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He's variational method for the Benjamin–Bona–Mahony equation and the Kawahara equation

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ABSTRACT

Variational principles for the generalized Benjamin–Bona–Mahony equation and the Kawahara equation are established using He's semi-inverse method. The solitary solutions are obtained using the Ritz method.

1. Introduction

In this paper we will consider the Benjamin–Bona–Mahony (BBM) equation [1,2]

$$u_t + 6uu_x + u_{xxx} = 0 \tag{1}$$

and the Kawahara equation [3]

$$u_t + au^2u_x + bu_{3x} - ku_{5x} = 0 \tag{2}$$

where a, b and k are constants.

The Benjamin–Bona–Mahony (BBM) equation describes the unidirectional propagation of small-amplitude long waves on the surface of water in a channel [1,2]. It is proposed as an alternative to the Korteweg–de Vries equation (KdV). The Kawahara equation, fifth-order KdV-type equation, is a model equation for plasma waves, capillary-gravity water waves [3]. Moreover, this equation describes water waves with surface tension [4]. The inclusion of a fifth-order term to KdV is necessary to model magneto-acoustic waves [5]. For details of Kawahara equation, the reader is advised to read [4] and the references therein.

Eqs. (1) and (2) were widely discussed by many authors using different methods, such as the exp-function method [6] by Yusufoglu and Bekir [7,8], the variational iteration method [9] and the homotopy perturbation method [10] by Tari and Ganji [11].

2. He's variational method

In his review article [12] and his monograph [13], Ji-Huan He suggested a preliminary but promising variational approach to the search for solitary solutions of nonlinear wave equations. Tao [14,15] found that He's variational method was very simple and effective. Other applications of He's variational method are available in Refs. [16,17].

In order to seek its travelling wave solution, we introduce a transformation

$$u(x,t) = v(\xi) \tag{3}$$

$$\xi = cx + \lambda t. \tag{4}$$

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Substituting Eqs. (3) and (4) into Eq. (1) yields

$$\lambda v' + 6cvv' + c^3 v''' = 0. \tag{5}$$

Integrating Eq. (5) once, we have

$$\lambda v + 3cv^2 + c^3 v'' = m \tag{6}$$

where m is the integration constant. We set $m = 0$ for simplicity:

$$\lambda v + 3cv^2 + c^3 v'' = 0. \tag{7}$$

By He's semi-inverse method [18], we can arrive at the following variational formulation:

$$J(v) = \int_0^\infty \left[\frac{1}{2}\lambda v^2 + cv^3 - \frac{c^3}{2}(v')^2\right] \mathrm{d}\xi. \tag{8}$$

We search for a soliton solution in the form

$$v(\xi) = A \sec h(\xi) \tag{9}$$

where A is an unknown constant to be further determined.

Substituting Eq. (9) into Eq. (8), we have

$$J = \frac{1}{2}\lambda A^2 - \frac{1}{6}c^3 A^2 + \frac{\pi}{4}cA^3. \tag{10}$$

Making J stationary with A results in

$$\frac{\partial J}{\partial A} = \lambda A - \frac{1}{3}c^3 A + \frac{3}{4}\pi c A^2 = 0. \tag{11}$$

From Eq. (11), we get

$$A = \frac{4c^3 - 12\lambda}{9\pi c}. \tag{12}$$

The solitary solution is, therefore, obtained as follows:

$$v(\xi) = \frac{4c^3 - 12\lambda}{9\pi c} \sec h(\xi). \tag{13}$$

By a similar manipulation, the Kawahara equation can be converted into the following ordinary differential equation:

$$\lambda v + \frac{ac}{3}v^3 + bc^3 v'' - kc^5 v^{(4)} = 0. \tag{14}$$

Its variational formulation reads

$$J(v) = \int_0^\infty \left[\frac{1}{2}\lambda v^2 + \frac{ac}{12}v^4 - \frac{bc^3}{2}(v')^2 + \frac{1}{2}kc^5(v'')^2\right] \mathrm{d}\xi. \tag{15}$$

We search for a soliton solution in the form

$$v(\xi) = A \sec h^2(\xi) \tag{16}$$

where A is an unknown constant to be further determined.

Substituting Eq. (16) into Eq. (15), we obtain

$$J = \frac{1}{3}\lambda A^2 + \frac{4}{105}acA^4 - \frac{4}{15}bc^3 A^2 + \frac{16}{21}kc^5 A^2. \tag{17}$$

Making J stationary with A results in

$$\frac{\partial J}{\partial A} = \frac{2}{3}\lambda + \frac{16}{105}acA^3 - \frac{16}{15}bc^3 A + \frac{32}{21}kc^5 A = 0. \tag{18}$$

From Eq. (18), we have

$$A = \sqrt{\frac{56bc^3 - 35\lambda + 80kc^5}{8ac}}. \tag{19}$$

The solitary solution is, therefore, obtained as follows:

$$v(\xi) = \sqrt{\frac{56bc^3 - 35\lambda + 80kc^5}{8ac}} \cdot \sec h^2(\xi). \tag{20}$$

3. Conclusion

In this study, we used He's variational method to search for solitary solutions. It is obvious that the employed approach is useful and manageable and remarkably simple to find various kinds of solitary solutions.

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