TN/N \le O_{p'}(G/N) = F/N$, so $T \le F$ and $[A, T] = 1$. Hence $T = A \times O_{p'}(T)$, which shows that $T = A$. Thus $O_{p'p}(G/A) = O_p(G/A) = N/A$. Therefore N/A is the intersection of the centralizers of the p-chief factors of G/A

(see [8, p. 270]). From this it follows that G/N is a subgroup of a direct product of copies of the cyclic group C_{p-1} if $p > 2$, or of copies of S_3 if $p = 2$. If $p > 2$, this means that $F = G$ and so $A \leq Z(G)$, a contradiction. Thus $p = 2$ and G/N is an extension of a 3-group by a 2-group. Since $F/N = O_{2'}(G/N)$, it follows that G/F is a 2-group. But G/F acts irreducibly on A, so again A must be contained in $Z(G)$.

We are now left with the situation where $G = LN$ and $L \cap A = 1$. Let U be chosen maximal subject to $L \leq U$ and $U \cap A = 1$. Since G has **P**, there is an element $g \in G \backslash U$ such that $\langle g \rangle U = U \langle g \rangle$. In fact we can assume that g is a p-element. For let $\langle g \rangle = \langle t \rangle \times \langle z \rangle$, where t is a p-element and z is a p'-element. Now $|\langle g \rangle U : U|$ divides $|G : U| = |UN : U| = |N : U \cap N|$, which is a power of p. Hence $\langle g \rangle / (U \cap \langle g \rangle)$ is a p-group, which shows that $z \in U \cap \langle g \rangle$. Therefore $\langle g \rangle U = \langle t \rangle U$ and $U \langle g \rangle = U \langle t \rangle$. Now replace g by t.

Next put $\overline{U} = \langle g \rangle U$. Then $\overline{U} \cap A \neq 1$ by maximality of U. But $\overline{U} \cap A \triangleleft \overline{U}N = G$, so that $A \leq \overline{U}$ and $UA \leq \overline{U}$. Consequently

$$UA = (UA) \cap \overline{U} = (UA \cap \langle g \rangle)U.$$

Now let Q be a Sylow p-subgroup of UA containing $UA \cap \langle g \rangle$. Since $A \leq Q$, we have

$$Q = Q \cap (UA) = (Q \cap U)A;$$

in addition

$$\begin{aligned} Q = Q \cap (UA) &= Q \cap ((UA \cap \langle g \rangle)U) \\ &= (UA \cap \langle g \rangle)(Q \cap U). \end{aligned}$$

Applying (3.2) to Q with $Q \cap U$ for X and A for N, we conclude that $|A| = p$ if $p > 2$ and $|A| = 4$ if $p = 2$, a final contradiction.

COROLLARY 1. *If G is a* **P**-*group, then every maximal subgroup M of G has index* 4 *or a prime. Also M has a cyclic supplement in G with prime power order.*

This follows easily from Theorems 2.1 and 3.3. Note that the second property of G is Kegel's condition *.

COROLLARY 2 ([9, p. 28]). *For groups of odd order the property* **P** *is equivalent to supersolubility.*

COROLLARY 3 (cf. [10]). *Let* **G** *be a* **P**-*group. Then*

(i) *G is supersoluble if and only if S_4 is not isomorphic with a quotient of G;*

(ii) *if the largest prime dividing* $|G|$ *is* $p > 3$, *then* G *has a normal Sylow* p-*subgroup*;

(iii) *if* Q *is a Sylow* 2-*subgroup of* G', *then* $Q \triangleleft G$ *and* G/Q *is supersoluble*.

These results follow easily from (3.1), Theorem 3.3, and Satz 9.1 of [2, p. 716] (for the first statement of Corollary 3 a minimal counterexample is considered).

Another criterion for a **P**-group to be supersoluble involves the derived subgroup.

THEOREM 3.4. *A group* G *is supersoluble if and only if* G *and* G' *satisfy* **P**.

Proof. Assume that G and G' satisfy **P** and that G has least order subject to being nonsupersoluble. Then G has a unique minimal normal subgroup A and G/A is supersoluble. By Theorem 3.3 we must have $|A| = 4$. If $\phi(G) \neq 1$, then $G/\phi(G)$ is supersoluble, as must be G by a well-known theorem of Huppert (see [8, p. 276]). Therefore $\phi(G) = 1$, which implies that A has a complement X in G (see [8, p. 135]). But $C_X(A) = 1$, so $G \simeq S_4$ or A_4 and $G' \simeq A_4$ or $|G'| = 4$. Both situations are impossible since A_4 does not have **P**.

The next result is an elementary remark.

(3.5) *Let* G *be a group each of whose chief factors has order* 4 *or a prime*. *Then* $G/O_{2'}(G)$ *is a* $\{2,3\}$-*group*.

Proof. Consider a chief series $1 = G_0 \leq G_1 \cdots \leq G_n = G$ and assume that G_{i+1}/G_i has order a prime $p > 3$ while G_i/G_{i-1} has order 2, 3, or 4. Then $G_i/G_{i-1} \leq Z(G_{i+1}/G_{i-1})$, so $G_{i+1}/G_{i-1} = G_i/G_{i-1} \times \overline{G}_i/G_{i-1}$ where $|\overline{G}_i/G_{i-1}| = p$ and $|G_{i+1}/\overline{G}_i| = 2$, 3, or 4. Replace G_i by $\overline{G}_i$. By a sequence of such replacements we can move chief factors with prime order greater than 3 down the series past factors of order 2, 3, or 4. Consequently $G/O_{2'}(G)$ is a $\{2,3\}$-group.

The next result sheds further light on the internal structure of **P**-groups. It is significant in that it distinguishes between S_4, which is a **P**-group, and A_4, which is not.

THEOREM 3.6. *Let* G *be a soluble group satisfying* **P** *and let* M *be a chief factor of* G *with order* 4. *Then* G *induces the full group of automorphisms in* M, *i.e.*, $G/C_G(M) \simeq S_3$.

Proof. We can assume that $O_{2'}(G) = 1$. Hence G is a $\{2,3\}$-group by Theorem 3.3 and (3.5). Thus chief factors of G have order 2, 3, or 4 and $F = \mathrm{Fit}(G)$ is a 2-group. Also $O_2(G/F) = 1$ and $C_G(F/F') = F$.

Consider a G-composition series in F/F'. If U is a factor, then $G/C_G(U)$ is isomorphic with a subgroup of S_3, and if D is the intersection of all the $C_G(U)$, then G/D is an extension of a 3-group by a 2-group. Also $F = C_G(F/F') \leq D$ and D/F is a 2-group. Hence $D = F$ and G/F has a normal Sylow 3-subgroup, say V/F.

Let $L < K < H$ be successive terms of a G-composition series of F/F' with $|K:L| = 4$ and $|H:K| = 2$. We claim that $\bar{K}/L = C_{H/L}(V) \neq 1$. This is clear if H/L is not elementary abelian since $[\phi(H/L), V] = 1$; in the elementary case use complete reducibility of H/L as a V/F-module. Since V acts irreducibly on K/L, it follows that $C_{H/L}(V) = \bar{K}/L$ has order 2. Note that $\bar{K} \lhd G$ and $|H:\bar{K}| = 4$. Now replace K by $\bar{K}$. By a sequence of such replacements G-composition factors of order 4 in F/F' may be moved up the G-composition series. Hence there is a G-submodule F_1/F' such that all G-composition factors of F/F_1 have order 4, while those of F_1/F' have order 2.

The next step is to prove that G/F_1 splits over F/F_1. With $V/F = O_3(G/F)$ as above, we have by the Schur−Zassenhaus theorem that $V = QF$, where $Q \cap F = F_1$ and Q/F_1 is a Sylow 3-subgroup of V/F_1. Then the Frattini argument shows that $G = N_G(Q)V = TF$, where $T = N_G(Q)$. Notice that Q has no nontrivial fixed points in F/F_1 since all G-composition factors of F/F_1 have order 4 and $|G:V|$ is a power of 2. Hence $T \cap F = F_1$ and G/F_1 splits over F/F_1 with complement T/F_1. Let $X_1 = T$.

Noting that chief factors of X_1 are also of order 2, 3, or 4, we may apply the argument just given to X_1, with F_1 in place of F, obtaining $X_1 = X_2 F_1$ and $X_2 \cap F_1 = F_2$. By repetition of the procedure we generate a sequence of subgroups X_i, F_i, $i = 1, 2, \ldots, r$, such that $F_i \lhd X_i$, $X_i = X_{i+1}F_i$, and $X_{i+1} \cap F_i = F_{i+1}$. Moreover by construction every X_i-composition factor of F_i/F_{i+1} has order 4. The sequence terminates at r if F_r is contained in the hypercentre of X_r.

Put $X = X_r$ and note that $G = XF$ and $|G:X|$ is a power of 2. By the property **P** there is a subgroup Y of G such that X is maximal in Y and X has a cyclic supplement in Y. Clearly $|Y:X| = 2$ or 4. Suppose that $|Y:X| = 2$. Then $X \lhd Y$ and $Y = X(Y \cap F)$; hence Y/X is isomorphic with $Y \cap F/F_r$. Thus $Y \cap F/F_r$ is centralized by X. However, each F_i/F_{i+1} has by construction no nontrivial X-fixed points. By this contradiction $|Y:X| = 4$.

Now put $V_1 = Y \cap F$, so that $Y = XV_1$ and $|V_1 : F_r| = 4$. Note that F_r is a maximal proper X-invariant subgroup of V_1 because X is maximal in Y. Assume that $V_1' \not\leq F_r$. Then $V_1 = \langle V_1', F_r \rangle$. But V_1 is a 2-group, so that $V_1' \leq \phi(V_1)$ and $V_1 = F_r$, a contradiction. Hence $V_1' \leq F_r$, $F_r \lhd XV_1 = Y$, and V_1/F_r is a chief factor of Y of order 4.

Next consider the group $Y/C_X(V_1/F_r)$; this is isomorphic with either S_4 or A_4. But in the latter case $X/C_X(V_1/F_r)$ would not have a cyclic supplement in $Y/C_X(V_1/F_r)$. Consequently X must induce the full group of automorphisms in V_1/F_r.

Put $Y_0 = X$, $V_0 = F_r$, and $Y_1 = Y = Y_0V_1$; then $Y_1 \cap F = V_1$. Repeat the preceding argument with Y_1 in place of X to obtain $V_2 > V_1$ with $Y_2 = Y_1V_2$, $V_2 \lhd Y_2$, and X inducing the full group of automorphisms in V_2/V_1. This procedure generates a chain of X-invariant subgroups $F_r = V_0 \lhd V_1 \lhd \cdots \lhd V_s = F$ such that X induces the full group of automorphisms in each of the elementary abelian groups V_{i+1}/V_i of order 4.

Now let M be a minimal normal subgroup of G with order 4. Then $M \leq Z(F)$ and, since $G = XF$, the subgroup X acts irreducibly on M. Now $M \cap F_r = 1$ since F_r lies in the hypercentre of X, so M must be X-isomorphic with some factor V_{i+1}/V_i. Therefore X induces the full group of automorphisms in M and $G/C_G(M) \simeq S_3$. By induction on $|G|$ the same holds for the quotient group G/M, so the theorem is proved.

COROLLARY. *Let G be a soluble group with the property* **P**. *If $O_{2'}(G) = 1$, then $G/\mathrm{Fit}(G)$ is a subdirect product of copies of S_3.*

For $\mathrm{Fit}(G)$ equals the intersection of the centralizers of its chief factors in G.

Necessary and Sufficient Conditions for **P**

We turn now to the problem of characterizing **P**-groups. For this purpose Kegel's D-subgroup, defined in the introduction, is well-suited. We begin by using the D-subgroup to give necessary and sufficient conditions for a normal subgroup of a **P**-group to be supersoluble.

(3.7) *Let N be a normal subgroup of a* **P**-*group G. Then the following statements are equivalent*:

(i) *N is supersoluble*;

(ii) *neither A_4 nor S_4 is an image of N*;

(iii) *$N \leq D(G)$.*

Proof. Obviously (i) implies (ii). Assume that N satisfies (ii) but not (iii). Then $N \nleq D_\lambda(G)$ for some surjective homomorphism $\lambda\colon G \to S_4$. Hence $N^\lambda = A_4$ or S_4, a contradiction. Finally, to see that (iii) implies (i), note that by Theorem 3.3 and Corollary 1, every maximal subgroup has a cyclic supplement of prime power order in G. Therefore $D(G)$ is supersoluble by [3, Proposition 2]. If $N \leq D(G)$, then N is supersoluble.

COROLLARY. *If G is a* **P**-*group, then $D(G)$ is the unique largest supersoluble normal subgroup of G.*

We will now establish Theorem B, the promised characterization of **P**-grous.

Proof of Theorem B. Assume first that G is a **P**-group. Then G satisfies (i) by Theorems 2.1 and 3.3. Let L be a proper self-normalizing subgroup of G containing $O_{2'}(G)$; clearly we can assume that $O_{2'}(G) = 1$. Then L is maximal in some subgroup X such that $X = L\langle x \rangle$ for some element x. It follows from Theorem 3.3 that $|X : L| = 3$ or 4. Now X/L_X cannot be abelian, so it must be isomorphic with S_3, A_4 or S_4. In the first and last cases $D(X/L_X)$ is supersoluble, as required. But X/L_X cannot be isomorphic with A_4 since in A_4 a subgroup of order 3 has no cyclic supplements.

Conversely, assume that G satisfies (i) and (ii). Then $O_{2'}(G) \leq \sigma(G)$ and so by (3.1) we can assume that $O_{2'}(G) = 1$. Let L be a proper self-normalizing subgroup of G; it will be shown that $\langle x \rangle L = L\langle x \rangle$ for some x in $G \setminus L$. By (ii) L is maximal in some subgroup X for which $D(X/L_X)$ is supersoluble. Now $|X : L| = 3$ or 4, and thus $X/L_X \simeq S_3$, A_4, or S_4. Since $D(A_4) = A_4$, we can exclude A_4 here. Thus $X/L_X \simeq S_3$ or S_4; in both cases L/L_X has a cyclic supplement in X/L_X, which completes the proof.

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