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Analysis of the Lack of Compactness in the Critical Sobolev Embeddings

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has arbitrary small norm in L^q $(1/q=(1/p)-(s/d))$. This generalizes a result obtained by Patrick Gérard for the case $p=2$.

Key Words: Sobolev embeddings; wavelet bases; compact embeddings; Besov spaces.

1. INTRODUCTION AND STATMENT OF RESULTS

We will analyze the lack of compactness of the embedding

$$\dot{H}^{s,p}(\mathbf{R}^d) \hookrightarrow L^q(\mathbf{R}^d) \qquad (s>0,\ 1<p<q<\infty) \tag{1}$$

in the critical case

$$\frac{s}{d}=\frac{1}{p}-\frac{1}{q}. \tag{2}$$

Recall that $\dot{H}^{s,p}$ is the homogeneous Sobolev space of functions u satisfying

$$\|(-\Delta)^{s/2}\, u\|_{L^p}<\infty.$$

If $(\Delta_j(f))_{j\in\mathbf{Z}}$ denotes a Littlewood–Paley decomposition of f, the homogeneous Besov space $\dot{B}^{\alpha,\infty}_{\infty}$ is defined by the condition

$$\|\Delta_j(f)\|_{L^\infty}\leqslant C2^{-\alpha j}.$$

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The following inequality due to Gérard, Meyer, and Oru (see [6]) can be interpreted as a sharpened form of the embedding (1)

$$\|f\|_{L^q} \leqslant C\, \|f\|^{p/q}_{\dot{H}^{s,p}} \|f\|^{1-p/q}_{\dot{B}_{\infty}^{-d/q,\infty}}. \tag{3}$$

If a function u belongs to $\dot{H}^{s,p}$, the functions

$$\left(\frac{1}{h}\right)^{d/q} u\left(\frac{x-x_0}{h}\right)$$

have $\dot{H}^{s,p}$ and L^q norms independent of h and x_0. These invariances induce a lack of compactness in the embedding $\dot{H}^{s,p}(\mathbf{R}^d) \hookrightarrow L^q(\mathbf{R}^d)$. Indeed, if $(\log h_n, x_n)$ is a sequence that tends to infinity, the sequence of functions

$$u_n(x) = \left(\frac{1}{h_n}\right)^{d/q} u\left(\frac{x-x_n}{h_n}\right) \tag{4}$$

converges weakly to 0 in $\dot{H}^{s,p}$, however it does not converge to 0 in L^q. Thus a sequence (u_n) that converges weakly to 0 in $\dot{H}^{s,p}$ can fail to converge to 0 in L^q because the u_n are translates and dilates of a single function, or more generally, because the u_n are sums of translates and dilates of a set of functions. The purpose of this paper is to show precisely in which sense this phenomena is responsible for the lack of compactness in the embedding (1).

Wavelet analysis will be used to elucidate this phenomena. Since the elements of a wavelet basis can be obtained by translations/dilations of 2^d-1 functions, wavelet bases are well suited for detecting sequences such as (4) or sums of them. This arguments will be made precise at the end of this section. We next introduce some notation that will facilitate the discussion.

Define the translation/dilation $\tau = \tau(a, b)$, $a > 0$, $b \in \mathbf{R}^d$, by

$$\tau(x) = \frac{x-b}{a}$$

and define the action on τ of a function $\phi\colon \mathbf{R}^d \to \mathbf{C}$ by

$$\tau(\phi)(x) = a^{-d/q}\phi(\tau(x)).$$

If $\tau_n = (x-b_n)/a_n$ is a sequence of translations/dilations, we say that τ_n tends to infinity if $|\log a_n| + |b_n| \to +\infty$ as $n \to \infty$. Two sequences τ_n and τ'_n are called *orthogonal* if $\tau_n(\tau'_n)^{-1}$ tends to infinity. We will prove the following theorem. This result generalizes in the $\dot{H}^{s,p}$ setting a result by Patrick Gérard in the case $p=2$, see [5].

THEOREM 1. *Let (u_n) be a bounded sequence in $\dot{H}^{s,p}(\mathbf{R}^d)$ Then there exists a subsequence (u'_n) of (u_n), functions ϕ_m in $\dot{H}^{s,p}$ and sequences $\tau_{n,m}$ of translations/dilations that have the following properties*:

(1) *If $m \neq m'$, then $\tau_{n,m}$ and $\tau_{n,m'}$ are orthogonal.*

(2) *For all $l \geqslant 1$, the u'_n can be decomposed as*

$$u'_n = \sum_{m=1}^{l} \tau_{n,m}(\phi_m) + r^l_n, \tag{5}$$

and

$$\limsup_{n \to \infty} \|r^l_n\|_{L^q} = \varepsilon_l \qquad \text{with} \quad \varepsilon_l \to 0 \quad \text{when} \quad l \to \infty. \tag{6}$$

(3) *The following estimates hold in $\dot{H}^{s,p}$,*

$$\sum_{m=1}^{\infty} \|\phi_m\|^p_{\dot{H}^{s,p}} \leqslant C \limsup_{N \to \infty} \|u'_N\|^p_{\dot{H}^{s,p}}, \tag{7}$$

$$\|r^l_n\|^p_{\dot{H}^{s,p}} \leqslant C \limsup_{N \to \infty} \|u'_N\|^p_{\dot{H}^{s,p}} \tag{8}$$

for all n and l.

Remarks.

• The series $\sum_{m=1}^{\infty} \tau_{n,m}(\phi_m)$ is usually not convergent for a given n; hence the necessity to write only finite sums in (5).

• We cannot expect the $\dot{H}^{s,p}$ norm of r^l_n to tend to 0 because there are other reasons why weak convergence in $\dot{H}^{s,p}$ does not imply strong convergence (such as oscillations, etc.).

• Estimate (7) can be interpreted as a kind of "L^p quasi-orthogonality" of the functions $\tau_{n,m}(\phi_m)$ that appear in (5). This property is a consequence of the orthogonality condition satisfied by the $\tau_{n,m}$, which implies that, for l fixed and n large, the functions $(\tau_{n,m}(\phi_m))$, $m = 1, ..., l$, live either at very different scales or very far from each other.

• Once a subsequence u'_n has been selected, the decomposition (5) is unique; indeed the ϕ_m are clearly characterized as being all possible weak limits in $\dot{H}^{s,p}$ of translations/dilations of u'_n. Thus, only the selection of the subsequence u'_n depends on the particular technique we will describe.

• The result of Gérard for $p = 2$ is more precise, since in this case (7) and (8) are replaced by

$$\sum_{m=1}^{l} \|\phi_m\|^2_{\dot{H}^{s,2}} + \|r^l_n\|^2_{\dot{H}^{s,2}} = \|u'_n\|^2_{\dot{H}^{s,2}} + o(1). \tag{9}$$

Note that the result of Gérard has applications in the description of solutions of nonlinear wave equations with critical exponent (see [1]) and implies the microlocal compacity-concentration criterium (see [4]).

We first show that (9) cannot hold in general if $p \neq 2$ by constructing an explicit counterexample for $p=4$. Take f and ω in $\mathscr{S}(\mathbf{R})$ such that $\hat{f}$ vanishes in a neighborhood of 0, $\hat{\omega}$ is compactly supported, and f and ω are both even and define

$$u_n(x) = f(x) + \frac{e^{inx}\omega(x)}{n^s}.$$

Here we have $l=1$, $\phi_1 = f$, and $\tau_{n,1}(x) = x$, thus $r_n^1 = e^{inx}\omega(x)/n^s$. Clearly, if Λ^s denotes the operator $(-\Delta)^{s/2}$,

$$\Lambda^s\left(\frac{e^{inx}\omega(x)}{n^s}\right) = e^{ins}\eta_n(x),$$

where η_n has the same regularity and localization estimates as ω (uniformly in n) and $\eta_n \to \omega$ when $n \to \infty$. Thus

$$\int |\Lambda^s u_n|^4 = \int |\Lambda^s f|^4 + \int |\eta_n|^4 + 2\int |\Lambda^s f|^2\, |\eta_n|^2\, (2 + \cos 2nx) + \cdots$$

$$+ 4\int ((\Lambda^s f)^3\, \eta_n + \Lambda^s f(\eta_n)^3) \cos nx$$

which converges to

$$\int |\Lambda^s f|^4 + \int |\omega|^4 + 4\int |\Lambda^s f|^2\, |\omega|^2.$$

The result is obtained by choosing f and ω such that the last term does not vanish.

Our main tool for proving Theorem 1 will be an orthonormal basis of compactly supported wavelets $(\psi^{(i)})_{i=1,\dots,2^d-1}$ that are sufficiently smooth (at least C^{s+1}). Such bases are described in [3], for instance. Thus the functions

$$2^{dj/2}\psi^{(i)}(2^j x - k), \qquad j \in \mathbf{Z}, k \in \mathbf{Z}^d \tag{10}$$

form an orthonormal basis of $L^2(\mathbf{R}^d)$. We index the wavelets in terms of the dyadic cubes: If λ is the cube

$$\lambda = \{x \in \mathbf{R}^d \colon 2^j x - k \in [0, 1]^d\},$$

we use the notation

$$\psi_\lambda^{(i)}(x) = 2^{dj/q}\psi^{(i)}(2^j x - k).$$

Thus

$$f(x) = \sum_{i,\lambda} c_\lambda^{(i)} \psi_\lambda^{(i)}(x), \tag{11}$$

where the wavelet coefficients of f are given by

$$c_\lambda^{(i)} = \int_{\mathbf{R}^d} 2^{dj(1-2/q)} \psi_\lambda^{(i)}(t)\, f(t)\, dt.$$

(Note that we do not use the usual L^2 normalization; the natural normalization for the problem we consider is the $\dot{H}^{s,p}$ normalization, which is the same as the L^q normalization.) We will call the functions $c_\lambda^{(i)}\psi_\lambda^{(i)}(x)$ that appear in (11) the *wavelet components* of f. Let us now come back to the model case supplied by (4) and sketch what a wavelet decomposition of this sequence looks like. Since the u_n in (4) are obtained by (correctly normalized) translations/dilations of a given function, the supremum of the moduli of the wavelet coefficients of u_n does not tend to 0 when $n \to \infty$. Conversely, if this is the case at least one wavelet component of u_n will be of the form

$$c_n \psi^{(i)}(2^{j_n} x - k_n), \tag{12}$$

where $c_n \to c \neq 0$ (after perhaps extracting a subsequence from (u_n)).

Our technique for proving Theorem 1 will be to substract from u_n wavelet components like (12), where c_n is the largest wavelet coefficient of u_n. This will reduce the $\dot{B}_\infty^{-d/q,\infty}$ norm of the remaining part of u_n, since the space $\dot{B}_\infty^{-d/q,\infty}$ has the following wavelet caracterization (see [7]):

$$f \in \dot{B}_\infty^{-d/q,\infty} \Leftrightarrow \sup_{\lambda, i} |c_\lambda^{(i)}| < \infty. \tag{13}$$

In view of (3), it will also reduce the L^q norm of the remainder. In other words, the proof of Theorem 1 will shed new light on the "Sobolev inequality made precise" (3), since it will show that the $\dot{B}_\infty^{-d/q,\infty}$ norm "controls" this lack of compactness.

2. EXTRACTION OF THE WAVELET COEFFICIENTS

Before starting the proof of Theorem 1, we review some functional spaces characterizations taken from [7] that will be useful. If χ_λ denotes the characteristic function of the cube λ, then, in view of (2) and the L^q wavelet normalization we chose, the characterizations in [7] are as follows:

$$f \in L^q \Leftrightarrow \left(\sum_{\lambda, i} |c_\lambda^{(i)}|^2 \, 2^{2dj/q} \chi_\lambda(x) \right)^{1/2} \in L^q, \tag{14}$$

$$f \in \dot{H}^{s, p} \Leftrightarrow \left(\sum_{\lambda, i} |c_\lambda^{(i)}|^2 \, 2^{2dj/p} \chi_\lambda(x) \right)^{1/2} \in L^p. \tag{15}$$

We denote by $\omega(f)$ the function

$$\omega(f)(x) = \left(\sum_{\lambda, i} |c_\lambda^{(i)}|^2 \, 2^{2dj/p} \chi_\lambda(x) \right)^{1/2}. \tag{16}$$

Let (u_n) be a bounded sequence of functions in $\dot{H}^{s, p}$. The wavelet coefficients of u_n, which will be denoted by $c_{\lambda, n}^{(i)}$, satisfy

$$\int \left(\sum_{\lambda, i} |c_{\lambda, n}^{(i)}|^2 \, 2^{2dj/p} \chi_\lambda(x) \right)^{1/2} dx \leqslant C. \tag{17}$$

Of course, by restricting the sum to one term, we see that, a fortiori,

$$\forall n, \lambda, i, \qquad |c_{\lambda, n}^{(i)}| \leqslant C$$

which simply means that (u_n) is a bounded sequence in $\dot{B}_\infty^{-d/q, \infty}$.

There are two cases: Either $\sup_{\lambda, i} |c_{\lambda, n}^{(i)}| \to 0$ when $n \to \infty$ (which just means that the sequence (u_n) tends to 0 in $\dot{B}_\infty^{-d/q, \infty}$), and, using (3), the result is proved; or, perhaps after extracting a subsequence, we can suppose that the sequence $\sup_{\lambda, i} |c_{\lambda, n}^{(i)}|$ converges to a non-vanishing limit $|c_1|$ when $n \to \infty$ (we will pick the sign of c_1 later). Note that, because of (17), for each n this supremum is attained for at least one cube λ_n^1. After, perhaps, extracting another subsequence, we may assume that the indices i for which this supremum is attained are all the same and that the corresponding coefficients $c_{\lambda, n}^{(i)}$ have the same sign (we require these coefficients to be of the same sign so that the sequence $c_{\lambda_n^1, n}^{(i_1)}$ converges). Finally, after these extractions, there exists an index i_1 and a sequence of cubes λ_n^1 such that for all n

$$\left. \begin{aligned} |c_{\lambda_n^1, n}^{(i_1)}| &= \sup_{\lambda, i} |c_{\lambda, n}^{(i)}| \\ \exists c_1 \neq 0 \qquad |c_{\lambda_n^1, n}^{(i_1)} - c_1| &\leqslant 2^{-n} |c_1|. \end{aligned} \right\} \tag{18}$$

Denote by $\tau_{n, 1}$ the translation/dilation such that

$$\tau_{n, 1}([0, 1]^d) = \lambda_n^1.$$

The functions $c^{(i_1)}_{\lambda^1_n, n}\psi^{(i_1)}_{\lambda^1_n}$ form the first sequence of wavelet components extracted from u_n, and by definition, they define the function $\phi^{1,1}_n$ (the first index 1 refers to the fact that we are dealing with the first sequence of translations/dilations and the second index 1 refers to the fact that we are dealing with the first extraction); $c^1\psi^{(i_1)}_{\lambda^1_n}$ is the first wavelet component of $\tau_{n,1}(\phi_1)$. (Following the terminology introduced by Patrick Gérard, the center of the cube λ^1_n and its width are respectively the first extracted sequences of *hearts* and *scales*.) To simplify (slightly) the notation, we will from now on drop the index (i) of the wavelets and wavelet coefficients. This is of no consequence provided that, in the following, each time that a sequence of wavelet components is extracted, it is understood that we perform a second extraction so that all the wavelets are indexed by the same (i). We thus denote our first sequence of wavelet components $c_{\lambda^1_n, n}\psi_{\lambda^1_n}$. The first remainder u^1_n is defined by

$$u_n = c_{\lambda^1_n, n}\psi_{\lambda^1_n} = u^1_n.$$

We now consider this sequence u^1_n, which is just the sequence u_n modified by replacing its largest coefficient $c_{\lambda^1_n, n}$ by zero. We will still denote the coefficients of u^1_n by $c_{\lambda, n}$. If the supremum of their moduli tends to 0, we conclude as before; otherwise, we obtain as before an index (i_2) and a new sequence of dyadic cubes λ^2_n such that (18) holds. The limit of the corresponding wavelet coefficients will be denoted by c_2. Let $\tau^{1,2}_n$ be the translation/dilation such that

$$\tau^{1,2}_n(\lambda^1_n) = \lambda^2_n.$$

Now, two cases can occur:

First Case. The sequence $\tau^{1,2}_n$ is bounded. After reextracting a subsequence, we can suppose that the respective positions of λ^1_n and λ^2_n remain the same, i.e., that there exists a unique translation/dilation $\tau_{1,2}$ such that

$$\forall n \qquad \tau_{1,2}(\lambda^1_n) = \lambda^2_n. \tag{19}$$

The sum $c_1\psi_{\lambda^1_n} + c_2\psi_{\lambda^2_n}$ is obtained by translations/dilations of a single function; the function $c_2\psi_{\lambda^2_n}$ thus becomes the second wavelet component of $\tau_{n,1}(\phi_1)$, and $c_{\lambda^2_n, n}\psi_{\lambda^2_n}$ is the second wavelet component of $\phi^{1,2}_n$.

Second Case. $\tau^{1,2}_n \to \infty$ (in which case, the two sequences of dyadic cubes λ^1_n and λ^2_n will be called orthogonal). We pick λ^2_n as a new sequence of dyadic cubes, which defines a new sequence of translations/dilations $\tau_{n,2}$ orthogonal to $\tau_{n,1}$. The coefficients $c_{\lambda^2_n, n}$ satisfy (18); the limit wavelet component $c_2\psi_{\lambda^2_n}$ is the first wavelet component of $\tau_{n,2}(\phi_2)$, and $c_{\lambda^2_n, n}\psi_{\lambda^2_n}$ is the first wavelet component of $\phi^{2,2}_n$.

In both cases, the remainder u_n^2 is defined by

$$u_n = c_{\lambda_n^1, n} \psi_{\lambda_n^1} + c_{\lambda_n^2, n} \psi_{\lambda_n^2} + u_n^2.$$

The extraction procedure that we initiated is continued, and after N extractions, we obtain the decomposition

$$u_n(x) = \sum_{m=1}^{l} \phi_n^{m, N}(x) + u_n^N(x), \tag{20}$$

where $l \leqslant N$ is the number of sequences of dyadic cubes orthogonal two by two, that have been extracted (the equality holds only for the values of n that remain after the N extractions). The wavelet components of $\phi_n^{m, N}$ are the extracted wavelet components indexed by the sequences of dyadic cubes that were not orthogonal to λ_n^m. Note that the decomposition (20) is nothing but a partition of the wavelet coefficients of u_n.

Each term $\phi_n^{m, N}$ can thus be written

$$\phi_n^{m, N} = c^{n, m}_{\lambda_1^{n, m}} \psi_{\lambda_1^{n, m}} + \cdots + c^{n, m}_{\lambda_k^{n, m}} \psi_{\lambda_k^{n, m}}. \tag{21}$$

(Here we have changed the notations concerning the indexing of these wavelet coefficients to show explicitly the sequences of dyadic cubes to which they correspond.) The cubes $\lambda_1^{n, m}$, ..., $\lambda_k^{n, m}$ keep the same respective positions, i.e., there exists translation/dilations $\tau_{k, m}$ such that

$$\forall n \qquad \lambda_k^{n, m} = \tau_{k, m}(\lambda_1^{n, m}), \tag{22}$$

and k is the number of wavelet components indexed by the sequences of dyadic cubes that were not orthogonal to λ_n^m; thus k depends on N.

The extraction at order $N+1$ is performed as follows:

- If $\|u_n^N\|_{\dot{B}_\infty^{-d/q, \infty}} \to 0$ when $n \to \infty$, then we stop the extractions at this point, and the theorem is proved.
- If not we extract as before a sequence of coefficients $c_{\lambda_n^{N+1}, n}$ that converges to a limit $c_{N+1} \neq 0$. There are two cases:

(1) The sequence λ_n^{N+1} of dyadic cubes thus defined remains in a bounded neighborhood of one of the sequences previously obtained (the p-th for instance), and the wavelet component $c_{\lambda_n^{N+1}, n} \psi_{\lambda_n^{N+1}}$ will contribute to the function $\phi_n^{p, N+1}$, and at the limit, $c_{N+1} \psi_{\lambda_n^{N+1}}$ will be a new wavelet component of $\tau_{n, p}(\phi^p)$.

(2) The sequence λ_n^{N+1} of dyadic cubes remains in a bounded neighborhood of none of the sequences previously obtained. It thus defines

a new sequence of translations/dilations $\tau_{n,\,l+1}$ that is orthogonal to the previous ones, and the wavelet component $c_{\lambda_n^{N+1},\,n}, \psi_{\lambda_n^{N+1}}$ will be a wavelet component of a new function $\phi_n^{l+1,\,N}$, and at the limit, $c_{N+1}\psi_{\lambda_n^{N+1}}$ will be the first wavelet component of $\tau_{n,\,l+1}(\phi^{l+1})$.

The subsequence of (u_n) that we will finally consider is the one obtained from the previous extraction procedures using a diagonal extraction. We still denote it by (u_n).

Furthermore, we will drop the index N in $\phi_n^{m,\,N}$, and write this function ϕ_n^m since, after the diagonal extraction, the index N becomes a function of n.

3. NORM ESTIMATES ON EXTRACTED SEQUENCES

If $u \in \dot{H}^{s,\,p}$ let $P_N(u)$ be defined as the "nonlinear projection" obtained by keeping the N wavelet components of u that have the N largest wavelet coefficients (in modulus), and let Q_N be the remainder, defined by $Q_N(U) = u - P_N(u)$. The following lemma shows how the $\dot{B}_\infty^{-d/q,\,\infty}$ seminorm of $Q_N(u)$ decays.

LEMMA 1. *Let* $r = \max(p, 2)$. *Then*

$$\sup_{\|u\|_{\dot{H}^{s,\,p}} \leqslant 1} \|Q_N(u)\|_{\dot{B}_\infty^{-d/q,\,\infty}} \leqslant \frac{C}{N^{1/r}}.$$

Proof of Lemma 1. The embedding $\dot{H}^{s,\,p} \hookrightarrow \dot{B}_r^{s,\,r}$ yields

$$\|u\|_{\dot{H}^{s,\,p}} \geqslant C\left(\sum_\lambda |c_\lambda|^r\right)^{1/r}.$$

If the N largest coefficients $|c_\lambda|$ are larger than a given value A,

$$1 \geqslant \|u\|_{\dot{H}^{s,\,p}} \geqslant C N^{1/r} A$$

so that $A \leqslant C^{-1} N^{-1/r}$, hence the lemma.

Unless otherwise mentioned, we use in the following the equivalent "wavelet norms" defined by the right-hand sides of (13), (14), and (15). The decomposition (20) of u_n is nothing but a partition of its wavelet coefficients; thus,

$$\forall n, m \qquad \|\phi_n^m\|_{\dot{H}^{s,\,p}} \leqslant \|u_n\|_{\dot{H}^{s,\,p}}.$$

Each coefficient $c^{n,m}_{\lambda^{n,m}_k}$ of (21) has a limit when $n \to \infty$. Using (22) and (15), there exists a function $\phi^m \in \dot{H}^{s,p}$ such that

$$\phi^m_n - \tau_{n,m}(\phi^m) \to 0 \qquad \text{in } \dot{H}^{s,p} \tag{23}$$

(recall that, if $\tau\colon x \to (x-b)/a$, $\tau(\phi)(x) = a^{-d/q}\phi(\tau(x))$). Note that the $\lim_{n\to\infty} c^{n,m}_{\lambda^{n,m}_k}$ are not necessarily the wavelet coefficients of ϕ^m; however, for each value of k, $\tau_{n,m}(\phi^m)$ has the wavelet coefficients $\lim_{n\to\infty} c^{n,m}_{\lambda^{n,m}_k}$ if n is large enough.

We used (20) for only one value of m. Let $L, N > 0$ be fixed. We will now reproduce the same argument for the whole set of functions $\phi^1_n, ..., \phi^L_n$. If we keep only the N largest wavelet coefficients of ϕ^m_n, we can write

$$u_n = \sum_{m=1}^{L} P_N(\phi^m_n) + s^l_n. \tag{24}$$

We pick n large enough so that the $N \times L$ wavelet coefficients of the ϕ^m_n satisfy (18), and hence

$$\forall m = 1, ..., L \qquad \|P_N(\phi^m_n) - P_N(\tau_{n,m}(\phi^m))\|_{\dot{H}^{s,p}} \to 0 \qquad \text{as} \quad n \to \infty. \tag{25}$$

Using again that (20) is a partition of the wavelet coefficients of u_n, and using (15) and (16),

$$\left\| \sum_{m=1}^{L} \omega(P_N(\phi^m_n))^2 \right\|_{L^{p/2}} \leqslant C$$

thus, because of (25),

$$\int \left(\sum_{m=1}^{L} \omega(P_N(\tau_{n,m}\phi^m))^2 \right)^{p/2} \leqslant C. \tag{26}$$

Since we use a basis of compactly supported wavelets, each function $P_N(\tau_{n,m}\phi^m))$ is supported in a cube $C\lambda^m_n$; the same property holds for $\omega(P_N(\tau_{n,m}\phi^m))$. Note also that for n large enough, the L^p norms of these functions is independent of n. We now decompose the integral (26) into a sum of L integrals on domains Ω^k_n defined as follows.

The domain Ω^k_n is the cube $C\lambda^k_n$ from which we have excluded the intersection with the other cubes $C\lambda^{k'}_n$ *of smaller size*. (Note that because of the orthogonality hypothesis, two cubes $C\lambda^k_n$ of comparable size cannot intersect for sufficiently large n.)

Since the sequences λ_n^m are orthogonal,

$$\forall k=1, ..., L \int_{\Omega_n^k} \left(\sum_{m=1}^{L} \omega(P_N(\tau_{n,m}\phi^m))^2 \right)^{p/2} \tag{27}$$

converges to

$$\int_{C\lambda_n^k} (\omega(P_N(\tau_{n,k}\phi^k))^2)^{p/2} = \|P_N(\tau_{n,k}\phi^k)\|^p_{\dot{H}^{s,p}}. \tag{28}$$

Indeed, one distinguishes two types of errors when comparing (27) and the left-hand side of (28): First, the error coming from the small cubes that were taken out in the definition of Ω_n^k. This error tends to 0 because the relative size of these small cubes becomes negligible. The second source of errors comes from the big cubes $C\lambda_n^{k'}$ that contain the Ω_n^k; but the functions which live at the scale of these big cubes have a negligible size on Ω_n^k. Thus, summing the contributions of all domains Ω_n^k,

$$\sum_{k=1}^{L} \|P_N(\tau_{n,k}\phi^k)\|^p_{\dot{H}^{s,p}} \leqslant \limsup_{l \to \infty} \|u_l\|^p_{\dot{H}^{s,p}},$$

since the left-hand term is independent of n for n large enough. The $\dot{H}^{s,p}$ norms of the $\tau_{n,k}\phi^k$ are all equivalent (because the "usual" $\dot{H}^{s,p}$ norm is invariant under $\tau_{n,k}$). Letting $N, L \to \infty$ we obtain

$$\sum_{k=1}^{\infty} \|\phi^k\|^p_{\dot{H}^{s,p}} \leqslant \limsup_{l \to \infty} \|u_l\|^p_{\dot{H}^{s,p}}.$$

When comparing (5) and (20), we see that the remainder r_n^l is composed of two terms:

- The term u_n^N in (20), which poses no problem. Indeed, since (20) is a partition of the wavelet coefficients, using again the equivalent norm (17), $\|u_n^N\|_{s,p} \leqslant \|u_n\|_{s,p}$ hence the bound (8) for that term. Furthermore, since u_n^N was obtained by replacing the N largest coefficients $c_{\lambda,n}$ of u_n by zero, Lemma 1 implies that the $\dot{B}_{\infty}^{-d/q,\infty}$ norm of u_n^N decays like $1/N^{1/r}$. Thus (6) follows from (3).

- The term $\sum_{m=1}^{l} \tau_{n,m}\phi_m - \phi_n^m$. But (23) implies that the $\dot{H}^{s,p}$ norm of each term of this sum tends to 0.

Remarks. The main ingredients in the proof above are two function spaces that are invariant under the same transformations (normalized the same way) and simultaneously that there exists a basis which is unconditional for both spaces, and whose elements are mapped to each other by these transformations. Thus the above techniques could certainly be adapted in other situations where this type of abstract setting is provided. We just illustrate this point by a particularly simple example.

The spaces $l^p(\mathbf{Z})$ and $l^q(\mathbf{Z})$ $(1<p<q<\infty)$ are invariant under the shifts, and the elements of the canonical basis, which is the same for $l^p(\mathbf{Z})$ and $l^q(\mathbf{Z})$, are mapped to each other by the shifts. Using this basis and the Hölder inequality

$$\|u\|_{l^q} \leqslant \|u\|_{l^p}^{p/q} \, \|u\|_{l^\infty}^{1-p/q}$$

(which, in this setting, plays the role of (3)), one obtains the following description of the lack of compactness of the $l^p \hookrightarrow l^q$ embedding. (If τ is the shift by l on the right, we write $A(\tau)=l$; two sequences τ_n and τ'_n of shifts are called orthogonal if $A(\tau_n(\tau'_n)^{-1}) \to \infty$).

PROPOSITION 1. *Let (u_n) be a bounded sequence in l^p. There exists a subsequence (u'_n) of (u_n), elements ϕ_m in l^p and sequences $\tau_{n,m}$ of shifts such that if $m \neq m'$, the sequences $\tau_{n,m}$ and $\tau_{n,m'}$ are orthogonal, and the following decomposition holds*

$$\forall l \geqslant 1 \qquad u'_n = \sum_{m=1}^{l} \tau_{n,m}(\phi_m) + r_n^l,$$

with the following l^p and l^q norm estimates

$$\limsup_{n\to\infty} \|r_n^l\|_{l^q} = \varepsilon_l \qquad \text{with} \quad \varepsilon_l \to 0 \quad \text{when} \quad l \to \infty,$$

$$\sum_{m=1}^{\infty} \|\phi_m\|_{l^p}^p \leqslant C \limsup_{N\to\infty} \|u_N\|_{l^p}^p,$$

$$\forall n, l \qquad \|r_n^l\|_{l^p}^p \leqslant C \limsup_{N\to\infty} \|u_N\|_{l^p}^p.$$

It is also clear that the above proof works for any of the critical Sobolev embeddings $\dot{H}^{s,p} \hookrightarrow \dot{H}^{s',p'}$ with $(s-s')/d = 1/p - 1/p'$, since (3) still holds when replacing L^q by $\dot{H}^{s',p'}$, with different weights on the right-hand side.

The case $p=1$ in Theorem 1 cannot be treated using similar ideas. Indeed, although Cohen, DeVore, Petrushev, and Xu proved that some Sobolev embeddings made precised hold in that case (see [2]), wavelets are not an unconditional basis for $H^{s,1}$ (in fact $H^{s,1}$ has no unconditional basis). This means that the technique of substracting the largest wavelet components might increase the $H^{s,1}$ norm.

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REFERENCES

1. H. Bahouri and P. Gérard, Generalized geometrical optics for the critical nonlinear wave equation, preprint.
2. A. Cohen, R. DeVore, P. Petrushev, and H. Xu, Nonlinear approximation and the space $BV(\mathbf{R}^2)$, preprint.
3. I. Daubechies, Orthonormal bases of compactly supported wavelets, *Comm. Pure Appl. Math* **41** (1988), 909–996.
4. P. Gérard, Oscillation and concentration effects in semilinear dispersive wave equations, *J. Funct. Anal.* **141** (1996), 60–98.
5. P. Gérard, Description du défaut de compacité de l'injection de Sobolev, *ESAIM COCV* **3** (1998), 213–233.
6. P. Gérard, Y. Meyer, and F. Oru, Inégalités de Sobolev précisées, *in* "Séminaire EDP 1996–1997, Exposé 4, Ecole Polytechnique, 1996."
7. Y. Meyer, "Ondelettes et opérateurs," Vol. 1, Hermann, Paris, 1990.