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Partial compact quantum groups

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ABSTRACT

Compact quantum groups of face type, as introduced by Hayashi, form a class of quantum groupoids with a classical, finite set of objects. Using the notions of weak multiplier bialgebras and weak multiplier Hopf algebras (resp. due to Böhm–Gómez-Torrecillas–López-Centella and Van Daele–Wang), we generalize Hayashi's definition to allow for an infinite set of objects, and call the resulting objects partial compact quantum groups. We prove a Tannaka–Kreĭn–Woronowicz reconstruction result for such partial compact quantum groups using the notion of partial fusion C*-categories. As examples, we consider the dynamical quantum $SU(2)$-groups from the point of view of partial compact quantum groups.

Introduction

The concept of *face algebra* was introduced by T. Hayashi in [13], motivated by the theory of solvable lattice models in statistical mechanics. It was further studied in [14–20], where for example associated *-structures and a canonical Tannaka duality were developed. This Tannaka duality allows one to construct a canonical face algebra from

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any (finite) fusion category. For example, a face algebra can be associated to the fusion category of a quantum group at root unity, for which no genuine quantum group implementation can be found.

In [32,36,37], it was shown that face algebras are particular kinds of $\times_R$-algebras [40] and of weak bialgebras [5,3,31]. More intuitively, they can be considered as quantum groupoids with classical, finite object set. In this article, we want to extend Hayashi's theory by allowing an *infinite* (but still discrete) object set. This requires passing from weak bialgebras to weak *multiplier* bialgebras [4]. At the same time, our structures admit a piecewise description by what we call a *partial bialgebra*, which is more in the spirit of Hayashi's original definition. In the presence of an antipode, an invariant integral and a compatible *-structure, we call our structures *partial compact quantum groups*.

The passage to the infinite object case requires extra arguments at certain points, as one has to impose the proper finiteness conditions on associated structures. However, once all conditions are in place, many of the proofs are similar in spirit to the finite object case.

Our main result is a Tannaka–Kreĭn–Woronowicz duality result which states that partial compact quantum groups (with finite hyperobject set) are, up to the appropriate notion of equivalence, in one-to-one correspondence with *concrete semisimple rigid tensor* C*-*categories*. Here we do not assume the unit of the tensor C*-category to be irreducible – this situation can also be dealt with using the notion of C*-bicategory introduced in [21]. By a *concrete* tensor C*-category we mean a tensor C*-category realized inside a category of (locally finite-dimensional) bigraded Hilbert spaces. Of course, Tannaka reconstruction is by now a standard procedure. For closely related results most relevant to our work, we mention [48,35,16,33,12,39,34,9,29] as well as the surveys [22] and [30, Section 2.3].

As an application, we generalize Hayashi's Tannaka duality [16] (see also [33]) by showing that any module C*-category over a semisimple rigid tensor C*-category has an associated canonical partial compact quantum group. By the results of [9], such data can be produced from ergodic actions of compact quantum groups. In particular, we consider the case of ergodic actions of $SU_q(2)$ for q a non-zero real. This will allow us to show that the construction of [14] generalizes to produce partial compact quantum group versions of the dynamical quantum $SU(2)$-group [11,24], see also [38] and the references therein. This construction will provide the right setting for the operator algebraic versions of these dynamical quantum $SU(2)$-groups, which was the main motivation for writing this paper. These operator algebraic details will be studied elsewhere [7].

The precise layout of the paper is as follows.

The *first section* introduces the basic theory of the structures which we will be concerned with. We introduce the notions of a *partial bialgebra*, *partial Hopf algebra* and *partial compact quantum group*, and show how they are related to the notion of a weak multiplier bialgebra [4], weak multiplier Hopf algebra [46,45] and compact quantum group of face type [14]. We also briefly recall the notions of *tensor category* and *tensor* C*-*category*.

In the next two sections, our main result is proven, namely the Tannaka–Kreĭn–Woronowicz duality. In the *second section* we develop the representation theory of partial compact quantum groups, and we show how it allows one to construct a concrete semisimple rigid tensor C*-category. In the *third* section, we show conversely how any concrete semisimple rigid tensor C*-category allows one to construct a partial compact quantum group, and we briefly show how the two constructions are inverses of each other.

In the final two sections, we provide some examples of our structures and applications of our main result. In the *fourth section*, we first consider the construction of a canonical partial compact quantum group from any module C*-category for a semisimple rigid tensor C*-category. We then introduce the notions of *Morita*, *co-Morita* and *weak Morita equivalence* [27] of partial compact quantum groups, and show that two partial compact quantum groups are weakly Morita equivalent if and only if they can be connected by a string of Morita and co-Morita equivalences. In the *fifth section*, we study in more detail a concrete example of a canonical partial compact quantum group, constructed from an ergodic action of quantum $SU(2)$. In particular, we obtain a partial compact quantum group version of the dynamical quantum $SU(2)$-group [11,24].

1. Partial compact quantum groups

1.1. Partial algebras

Definition 1.1. An *I-partial algebra* $\mathscr{A}$ is a set $I = \{k, l, \cdots\}$, the *object* set, together with $\mathbb{C}$-vector spaces ${}_kA_l$, multiplication maps

$$M = M_{k,l,m}\colon {}_kA_l \otimes {}_lA_m \to {}_kA_m, \qquad a \otimes b \mapsto ab$$

and *unit elements* $\mathbf{1}_k \in {}_kA_k$ such that obvious associativity and unit conditions are satisfied.

We emphasize that $\mathbf{1}_k = 0$ is allowed, in which case for example ${}_kA_k = \{0\}$. Note that I-partial algebras can be seen as small $\mathbb{C}$-linear categories, but we will use a different notion of morphisms for them than the usual one of functor. Other names for I-partial algebra would be $\mathbb{C}$-*algebroid* (with object set I) or $\mathbb{C}[I]$-*algebra*.

By making M the zero map on all other tensor products, we can turn $A = \oplus_{k,l}\, {}_kA_l$ into an associative algebra, the *total algebra* of $\mathscr{A}$. It is a locally unital algebra by the orthogonal idempotents $\mathbf{1}_k$.

For example, for any set I we can define a partial algebra $\mathscr{M}at_I$ with ${}_k\mathrm{Mat}_l = \mathbb{C}$ for all k, l and each $M_{k,l,m}$ scalar multiplication. The associated total algebra is the algebra of all finitely supported matrices based over I. For a general $\mathscr{A}$, one can identify A with finite support I-indexed matrices $(a_{kl})_{k,l}$ with $a_{kl} \in {}_kA_l$, equipped with the natural matrix multiplication.

Working with non-unital algebras necessitates the use of their *multiplier algebra* [6,43]. Recall that a multiplier m for an algebra A consists of a couple of linear maps

$a \mapsto L_m(a) = ma$ and $a \mapsto R_m(a) = am$ such that $a(bm) = (ab)m$, $(ma)b = m(ab)$ and $(am)b = a(mb)$ for all $a, b \in A$. They form an algebra, the *multiplier algebra* $M(A)$, under composition for the L-maps and anti-composition for the R-maps. In case A is the total algebra of an I-partial algebra $\mathscr{A}$, the natural homomorphism $A \to M(A)$ is injective, and $M(A)$ can be identified with matrices $(m_{kl})_{kl}$ which are rcf in the sense of the following definition.

Definition 1.2. An assignment $(k,l) \to m_{kl}$ into a set with distinguished zero element is called *row-and column-finite* (rcf) if it has finite support in either one of the variables when the other variable has been fixed.

When $m_i \in M(A)$ are such that for each $a \in A$ one has $m_i a = 0 = a m_i$ for all but a finite set of i, one can define a multiplier $\sum_i m_i$ in the obvious way. One says that the sum $\sum_i m_i$ converges in the *strict* topology. We will often use this kind of limit implicitly, for example in the next definition.

Definition 1.3. Let $\mathscr{A}$ and $\mathscr{B}$ be respectively I and J-partial algebras. A *φ-morphism* from $\mathscr{A}$ to $\mathscr{B}$ consists of a map $\varphi: I \to \mathscr{P}(J)$, the power set of J, and a homomorphism $f: A \to M(B)$ such that $f(\mathbf{1}_k) = \sum_{r\in\varphi(k)} \mathbf{1}_r$.

Note that a φ-morphism f splits up into linear maps $f_{rs}: {}_kA_l \to {}_rB_s$ for all $r \in \varphi(k), s \in \varphi(l)$, which completely determine f. These components satisfy

(a) $f_{rt}(\mathbf{1}_k) = \delta_{rt}\mathbf{1}_r$ for $r, t \in \varphi(k)$, and
(b) $f_{rt}(ab) = \sum_{s\in\varphi(l)} f_{rs}(a) f_{st}(b)$ for all $a \in {}_kA_l, b \in {}_lA_m, r \in \varphi(k)$ and $t \in \varphi(m)$,

where the last sum is finite by the rcf property of multipliers. Conversely, if one has a family of linear maps f_{rs} with $(r,s) \to f_{rs}(a)$ rcf on $\phi(k) \times \phi(l)$ for each $a \in {}_kA_l$, then one can meaningfully impose conditions (a) and (b) on these maps, which then determine a unique φ-morphism f. Note that if the images of φ are all singletons, we find back the notion of ($\mathbb{C}$-linear) functor.

1.2. Partial coalgebras

The notion of a partial algebra dualizes as follows.

Definition 1.4. An *I-partial coalgebra* $\mathscr{A}$ consists of a set $I = \{k, l, \ldots\}$, the *object set*, together with vector spaces A^k_l, comultiplication maps

$$\Delta_l = \Delta\begin{pmatrix} k \\ l \\ m \end{pmatrix}: A^k_m \to A^k_l \otimes A^l_m, \qquad a \mapsto a_{(l;1)} \otimes a_{(l;2)},$$

and counit maps $\epsilon = \epsilon_k \colon A^k_k \to \mathbb{C}$ satisfying obvious coassociativity and counitality conditions.

In the following, we will extend ϵ as the zero functional on A^k_l when $k \neq l$.

1.3. Partial bialgebras

We write I^2 for $I \times I$ seen as column vectors. To superimpose the notions of partial algebra and partial coalgebra into that of a partial bialgebra, we need the maps

$$\varphi_\Delta \colon I^2 \to \mathscr{P}(I^2 \times I^2), \quad \varphi_\Delta(\left(\begin{smallmatrix} k \\ m \end{smallmatrix}\right)) = \{(\left(\begin{smallmatrix} k \\ l \end{smallmatrix}\right), \left(\begin{smallmatrix} l \\ m \end{smallmatrix}\right)) \mid l \in I\},$$
$$\varphi_\epsilon \colon I^2 \to \mathscr{P}(I), \quad \varphi_\epsilon(\left(\begin{smallmatrix} k \\ l \end{smallmatrix}\right)) = \begin{cases} \{k\} & \text{if } k = l, \\ \emptyset & \text{if } k \neq l. \end{cases}$$

We also use the natural tensor product of an I-partial algebra $\mathscr{A}$ and J-partial algebra $\mathscr{B}$, which is an $I \times J$-partial algebra $\mathscr{A} \otimes \mathscr{B}$ with total algebra $A \otimes B$. This corresponds to the usual tensor product of (small) $\mathbb{C}$-linear categories.

Definition 1.5. A *partial bialgebra* $\mathscr{A}$ consists of a set I, the *object set*, a collection of vector spaces ${}^k_m A^l_n$ with I^2-partial algebra structure on the ${}_{\left(\begin{smallmatrix} k \\ l \end{smallmatrix}\right)} A_{\left(\begin{smallmatrix} m \\ n \end{smallmatrix}\right)} = {}^k_m A^l_n$, and a φ_Δ-homomorphism $\mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$ and φ_ϵ-homomorphism $\epsilon \colon \mathscr{A} \to \mathscr{M}at_I$ whose components turn $A^{(k\ l)}_{(m\ n)} = {}^k_m A^l_n$ into an $I \times I$-partial coalgebra.

Spelled out, this means we have maps and elements

$$M \colon {}^k_r A^l_s \otimes {}^l_s A^m_t \to {}^k_r A^m_t, \quad \Delta_{rs} \colon {}^k_m A^l_n \to {}^k_r A^l_s \otimes {}^r_m A^s_n, \quad \mathbf{1}\left(\begin{smallmatrix} k \\ l \end{smallmatrix}\right) \in {}^k_l A^k_l, \quad \epsilon \colon {}^k_k A^l_l \to \mathbb{C}$$

satisfying (co)associativity and (co)unitality, and such that moreover

(a) $\epsilon(\mathbf{1}\left(\begin{smallmatrix} k \\ k \end{smallmatrix}\right)) = 1$,
(b) $\epsilon(ab) = \epsilon(a)\epsilon(b)$ whenever a, b are composable,
(c) $\Delta_{ll'}(\mathbf{1}\left(\begin{smallmatrix} k \\ m \end{smallmatrix}\right)) = \delta_{l,l'} \mathbf{1}\left(\begin{smallmatrix} k \\ l \end{smallmatrix}\right) \otimes \mathbf{1}\left(\begin{smallmatrix} l \\ m \end{smallmatrix}\right)$, and
(d) $\Delta_{rs}(ab) = \sum_t \Delta_{rt}(a)\Delta_{ts}(b)$ whenever a, b are composable.

Note that this sum in the last entry is finite, as it is implicit in the definition that the applications $(r, s) \to \Delta_{rs}(a)$ are rcf for each a. We will use the Sweedler notation $\Delta(a) = a_{(1)} \otimes a_{(2)}$ for the total comultiplication, and $\Delta_{rs}(a) = a_{(rs;1)} \otimes a_{(rs;2)}$ for its components.

It will be convenient to consider the multipliers

$$\lambda_k = \sum_l \mathbf{1}\left(\begin{smallmatrix} k \\ l \end{smallmatrix}\right) \in M(A), \qquad \rho_l = \sum_k \mathbf{1}\left(\begin{smallmatrix} k \\ l \end{smallmatrix}\right) \in M(A).$$

Then by [45, Proposition A.3], there is a unique homomorphism $\Delta\colon M(A) \to M(A \otimes A)$ extending Δ on A and satisfying $\Delta(1) = \sum_k \rho_k \otimes \lambda_k$. It follows by elementary calculations that the total objects $(A, M, \Delta, \epsilon, \Delta(1))$ form a *regular weak multiplier bialgebra* [4, Definition 2.1 and Definition 2.3]. We will call (A, Δ) the *total weak multiplier bialgebra* associated to $\mathscr{A}$.

Recall from [4, Section 3] that a regular weak multiplier bialgebra admits four projections $\Pi^L, \Pi^R, \overline{\Pi}^L, \overline{\Pi}^R\colon A \to M(A)$, where for example $\overline{\Pi}^L(a) = (\epsilon \otimes \mathrm{id})((a \otimes 1)\Delta(1))$. One computes that for $a \in {}^k_m A^l_n$, one has

$$\Pi^L(a) = \epsilon(a)\lambda_m = \epsilon(a)\lambda_k, \qquad \overline{\Pi}^L(a) = \epsilon(a)\lambda_n = \epsilon(a)\lambda_l,$$

$$\Pi^R(a) = \epsilon(a)\rho_l = \epsilon(a)\rho_n, \qquad \overline{\Pi}^R(a) = \epsilon(a)\rho_k = \epsilon(a)\rho_m.$$

The *base algebra* of (A, Δ) is therefore the algebra $\mathrm{Fun}_f(I)$ of finite support functions on I. By [4, Theorem 3.13], the comultiplication of A is left and right *full* ('the legs of $\Delta(A)$ span A').

It is of more interest to consider the converse question. If (A, Δ) is a regular left and right full weak multiplier bialgebra, let us write $A^L = \Pi^L(A) = \overline{\Pi}^L(A)$ and $A^R = \Pi^R(A) = \overline{\Pi}^R(A)$ for the base algebras. By [4, Lemma 4.8], the algebra A^L is anti-isomorphic to A^R by the map $\sigma\colon A^L \to A^R$ sending $\overline{\Pi}^L(a)$ to $\Pi^R(a)$. We then refer to A^L as *the* base algebra.

Proposition 1.6. *Let (A, Δ) be a regular left and right full weak multiplier bialgebra whose base algebra is isomorphic to $\mathrm{Fun}_f(I)$ for some set I, and such that moreover $A^L A^R \subseteq A$. Then (A, Δ) is the total weak multiplier bialgebra of a uniquely determined partial bialgebra $\mathscr{A}$ over I.*

The condition $A^L A^R \subseteq A$ is essential, and should be considered as a *properness* condition. Indeed, this condition can be interpreted as saying that morphism spaces of the 'quantum category' associated to A are compact, which coincides with the notion of properness for a groupoid with discrete object set, cf. [42].

Proof. Write $\lambda_k \in A^L$ for the function $\lambda_k(l) = \delta_{kl}$, and write $\sigma(\lambda_k) = \rho_k \in A^R$. By assumption, $\mathbf{1}\binom{k}{l} = \lambda_k \rho_l \in A$. Further $A = AA^R = AA^L = A^L A = A^R A$, cf. the proof of [4, Theorem 3.13]. Hence the $\mathbf{1}\binom{k}{l}$ make A into the total algebra of an I^2-partial algebra, as A^L and A^R elementwise commute by [4, Lemma 3.5].

Let us show that $\Delta(1) = \sum_k \rho_k \otimes \lambda_k$. By [4, Lemma 3.9], we have $(\rho_k \otimes 1)\Delta(a) = (1 \otimes \lambda_k)\Delta(a)$ for all $a \in A$. By [4, Lemma 4.10] and the fact that $\Delta(1)$ is an idempotent, we can then write $\Delta(1) = \sum_{k \in I'} \rho_k \otimes \lambda_k$ for some subset $I' \subseteq I$. As by definition $\overline{\Pi}^L(A) = \mathrm{Fun}_f(I)$, we deduce that $I = I'$. We then have as well that $\Delta(\mathbf{1}\binom{k}{m}) = \sum_l \mathbf{1}\binom{k}{l} \otimes \mathbf{1}\binom{l}{m}$ by [4, Lemma 3.3], and it follows that Δ is a φ_Δ-homomorphism in the sense of Definition 1.5.

For $a \in {}^{k}_{p}A^{l}_{q}$ and $b \in {}^{l}_{q}A^{m}_{r}$, we then have $\epsilon(ab) = \epsilon(a\mathbf{1}\binom{l}{q}b) = \epsilon(a)\epsilon(b)$ by [4, Proposition 2.6.(4)]. The counitality of ϵ gives $\delta_{kl}\mathbf{1}\binom{l}{m} = \mathbf{1}\binom{k}{m}\mathbf{1}\binom{l}{m} = (\epsilon \otimes \mathrm{id})\left(\Delta(\mathbf{1}\binom{k}{m})(1 \otimes \mathbf{1}\binom{l}{m})\right) = \epsilon(\mathbf{1}\binom{k}{l})\mathbf{1}\binom{l}{m}$ for all k, l, m. Summing over m and using $\lambda_l \neq 0$, we deduce $\epsilon(\mathbf{1}\binom{k}{l}) = \delta_{kl}$. This shows that ϵ is a φ_ϵ-homomorphism.

As Δ is coassociative and ϵ satisfies the counit property, it is clear that the components of Δ and ϵ satisfy the conditions for a partial coalgebra, which finishes the proof. $\square$

1.4. Partial Hopf algebras

Definition 1.7. A partial bialgebra $\mathscr{A}$ is called a *partial Hopf algebra* if the total weak multiplier bialgebra (A, Δ) admits an antipode $S: A \to M(A)$ in the sense of [4, Theorem 6.8].

By [4, Theorem 6.8, Theorem 6.12 and Proposition 6.13], the antipode is uniquely determined, anti-multiplicative and non-degenerate, and by [4, Corollary 6.16] moreover anti-comultiplicative, $\Delta(S(a)) = (S \otimes S)\Delta^{\mathrm{op}}(a)$ for all $a \in A$. We will call S the *total antipode* of $\mathscr{A}$. The next proposition will show that we have in fact in particular that $S(A) \subseteq A$.

Proposition 1.8. *Let $\mathscr{A}$ be a partial Hopf algebra. Then S maps ${}^{k}_{m}A^{l}_{n}$ into ${}^{n}_{l}A^{m}_{k}$, and*

$$\sum_s a_{(rs;1)}S(a_{rs;2}) = \epsilon(a)\mathbf{1}\binom{k}{r}, \qquad \sum_r S(a_{(rs;1)})a_{(rs;2)} = \epsilon(a)\mathbf{1}\binom{s}{n}, \qquad a \in {}^{k}_{m}A^{l}_{n}. \tag{1.1}$$

Conversely, if $\mathscr{A}$ is a partial bialgebra with maps $S: {}^{k}_{m}A^{l}_{n} \to {}^{n}_{l}A^{m}_{k}$ satisfying the above identities, then the latter extend by linearity to an antipode of (A, Δ) and $\mathscr{A}$ is a partial Hopf algebra.

Proof. Let $\mathscr{A}$ be a partial Hopf algebra. From [4, Lemma 6.14], applied with one of the elements of the form $\mathbf{1}\binom{k}{k}$, we deduce $S(\mathbf{1}\binom{k}{m}a\mathbf{1}\binom{l}{m}) = \mathbf{1}\binom{m}{l}S(a)\mathbf{1}\binom{m}{k}$, hence $S\left({}^{k}_{m}A^{l}_{n}\right) \subseteq {}^{n}_{l}A^{m}_{k}$. From the identities (6.14) in [4], we obtain $ab_{(1)}S(b_{(2)}) = a\Pi^L(b)$ for all $a, b \in A$. Taking $a = \mathbf{1}\binom{k}{r}$, we find $\sum_s b_{(rs;1)}S(b_{rs;2)}) = \epsilon(b)\mathbf{1}\binom{k}{r}$ for all $b \in {}^{k}_{m}A^{l}_{n}$. The other antipode identity in (1.1) follows similarly.

Conversely, assume that $\mathscr{A}$ is a partial bialgebra and S is defined on components and satisfies (1.1). Then the linear extension of S satisfies

$$ba_{(1)}S(a_{(2)}) = b\Pi^L(a), \qquad S(a_{(1)})a_{(2)}b = \Pi^R(a)b, \qquad \forall a, b \in A. \tag{1.2}$$

Hence $a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)}S(b_{(3)})c = a_{(1)}b_{(1)} \otimes a_{(2)}\Pi^L(b_{(2)})c$ for all $a, b, c \in A$. As $b_{(1)} \otimes \Pi^L(b_{(2)}) = \Delta(1)(b \otimes 1)$ by an easy computation, identity [4, Theorem 6.8.(2)(vii)] holds. In the same way one proves the identity in [4, Theorem 6.8.(2)(viii)]. Finally, [4, Theorem 6.8.(2)(ix)] requires $\sum_k S(\rho_m a)\lambda_m = S(a)$ for all $a \in A$, but this is immediate from the condition $S({}^{k}_{m}A^{l}_{n}) = {}^{n}_{l}A^{m}_{k}$. $\square$

Lemma 1.9. *Let $(\mathscr{A}, \Delta)$ be a partial Hopf algebra. Then $\epsilon \circ S = \epsilon$.*

Proof. As both $\epsilon \circ S$ and ϵ vanish on ${}^{k}_{m}A^{l}_{n}$ when $k \neq m$ or $l \neq n$, it suffices to check the identity on $a \in {}^{k}_{k}A^{l}_{l}$. But then $\epsilon(S(a)) = \epsilon(a_{(kl;1)})\epsilon(S(a_{(kl;2)}) = \sum_s \epsilon(a_{(ks;1)})\epsilon(S(a_{(ks;2)}))$. By partial multiplicativity of ϵ and (1.1), this equals $\epsilon\left(\sum_s a_{(ks;1)}S(a_{(ks;2)})\right) = \epsilon(a)\epsilon(\mathbf{1}\binom{k}{k}) = \epsilon(a)$. $\square$

Define now on I the relation $k \sim l \iff \mathbf{1}\binom{k}{l} \neq 0$. As $\epsilon(\mathbf{1}\binom{k}{k}) = 1$, we have in particular $\mathbf{1}\binom{k}{k} \neq 0$, hence this relation is reflexive. As $S(\mathbf{1}\binom{k}{l}) = \mathbf{1}\binom{l}{k}$, this relation is symmetric. As $\Delta_{ll}(\mathbf{1}\binom{k}{m}) = \mathbf{1}\binom{k}{l} \otimes \mathbf{1}\binom{l}{m}$, this relation is transitive. Hence $\sim$ is an equivalence relation.

Definition 1.10. The *hyperobject* set of a partial Hopf algebra $\mathscr{A}$ is the set $I/\sim$.

For reasons of technical simplicity, and since it will be sufficient for our purposes, we will later on assume that the hyperobject set is finite. This is no real restriction, as results for infinite hyperobject sets can then easily be derived by inductive limit arguments.

Definition 1.11. A partial Hopf algebra $\mathscr{A}$ will be called *regular* if the antipode $S\colon A \to A$ is invertible.

We can characterize regularity in terms of the coopposite partial bialgebra $\mathscr{A}^{\mathrm{cop}}$, for which the grading is ${}^{k}_{m}(A^{\mathrm{cop}})^{l}_{n} = {}^{m}_{k}A^{n}_{l}$ with the same multiplication maps but $\Delta_{rs}(a) = a_{(rs;2)} \otimes a_{(rs;1)}$.

Lemma 1.12. *A partial Hopf algebra $\mathscr{A}$ is regular if and only if the partial bialgebra $\mathscr{A}^{\mathrm{cop}}$ is a partial Hopf algebra. In this case, S^{-1} is the antipode of $\mathscr{A}^{\mathrm{cop}}$.*

Proof. Note that $\overline{\Pi}^{R/L}$ is the $\Pi^{L/R}$-map for $\mathscr{A}^{\mathrm{cop}}$. Hence, by Proposition 1.8, $\mathscr{A}^{\mathrm{cop}}$ is a partial Hopf algebra if and only if there exists a map $T\colon A \to A$ mapping ${}^{k}_{m}A^{l}_{n}$ into ${}^{n}_{l}A^{m}_{k}$ and such that $ba_{(2)}T(a_{(1)}) = b\overline{\Pi}^{R}(a)$ and $T(a_{(2)})a_{(1)}b = \overline{\Pi}^{L}(a)b$ for all $a, b \in A$.

Now if S is invertible, we see that these identities indeed hold with $T = S^{-1}$ by applying S to them and using $S \circ \overline{\Pi}^{R/L} = \Pi^{L/R}$. Conversely, if such a T exists, take $a \in {}^{k}_{m}A^{l}_{n}$ and put $e = \mathbf{1}\binom{k}{m}$ and $f = \mathbf{1}\binom{l}{n}$. Then, using that we know the behavior of S and T on the local units, we compute on the one hand

$$ST(ea_{(1)})S(a_{(2)})a_{(3)}f = ST(ea_{(1)})\Pi^{R}(a_{(2)})f = ST(ea_{(1)}\Pi^{R}(a_{(2)}))f = ST(a),$$

while on the other

$$\begin{aligned} ST(ea_{(1)})S(a_{(2)})a_{(3)}f &= S(a_{(2)}T(ea_{(1)}))a_{(3)}f = S(\overline{\Pi}^{R}(a_{(1)})T(e))a_{(2)}f \\ &= e\Pi^{L}(a_{(1)})a_{(2)}f = a. \end{aligned}$$

Hence $ST(a) = a$ for all $a \in A$. $\square$

1.5. Invariant integrals

Invariant integrals for I-partial bialgebras over a finite set I were introduced in [14]. A more general definition of invariant integrals for regular left and right full weak multiplier bialgebras was developed in [23], see also [2, Section 3] and the unpublished work [47]. We will base our definition on the characterization obtained in [47, Proposition 2.7], but as in [14] we will assume that our integral has been normalized on the base algebra. The normalization will however be different from the one in [14], where the Dirac functions λ_k and ρ_m were assigned weight one, as this normalization does not make sense in case I is infinite.

Definition 1.13. Let $\mathscr{A}$ be an I-partial bialgebra. A functional $\phi\colon A \to \mathbb{C}$ is called an *invariant integral* if $\phi(\mathbf{1}\binom{k}{k}) = 1$ for all k and

$$(\mathrm{id} \otimes \phi)\Delta(a) = \sum_k \phi\left(\lambda_k a \lambda_k\right) \lambda_k, \qquad (\phi \otimes \mathrm{id})\Delta(a) = \sum_m \phi\left(\rho_m a \rho_m\right) \rho_m$$

as multipliers in $M(A)$.

It follows from the Larson–Sweedler theorem, [47, Theorem 2.14], that if one moreover assumes ϕ to be *faithful*, in the sense that $\phi(ab) = 0$ for all b (resp. all a) then $a = 0$ (resp. $b = 0$), then $\mathscr{A}$ is automatically a regular I-partial Hopf algebra. Conversely, we will show in the next section that an invariant integral on a regular I-partial Hopf algebra is automatically faithful.

Note that if ϕ is an invariant integral on an I-partial bialgebra, then $\phi(\mathbf{1}\binom{k}{m}) = 1$ *whenever* $\mathbf{1}\binom{m}{k} \neq 0$, by applying $(\mathrm{id} \otimes \phi)$ to $\Delta_{kk}(\mathbf{1}\binom{m}{m})$. In particular, also in this case the relation $k \sim l \Leftrightarrow \mathbf{1}\binom{k}{l} \neq 0$ is an equivalence relation.

An invariant integral will have support on the homogeneous components of the form ${}^{k}_{m}A^{k}_{m}$.

Lemma 1.14. *Let $\mathscr{A}$ be an I-partial bialgebra with invariant integral ϕ. Then for all $a \in A$ and all $k, m \in I$, one has $\phi(\mathbf{1}\binom{k}{m}a) = \phi(a\mathbf{1}\binom{k}{m})$.*

Proof. Cf. the discussion following [47, Proposition 2.7]. Namely, if $a \in A$ and $s \in I$, we compute

$$(\mathrm{id} \otimes \phi)((1 \otimes \lambda_s)\Delta(a)) = \rho_s(\mathrm{id} \otimes \phi)(\Delta(a)) = (\mathrm{id} \otimes \phi)(\Delta(a))\rho_s = (\mathrm{id} \otimes \phi)(\Delta(a)(1 \otimes \lambda_s)).$$

As both sides lie in A since $(r, s) \to \Delta_{rs}(a)$ is rcf, we can apply ϵ to conclude $\phi(\lambda_s a) = \phi(a\lambda_s)$. Similarly the identity $\phi(\rho_s a) = \phi(a\rho_s)$ can be derived. $\square$

One easily concludes from this that an invariant integral ϕ is uniquely determined, since any other invariant integral ψ satisfies

$$\phi(a) = \psi\left(\mathbf{1}\left(\begin{smallmatrix} k \\ k \end{smallmatrix}\right)\right)\phi(a) = (\psi \otimes \phi)(\Delta_{kk}(a)) = \psi(a)\phi(\mathbf{1}\left(\begin{smallmatrix} k \\ m \end{smallmatrix}\right)) = \psi(a), \qquad a \in {}^{k}_{m}A^{k}_{m}.$$

1.6. Partial compact quantum groups

Definition 1.15. A *partial $*$-algebra* is a partial algebra $\mathscr{A}$ whose total algebra A is equipped with an anti-linear, anti-multiplicative involution $a \mapsto a^*$ with $\mathbf{1}_k^* = \mathbf{1}_k$ for all objects k.

This implies that $*$ restricts to anti-linear maps ${}_kA_l \to {}_lA_k$.

Definition 1.16. A *partial $*$-bialgebra* is a partial bialgebra $\mathscr{A}$ whose underlying partial algebra has been endowed with a partial $*$-algebra structure such that $\Delta_{rs}(a)^* = \Delta_{sr}(a^*)$ for all $a \in {}^{k}_{m}A^{l}_{n}$. A *partial Hopf $*$-algebra* is a partial bialgebra which is at the same time a partial $*$-bialgebra and a partial Hopf algebra.

Proposition 1.17. *An I-partial $*$-bialgebra $\mathscr{A}$ is an I-partial Hopf $*$-algebra if and only if the weak multiplier $*$-bialgebra (A, Δ) is a weak multiplier Hopf $*$-algebra. In that case, the counit and antipode satisfy $\epsilon(a^*) = \overline{\epsilon(a)}$ and $S(S(a)^*)^* = a$ for all $a \in A$.*

Note that $\mathscr{A}$ is then automatically regular.

Proof. The if and only if part follows immediately from Proposition 1.8, the relation for the counit from uniqueness of the counit [4, Theorem 2.8], and the relation for the antipode from [46, Proposition 4.11]. □

Definition 1.18. A *partial compact quantum group* consists of a partial Hopf $*$-algebra $\mathscr{A}$ with an invariant integral ϕ that is *positive* in the sense that $\phi(a^*a) \geq 0$ for all $a \in A$. We then write $\mathscr{A} = \mathscr{P}(\mathscr{G})$, where we refer to $\mathscr{G}$ as the partial compact quantum group defined by $\mathscr{A}$, and to $\mathscr{A}$ as the algebra of (regular) functions on $\mathscr{G}$.

It will follow from Theorem 2.14 and [14, Theorem 3.3 and Theorem 4.4] that for I finite, a partial compact quantum group is precisely a compact quantum group of face type [14, Definition 4.1]. As the total structure should not be considered compact for I infinite, we have changed the terminology to *partial* compact quantum group to reflect that only the parts should be considered compact.

1.7. Tensor categories

We assume that the reader is familiar with the basic notions concerning tensor categories – we refer to [28] for an overview. We will always assume that our tensor categories

$(\mathcal{C}, \otimes)$ are $\mathbb{C}$-linear and strict. If moreover $\mathcal{C}$ is semisimple, then each object has a finite-dimensional endomorphism algebra, and will be isomorphic to a direct sum of irreducible (or, equivalently, simple) objects. We will in general *not* assume that the unit object $\mathbb{1}$ of $\mathcal{C}$ is simple. A tensor category will be called *rigid* if each object has a left and right dual object. By *tensor functor* we will mean a strongly monoidal functor.

If $\mathcal{C}$ is semisimple, we know by the Eckmann–Hilton argument that $\mathrm{End}(\mathbb{1}) \cong \mathbb{C}^{\mathscr{I}}$ for some finite set $\mathscr{I}$. If an identification with such a set $\mathscr{I}$ has been made, we will refer to $\mathscr{I}$ as the *hyperobject set* of $\mathcal{C}$. To each $\alpha \in \mathscr{I}$ then corresponds a simple subobject $\mathbb{1}_\alpha$ of $\mathbb{1}$. If $\alpha, \beta \in \mathscr{I}$, we can then construct full subcategories $\mathcal{C}_{\alpha\beta}$ of $\mathcal{C}$ consisting of all objects X for which $\mathbb{1}_\alpha \otimes X \otimes \mathbb{1}_\beta \cong X$. The collection $\{\mathcal{C}_{\alpha\beta}\}$ can be looked upon as a particular kind of 2-category, or, seeing it as a categorification of the notion of partial algebra, as a *partial tensor category*. In this way, one could also treat the case where $\mathscr{I}$ is infinite, in which case the associated total tensor category would be 'non-unital with local units'. However, to be able to stick to more familiar terminology and to avoid some technical points, we will not discuss this more general setting which would bring nothing essentially new to the discussion.

We will also need the more structured notion of *tensor* C*-*category*, by which we will understand a tensor category with each $\mathrm{Mor}(X, Y)$ a Banach space equipped with an anti-linear map $*\colon \mathrm{Mor}(X, Y) \to \mathrm{Mor}(Y, X)$ satisfying the appropriate submultiplicativity and C*-conditions – see [26] or again [28]. Again, we stress that we do not assume the unit of a tensor C*-category to be simple. In a rigid tensor C*-category, left and right duals are isomorphic, and given an object X we will simply pick a dual object and denote it by $\overline{X}$.

2. Representation theory of partial compact quantum groups

2.1. Corepresentations of partial bialgebras

A notion of *full comodule* for a general weak multiplier bialgebra was introduced in [2, Definition 2.1 and Definition 4.1]. It was then shown in [2, Theorem 5.1] that the category of full comodules forms a (possibly not semisimple) tensor category, and in [2, Theorem 6.2] that the category of *finite-dimensional* full comodules for a weak multiplier Hopf algebra forms a rigid (but possibly not semisimple) tensor category. However, in many situations the class of finite-dimensional comodules will be too small. We introduce here for partial bialgebras a class of 'intermediate size' comodules, which in the case of partial Hopf algebras with invariant integral will turn out to be large enough. We will however phrase the result in terms of corepresentations in stead of comodules, as this will be more appropriate when discussing matrix coefficients. We then make the connection with the comodules from [2] in Lemma 2.6.

Definition 2.1. Let I be a set. An $I \times I$-graded vector space $V = \bigoplus_{k,l \in I} {}_kV_l$ will be called *row-and column finite-dimensional* (rcfd) if the $\oplus_{l \in I}\, {}_kV_l$ (resp. $\oplus_{k \in I}\, {}_kV_l$) are finite-dimensional for each k (resp. l) fixed.

We denote by $\mathrm{Vect}_{\mathrm{rcfd}}^{I\times I}$ the category whose objects are rcfd $I\times I$-graded vector spaces. Morphisms are linear maps T that preserve the grading, and can therefore be written as $T=\prod_{k,l\in I} {}_kT_l$.

Definition 2.2. Let $\mathscr{A}$ be an I-partial bialgebra. A *corepresentation* $\mathscr{X}$ of $\mathscr{A}$ on an rcfd $I\times I$-graded vector space V is a family of elements ${}^{k}_{m}X^{l}_{n} \in {}^{k}_{m}A^{l}_{n} \otimes \mathrm{Hom}_{\mathbb{C}}({}_mV_n, {}_kV_l)$ satisfying $(\Delta_{pq}\otimes \mathrm{id})({}^{k}_{m}X^{l}_{n}) = \left({}^{k}_{p}X^{l}_{q}\right)_{13}\left({}^{p}_{m}X^{q}_{n}\right)_{23}$ and $(\epsilon\otimes\mathrm{id})({}^{k}_{m}X^{l}_{n}) = \delta_{k,m}\delta_{l,n}\,\mathrm{id}_{{}_kV_l}$.

We use here the standard leg numbering notation, e.g. $a_{23}=1\otimes a$.

Example 2.3.

1. Equip the vector space $\mathbb{C}^{(I)}=\bigoplus_{k\in I}\mathbb{C}$ with the diagonal $I\times I$-grading. Then the family $\mathscr{U}$ given by ${}^{k}_{m}U^{l}_{n} = \delta_{k,l}\delta_{m,n}\mathbf{1}\binom{k}{m} \in {}^{k}_{m}A^{l}_{n}$ is a corepresentation of $\mathscr{A}$ on $\mathbb{C}^{(I)}$, called the *trivial corepresentation*.
2. Assume given an rcfd family of subspaces ${}_mV_n \subseteq \bigoplus_{k,l} {}^{k}_{m}A^{l}_{n}$ with $\Delta_{pq}({}_mV_n) \subseteq {}_pV_q \otimes {}^{p}_{m}A^{q}_{n}$ for all indices. Then the elements ${}^{k}_{m}X^{l}_{n} \in {}^{k}_{m}A^{l}_{n}\otimes \mathrm{Hom}_{\mathbb{C}}({}_mV_n, {}_kV_l)$ defined by

$$ {}^{k}_{m}X^{l}_{n}(1\otimes b) = \Delta^{\mathrm{op}}_{kl}(b) \in {}^{k}_{m}A^{l}_{n}\otimes {}_kV_l \quad \text{for all } b\in {}_mV_n $$

   form a corepresentation $\mathscr{X}$ of $\mathscr{A}$ on V. Corepresentations of this form will be called *regular*.

A morphism T between corepresentations $(V,\mathscr{X})$ and $(W,\mathscr{Y})$ of $\mathscr{A}$ will be a morphism T from V to W satisfying the intertwiner property $(1\otimes {}_kT_l)\,{}^{k}_{m}X^{l}_{n} = {}^{k}_{m}Y^{l}_{n}(1\otimes {}_mT_n)$. In this way, corepresentations form a category which we will denote $\mathrm{Corep}_{\mathrm{rcfd}}(\mathscr{A})$.

We next consider the total form of a corepresentation. Let V be an rcfd $I\times I$-graded vector space, and write p_{kl} for the projections on the component ${}_kV_l$. Write $\mathrm{End}_0(V)$ for the algebra of endomorphisms on V having finite-dimensional support.

Definition 2.4. Let $\mathscr{A}$ be a partial bialgebra, and let V be an rcfd $I\times I$-graded vector space. An element $X\in M(A\otimes \mathrm{End}_0(V))$ is called a *total corepresentation* if for all $k,l,m,n\in I$

$$ (\mathbf{1}\tbinom{k}{m}\otimes \mathrm{id})X(\mathbf{1}\tbinom{l}{n}\otimes\mathrm{id}) = (1\otimes p_{kl})X(1\otimes p_{mn}) \in A\otimes \mathrm{End}_0(V), \tag{2.1} $$

and

$$ (\Delta\otimes\mathrm{id})(X) = X_{13}X_{23}, \qquad (\epsilon\otimes\mathrm{id})(X) = \mathrm{id}_V\,. \tag{2.2} $$

Here $(\epsilon\otimes\mathrm{id})(X)$ makes sense as a multiplier by (2.1) and the fact that ϵ has support on the ${}^{k}_{k}A^{l}_{l}$.

The following lemma is straightforward.

Lemma 2.5. *Let X be a total corepresentation. Then the ${}^{k}_{m}X^{l}_{n} = (\mathbf{1}\binom{k}{m}\otimes\mathrm{id})X(\mathbf{1}\binom{l}{n}\otimes\mathrm{id})$ form a corepresentation $\mathscr{X}$ of $\mathscr{A}$, and every corepresentation arises in this way from a unique total corepresentation.*

We will in the following call X the *corepresentation multiplier* of $\mathscr{X}$.

Recall now the notion of *full right comodule* for a regular weak multiplier bialgebra (A, Δ) [2, Definition 2.1 and Definition 4.1]. It consists of a vector space V, equipped with two maps $\lambda, \rho: V \otimes A \to V \otimes A$ satisfying certain assumptions. It then follows from [2, Theorem 4.1] that V is a firm bimodule over the base algebra. In particular, if (A, Δ) is the total weak multiplier bialgebra associated to a partial bialgebra, $\mathscr{A}$, then V becomes in a natural way an I-bigraded vector space. Whenever considering full right comodules, we will consider V with this natural bigrading.

Proposition 2.6. *Let $\mathscr{A}$ be a partial bialgebra. There is a natural one-to-one correspondence between corepresentations of $\mathscr{A}$ and full right comodules for (A, Δ) on rcfd I-bigraded vector spaces.*

Proof. Let (λ, ρ) be a full right comodule of (A, Δ) on an rcfd vector space V. Let us write ${}_{m}V = \oplus_{n}\, {}_{m}V_{n}$ etc.

It follows from [2, Lemma 4.2.(1) and (7)] that λ maps ${}_{m}V \otimes {}^{l}A$ into $V_{l} \otimes {}_{m}A$. From [2, Lemma 4.2.(8)] it follows that λ has range in $\oplus_{k}\, {}_{k}V \otimes {}^{k}A$. As $\lambda(v \otimes ab) = \lambda(v \otimes a)(1 \otimes b)$ by the multiplier property with respect to ρ, and as V is rcfd, all this implies that we can define inside the tensor product ${}^{k}_{m}A^{l}_{n} \otimes \mathrm{Hom}_{\mathbb{C}}({}_{m}V_{n}, {}_{k}V_{l}) = \mathrm{Hom}_{\mathbb{C}}({}_{m}V_{n}, {}^{k}_{m}A^{l}_{n} \otimes {}_{k}V_{l})$ an element ${}^{k}_{m}X^{l}_{n}$ by the formula

$$v \mapsto (\mathbf{1}\binom{k}{m} \otimes \mathrm{id}_{V})\sigma\lambda(v \otimes \mathbf{1}\binom{l}{n}), \qquad v \in {}_{m}V_{n},$$

where σ is the flip map. Moreover, as λ has support on $\oplus_{n} V_{n} \otimes {}_{n}A$ by [2, Lemma 4.2.(2)], the maps ${}^{k}_{m}X^{l}_{n}$ completely determine λ.

The defining identity [2, (2.12)] for λ, applied to $v \otimes \mathbf{1}\binom{l}{n} \otimes \mathbf{1}\binom{q}{n}$ with $v \in V_{n}$, leads immediately to the corepresentation identities for the ${}^{k}_{m}X^{l}_{n}$. Finally, [2, Lemma 4.2] and the definition of the right action of the base algebra A^{R} on V implies that for $v \in {}_{k}V_{l}$,

$$\begin{aligned}(\epsilon \otimes \mathrm{id})({}^{k}_{k}X^{l}_{l})v &= (\mathrm{id} \otimes \epsilon)((1 \otimes \mathbf{1}\binom{k}{k})\lambda(v \otimes \mathbf{1}\binom{l}{l})) \\ &= \rho_{k}(\mathrm{id} \otimes \epsilon)(\lambda(\rho_{k}v \otimes \mathbf{1}\binom{l}{l})) = \rho_{k}v\rho_{l} = v.\end{aligned}$$

We leave it to the reader to check that conversely, each total corepresentation (V, X) leads to a right comodule (V, λ, ρ) by the formulas $\lambda(v \otimes a) = \sigma X(a \otimes v)$, and $\rho(v \otimes a)(\rho_{l} \otimes 1) = (1 \otimes a)\sigma X(\mathbf{1}\binom{l}{n} \otimes v)$ for $v \in V_{n}$. $\square$

If $\mathscr{A}$ is a partial bialgebra, the category $\mathrm{Corep}_{\mathrm{rcfd}}(\mathscr{A})$ is easily seen to be an abelian category with a faithful functor into $\mathrm{Vect}_{\mathrm{rcfd}}^{I\times I}$ lifting kernels, cokernels and biproducts, cf. [2, Lemma 5.3]. In particular, one can call a corepresentation V *irreducible* if any morphism T from (resp. into) V either has all T_{kl} zero or injective (resp. surjective).

Moreover, from [2, Theorem 5.1] we know that the category of all full comodules over (A, Δ) forms a monoidal category with a strict monoidal imbedding into the tensor category of firm A^R-bimodules, that is, $I \times I$-graded vector spaces. The latter tensor product, which we will denote by $\otimes_I$, can easily be identified concretely as follows:

$$_k(V \otimes_I W)_m = \oplus_l \, {}_kV_l \otimes {}_lW_m.$$

It follows immediately that $\mathrm{Corep}_{\mathrm{rcfd}}(\mathscr{A})$ is a tensor subcategory, represented faithfully inside $\mathrm{Vect}_{\mathrm{rcfd}}^{I\times I}$. In terms of total corepresentations, the tensor product of $\mathscr{X}$ and $\mathscr{Y}$ is given by $X \textcircled{\scriptsize T} Y = X_{12}Y_{13}$, with associated components

$$^k_m(X \textcircled{\scriptsize T} Y)^p_q = \sum_{l,n} \left({}^k_m X^l_n\right)_{12} \left({}^l_n Y^p_q\right)_{13},$$

where the sum is actually finite because of the rcfd condition. The unit is given by the trivial corepresentation $(\mathbb{C}^{(I)}, \mathscr{U})$.

2.2. Corepresentations of partial Hopf algebras

Assume now that $\mathscr{A}$ is a partial Hopf algebra. If V is an rcfd $I \times I$-graded vector space, we will denote by λ_k^V (resp. ρ_k^V) the projection in $\mathrm{End}_0(V)$ onto elements with left (resp. right) grading equal to k.

Lemma 2.7. *Let $(V, \mathscr{X})$ be a corepresentation of $\mathscr{A}$ on an rcfd vector space. Then the element*

$$X^{-1} = (S \otimes \mathrm{id})(X) \in M(A \otimes \mathrm{End}_0(V))$$

is a generalized inverse of X in the sense that $XX^{-1}X = X$ and $X^{-1}XX^{-1} = X^{-1}$. More precisely, one has $XX^{-1} = \sum_k \lambda_k \otimes \lambda_k^V$ and $X^{-1}X = \sum_l \rho_l \otimes \rho_l^V$ (w.r.t. strict convergence).

Proof. This follows immediately from Proposition 1.8 and the corepresentation identity. $\square$

Given a corepresentation $\mathscr{X}$, we then write

$$^k_m(X^{-1})^l_n = (S \otimes \mathrm{id})({}^n_l X^m_k) \in {}^k_m A^l_n \otimes \mathrm{Hom}_{\mathbb{C}}({}_lV_k, {}_nV_m)$$

for the components of X^{-1}.

The following easy lemma will be very useful.

Lemma 2.8. *Let $\mathscr{A}$ be a partial Hopf algebra. A bigraded map T defines a morphism from $(V,\mathscr{X})$ to $(W,\mathscr{Y})$ if and only if one (and hence both) of the following relations hold:*

$$Y^{-1}(1\otimes T)X = \sum_{m,n} \rho_n \otimes {}_mT_n, \qquad Y(1\otimes T)X^{-1} = \sum_{k,l} \lambda_k \otimes {}_kT_l.$$

Proposition 2.9. *Let $\mathscr{A}$ be a partial Hopf algebra. Then* $\mathrm{Corep}_{\mathrm{rcfd}}(\mathscr{A})$ *has left duals. If $\mathscr{A}$ is regular, then* $\mathrm{Corep}_{\mathrm{rcfd}}(\mathscr{A})$ *also has right duals.*

Proof. Let $(V,\mathscr{X})$ be a corepresentation. Denote the dual of vector spaces V and linear maps T by V^* and T^{tr}, respectively, and define the dual of an $I\times I$-graded vector space $V=\bigoplus_{k,l}{}_kV_l$ to be the space $V^\wedge=\bigoplus_{k,l}{}_k(V^\wedge)_l$ where ${}_k(V^\wedge)_l=({}_lV_k)^*$. Then using anti-comultiplicativity of S and Lemma 1.9, we see that $V^\wedge$ and the family $\hat{\mathscr{X}}$ given by

$${}^k_m\hat{X}^l_n := (S\otimes -^{\mathrm{tr}})({}^n_lX^m_k)$$

form a corepresentation of $\mathscr{A}$. To see that it is a left dual of $\mathscr{X}$, consider the natural evaluation and coevaluation maps

$$\begin{aligned}\mathrm{ev}\colon V^\wedge\otimes_I V &\to \mathbb{C}^{(I)}, \quad \omega\otimes v\mapsto \omega(v),\\ \mathrm{coev}\colon \mathbb{C}^{(I)} &\to V\otimes_I V^\wedge, \quad \delta_k\mapsto \sum_{l,i} e_i^{(kl)}\otimes\omega_i^{(lk)},\end{aligned}$$

where $\{e_i^{(kl)}\}$ and $\{\omega_i^{(lk)}\}$ are a basis and its dual basis for ${}_kV_l$. Note that the right hand sum is finite by the rcfd condition. These maps provide the duality between V and $V^\wedge$ in $\mathrm{Vect}^{I\times I}_{\mathrm{rcfd}}$. It then suffices to show that they are also morphisms from the trivial corepresentation to the tensor product representations of $\mathscr{X}$ with $\hat{\mathscr{X}}$. But for example the intertwining property of ev follows from

$$\begin{aligned}(1\otimes {}_k\mathrm{ev}_k)\sum_{l,n}\left({}^k_m\hat{X}^l_n\right)_{12}\left({}^l_nX^k_q\right)_{13} &= (1\otimes {}_k\mathrm{ev}_k)\sum_{l,n}(S\otimes -^{\mathrm{tr}})({}^n_lX^m_k)_{12}({}^l_nX^k_q)_{13}\\ &= \delta_{m,q}(1\otimes {}_m\mathrm{ev}_m)\sum_{l,n}(S\otimes\mathrm{id})({}^n_lX^m_k)_{13}({}^l_nX^k_q)_{13}\\ &= \delta_{m,q}\mathbf{1}\binom{k}{q}\otimes {}_m\mathrm{ev}_m\\ &= {}^k_mU^k_q(1\otimes {}_m\mathrm{ev}_m).\end{aligned}$$

If $\mathscr{A}$ is a regular partial Hopf algebra, it follows that $\mathrm{Corep}_{\mathrm{rcfd}}(\mathscr{A})$ has right duals since $\mathscr{X}\mapsto\hat{\mathscr{X}}$ is then essentially surjective, with inverse $\mathscr{X}\mapsto\check{\mathscr{X}}$ for

$${}^k_m\check{X}^l_n := (S^{-1}\otimes -^{\mathrm{tr}})({}^n_lX^m_k). \qquad \square$$

Remark 2.10. The previous proposition was proven for general weak multiplier Hopf algebras in [2, Theorem 6.2] with respect to *finite-dimensional* full comodules.

2.3. Corepresentations of partial Hopf algebras with invariant integral

In the presence of an invariant integral, one can integrate morphisms of bigraded vector spaces to obtain morphisms of corepresentations.

Lemma 2.11. *Let $(V, \mathscr{X})$ and $(W, \mathscr{Y})$ be rcfd corepresentations of a partial Hopf algebra $\mathscr{A}$ with invariant integral ϕ. Fix $m, n \in I$, and let $T\colon {}_mV_n \to {}_mV_n$. Then the families*

$$
\begin{aligned}
{}_k\check{T}_l &:= (\phi \otimes \mathrm{id})\big({}^{n}_{l}(Y^{-1})^{m}_{k}(1 \otimes T)\,{}^{m}_{k}X^{n}_{l}\big),\\
{}_k\hat{T}_l &:= (\phi \otimes \mathrm{id})\big({}^{k}_{m}Y^{l}_{n}(1 \otimes T)\,{}^{l}_{n}(X^{-1})^{k}_{m}\big)
\end{aligned}
$$

form morphisms $\check{T}$ and $\hat{T}$ from $(V, \mathscr{X})$ to $(W, \mathscr{Y})$.

Proof. Viewing T as a linear map $V \to W$ concentrated at the component at (m, n), we can consider the total forms $\check{T} = (\phi\otimes\mathrm{id})(Y^{-1}(1\otimes T)X)$ and $\hat{T} = (\phi\otimes\mathrm{id})(Y(1\otimes T)X^{-1})$. We compute

$$
\begin{aligned}
Y^{-1}(1 \otimes \check{T})X &= (\phi \otimes \mathrm{id} \otimes \mathrm{id})((Y^{-1})_{23}(Y^{-1})_{13}(1 \otimes 1 \otimes T)X_{13}X_{23})\\
&= ((\phi \otimes \mathrm{id})\Delta \otimes \mathrm{id})(Y^{-1}(1 \otimes T)X)\\
&= \sum_{l} \rho_l \otimes (\phi \otimes \mathrm{id})((\rho_l \otimes 1)Y^{-1}(1 \otimes T)X(\rho_l \otimes 1))\\
&= \sum_{k,l} \rho_l \otimes {}_k\check{T}_l.
\end{aligned}
$$

Hence $\check{T}$ is a morphism from $\mathscr{X}$ to $\mathscr{Y}$ by Lemma 2.8. The assertion for $\hat{T}$ follows similarly. $\square$

Lemma 2.12. *Let $\mathscr{A}$ be a partial Hopf algebra with invariant integral ϕ. Let $(V, \mathscr{X})$ be an rcfd corepresentation, and ${}_kW_l \subseteq {}_kV_l$ an invariant family of subspaces. Then there exists an idempotent endomorphism T of $(V, \mathscr{X})$ such that ${}_kW_l = \mathrm{img}\ {}_kT_l$ for all k, l.*

Proof. We can decompose $V = \oplus {}_\alpha V_\beta$, where for α, β in the hyperobject set $\mathscr{I}$ we write ${}_\alpha V_\beta$ for the direct sum of all ${}_kV_l$ with $k \in \alpha, l \in \beta$. As each ${}_\alpha V_\beta$ is invariant, we may assume that $V = {}_\alpha V_\beta$ for some α, β.

Let $\mathscr{Y}$ be the restriction of $\mathscr{X}$ to W. Fix $n \in \beta$. Let T be a bigraded idempotent endomorphism of V with image W, and write $T^{(m)} = {}_mT_n$. By Lemma 2.11, we obtain endomorphisms $\check{T}^{(m)}$ of $(V, \mathscr{X})$. Using column-finiteness of V, we can define $\check{T} = \sum_m \check{T}^{(m)}$. We claim that W is the image of $\check{T}$.

Invariance of W implies $(1\otimes T)X(1\otimes T) = X(1\otimes T)$. Applying $(S\otimes \mathrm{id})$, we get $(1\otimes T)X^{-1}(1\otimes T) = X^{-1}(1\otimes T)$. Then

$$\begin{aligned}\check{T}T &= (\phi\otimes \mathrm{id})(X^{-1}(1\otimes \rho_n^V T)X(1\otimes T))\\ &= (\phi\otimes \mathrm{id})(X^{-1}(1\otimes \rho_n^V)X(1\otimes T))\\ &= (\phi\otimes \mathrm{id})(X^{-1}X(\lambda_n\otimes T))\\ &= \sum_l \phi(\mathbf{1}\tbinom{n}{l})\rho_l^V T = \sum_{l\in\beta}\rho_l^V T = T.\end{aligned}$$

As ${}_k\check{T}_l$ sends ${}_kV_l$ into ${}_kW_l$, it follows that $\operatorname{img}\check{T} = W$, which proves the claim. □

Lemma 2.13. *Let $\mathscr{A}$ be a partial Hopf algebra with hyperobject set $\mathscr{I}$, and fix representatives $l_\beta\in\beta$ for each β. If T is a morphism in* $\mathrm{Corep}_{\mathrm{rcfd}}(\mathscr{A})$ *and ${}_kT_{l_\beta} = 0$ for all k and β, then $T = 0$.*

Proof. This follows from the equations in Lemma 2.8. □

Theorem 2.14. *Let $\mathscr{A}$ be a partial Hopf algebra with invariant integral. Assume that the hyperobject set $\mathscr{I}$ is finite. Then* $\mathrm{Corep}_{\mathrm{rcfd}}(\mathscr{A})$ *is a semisimple tensor category with left duals and hyperobject set $\mathscr{I}$.*

Proof. It is easy to verify that, for each hyperobject α, the space $\mathbb{C}^{(\alpha)}$ is an invariant subspace of $\mathbb{C}^{(I)}$ w.r.t. the trivial representation $\mathscr{U}$, and that the corresponding subrepresentations $\mathscr{U}^{(\alpha)}$ provide a decomposition of $\mathscr{U}$ into irreducible components.

By Lemma 2.12 and Proposition 2.9, it now suffices to show that each endomorphism space of $\mathrm{Corep}_{\mathrm{rcfd}}(\mathscr{A})$ is finite-dimensional.

Choose a representative l_β for each $\beta\in\mathscr{I}$. Assume that T is an endomorphism of some rcfd corepresentation $(V,\mathscr{X})$ with $\sum_{k\in I} {}_kT_{l_\beta} = 0$ for all β. From Lemma 2.13, it follows that $T = 0$. Hence the map $T\mapsto \sum_{k,\beta} {}_kT_{l_\beta}$ is an injective map from $\mathrm{End}(V,\mathscr{X})$ into a finite-dimensional space of linear maps. □

In fact, as $\mathrm{Corep}_{\mathrm{rcfd}}(\mathscr{A})$ is semisimple, it will automatically have right duals as well. This will also follow more concretely from Corollary 2.25 and Proposition 2.9.

For general partial Hopf algebras with invariant integrals, we have a weak form of semisimplicity.

Corollary 2.15. *Let $\mathscr{A}$ be a partial Hopf algebra with invariant integral. Then every rcfd corepresentation of $\mathscr{A}$ decomposes into a (possibly infinite) direct sum of rcfd irreducible corepresentations.*

Proof. Any rcfd corepresentation is a (possibly infinite) direct sum of rcfd corepresentations with a singleton as left and right hyperobject support. As in Theorem 2.14, we conclude by Lemma 2.13 that such rcfd corepresentations have a finite-dimensional space of self-intertwiners. The corollary then follows again from Lemma 2.12. □

2.4. Schur orthogonality

Our next goal is to obtain Schur orthogonality for matrix coefficients of corepresentations. We give a little more detail than provided in Hayashi's paper [14].

Given finite-dimensional vector spaces V and W, the dual space of $\mathrm{Hom}_{\mathbb{C}}(V,W)$ is linearly spanned by functionals of the form $\omega_{f,v}(T) = (f|Tv)$, where $v \in V$, $f \in W^*$, and $(-|-)$ denotes the natural pairing of W^* with W.

Definition 2.16. Let $\mathscr{A}$ be a partial bialgebra. The space of *matrix coefficients* $\mathcal{C}(\mathscr{X})$ of an rcfd corepresentation $(V,\mathscr{X})$ is the sum of the subspaces

$$ {}^{k}_{m}\mathcal{C}(\mathscr{X})^{l}_{n} = \mathrm{span}\left\{(\mathrm{id}\otimes\omega_{f,v})({}^{k}_{m}X^{l}_{n}) \mid v \in {}_{m}V_{n}, f \in ({}_{k}V_{l})^*\right\} \subseteq {}^{k}_{m}A^{l}_{n}. $$

As $\Delta_{pq}({}^{k}_{m}\mathcal{C}(\mathscr{X})^{l}_{n}) \subseteq {}^{k}_{p}\mathcal{C}(\mathscr{X})^{l}_{q} \otimes {}^{p}_{m}\mathcal{C}(\mathscr{X})^{q}_{n}$, the ${}^{k}_{m}\mathcal{C}(\mathscr{X})^{l}_{n}$ form a partial coalgebra with respect to Δ and ϵ. Moreover, for each k,l the $I\times I$-graded vector space ${}^{k}\mathcal{C}(\mathscr{X})^{l} := \bigoplus_{m,n} {}^{k}_{m}\mathcal{C}(\mathscr{X})^{l}_{n}$ is rcfd, and the inclusion above shows that it supports a regular corepresentation in the sense of Example 2.3.2.

Lemma 2.17. *Let $(V,\mathscr{X})$ be an rcfd corepresentation of a partial bialgebra, and let $f \in ({}_{k}V_{l})^*$. Then the family of maps*

$$ {}_{m}T^{(f)}_{n}\colon {}_{m}V_{n} \to {}^{k}_{m}\mathcal{C}(\mathscr{X})^{l}_{n},\ w \mapsto (\mathrm{id}\otimes\omega_{f,w})({}^{k}_{m}X^{l}_{n}) = (\mathrm{id}\otimes f)({}^{k}_{m}X^{l}_{n}(1\otimes w)), $$

is a morphism from $\mathscr{X}$ to the regular corepresentation on ${}^{k}\mathcal{C}(\mathscr{X})^{l}$.

Proof. Denote by $\mathscr{Y}$ the regular corepresentation on $\bigoplus_{m,n} {}^{k}_{m}\mathcal{C}(\mathscr{X})^{l}_{n}$. Then for all $v \in {}_{m}V_{n}$,

$$ \begin{aligned} {}^{p}_{m}Y^{q}_{n}(1\otimes {}_{m}T^{(f)}_{n}(v)) &= (\Delta^{\mathrm{op}}_{pq}\otimes\omega_{f,v})({}^{k}_{m}X^{l}_{n}) = (\mathrm{id}\otimes\mathrm{id}\otimes f)(({}^{k}_{p}X^{l}_{q})_{23}({}^{p}_{m}X^{q}_{n})_{13}(1\otimes 1\otimes v)) \\ &= (1\otimes {}_{p}T^{(f)}_{q})\,{}^{p}_{m}X^{q}_{n}(1\otimes v). \qquad \square \end{aligned} $$

As before, we denote by V^* the dual of a vector space V.

Lemma 2.18. *Let $\mathscr{A}$ be a partial Hopf algebra. Then for any $a \in \oplus_{k,l}{}^{k}_{m}A^{l}_{n}$, the family of subspaces*

$$ {}_{p}V^{(a)}_{q} = \{(\mathrm{id}\otimes f)(\Delta_{pq}(a))\colon f \in ({}^{p}_{m}A^{q}_{n})^*\} $$

supports a regular corepresentation such that $a \in {}_{m}V^{(a)}_{n}$. If further $(W,\mathscr{Y})$ is an irreducible *regular corepresentation, then ${}_{k}W_{l} = {}_{k}V^{(a)}_{l}$ for all k,l and any non-zero $a \in {}_{m}W_{n}$.*

Proof. Assume that a and $V^{(a)}$ are as above. Taking $f = \epsilon$, one finds $a \in {}_mV_n^{(a)}$. Next, write $\Delta_{pq}(a) = \sum_i b_{pq}^i \otimes c_{pq}^i$ with linearly independent $(c_{pq}^i)_i$. Then ${}_pV_q^{(a)} = \operatorname{span}\{b_{pq}^i : i\}$, and $\Delta_{rs}({}_pV_q^{(a)}) \subseteq {}_rV_s^{(a)} \otimes {}^r_pA^s_q$ as $\sum_i \Delta_{rs}(b_{pq}^i) \otimes c_{pq}^i = \sum_j b_{rs}^j \otimes \Delta_{pq}(c_{rs}^j)$ by coassociativity.

If now W is an irreducible regular corepresentation and $a \in {}_mW_n$ non-zero, then ${}_pV_q^{(a)}$ is included in ${}_pW_q$. As $a \in {}_mV_n^{(a)}$, it follows by irreducibility and Lemma 2.12 that ${}_pV_q^{(a)} = {}_pW_q$. $\square$

Proposition 2.19. *Let $\mathscr{A}$ be a partial Hopf algebra with invariant integral. Then the total algebra A is the sum of the matrix coefficients of irreducible rcfd corepresentations.*

Proof. Let $a \in {}^k_mA^l_n$, define ${}_pV_q^{(a)}$ as in Lemma 2.18 and let $\mathscr{X}$ be the regular corepresentation on $V^{(a)}$. Then $a = (\mathrm{id} \otimes \epsilon)(\Delta_{kl}^{\mathrm{op}}(a)) = (\mathrm{id} \otimes \epsilon)({}^k_mX^l_n(1 \otimes a)) \in {}^k_m\mathcal{C}(\mathscr{X})^l_n$. Decomposing $(V^{(a)}, \mathscr{X})$ by Corollary 2.15, we find that a is contained in the sum of matrix coefficients of irreducible rcfd corepresentations. $\square$

Proposition 2.20. *Let $\mathscr{A}$ be a partial Hopf algebra with invariant integral ϕ, and let $(V, \mathscr{X})$ and $(W, \mathscr{Y})$ be inequivalent irreducible rcfd corepresentations. Then for all $a \in \mathcal{C}(X), b \in \mathcal{C}(Y)$,*

$$\phi(S(b)a) = \phi(bS(a)) = 0.$$

Proof. Since ϕ vanishes on $S({}^k_mA^l_n)\,{}^p_rA^q_s$ and on ${}^p_rA^q_sS({}^k_mA^l_n)$ unless $(p, q, r, s) = (m, n, k, l)$, it suffices to prove the assertion for elements of the form $a = (\mathrm{id} \otimes \omega_{f,v})({}^k_mX^l_n)$ and $b = (\mathrm{id} \otimes \omega_{g,w})({}^m_kY^n_l)$ where $f \in ({}_kV_l)^*, v \in {}_mV_n$ and $g \in ({}_mW_n)^*, w \in {}_kW_l$. Applying Lemma 2.11 to the map $T\colon {}_kV_l \to {}_kW_l$ with $T(u) = f(u)w$ yields morphisms $\check{T}, \hat{T}$ from $(V, \mathscr{X})$ to $(W, \mathscr{Y})$ which are necessarily 0. Inserting the definition of $\check{T}$, we find

$$\begin{aligned} \phi(S(b)a) &= \phi\big((S \otimes \omega_{g,w})({}^m_kY^n_l) \cdot (\mathrm{id} \otimes \omega_{f,v})({}^k_mX^l_n)\big) \\ &= (\phi \otimes \omega_{g,v})\left({}^l_n(Y^{-1})^k_m(1 \otimes T)\,{}^k_mX^l_n\right) = \omega_{g,v}({}_m\check{T}_n) = 0. \end{aligned}$$

A similar calculation involving $\hat{T}$ shows that $\phi(bS(a)) = 0$. $\square$

Corollary 2.21. *Let $\mathscr{A}$ be a partial Hopf algebra with invariant integral ϕ. Then $\phi = \phi \circ S$.*

Proof. Assume that $a \in A$ lies in an irreducible regular corepresentation which is not a direct summand of the trivial corepresentation. Then $\phi(a) = \phi(S(a)) = 0$ by Proposition 2.20. As such a together with the $\mathbf{1}\binom{k}{m}$ linearly span A, this proves the corollary. $\square$

For the following theorem, recall from the beginning of the proof of Lemma 2.12 that any rcfd corepresentation $(V, \mathscr{X})$ can be decomposed into a direct sum $V = \oplus_\alpha V_\beta$. Hence if V is irreducible, we have $V = {}_\alpha V_\beta$ for unique α, β in the hyperobject set $\mathscr{I}$.

We will call α the *left* and β the *right hyperobject support* of V. Recall further that $\hat{\mathscr{X}}$ denotes the dual left corepresentation of an rcfd corepresentation $\mathscr{X}$.

Theorem 2.22. *Let $\mathscr{A}$ be a partial Hopf algebra with invariant integral ϕ. Let $(V, \mathscr{X})$ be an irreducible rcfd corepresentation of $\mathscr{A}$ with left hyperobject support α and right hyperobject support β. Then there exists an isomorphism G from $(V, \hat{\hat{\mathscr{X}}})$ to $(V, \mathscr{X})$. Moreover, with F denoting the inverse of G, the following hold.*

(1) *For $l \in \beta$ and $m \in \alpha$, the numbers $d_G := \sum_k \mathrm{Tr}({}_kG_l)$ and $d_F := \sum_n \mathrm{Tr}({}_mF_n)$ are non-zero and do not depend on the choice of l or m.*
(2) *For all $k, m \in \alpha$ and $l, n \in \beta$,*

$$(\phi \otimes \mathrm{id})({}^{\,l}_{\,n}(X^{-1})^{k}_{m}\,{}^{k}_{m}X^{l}_{n}) = d_G^{-1}\,\mathrm{Tr}({}_kG_l)\,\mathrm{id}_{\,{}_mV_n},$$
$$(\phi \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n}\,{}^{\,l}_{\,n}(X^{-1})^{k}_{m}) = d_F^{-1}\,\mathrm{Tr}({}_mF_n)\,\mathrm{id}_{\,{}_kV_l}\,.$$

(3) *Denote by Σ_{klmn} the flip map ${}_kV_l \otimes {}_mV_n \to {}_mV_n \otimes {}_kV_l$. Then*

$$(\phi \otimes \mathrm{id} \otimes \mathrm{id})(({}^{\,l}_{\,n}(X^{-1})^{k}_{m})_{12}({}^{k}_{m}X^{l}_{n})_{13}) = d_G^{-1}(\mathrm{id}_{\,{}_mV_n} \otimes {}_kG_l) \circ \Sigma_{klmn},$$
$$(\phi \otimes \mathrm{id} \otimes \mathrm{id})(({}^{k}_{m}X^{l}_{n})_{13}({}^{\,l}_{\,n}(X^{-1})^{k}_{m})_{12}) = d_F^{-1}({}_mF_n \otimes \mathrm{id}_{\,{}_kV_l}) \circ \Sigma_{klmn}.$$

Proof. We prove the assertions and equations involving d_G in (1), (2) and (3) simultaneously; the assertions involving d_F follow similarly.

Consider the following endomorphism F_{mnkl} of ${}_mV_n \otimes {}_kV_l$,

$$\begin{aligned} F_{mnkl} &:= (\phi \otimes \mathrm{id} \otimes \mathrm{id})\left(({}^{\,l}_{\,n}(X^{-1})^{k}_{m})_{12}({}^{k}_{m}X^{l}_{n})_{13}\right) \circ \Sigma_{mnkl} \\ &= (\phi \otimes \mathrm{id} \otimes \mathrm{id})\left(({}^{\,l}_{\,n}(X^{-1})^{k}_{m})_{12}\Sigma_{klkl,23}({}^{k}_{m}X^{l}_{n})_{12}\right). \end{aligned}$$

By applying Lemma 2.11 with respect to the flip map Σ_{klkl}, we see that the family $(F_{mnkl})_{m,n}$ is an endomorphism of $(V \otimes {}_kV_l, \mathscr{X}_{12})$ and hence $F_{mnkl} = \mathrm{id}_{\,{}_mV_n} \otimes {}_kR_l$ with some ${}_kR_l \in \mathrm{End}_{\mathbb{C}}({}_kV_l)$ not depending on m, n.

On the other hand, since $\phi = \phi \circ S$ by Corollary 2.21,

$$\begin{aligned} F_{mnkl} &= (\phi \otimes \mathrm{id} \otimes \mathrm{id})((S \otimes \mathrm{id})({}^{m}_{k}X^{n}_{l})_{12}({}^{k}_{m}X^{l}_{n})_{13}) \circ \Sigma_{mnkl} \\ &= (\phi \otimes \mathrm{id} \otimes \mathrm{id})\left(((S \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n}))_{13}((S^2 \otimes \mathrm{id})({}^{m}_{k}X^{n}_{l}))_{12}\right) \circ \Sigma_{mnkl} \\ &= (\phi \otimes \mathrm{id} \otimes \mathrm{id})\left(({}^{\,n}_{\,l}(X^{-1})^{m}_{k})_{13}(\Sigma_{mnmn})_{23}({}^{m}_{k}(X^{\wedge\wedge})^{n}_{l})_{13}\right). \end{aligned}$$

Hence we can again apply Lemma 2.11 and find that the family $(F_{mnkl})_{k,l}$ is a morphism from $({}_mV_n \otimes V, \hat{\hat{\mathscr{X}}}_{13})$ to $({}_mV_n \otimes V, \mathscr{X}_{13})$. Now $\hat{\hat{\mathscr{X}}}$ is also irreducible, since otherwise its double right dual $\mathscr{X}$ would split. Hence the space $\mathrm{Mor}(\mathscr{X}, \hat{\hat{\mathscr{X}}})$ is linearly spanned by a single element G. Therefore $F_{mnkl} = {}_mT_n \otimes {}_kG_l$ with some ${}_mT_n \in \mathrm{End}_{\mathbb{C}}({}_mV_n)$ not depending on k, l.

We conclude from the above calculations that ${}_mT_n \otimes {}_kG_l = {}_m\mathrm{id}_n \otimes {}_kR_l$ for all indices. This implies that for all m, n with ${}_mV_n \neq 0$, there exists $\lambda_{mn} \in \mathbb{C}$ with ${}_kR_l = \lambda_{mn}\, {}_kG_l$ for all k, l. As however ${}_kR_l$ does not depend on m, n, we see $\lambda_{mn} = \lambda \in \mathbb{C}$. In the end, we deduce $F_{mnkl} = \lambda(\mathrm{id}_{{}_mV_n} \otimes {}_kG_l)$.

Choose now dual bases $(v_i)_i$ for ${}_kV_l$ and $(f_i)_i$ for $({}_kV_l)^*$. Then

$$\lambda \operatorname{Tr}({}_kG_l)\, \mathrm{id}_{{}_mV_l} = \sum_i (\mathrm{id} \otimes \omega_{f_i,v_i})(F_{mlkl}) = (\phi \otimes \mathrm{id})((\tbinom{l}{l}(X^{-1})^{k}_{m})\, {}^{k}_{m}X^{l}_{l}).$$

By Lemma 2.13, we can choose for each l some $m \in \alpha$ with ${}_mV_l \neq 0$. Then summing the previous relation over k, the relations $\sum_k (\tbinom{l}{l}(X^{-1})^{k}_{m})\, {}^{k}_{m}X^{l}_{l} = \mathbf{1}\tbinom{l}{l} \otimes \mathrm{id}_{{}_mV_l}$ and $\phi(\mathbf{1}\tbinom{l}{l}) = 1$ give $\lambda \cdot \sum_k \operatorname{Tr}({}_kG_l) = 1$. It follows at once that G is not the zero morphism, and hence $\mathscr{X}$ is isomorphic to $\hat{\hat{\mathscr{X}}}$. Now all assertions in (1)–(3) concerning d_G follow. $\square$

Corollary 2.23. *Let $\mathscr{A}$ be a partial Hopf algebra with invariant integral ϕ, let $(V, \mathscr{X})$ be an irreducible corepresentation of $\mathscr{A}$, let F be an isomorphism from $(V, \mathscr{X})$ to $(V, \hat{\hat{\mathscr{X}}})$ and $G = F^{-1}$, and let $a = (\mathrm{id} \otimes \omega_{f,v})({}^{k}_{m}X^{l}_{n})$ and $b = (\mathrm{id} \otimes \omega_{g,w})({}^{m}_{k}X^{n}_{l})$ for $f \in ({}_kV_l)^*$, $v \in {}_mV_n$, $g \in ({}_mV_n)^*$, $w \in {}_kV_l$. Then*

$$\phi(S(b)a) \cdot \sum_r \operatorname{Tr}({}_rG_n) = (g|v)(f|Gw), \qquad \phi(aS(b)) \cdot \sum_s \operatorname{Tr}({}_mF_s) = (g|Fv)(f|w).$$

Proof. Apply $\omega_{g,w} \otimes \omega_{f,v}$ to the formulas in Theorem 2.22.(c). $\square$

Corollary 2.24. *Let $\mathscr{A}$ be a partial Hopf algebra with invariant integral, and let $((V^{(x)}, \mathscr{X}_x))_{x \in \mathcal{I}}$ be a maximal family of mutually non-isomorphic irreducible rcfd corepresentations of $\mathscr{A}$. Denote the regular corepresentation on ${}^k\mathcal{C}(\mathscr{X}_x)^l$ by ${}^k\mathscr{Y}^l_x$. Then there exists a linear isomorphism*

$$({}_kV^{(x)}_l)^* \to \mathrm{Mor}(\mathscr{X}_x, {}^k\mathscr{Y}^l_x)$$

assigning to each $f \in ({}_kV^{(x)}_l)^*$ the morphism $T^{(f)}$ of Lemma 2.17, and with inverse the map $T \mapsto \epsilon \circ {}_kT_l$. Furthermore, the map

$$\bigoplus_{x \in \mathcal{I}} \bigoplus_{k,l,m,n} (({}_kV^{(x)}_l)^* \otimes {}_mV^{(x)}_n) \to A, \quad f \otimes w \mapsto T^{(f)}(w)$$

is a linear isomorphism.

Proof. The injectivity of the first map follows from Corollary 2.23. Its surjectivity follows immediately by checking that the map $T \mapsto \epsilon \circ {}_kT_l$ is an inverse.

The injectivity of the second map follows again from Corollary 2.23, and its surjectivity from Proposition 2.19. $\square$

In particular, it follows from the previous two corollaries that an invariant integral ϕ is faithful, i.e. $\phi(ab) = 0$ or $\phi(ba) = 0$ for all b implies $a = 0$. This can also be proven more directly along the lines of [44, Proposition 3.4].

The following corollary generalizes [25, Theorem 3.3] and [14, Corollary 3.6].

Corollary 2.25. *Let $\mathscr{A}$ be a partial Hopf algebra with invariant integral ϕ. Then $\mathscr{A}$ is regular.*

Proof. Injectivity of S follows from Corollary 2.23. As further $\mathscr{X} \cong \hat{\hat{\mathscr{X}}}$ for any irreducible corepresentation $\mathscr{X}$, it follows that $S^2({}^{k}_{m}\mathcal{C}(\mathscr{X})^{l}_{n}) \subseteq {}^{k}_{m}\mathcal{C}(\mathscr{X})^{l}_{n}$, hence S is surjective by Proposition 2.19 and finite dimensionality of each ${}^{k}_{m}\mathcal{C}(\mathscr{X})^{l}_{n}$. $\square$

2.5. Unitary corepresentations of partial compact quantum groups

Let us write $B(\mathcal{H}, \mathcal{G})$ for the space of bounded morphisms between Hilbert spaces $\mathcal{H}$ and $\mathcal{G}$.

Definition 2.26. Let $\mathscr{G}$ be a partial compact quantum group. A corepresentation $\mathscr{X}$ of $\mathscr{P}(\mathscr{G})$ on a rcfd collection of Hilbert spaces ${}_{k}\mathcal{H}_{l}$ is called *unitary* if ${}^{k}_{m}(X^{-1})^{l}_{n} = ({}^{l}_{n}X^{k}_{m})^{*}$ as elements inside ${}^{k}_{m}P(\mathscr{G})^{l}_{n} \otimes B({}_{l}\mathcal{H}_{k}, {}_{n}\mathcal{H}_{m})$. We also refer to $\mathscr{X}$ as a *unitary representation of $\mathscr{G}$*.

For example, viewing $\mathbb{C}^{(I)}$ as a direct sum of the trivial Hilbert spaces $\mathbb{C}$, the trivial corepresentation $\mathscr{U}$ on $\mathbb{C}^{(I)}$ is unitary.

It is easily seen that the tensor product of corepresentations lifts to a tensor product of unitary corepresentations. We hence obtain a tensor C*-category $\mathrm{Rep}_{u,\mathrm{rcfd}}(\mathscr{G}) = \mathrm{Corep}_{u,\mathrm{rcfd}}(\mathscr{P}(\mathscr{G}))$ of unitary corepresentations, where intertwiners are *bounded* bigraded maps $T: \mathcal{H} \to \mathcal{K}$ commuting with the corepresentations. Our aim is to show that, in case the hyperobject set $\mathscr{I}$ is finite, it is a semisimple rigid tensor C*-category.

In the following, we use the physicist convention that inner products on Hilbert spaces are anti-linear in their *first* argument.

Lemma 2.27. *Let $\mathscr{A}$ define a partial compact quantum group with positive invariant integral ϕ, and let ${}_{m}V_{n} \subseteq \bigoplus_{k,l} {}^{k}_{m}A^{l}_{n}$ define a regular corepresentation $\mathscr{X}$. Then with respect to the inner product given by $\langle a|b\rangle := \phi(a^{*}b)$ each ${}_{k}V_{l}$ is a Hilbert space and $\mathscr{X}$ unitary.*

Proof. Let $a \in {}_{m}V_{n}$, $b \in {}_{m}V_{n'}$ and define $\omega_{b,a}: \mathrm{Hom}_{\mathbb{C}}({}_{m}V_{n}, {}_{m}V_{n'}) \to \mathbb{C}$ by $T \mapsto \langle b|Ta\rangle$. Then

$$\sum_k (\mathrm{id} \otimes \omega_{b,a})(({}^k_m X^l_{n'})^* \, {}^k_m X^l_n)) = \sum_k (\mathrm{id} \otimes \phi)(\Delta^{\mathrm{op}}_{kl}(b)^* \Delta^{\mathrm{op}}_{kl}(a))$$
$$= \sum_k (\phi \otimes \mathrm{id})(\Delta_{lk}(b^*)\Delta_{kl}(a))$$
$$= (\phi \otimes \mathrm{id})(\Delta_{ll}(b^* a)) = \phi(b^* a)\mathbf{1}\binom{l}{n} = \delta_{n',n}\mathbf{1}\binom{l}{n} \otimes \langle b|a\rangle.$$

Hence $\sum_k ({}^k_m X^l_{n'})^* \, {}^k_m X^l_n = \delta_{n,n'}\mathbf{1}\binom{l}{n} \otimes \mathrm{id}_{{}_m V_n}$, and $\mathscr{X}$ is unitary by definition of the generalized inverse of a corepresentation. □

Proposition 2.28. *Let $\mathscr{A}$ define a partial compact quantum group. Then every rcfd corepresentation of $\mathscr{A}$ is isomorphic to a unitary corepresentation.*

Proof. By Theorem 2.14 and Corollary 2.24, every corepresentation is isomorphic to a direct sum of irreducible regular corepresentations, which are unitary by Lemma 2.27. □

From Theorem 2.14, we deduce the following corollary.

Corollary 2.29. *Let $\mathscr{G}$ be a partial compact quantum group with finite hyperobject set $\mathscr{I}$. Then $\mathrm{Rep}_{u,\mathrm{rcfd}}(\mathscr{G})$ is a semisimple rigid tensor C^*-category with hyperobject set $\mathscr{I}$.*

Proof. Write $\mathscr{A} = \mathscr{P}(\mathscr{G})$. We can immediately derive the corollary from Theorem 2.14 once we know that the forgetful functor $F\colon \mathrm{Corep}_{u,\mathrm{rcfd}}(\mathscr{A}) \to \mathrm{Corep}_{\mathrm{rcfd}}(\mathscr{A})$ is an equivalence of categories.

Clearly F is faithful, and by Proposition 2.28 it is essentially surjective. To see that it is full, it is sufficient to show that $F(X) \cong F(Y)$ for irreducibles X, Y iff $X \cong Y$. But if $T\colon F(X) \to F(Y)$ is an isomorphism, it follows from the unitarity of X and Y that also the component-wise adjoint $T^*\colon F(Y) \to F(X)$ is an isomorphism, and then also $|T| = (T^*T)^{1/2}$ is an isomorphism from $F(X)$ to $F(X)$. It follows that if $T = U|T|$, then U is a unitary intertwiner from $F(X)$ to $F(Y)$, hence U is an intertwiner from X to Y in $\mathrm{Corep}_{u,\mathrm{rcfd}}(\mathscr{A})$. □

2.6. Schur orthogonality for partial compact quantum groups

Proposition 2.30. *Let $\mathscr{G}$ be a partial compact quantum group and let $(\mathcal{H}, \mathscr{X})$ be an irreducible unitary rcfd representation of $\mathscr{G}$. Then one can find an isomorphism F in $\mathrm{Rep}_{\mathrm{rcfd}}(\mathscr{G})$ between $(\mathcal{H}, \mathscr{X})$ and $(\mathcal{H}, \hat{\hat{\mathscr{X}}})$ such that each ${}_kF_l$ is positive.*

Proof. By Proposition 2.28, there exists an isomorphism $T\colon \hat{\mathscr{X}} \to \mathscr{Y}$ for some unitary corepresentation $\mathscr{Y}$ on $\mathcal{H}^*$, so that in total form $(1 \otimes T)\hat{X} = Y(1 \otimes T)$. We apply $S \otimes -^{\mathrm{tr}}$ and $-^* \otimes -^{*\,\mathrm{tr}}$, respectively to find $\hat{\hat{X}}(1 \otimes T^{\mathrm{tr}}) = (1 \otimes T^{\mathrm{tr}})\hat{Y}$ and $(1 \otimes T^{*\,\mathrm{tr}})X = \hat{Y}(1 \otimes T^{*\,\mathrm{tr}})$.

Combining both equations, we find $\hat{\hat{X}}(1 \otimes T^{\mathrm{tr}} T^{*\,\mathrm{tr}}) = (1 \otimes T^{\mathrm{tr}} T^{*\,\mathrm{tr}})X$. Thus, we can take $F := T^{\mathrm{tr}} T^{*\,\mathrm{tr}}$. □

The Schur orthogonality relations in Corollary 2.23 can be rewritten using the involution. If $v \in {}_k\mathcal{H}_l$, $v' \in {}_m\mathcal{H}_n$ and $\omega_{v,v'}(T) = \langle v|Tv'\rangle$, then

$$\begin{aligned} S((\mathrm{id} \otimes \omega_{v,v'})({}^{k}_{m}X^{l}_{n})) &= (\mathrm{id} \otimes \omega_{v,v'})({}^{n}_{l}(X^{-1})^{m}_{k})) \\ &= (\mathrm{id} \otimes \omega_{v,v'})(({}^{m}_{k}X^{n}_{l})^*) = (\mathrm{id} \otimes \omega_{v',v})({}^{m}_{k}X^{n}_{l})^*. \end{aligned}$$

This equation and Corollary 2.23 imply the following corollary.

Corollary 2.31. *Let $\mathscr{G}$ be a partial compact quantum group with associated partial Hopf *-algebra $\mathscr{A}$ and positive invariant integral ϕ. Let $(\mathcal{H}, \mathscr{X})$ be an irreducible unitary rcfd representation of $\mathscr{G}$, let F be a positive isomorphism from $(\mathcal{H}, \mathscr{X})$ to $(\mathcal{H}, \hat{\hat{\mathscr{X}}})$ in $\mathrm{Corep}_{\mathrm{rcfd}}(\mathscr{A})$ and $G = F^{-1}$, and let $a = (\mathrm{id} \otimes \omega_{v,v'})({}^{k}_{m}X^{l}_{n})$ and $b = (\mathrm{id} \otimes \omega_{w,w'})({}^{k}_{m}X^{l}_{n})$, where $v, w \in {}_k\mathcal{H}_l$ and $v', w' \in {}_m\mathcal{H}_n$. Then*

$$\phi(b^*a) = \frac{\langle w|v'\rangle\langle v|Gw'\rangle}{\sum_r \mathrm{Tr}({}_rG_n)}, \qquad \phi(ab^*) = \frac{\langle w|Fv'\rangle\langle v|w'\rangle}{\sum_s \mathrm{Tr}({}_mF_s)}.$$

Remark 2.32. In fact, Proposition 2.30 and Corollary 2.31 show the following. Let $\mathscr{A}$ be a partial Hopf *-algebra admitting an invariant integral ϕ, which a priori we do not assume to be positive. Suppose however that each irreducible corepresentation of $\mathscr{A}$ is equivalent to a unitary corepresentation. Then ϕ is necessarily positive.

3. Tannaka–Kreĭn–Woronowicz duality for partial compact quantum groups

In the previous section, we showed how any partial compact quantum group with finite hyperobject set $\mathscr{I}$ gave rise to a semisimple rigid tensor C*-category with hyperobject set $\mathscr{I}$ and a faithful morphism into the tensor C*-category of rcfd Hilbert spaces. In this section we reverse this construction, and show that the two structures are in duality with each other. We first deal with the purely algebraic setting without C*-structures.

Fix a (strict) semisimple tensor category $\mathcal{C}$ with (necessarily finite) hyperobject set $\mathscr{I}$, and fix a faithful tensor functor $F: \mathcal{C} \to \mathrm{Vect}^{I\times I}_{\mathrm{rcfd}}$ with product and unit constraints ι and μ. Then $F(\mathbb{1}_\alpha) \cong l^2(I_\alpha)$ for some non-empty subset $I_\alpha \subseteq I$, and $\{I_\alpha\}$ is a partition of I. We write $k' = \alpha$ if $k \in I_\alpha$. We write $F_{kl}(X) = {}_kF(X)_l$ for $k, l \in I$ and $X \in \mathcal{C}$, so that each F_{kl} is a $\mathbb{C}$-linear functor from $\mathcal{C}$ into the category $\mathrm{Vect}_{\mathrm{fd}}$ of finite-dimensional vector spaces.

For $X, Y \in \mathcal{C}$, we write the inclusion maps associated to ι as

$$\begin{aligned} \iota^{(klm)}_{X,Y}: F_{kl}(X) \otimes F_{lm}(Y) &\hookrightarrow \oplus_r F_{kr}(X) \otimes F_{rm}(Y) \\ &= {}_k(F(X) \otimes_I F(Y))_m \to^{\cong} F_{km}(X \otimes Y), \end{aligned}$$

and we write the associated projection maps as

$$\pi^{(klm)}_{X,Y}: F_{km}(X \otimes Y) \to F_{kl}(X) \otimes F_{lm}(Y).$$

We choose a set $\mathcal{I}$ parametrizing a maximal family of mutually inequivalent irreducible objects $\{u_a\}_{a\in\mathcal{I}}$ in $\mathcal{C}$. We assume that the u_a include the unit objects $\mathbb{1}_\alpha$ for $\alpha \in \mathscr{I}$, so that we may identify $\mathscr{I} \subseteq \mathcal{I}$. For $a \in \mathcal{I}$, there exist unique $\lambda_a, \rho_a \in \mathscr{I}$ with $u_a \in \mathcal{C}_{\lambda_a\rho_a}$. For $\alpha, \beta \in \mathscr{I}$ fixed, we write $\mathcal{I}_{\alpha\beta}$ for the set of all $a \in \mathcal{I}$ with $\lambda_a = \alpha$ and $\rho_a = \beta$. Note that $u_a \otimes u_b = 0$ if $\rho_a \neq \lambda_b$. When $a, b, c \in \mathcal{I}$, we write $c \leq a \cdot b$ if $\mathrm{Mor}(u_c, u_a \otimes u_b) \neq \{0\}$. Note that with a, b fixed, there is only a finite set of c with $c \leq a \cdot b$. We also use this notation for multiple products.

Recall that we write V^* for the dual of a (finite-dimensional) vector space V.

Definition 3.1. For $a \in \mathcal{I}$ and $k, l, m, n \in I$, define vector spaces

$$ {}^{k}_{m}A^{l}_{n}(a) = \mathrm{Hom}_{\mathbb{C}}(F_{mn}(u_a), F_{kl}(u_a))^*, $$

and write ${}^{k}_{m}A^{l}_{n} = \oplus_{a\in\mathcal{I}}\, {}^{k}_{m}A^{l}_{n}(a)$, $A(a) = \oplus_{k,l,m,n}\, {}^{k}_{m}A^{l}_{n}(a)$, $A = \oplus_{k,l,m,n}\, {}^{k}_{m}A^{l}_{n}$.

Note that ${}^{k}_{m}A^{l}_{n}(a) = 0$ unless $k' = m' = \lambda_a$ and $l' = n' = \rho_a$. We further write $\mathscr{A} = \{\, {}^{k}_{m}A^{l}_{n} \mid k, l, m, n\}$.

Definition 3.2. For $r, s \in I$, we define Δ_{rs}: ${}^{k}_{m}A^{l}_{n} \to {}^{k}_{r}A^{l}_{s} \otimes {}^{r}_{m}A^{s}_{n}$ as the direct sum over a of the duals of the composition

$$ \mathrm{Hom}_{\mathbb{C}}(F_{rs}(u_a), F_{kl}(u_a)) \otimes \mathrm{Hom}_{\mathbb{C}}(F_{mn}(u_a), F_{rs}(u_a)) \to \mathrm{Hom}_{\mathbb{C}}(F_{mn}(u_a), F_{kl}(u_a)), $$

sending $x \otimes y$ to $x \circ y$.

Lemma 3.3. *The couple $(\mathscr{A}, \Delta)$ is an $I \times I$-partial coalgebra with counit map ϵ: ${}^{k}_{k}A^{l}_{l}(a) \to \mathbb{C}$ sending f to $f(\mathrm{id}_{F_{kl}(u_a)})$. Moreover, for each fixed $f \in {}^{k}_{m}A^{l}_{n}(a)$, the matrix $(\Delta_{rs}(f))_{rs}$ is rcf.*

Proof. Coassociativity and counitality are immediate by duality, as the $\mathrm{Hom}_{\mathbb{C}}(F_{mn}(u_a), F_{kl}(u_a))$ form a partial algebra with units $\mathrm{id}_{F_{kl}(u_a)}$ for each fixed a. The rcf condition follows from the fact that the total $F(u_a)$ is rcfd. $\square$

In the next step, we define a partial algebra structure on $\mathscr{A}$. First note that we can identify

$$ \mathrm{Nat}(F_{mn}, F_{kl}) \cong \prod_a \mathrm{Hom}_{\mathbb{C}}(F_{mn}(u_a), F_{kl}(u_a)), $$

where $\mathrm{Nat}(F_{mn}, F_{kl})$ denotes the space of natural transformations from F_{mn} to F_{kl}. Similarly, we can identify

$$ \mathrm{Nat}(F_{mn} \otimes F_{pq}, F_{kl} \otimes F_{rs}) \cong \prod_{b,c} \mathrm{Hom}_{\mathbb{C}}(F_{mn}(u_b) \otimes F_{pq}(u_c), F_{kl}(u_b) \otimes F_{rs}(u_c)), $$

where $F_{kl} \otimes F_{rs}: \mathcal{C} \times \mathcal{C} \to \mathrm{Vect}_{\mathrm{fd}}$ sends (X,Y) to $F_{kl}(X) \otimes F_{rs}(Y)$. As such, there is a natural pairing of these Nat-spaces with resp. ${}^{k}_{m}A^{l}_{n}$ and ${}^{k}_{m}A^{l}_{n} \otimes {}^{r}_{p}A^{s}_{q}$. For example, ${}^{k}_{m}A^{l}_{n}$ can be identified with the subspace of functionals on $\mathrm{Nat}(F_{mn}, F_{kl})$ of finite support with respect to $\mathcal{I}$.

Definition 3.4. We define a product map

$$M: {}^{k}_{r}A^{l}_{s} \otimes {}^{l}_{s}A^{m}_{t} \to {}^{k}_{r}A^{m}_{t}, \quad (f \cdot g)(x) = (f \otimes g)(\hat{\Delta}^{l}_{s}(x)) \qquad \text{for } x \in \mathrm{Nat}(F_{rt}, F_{km}),$$

where $\hat{\Delta}^{l}_{s}(x)$ is the natural transformation

$$\hat{\Delta}^{l}_{s}(x): F_{rs} \otimes F_{st} \to F_{kl} \otimes F_{lm}, \quad \hat{\Delta}^{l}_{s}(x)_{X,Y} = \pi^{(klm)}_{X,Y} \circ x_{X\otimes Y} \circ \iota^{(rst)}_{X,Y} \qquad \text{for } X, Y \in \mathcal{C}.$$

Note that indeed $f \cdot g$ has finite support as a functional on $\mathrm{Nat}(F_{rt}, F_{km})$: if f is supported at $b \in \mathcal{I}$ and g at $c \in \mathcal{I}$, then $f \cdot g$ has support in the finite set of $a \in \mathcal{I}$ with $a \leq b \cdot c$, since if x is a natural transformation with support outside this set, one has $x_{u_b \otimes u_c} = 0$, and hence any of the $\left(\hat{\Delta}^{l}_{s}(x)\right)_{u_b, u_c} = 0$.

Lemma 3.5. *The above product maps turn $(\mathscr{A}, M)$ into an $I \times I$-partial algebra.*

Proof. We can extend the map $(\hat{\Delta}^{l}_{s} \otimes \mathrm{id})$ on $\mathrm{Nat}(F_{rt}, F_{km}) \otimes \mathrm{Nat}(F_{tu}, F_{mn})$ to a map

$$(\hat{\Delta}^{l}_{s} \otimes \mathrm{id}): \mathrm{Nat}(F_{rt} \otimes F_{tu}, F_{km} \otimes F_{mn}) \to \mathrm{Nat}(F_{rs} \otimes F_{st} \otimes F_{tu}, F_{kl} \otimes F_{lm} \otimes F_{mn}),$$

with

$$(\hat{\Delta}^{l}_{s} \otimes \mathrm{id})(x)_{X,Y,Z} = \left(\pi^{(klm)}_{X,Y} \otimes \mathrm{id}_{F_{mn}(Z)}\right) x_{X\otimes Y, Z} \left(\iota^{(rst)}_{X,Y} \otimes \mathrm{id}_{F_{tu}(Z)}\right).$$

By finite support, we then have that

$$((f \cdot g) \cdot h)(x) = (f \otimes g \otimes h)((\hat{\Delta}^{l}_{s} \otimes \mathrm{id})\hat{\Delta}^{m}_{t}(x)),$$
$$\forall f \in {}^{k}_{r}A^{l}_{s}, g \in {}^{l}_{s}A^{m}_{t}, h \in {}^{m}_{t}A^{n}_{u}, x \in \mathrm{Nat}(F_{ru}, F_{kn}).$$

Similarly, $((f \cdot g) \cdot h)(x) = (f \otimes g \otimes h)((\mathrm{id} \otimes \hat{\Delta}^{m}_{t})\hat{\Delta}^{l}_{s}(x))$. The associativity then follows from the 2-cocycle condition for the ι- and π-maps.

By a similar argument, one sees that the units are given by $\mathbf{1}\binom{k}{l} \in {}^{k}_{l}A^{k}_{l}(\mathbb{1}_\alpha)$, which for $\alpha = k' = l'$ correspond to 1 in the canonical identifications

$${}^{k}_{l}A^{k}_{l}(\alpha) = \mathrm{Hom}_{\mathbb{C}}(F_{ll}(\mathbb{1}_\alpha), F_{kk}(\mathbb{1}_\alpha))^* \cong \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C})^* \cong \mathbb{C},$$

and which are zero otherwise. □

Proposition 3.6. *The partial algebra and coalgebra structures on $\mathscr{A}$ define a partial bialgebra structure on $\mathscr{A}$.*

Proof. We only show multiplicativity, which means showing that for each $x \in \mathrm{Nat}(F_{uw}, F_{km})$ and $y \in \mathrm{Nat}(F_{rt}, F_{uw})$, one has pointwise that $\hat{\Delta}^l_s(x \circ y) = \sum_v \hat{\Delta}^l_v(x) \circ \hat{\Delta}^v_s(y)$. This follows from the fact that $\sum_v \iota^{(uvw)}_{X,Y} \pi^{(uvw)}_{X,Y} \cong \mathrm{id}_{u(F(X) \otimes_I F(Y))_w}$, where we again note that the left hand side sum is in fact finite. □

Lemma 3.7. *Define $\phi: A \to \mathbb{C}$ as the functional which is zero on ${}^k_m A^k_m(a)$ with $a \neq \mathbb{1}_{k'}$, and which coincides with the canonical identification ${}^k_m A^k_m(\mathbb{1}_{k'}) \cong \mathbb{C}$ on the unit component for $k' = m'$. Then the functional ϕ is an invariant integral.*

Proof. The normalization condition $\phi(\mathbf{1}\binom{k}{k}) = 1$ is immediate by construction. Let $\hat{\phi}^k_m$ be the natural transformation from F_{mm} to F_{kk} which has support on direct sums of $\mathbb{1}_{k'}$ with itself, and with $(\hat{\phi}^k_m)_{\mathbb{1}_{k'}} = 1$ for $k' = m'$ and 0 otherwise. Then for $f \in {}^k_m A^k_m$, we have $\phi(f) = f(\hat{\phi}^k_m)$. The invariance of ϕ then follows from the easy verification that for $x \in \mathrm{Nat}(F_{rr}, F_{kl})$ one has for example $x \circ \hat{\phi}^r_m = \delta_{k,l} \mathbf{1}\binom{k}{r}(x) \hat{\phi}^k_m$. □

Let us further impose for the rest of this section that $\mathcal{C}$ also admits left (and hence right) duality. The following lemma just writes out how the tensor functor F preserves duality.

Lemma 3.8. *For all k, l and $X \in \mathcal{C}$, the maps*

$$\mathrm{coev}^{kl}_X := \pi^{(klk)}_{X,\hat{X}} \circ F_{kk}(\mathrm{coev}_X): \mathbb{C} \to F_{kl}(X) \otimes F_{lk}(\hat{X}),$$

$$\mathrm{ev}^{kl}_X := F_{ll}(\mathrm{ev}_X) \circ \iota^{(lkl)}_{\hat{X},X}: F_{lk}(\hat{X}) \otimes F_{kl}(X) \to \mathbb{C}$$

define a duality between $F_{kl}(X)$ and $F_{lk}(\hat{X})$.

Proposition 3.9. *The partial bialgebra $\mathscr{A}$ is a regular partial Hopf algebra with $\mathscr{I}$ as its hyperobject set.*

Proof. The statement concerning the hyperobject set $\mathscr{I}$ is clear by construction of the units $\mathbf{1}\binom{k}{l}$. By Corollary 2.25, it is now sufficient to prove that $\mathscr{A}$ is a partial Hopf algebra.

For any $x \in \mathrm{Nat}(F_{mn}, F_{kl})$, let us define $\hat{S}(x) \in \mathrm{Nat}(F_{lk}, F_{nm})$ by

$$\hat{S}(x)_X = (\mathrm{id} \otimes \mathrm{ev}^{lk}_X) \circ (\mathrm{id} \otimes x_{\hat{X}} \otimes \mathrm{id}) \circ (\mathrm{coev}^{nm}_X \otimes \mathrm{id}).$$

Then the assignment $\hat{S}$ dualizes to maps $S: {}^k_m A^l_n \to {}^n_l A^m_k$ by $S(f)(x) = f(\hat{S}(x))$. We claim that S is an antipode for $\mathscr{A}$.

Let us check for example the formula $\sum_r f_{(nr;1)} S(f_{(nr;2)}) = \delta_{k,m} \epsilon(f) \mathbf{1}\binom{k}{n}$ for $f \in {}^k_m A^l_l$. By duality, this is equivalent to the pointwise identity of natural transformations $\sum_r \hat{M}^n_r (\mathrm{id} \otimes \hat{S}) \hat{\Delta}^l_r(x) = \delta_{k,m} \mathbf{1}\binom{k}{n}(x)\, \mathrm{id}_{F_{kl}}$ for $x \in \mathrm{Nat}(F_{nn}, F_{km})$, where $\hat{M}^n_r$ and $(\mathrm{id} \otimes \hat{S})$ are dual to respectively Δ_{nr} and $\mathrm{id} \otimes S$.

Let us fix $X \in \mathcal{C}$. Then for any $x \in \mathrm{Nat}(F_{nr}, F_{kl})$, $y \in \mathrm{Nat}(F_{rn}, F_{lm})$, we have

$$\left(\hat{M}^n_r(\mathrm{id} \otimes \hat{S})(x \otimes y)\right)_X = \left(\mathrm{id} \otimes \mathrm{ev}^{ml}_X\right)\left(x_X \otimes y_{\hat{X}} \otimes \mathrm{id}\right)\left(\mathrm{coev}^{nr}_X \otimes \mathrm{id}\right).$$

For any $x \in \mathrm{Nat}(F_{nn}, F_{km})$, we therefore have

$$\left(\hat{M}^n_r(\mathrm{id} \otimes \hat{S})\hat{\Delta}^l_r(x)\right)_X = \left(\mathrm{id} \otimes \mathrm{ev}^{ml}_X\right)\left(\pi^{(klm)}_{X,\hat{X}} x_{X \otimes \hat{X}} \iota^{(nrm)}_{X,\hat{X}} \otimes \mathrm{id}\right)\left(\mathrm{coev}^{nr}_X \otimes \mathrm{id}\right).$$

We sum over r, use naturality of x, and obtain

$$\begin{aligned}
\sum_r \left(\hat{M}^n_r(\mathrm{id} \otimes \hat{S})\hat{\Delta}^l_r(x)\right)_X &= \left(\mathrm{id} \otimes \mathrm{ev}^{ml}_X\right)\left(\pi^{(klm)}_{X,\hat{X}} x_{X \otimes \hat{X}} F_{nn}(\mathrm{coev}_X) \otimes \mathrm{id}\right) \\
&= \delta_{k,m}\mathbf{1}\left(\begin{smallmatrix} k \\ n \end{smallmatrix}\right)(x)\left(\mathrm{id} \otimes \mathrm{ev}^{ml}_X\right)\left(\pi^{(mlm)}_{X,\hat{X}} F_{mm}(\mathrm{coev}_X) \otimes \mathrm{id}\right) \\
&= \delta_{k,m}\mathbf{1}\left(\begin{smallmatrix} k \\ n \end{smallmatrix}\right)(x)\left(\mathrm{id} \otimes \mathrm{ev}^{ml}_X\right)\left(\mathrm{coev}^{ml}_X \otimes \mathrm{id}\right) \\
&= \delta_{k,m}\mathbf{1}\left(\begin{smallmatrix} k \\ n \end{smallmatrix}\right)(x)\,\mathrm{id}_{F_{kl}(X)}. \qquad \square
\end{aligned}$$

Assume now that $\mathcal{C}$ is a semisimple rigid tensor C*-category, and F a *-functor from $\mathcal{C}$ to $\{\mathrm{Hilb}_{\mathrm{fd}}\}_{I \times I}$. Let us show that $\mathscr{A}$, as constructed above, becomes a partial Hopf *-algebra with positive invariant integral. In the following definition, we borrow the notation used in the proof of Proposition 3.9, and we write the (left and right) dual of an object X as $\overline{X}$.

Definition 3.10. We define $*$: ${}^k_m A^l_n \to {}^l_n A^k_m$ by the formula $f^*(x) = \overline{f(\hat{S}(x)^*)}$ for $x \in \mathrm{Nat}(F_{nm}, F_{lk})$.

Lemma 3.11. *The operation * is an anti-linear, anti-multiplicative, comultiplicative involution.*

Proof. Anti-linearity is clear. Comultiplicativity follows from the fact that $(xy)^* = y^*x^*$ and $\hat{S}(xy) = \hat{S}(y)\hat{S}(x)$ for natural transformations. To see anti-multiplicativity of *, note first that, since S is anti-multiplicative for $\mathscr{A}$, the map $\hat{S}$ is anti-comultiplicative on natural transformations. Now as $(\iota^{(klm)}_{X,Y})^* = \pi^{(klm)}_{X,Y}$ by assumption, we also have $\hat{\Delta}^l_s(x)^* = \hat{\Delta}^s_l(x^*)$, which proves anti-multiplicativity of * on $\mathscr{A}$. Finally, involutivity follows from the involutivity of $x \mapsto \hat{S}(x)^*$, which is a consequence of the fact that one can choose $\mathrm{ev}^{kl}_{\overline{X}} = (\mathrm{coev}^{lk}_X)^*$ and $\mathrm{coev}^{kl}_{\overline{X}} = (\mathrm{ev}^{lk}_X)^*$. $\square$

Proposition 3.12. *With the above *-structure, $(\mathscr{A}, \Delta)$ defines a partial compact quantum group.*

Proof. The only thing which is left to prove is that our invariant integral ϕ is a positive. We will use the notation from the proof of Lemma 3.7. Now it is easily seen from the definition of ϕ that the ${}_m^kA_n^l(a)$ are all mutually orthogonal. It is furthermore clear that $\phi(f^*) = \overline{\phi(f)}$, since $\hat{S}(\hat{\phi}_m^k) = (\hat{\phi}_m^k)^* = \hat{\phi}_k^m$. Hence it suffices to prove that the hermitian form $\langle f|g\rangle = \phi(f^*g)$ on ${}_m^kA_n^l(a)$ is positive-definite.

Let us write $\bar{f}(x) = \overline{f(x^*)}$. By definition, $\phi(f^*g) = (\bar{f}\otimes g)((\hat{S}\otimes \mathrm{id})\hat{\Delta}_m^k(\hat{\phi}_n^l))$.

Assume that $f(x) = \langle v'|x_a v\rangle$ and $g(x) = \langle w'|x_a w\rangle$ for $v, w \in F_{mn}(u_a)$ and $v', w' \in F_{kl}(u_a)$. Then $\bar{f}(x) = \langle v|x_a v'\rangle$, and using the expression for $\hat{S}$ as in Proposition 3.9 we find that

$$\phi(f^*g) = \langle v\otimes w'|(\mathrm{ev}_a^{kl})_{23}(\hat{\Delta}_m^k(\hat{\phi}_n^l)_{\bar{a},a})_{24}(\mathrm{coev}_a^{mn})_{12}(v'\otimes w)\rangle.$$

However, up to a positive non-zero scalar, which we may assume to be 1 by proper rescaling, we have $\hat{\Delta}_m^k(\hat{\phi}_n^l)_{\bar{a},a} = (\mathrm{ev}_a^{kl})^*(\mathrm{ev}_a^{kl})$. Hence

$$\begin{aligned}\phi(f^*g) &= \langle v\otimes w'|(\mathrm{ev}_a^{kl})_{23}((\mathrm{ev}_a^{kl})^*(\mathrm{ev}_a^{kl}))_{24}(\mathrm{coev}_a^{mn})_{12}(v'\otimes w)\rangle \\ &= \langle v\otimes w'|(\mathrm{ev}_a^{kl})_{23}(\mathrm{ev}_a^{kl})_{24}^*(w\otimes v')\rangle \\ &= \langle v|w\rangle(\mathrm{ev}_a^{kl}\,|v'\rangle_2)(\mathrm{ev}_a^{kl}\,|w'\rangle_2)^*,\end{aligned}$$

where $\mathrm{ev}_a^{kl}\,|z\rangle_2$ denotes the map $y\mapsto \mathrm{ev}_a^{kl}(y\otimes z)$. This clearly defines a positive definite inner product on ${}_m^kA_n^l(a) \cong F_{mn}(u_a)\otimes F_{kl}(u_a)^*$. $\square$

For $\mathscr{G}$ an I-partial compact quantum group with finite hyperobject set $\mathscr{I}$, let us write $F_{\mathscr{G}}$ for the forgetful functor $\mathrm{Rep}_{u,\mathrm{rcfd}}(\mathscr{G}) \to \mathrm{Hilb}_{\mathrm{rcfd}}^{I\times I}$.

Theorem 3.13. *Fix a set I and a finite set $\mathscr{I}$. Then the assignment $\mathscr{G} \to (\mathrm{Rep}_{u,\mathrm{rcfd}}(\mathscr{G}), F_{\mathscr{G}})$ is (up to isomorphism/equivalence) a one-to-one correspondence between I-partial compact quantum groups with hyperobject set $\mathscr{I}$ and semisimple rigid tensor C^*-categories $\mathcal{C}$ with hyperobject set $\mathscr{I}$ and faithful tensor $*$-functor into $\mathrm{Hilb}_{\mathrm{rcfd}}^{I\times I}$.*

Proof. By Corollary 2.29, $\mathrm{Rep}_{u,\mathrm{rcfd}}(\mathscr{G})$ is a semisimple rigid tensor C*-category with hyperobject set $\mathscr{I}$, and $F_{\mathscr{G}}$ a faithful tensor $*$-functor into $\mathrm{Hilb}_{\mathrm{rcfd}}^{I\times I}$. Conversely, the results of this section assign to any semisimple rigid tensor C*-category with hyperobject set $\mathscr{I}$ and faithful tensor $*$-functor into $\mathrm{Hilb}_{\mathrm{rcfd}}^{I\times I}$ an I-partial compact quantum group with finite hyperobject set $\mathscr{I}$. Let us now show that these two maps are inverses of each other, up to isomorphism/equivalence.

Fix now $\mathscr{A} = P(\mathscr{G})$, and let $\mathscr{B}$ be the partial Hopf *-algebra with invariant integral constructed from $\mathrm{Corep}_{u,\mathrm{rcfd}}(\mathscr{A})$ with its natural forgetful functor. Then we have a map $\mathscr{B} \to \mathscr{A}$ which piecewise goes from ${}_m^kB_n^l(a) = \mathrm{Hom}({}_mV_n^{(a)}, {}_kV_l^{(a)})^*$ to ${}_m^kA_n^l(a)$ sending f to $(\mathrm{id}\otimes f)(X_a)$, where the $(V^{(a)}, \mathscr{X}_a)$ run over a maximal family of non-equivalent irreducible unitary corepresentations of $\mathscr{A}$. It is easy to check from the definition of $\mathscr{B}$ that this map is a morphism of partial Hopf *-algebras. By Corollary 2.24, it is bijective.

Conversely, let $\mathcal{C}$ be a semisimple rigid tensor C*-category with hyperobject set $\mathscr{I}$ and faithful tensor $*$-functor F into $\mathrm{Hilb}_{\mathrm{rcfd}}^{I\times I}$. Let $\mathscr{A}$ be the associated partial Hopf *-algebra. For each irreducible $u_a \in \mathcal{C}$, let $V^{(a)} = F(u_a)$, and

$$ {}^{k}_{m}(X_a)^{l}_{n} = \sum_i e_i^* \otimes e_i \quad \in {}^{k}_{m}A^{l}_{n} \otimes \mathrm{Hom}_{\mathbb{C}}(F_{mn}(u_a), F_{kl}(u_a)), $$

where e_i is a basis of $\mathrm{Hom}_{\mathbb{C}}(F_{mn}(u_a), F_{kl}(u_a))$ and e_i^* a dual basis. From the definition of $\mathscr{A}$ it easily follows that each $\mathscr{X}_a$ is a unitary corepresentation for $\mathscr{A}$. Clearly, $\mathscr{X}_a$ is irreducible. As the matrix coefficients of the $\mathscr{X}_a$ span $\mathscr{A}$, it follows that the $\mathscr{X}_a$ form a maximal class of non-isomorphic unitary corepresentations of $\mathscr{A}$. Hence we can find a unique equivalence $\mathcal{C} \to \mathrm{Corep}_{\mathrm{rcfd},u}(\mathscr{A})$ sending X to $(F(X), \mathscr{X}_X)$ and such that $u_a \to \mathscr{X}_a$. From the definitions of the coproduct and product in $\mathscr{A}$, it is readily verified that the natural morphisms $\iota^{(klm)}_{X,Y} : F_{kl}(X) \otimes F_{lm}(Y) \to F_{km}(X \otimes Y)$ turn it into a monoidal equivalence. $\square$

4. Examples

4.1. Hayashi's canonical partial compact quantum groups

Let $\mathcal{C}$ be a semisimple rigid tensor C*-category. A *semisimple module* C*-*category* $\mathcal{D}$ consists of a semisimple C*-category $\mathcal{D}$ and a bifunctor $\mathcal{C} \times \mathcal{D} \to \mathcal{D}$ with natural coherence maps such that the obvious module axioms are satisfied [33,28].

Choose a labeling $\mathcal{I}$ for a distinguished maximal set $\{u_a\}$ of mutually non-isomorphic irreducible objects of $\mathcal{D}$. Then for $a, b \in \mathcal{I}$, we can define

$$ F(X) = \oplus_{a,b} F_{ab}(X), \quad F_{ab}(X) = \mathrm{Hom}(u_a, X \otimes u_b), \qquad X \in \mathcal{C}, $$

where $F_{ab}(X)$ is a Hilbert space (possibly zero) for the inner product $\langle f|g\rangle = f^*g$. It is easy to check that one obtains in a natural way a tensor $*$-functor F from $\mathcal{C}$ to $\mathrm{Hilb}_{\mathrm{rcfd}}^{\mathcal{I}\times\mathcal{I}}$, where the rcfd condition follows from $\mathrm{Hom}(u_a, X \otimes u_b) \cong \mathrm{Hom}(\overline{X} \otimes u_a, u_b)$, by Frobenius reciprocity. It is not necessarily faithful, as some $\mathbb{1}_\alpha$ may act as the zero functor, but using duality one sees that the tensor functor will be faithful on the full tensor C*-subcategory of all X with $X \otimes \mathbb{1}_\alpha \cong \mathbb{1}_\alpha \otimes X \cong 0$ whenever $F(\mathbb{1}_\alpha) = 0$. The associated partial Hopf *-algebra $\mathscr{A}_{(\mathcal{C},\mathcal{D})}$ will be called the *canonical partial compact quantum group* associated with $(\mathcal{C}, \mathcal{D})$.

For example, given a faithful tensor $*$-functor $F: \mathscr{C} \to \mathrm{Hilb}_{\mathrm{rcfd}}^{I\times I}$ and defining $\mathcal{D} = \mathrm{Hilb}_{\mathrm{fd}}^{I}$ as the category of I-graded finite-dimensional Hilbert spaces $V = \oplus_k {}_kV$ with

$$ {}_k(X \otimes V) = \oplus_l F_{kl}(X) \otimes {}_lV, \quad X \in \mathcal{C}, V \in \mathrm{Hilb}_{\mathrm{fd}}^{I}, $$

we get back the reconstruction obtained in the previous section.

If $\mathcal{C}$ is a semisimple rigid tensor C*-category, one can take $\mathcal{D} = \mathcal{C}$ endowed with the module structure coming from the tensor product of $\mathcal{C}$. The associated partial Hopf *-algebra $\mathscr{A}_{\mathcal{C}}$ coincides with Hayashi's construction [16] in case $\mathcal{C}$ has only finitely many irreducible object classes.

As an example coming from compact quantum group theory, let $\mathbb{G}$ be a compact quantum group with ergodic action on a unital C*-algebra $C(\mathbb{X})$. Then the collection of finitely generated $\mathbb{G}$-equivariant $C(\mathbb{X})$-Hilbert modules forms a semisimple module C*-category over $\mathrm{Rep}_u(\mathbb{G})$, cf. [9].

4.2. Morita equivalence

Definition 4.1. Two partial compact quantum groups $\mathscr{G}$ and $\mathscr{H}$ with finite hyperobject set are said to be *Morita equivalent* if there exists a monoidal $*$-equivalence $\mathrm{Rep}_{u,\mathrm{rcfd}}(\mathscr{G}) \to \mathrm{Rep}_{u,\mathrm{rcfd}}(\mathscr{H})$.

In particular, if $\mathscr{G}$ and $\mathscr{H}$ are Morita equivalent they have the same hyperobject set, but they need not share the same object set.

Definition 4.2. A *linking partial compact quantum group* consists of a partial compact quantum group $\mathscr{G}$ defined by a partial Hopf *-algebra $\mathscr{A}$ over a set I with a distinguished partition $I = I_1 \sqcup I_2$ such that the idempotents $\mathbf{1}\binom{i}{j} = \sum_{k\in I_i, l\in I_j} \mathbf{1}\binom{k}{l} \in M(A)$ are central, and such that for each $r \in I_i$, there exists $s \in I_{i+1}$ such that $\mathbf{1}\binom{r}{s} \neq 0$ (with the indices i considered modulo 2).

If $\mathscr{A}$ defines a linking partial compact quantum group $\mathscr{G}$, we can split the total algebra A into four component algebras $A^i_j = A\mathbf{1}\binom{i}{j} = \mathbf{1}\binom{i}{j}A$. It is readily verified that for equal indices, the A^i_i together with all Δ_{rs} with $r, s \in I_i$ define themselves partial compact quantum groups $\mathscr{G}_i$, called the *corner* partial compact quantum groups of $\mathscr{G}$. It is clear from the conditions on a linking partial compact quantum group that the partial compact quantum groups $\mathscr{G}, \mathscr{G}_1$ and $\mathscr{G}_2$ all share the same hyperobject set.

Proposition 4.3. *Two partial compact quantum groups with finite hyperobject set are Morita equivalent iff they arise as the corners of a linking partial compact quantum group.*

Proof. Suppose first that $\mathscr{G}_1$ and $\mathscr{G}_2$ are Morita equivalent partial compact quantum groups with associated partial Hopf *-algebras $\mathscr{A}_1$ and $\mathscr{A}_2$ over respective sets I_1 and I_2. Then we may identify their corepresentation categories with the same abstract tensor C*-category $\mathcal{C}$ based over their common hyperobject set $\mathscr{I}$. This $\mathcal{C}$ comes endowed with two forgetful functors F_i to $\mathrm{Hilb}_{\mathrm{rcfd}}^{I_i\times I_i}$ corresponding to the respective $\mathscr{A}_i$.

With $I = I_1 \sqcup I_2$, we can combine the F_i into a global (faithful) tensor *-functor $F: \mathcal{C} \to \mathrm{Hilb}_{\mathrm{rcfd}}^{I\times I}$, with $F(X) = F_1(X) \oplus F_2(X)$. Let $\mathscr{A}$ be the associated partial Hopf *-algebra constructed from the Tannaka–Kreĭn–Woronowicz reconstruction procedure.

From the precise form of this reconstruction, it follows immediately that ${}^{k}_{m}A^{l}_{n} = 0$ if either k, l or m, n do not lie in the same I_i. Hence the $\mathbf{1}\binom{i}{j} = \sum_{k \in I_i, l \in I_j} \mathbf{1}\binom{k}{l}$ are central.

Moreover, fix $k \in I_i$ and any $l \in I_{i+1}$ with $k' = l'$. Then $\mathrm{Nat}(F_{ll}, F_{kk}) \neq \{0\}$. It follows that $\mathbf{1}\binom{k}{l} \neq 0$. Hence $\mathscr{A}$ defines a linking partial compact quantum group. It is clear that $\mathscr{A}_1$ and $\mathscr{A}_2$ are the corners of $\mathscr{A}$.

Conversely, suppose that $\mathscr{A}_1$ and $\mathscr{A}_2$ arise from the corners of a linking partial compact quantum group defined by $\mathscr{A}$ with invariant integral ϕ. We will show that the associated partial compact quantum groups $\mathscr{G}$ and $\mathscr{G}_1$ are Morita equivalent. Then by symmetry $\mathscr{G}$ and $\mathscr{G}_2$ are Morita equivalent, and hence also $\mathscr{G}_1$ and $\mathscr{G}_2$.

For $(V, \mathscr{X}) \in \mathrm{Rep}_{u,\mathrm{rcfd}}(\mathscr{G})$, let $F(V, \mathscr{X}) = (W, \mathscr{Y})$ be the pair obtained from $(V, \mathscr{X})$ by restricting all indices to those belonging to I_1. It is immediate that $(W, \mathscr{Y})$ is a unitary corepresentation of $\mathscr{A}_1$, and that the functor F is a tensor *-functor from $\mathrm{Rep}_{u,\mathrm{rcfd}}(\mathscr{G})$ to $\mathrm{Rep}_{u,\mathrm{rcfd}}(\mathscr{G}_1)$. What remains to show is that F is an equivalence of categories, i.e. that F is faithful, full and essentially surjective.

By assumption, the hyperobject set of a linking partial compact group coincides with the hyperobject sets of its corners. Hence, using the assumed rigidity, we obtain that F is faithful since $\mathrm{End}(\mathbb{1}) \cong \mathrm{End}(F(\mathbb{1}))$.

To complete the proof, it is sufficient to show that F induces a bijection between isomorphism classes of irreducible unitary corepresentations of $\mathscr{A}$ and of $\mathscr{A}_1$. Note that by Theorem 2.14 and Lemma 2.17, each such class can be represented by a restriction of the regular corepresentation of $\mathscr{A}$ or $\mathscr{A}_1$, respectively.

So, let $(W, \mathscr{Y})$ be an irreducible restriction of the regular corepresentation of $\mathscr{A}_1$. Pick a non-zero $a \in {}_{m}W_{n}$, define ${}_{p}V_{q}^{(a)} \subseteq \bigoplus_{k,l} {}^{k}_{p}A^{l}_{q}$ as in (2.18) and form the regular corepresentation $(V^{(a)}, \mathscr{X})$ of $\mathscr{A}$. Then ${}_{p}V_{q}^{(a)} = {}_{p}W_{q}^{(a)}$ for all $p, q \in I_1$ by Lemma 2.18 2. and hence $F(V, \mathscr{X}) = (W, \mathscr{Y})$. Since F is faithful, $(V, \mathscr{X})$ must be irreducible.

Conversely, let $(V, \mathscr{X})$ be an irreducible restriction of the regular corepresentation of $\mathscr{A}$. Since F is faithful, there exist $k, l \in I_1$ such that ${}_{k}V_{l} \neq 0$. Applying Corollary 2.24, we may assume that ${}_{p}V_{q} \subseteq {}^{k}_{p}A^{l}_{q}$ for some $k, l \in I_1$ and all $p, q \in I$. But then $F(V, \mathscr{X})$ is a restriction of the regular corepresentation of $\mathscr{A}_1$. If $F(V, \mathscr{X})$ would decompose into a direct sum of several irreducible corepresentations, then the same would be true for $(V, \mathscr{X})$ by the argument above. Thus, $F(V, \mathscr{X})$ is irreducible.

Finally, assume that $(V, \mathscr{X})$ and $(W, \mathscr{Y})$ are inequivalent irreducible unitary corepresentations of $\mathscr{A}$. Then $\phi(\mathcal{C}(V, \mathscr{X})^* \mathcal{C}(W, \mathscr{Y})) = 0$ by Corollary 2.31. Since ϕ is faithful by Corollary 2.24, $\mathcal{C}(V, \mathscr{X}) \cap \mathcal{C}(W, \mathscr{Y}) = 0$, and hence $\mathcal{C}(F(V, \mathscr{X})) \cap \mathcal{C}(F(W, \mathscr{Y})) = 0$. So $F(V, \mathscr{X})$ and $F(W, \mathscr{Y})$ are inequivalent. $\square$

If $\mathscr{G}_1$ and $\mathscr{G}_2$ are Morita equivalent compact quantum groups, the total partial compact quantum group coincides with the co-groupoid $\mathscr{G}$ constructed in [1].

4.3. Weak Morita equivalence

Definition 4.4. A semisimple rigid *linking* tensor C*-category consists of a semisimple rigid tensor C*-category $\mathcal{C}$ with a distinguished partition $\mathscr{I} = \mathscr{I}_1 \cup \mathscr{I}_2$ of its hyperobject set such that for each $\alpha \in \mathscr{I}_1$, there exists $\beta \in \mathscr{I}_2$ with $\mathcal{C}_{\alpha\beta} \neq \{0\}$.

The *corners* $\mathcal{C}_i$ of $\mathcal{C}$ are the full semisimple rigid tensor C*-subcategories of objects X with $X \cong \mathbb{1}_i \otimes X \otimes \mathbb{1}_i$, where $\mathbb{1}_i = \oplus_{\alpha\in\mathscr{I}_i}\mathbb{1}_\alpha$.

The following notion is essentially the same as the one by M. Müger [27].

Definition 4.5. Two semisimple rigid tensor C*-categories $\mathcal{C}_1$ and $\mathcal{C}_2$ over respective sets $\mathscr{I}_1$ and $\mathscr{I}_2$ are called *Morita equivalent* if there exists a semisimple rigid linking tensor C*-category $\mathcal{C}$ over the set $\mathscr{I} = \mathscr{I}_1 \sqcup \mathscr{I}_2$ whose corners are isomorphic to $\mathcal{C}_1$ and $\mathcal{C}_2$.

We say two partial compact quantum groups $\mathscr{G}_1$ and $\mathscr{G}_2$ with finite hyperobject set are *weakly Morita equivalent* if their representation categories $\mathrm{Rep}_{u,\mathrm{rcfd}}(\mathscr{G}_i)$ are Morita equivalent.

One can prove directly that this is indeed an equivalence relation, but it will follow indirectly from the discussion below.

The following notion is dual to that of linking partial compact quantum group.

Definition 4.6. A *co-linking partial compact quantum group* consists of a partial compact quantum group $\mathscr{G}$ defined by a partial Hopf *-algebra $\mathscr{A}$ over an index set I, together with a distinguished partition $I = I_1 \cup I_2$ such that $\mathbf{1}\binom{k}{l} = 0$ whenever $k \in I_i$ and $l \in I_{i+1}$, and such that for each $k \in I_i$, there exists $l \in I_{i+1}$ with ${}^{k}_{k}A^{l}_{l} \neq 0$.

It is again easy to see that if we restrict all indices of a co-linking partial compact quantum group to one of the distinguished sets I_i, we obtain a partial compact quantum group $\mathscr{A}_i$ which we will call a corner. If we write $e_i = \sum_{k,l\in I_i}\mathbf{1}\binom{k}{l}$, we can decompose the total algebra A into components $A_{ij} = e_iAe_j$, and correspondingly write A in matrix notation $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{pmatrix}$, where $A_{ii} = A_i$.

Lemma 4.7. *If $\mathscr{A}$ is a co-linking partial compact quantum group, then $A_{ij}A_{jk} = A_{ik}$.*

Proof. It suffices to show $A_{12}A_{21} = A_{11}$. Take $k \in I_1$, and pick $l \in I_2$ with ${}^{k}_{k}A^{l}_{l} \neq \{0\}$. Then in particular, we can find an $a \in {}^{k}_{k}A^{l}_{l}$ with $\epsilon(a) = 1$. Hence for any $m \in I_1$, we have $\mathbf{1}\binom{k}{m} = \mathbf{1}\binom{k}{m}a_{(1)}S(a_{(2)}) \in A_{12}A_{21}$. Hence this latter space contains all local units of A_{11}. As it is a right A_{11}-module, it follows that it is in fact equal to A_{11}. $\square$

It follows that A_{11} and A_{22} are Morita equivalent algebras, where for non-unital algebras one can define Morita equivalence by asking for the existence of a (non-unital) linking algebra satisfying the conclusion of the previous lemma.

Definition 4.8. We call two partial compact quantum groups *co-Morita equivalent* if there exists a *co-linking partial compact quantum group* having these partial compact quantum groups as its corners.

Lemma 4.9. *Co-Morita equivalence is an equivalence relation.*

Proof. The idea is standard, and consists in concretely building the appropriate co-linking partial compact quantum groups. Let us illustrate this for transitivity. In the proof, we write $\sim$ for the relation of co-Morita equivalence.

Let $\mathscr{G}_1, \mathscr{G}_2$ and $\mathscr{G}_3$ be three partial compact quantum groups with $\mathscr{G}_1 \sim \mathscr{G}_2$ and $\mathscr{G}_2 \sim \mathscr{G}_3$. Then we can build a 3 by 3 matrix algebra $A_{\{1,2,3\}} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$ having in the upper left and lower right 2 by 2 corners the two co-linking partial compact quantum groups between resp. the partial compact quantum groups $\mathscr{G}_1$ and $\mathscr{G}_2$, and $\mathscr{G}_2$ and $\mathscr{G}_3$. For example, A_{13} is constructed as $A_{12} \otimes_{A_{22}} A_{23}$. It is straightforward to define a regular weak multiplier Hopf *-algebra structure on $A_{\{1,2,3\}}$ satisfying the conditions of Proposition 1.6 and restricting to the given structures on the 2 by 2 corners.

Let now ϕ be the functional which is zero on the off-diagonal entries A_{ij} and which coincides with the invariant positive integrals on the A_{ii}. Then it is readily checked that ϕ is invariant. To show that ϕ is positive, we invoke Remark 2.32. Indeed, any irreducible corepresentation of $A_{\{1,2,3\}}$ has coefficients in a single A_{ij}. For those i, j with $|i-j| \leq 1$, we know that the corepresentation is unitarizable by restricting to a corner 2×2-block. If however the corepresentation $\mathscr{X}$ has coefficients living in (say) A_{13}, it follows from the identity $A_{12}A_{23} = A_{13}$ that the corepresentation is a direct summand of a product $\mathscr{Y} \textcircled{\scriptsize T} \mathscr{Z}$ of corepresentations with coefficients in respectively A_{12} and A_{23}. This proves unitarizability of $\mathscr{X}$. It follows from Remark 2.32 that ϕ is positive, and hence $\mathscr{A}_{\{1,2,3\}}$ defines a partial compact quantum group.

We claim that the subspace $\mathscr{A}_{\{1,3\}}$ (in the obvious notation) defines a co-linking compact quantum group between $\mathscr{G}_1$ and $\mathscr{G}_3$. In fact, it is clear that the $\mathscr{A}_{11}$ and $\mathscr{A}_{33}$ are corners of $\mathscr{A}_{\{1,3\}}$, and that $\mathbf{1}\binom{k}{l} = 0$ for k, l not both in I_1 and I_3. To finish the proof, it is sufficient to show now that for each $k \in I_1$, there exists $l \in I_3$ with ${}^k_kA^l_l \neq 0$, as the other case follows by symmetry using the antipode. But there exists $m \in I_2$ with ${}^k_kA^m_m \neq \{0\}$, and $l \in I_3$ with ${}^m_mA^l_l \neq \{0\}$. As in the discussion following Definition 4.6, this implies that there exists $a \in {}^k_kA^m_m$ and $b \in {}^m_mA^l_l$ with $\epsilon(a) = \epsilon(b) = 1$. Hence $\epsilon(ab) = 1$, showing ${}^k_kA^l_l \neq \{0\}$. □

Proposition 4.10. *Assume that two partial compact quantum groups $\mathscr{G}_1$ and $\mathscr{G}_2$ with finite hyperobject set are co-Morita equivalent. Then they are weakly Morita equivalent.*

Proof. Consider the corepresentation category $\mathcal{C}$ of a co-linking partial compact quantum group $\mathscr{A}$ over $I = I_1 \cup I_2$. Let $\varphi: I \to \mathscr{I}$ be the partition of I along the hyperobject set. Then by the defining property of a co-linking partial compact quantum group, also $\mathscr{I} =$

$\mathscr{I}_1 \cup \mathscr{I}_2$ with $\mathscr{I}_i = \varphi(I_i)$ is a partition. In particular, writing again $\mathbb{1}_i = \oplus_{\alpha \in \mathscr{I}_i} \mathbb{1}_\alpha$, we have that $\mathbb{1}_i \otimes \mathcal{C} \otimes \mathbb{1}_i \cong \mathrm{Rep}_{u,\mathrm{rcfd}}(\mathscr{G}_i)$, since any (irreducible) unitary rcfd corepresentation of $\mathscr{A}_i$ is automatically also a(n irreducible) corepresentation of $\mathscr{A}$.

Fix now $\alpha \in \mathscr{I}_1$ and $k \in \alpha$. As $\mathscr{A}$ is co-linking, there exists $l \in I_2$ with ${}^k_k A^l_l \neq \{0\}$. By Lemma 2.18 there exists a non-zero regular unitary corepresentation supported inside $\oplus_{m,n} {}^k_m A^l_n$. If then $l \in I_\beta$ with $\beta \in \mathscr{I}_2$, it follows that $\mathcal{C}_{\alpha\beta} \neq 0$. By symmetry, we also have that for each $\alpha \in \mathscr{I}_2$ there exists $\beta \in \mathscr{I}_1$ with $\mathcal{C}_{\alpha\beta} \neq \{0\}$. This proves that the $\mathcal{C}$ forms a linking partial tensor C*-category over $\mathscr{I} = \mathscr{I}_1 \cup \mathscr{I}_2$. □

Proposition 4.11. *Let $\mathcal{C}$ be a semisimple rigid linking tensor C*-category over $\mathscr{I} = \mathscr{I}_1 \cup \mathscr{I}_2$. Let $\mathcal{I}$ be a set parametrizing a maximal family of non-equivalent irreducible unitary rcfd corepresentations $\mathcal{C}$, and let $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$ be the partition of $\mathcal{I}$ corresponding to the one of $\mathscr{I}$. Then the associated canonical partial compact quantum group is a co-linking partial compact quantum group over $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$.*

To be clear, to $a \in \mathcal{I}$ one assigns the unique $\alpha \in \mathscr{I}$ such that $\mathbb{1}_\alpha \otimes u_a \cong u_a$, and this provides the corresponding partition of $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2 = \cup_{\alpha \in \mathscr{I}} \mathcal{I}_\alpha$.

Proof. Let $\mathscr{A} = \mathscr{A}_\mathcal{C}$ define the canonical partial compact quantum group with object set $\mathcal{I}$. A fortiori, $\mathbf{1}\binom{a}{b} = 0$ if a and b are not both in $\mathcal{I}_1$ or $\mathcal{I}_2$.

Fix now $a \in \mathcal{I}_\alpha$ for some $\alpha \in \mathscr{I}_i$. Pick $\beta \in \mathscr{I}_{i+1}$ with $\mathcal{C}_{\alpha\beta} \neq \{0\}$, and let $(V, \mathscr{X})$ be a non-zero irreducible unitary rcfd corepresentation inside $\mathcal{C}_{\alpha\beta}$. Applying Lemma 2.13 with respect to the identity morphism, we get that there exists $b \in \mathcal{I}_\beta$ with ${}_aV_b \neq \{0\}$. As $(\epsilon \otimes \mathrm{id})\, {}^a_a X^b_b = \mathrm{id}_{\,{}_aV_b}$, we find that ${}^a_a A^b_b \neq 0$. This proves that $\mathscr{A}$ defines a co-linking partial compact quantum group. □

Note that the corners of the canonical partial compact quantum group associated with the semisimple rigid linking tensor C*-category *are not* the canonical partial compact quantum groups associated to the corners of the linking tensor C*-category, the reason being that the canonical construction is based only on left tensor multiplication and does not 'cut down on the right'. Rather, the two corner constructions will be related by a Morita equivalence.

Theorem 4.12. *Two partial compact quantum groups $\mathscr{G}_1$ and $\mathscr{G}_2$ with finite hyperobject set are weakly Morita equivalent if and only if they are connected by a string of Morita and co-Morita equivalences.*

Proof. Clearly if two partial compact quantum groups are Morita equivalent, they are weakly Morita equivalent. By Proposition 4.10, the same is true for co-Morita equivalence. This proves one direction of the theorem.

Conversely, assume $\mathscr{G}_1$ and $\mathscr{G}_2$ are weakly Morita equivalent. Let $\mathcal{C}$ be a semisimple rigid linking tensor C*-category between $\mathrm{Rep}_{u,\mathrm{rcfd}}(\mathscr{G}_1)$ and $\mathrm{Rep}_{u,\mathrm{rcfd}}(\mathscr{G}_2)$. Then the $\mathscr{G}_i$ are Morita equivalent with the corners of the canonical partial compact quantum group associated to $\mathcal{C}$. But Proposition 4.11 shows that these corners are co-Morita equivalent. □

5. Partial compact quantum groups from reciprocal random walks

5.1. Reciprocal random walks and the Temperley–Lieb category

In this section, we investigate a special class of partial compact quantum groups constructed from *t-reciprocal random walks* [9]. We first recall this notion, slightly changing the terminology for the sake of convenience.

Definition 5.1. Let $t \in \mathbb{R}_0$. A *t-reciprocal random walk* consists of a quadruple $(\Gamma, w, \mathrm{sgn}, i)$ where $\Gamma = (\Gamma^{(0)}, \Gamma^{(1)}, s, t)$ is a graph with source and target maps s and t, where w is a weight function $w: \Gamma^{(1)} \to \mathbb{R}_0^+$ and sgn a sign function $\mathrm{sgn}: \Gamma^{(1)} \to \{\pm 1\}$, and where i is an involution $e \mapsto \bar{e}$ on $\Gamma^{(1)}$ interchanging source and target, satisfying for all e the weight reciprocality $w(e)w(\bar{e}) = 1$, the sign reciprocality $\mathrm{sgn}(e)\,\mathrm{sgn}(\bar{e}) = \mathrm{sgn}(t)$, and the random walk property $\sum_{s(e)=v} \frac{1}{|t|} w(e) = 1$ for all $v \in \Gamma^{(0)}$.

By [9, Proposition 3.1], there is a uniform bound on the number of edges leaving from any given vertex v, i.e. Γ has a finite degree. For examples of t-reciprocal random walks, we refer to [9].

Let now $0 < |q| \leq 1$, and let $\mathcal{T}_q$ be the Temperley–Lieb C*-tensor category, which is the universal tensor C*-category with irreducible unit and duality, generated by a single self-adjoint object X and duality morphism $R: \mathbb{1} \to X \otimes X$ satisfying

$$R^*R = |q| + |q|^{-1}, \;\; (R^* \otimes \mathrm{id}_X)(\mathrm{id}_X \otimes R) = -\,\mathrm{sgn}(q)\,\mathrm{id}_X\,.$$

Then if $\Gamma = (\Gamma, w, \mathrm{sgn}, i)$ is a $-(q+q^{-1})$-reciprocal random walk, we have a $*$-functor F_Γ from $\mathcal{T}_q$ into $\mathrm{Hilb}_{\mathrm{rcfd}}^{I\times I}$ with $I = \Gamma^{(0)}$, by sending X to the bigraded Hilbert space $\mathcal{H}^\Gamma = l^2(\Gamma^{(1)})$, where the $\Gamma^{(0)}$-bigrading is given by $\delta_e \in {}_{s(e)}\mathcal{H}^\Gamma{}_{t(e)}$, and R to the morphism

$$R_\Gamma: l^2(\Gamma^{(0)}) \to \mathcal{H}^\Gamma \otimes_{\Gamma^{(0)}} \mathcal{H}^\Gamma, \quad R_\Gamma \delta_v = \sum_{e, s(e)=v} \mathrm{sgn}(e)\sqrt{w(e)}\delta_e \otimes \delta_{\bar{e}}.$$

Note that $\mathcal{H}^\Gamma$ is rcfd as Γ has finite degree. Up to equivalence, F_Γ only depends upon the isomorphism class of (Γ, w), and is independent of the chosen involution or sign structure. Conversely, every tensor $*$-functor from $\mathcal{T}_q$ into $\mathrm{Hilb}_{\mathrm{rcfd}}^{I\times I}$ for some set I arises in this way [8].

5.2. Partial compact quantum groups from reciprocal random walks

Let $\Gamma = (\Gamma, w, \mathrm{sgn}, i)$ be a $-(q+q^{-1})$-reciprocal random walk. Let us denote by $\mathscr{A}(\Gamma)$ the I-partial compact quantum group associated to the functor F_Γ by the Tannaka–

Kreĭn–Woronowicz reconstruction result. Our aim is to give a direct representation of the associated algebra $A(\Gamma)$ by generators and relations. We will write $\Gamma_{vw} \subseteq \Gamma^{(1)}$ for the set of edges with source v and target w.

Theorem 5.2. *The* **-algebra* $A(\Gamma)$ *is the universal* **-algebra generated by self-adjoint orthogonal idempotents* $\mathbf{1}\binom{v}{w}$ *for* $v, w \in \Gamma^{(0)}$ *and elements* $(u_{e,f})_{e,f\in\Gamma^{(1)}}$ *where the* $u_{e,f} \in {}^{s(e)}_{s(f)}A(\Gamma)^{t(e)}_{t(f)}$ *satisfy* $u^*_{e,f} = \operatorname{sgn}(e)\operatorname{sgn}(f)\sqrt{\frac{w(f)}{w(e)}}u_{\bar{e},\bar{f}}$ *and*

$$\sum_{t(g)=w} u^*_{g,e}u_{g,f} = \delta_{e,f}\mathbf{1}\binom{w}{t(e)}, \qquad \forall w \in \Gamma^{(0)}, e, f \in \Gamma^{(1)}, \tag{5.1}$$

$$\sum_{s(g)=v} u_{e,g}u^*_{f,g} = \delta_{e,f}\mathbf{1}\binom{s(e)}{v} \qquad \forall v \in \Gamma^{(0)}, e, f \in \Gamma^{(1)}. \tag{5.2}$$

The partial Hopf **-algebra structure is given by* $\Delta_{vw}(u_{e,f}) = \sum_{\substack{s(g)=v\\ t(g)=w}} u_{e,g} \otimes u_{g,f}$, $\varepsilon(u_{e,f}) = \delta_{e,f}$ *and* $S(u_{e,f}) = u^*_{f,e}$.

Note that the sums in (5.1) and (5.2) are in fact finite, as Γ has finite degree.

Proof. Let $(\mathcal{H}, V)$ be the generating unitary corepresentation of $A(\Gamma)$ on $\mathcal{H} = l^2(\Gamma^{(1)})$. Then V decomposes into parts ${}^{k}_{m}V^{l}_{n} = \sum_{e,f} v_{e,f} \otimes E_{e,f} \in {}^{k}_{m}A^{l}_{n} \otimes B({}_m\mathcal{H}_n, {}_k\mathcal{H}_l)$, where the $E_{e,f}$ are the natural matrix units and with the sum over all e with $s(e) = k, t(e) = l$ and all f with $s(f) = m, t(f) = n$. By construction V defines a unitary corepresentation of $A(\Gamma)$, hence the relations (5.1) and (5.2) are satisfied for the $v_{e,f}$. Now as R_Γ is an intertwiner between the trivial representation on $\mathbb{C}^{(\Gamma^{(0)})} = \oplus_{v\in\Gamma^{(0)}}\mathbb{C}$ and $V \circledT_{\Gamma^{(0)}} V$, we have for all $v \in \Gamma^{(0)}$ that

$$\sum_{\substack{e,f,g,h\in\Gamma^{(1)}\\ t(f)=s(h),t(e)=s(g)}} v_{e,f}v_{g,h} \otimes ((E_{e,f} \otimes E_{g,h})R_\Gamma\delta_v) = \sum_w \mathbf{1}\binom{w}{v} \otimes R_\Gamma\delta_v, \tag{5.3}$$

hence

$$\begin{aligned}\sum_{\substack{e,g,k\\ t(e)=s(g),s(k)=v}} &\operatorname{sgn}(k)\sqrt{w(k)}\left(v_{e,k}v_{g,\bar{k}} \otimes \delta_e \otimes \delta_g\right)\\ &= \sum_{\substack{w,k\\ s(k)=w}} \operatorname{sgn}(k)\sqrt{w(k)}\left(\mathbf{1}\binom{w}{v} \otimes \delta_k \otimes \delta_{\bar{k}}\right).\end{aligned}$$

So if $t(e) = s(g) = z$, we have $\sum_{k,s(k)=v} \operatorname{sgn}(k)\sqrt{w(k)}v_{e,k}v_{g,\bar{k}} = \delta_{e,\bar{g}}\operatorname{sgn}(e)\sqrt{w(e)}\mathbf{1}\binom{s(e)}{v}$. Multiplying to the left with $v^*_{e,l}$ and summing over all e with $t(e) = z$, we see from (5.1) that also $v_{e,f}^* = \operatorname{sgn}(e)\operatorname{sgn}(f)\sqrt{\frac{w(f)}{w(e)}}v_{\bar{e},\bar{f}}$ holds. Hence the $v_{e,f}$ satisfy the universal relations in the statement of the theorem. The formulas for comultiplication, counit and antipode then follow immediately from the fact that V is a unitary corepresentation.

Let us now a priori denote by $B(\Gamma)$ the *-algebra determined by the relations (5.1), (5.2) and the relation for the adjoint as above, and write $\mathscr{B}(\Gamma)$ for the associated $\Gamma^{(0)} \times \Gamma^{(0)}$-partial *-algebra induced by the local units $\mathbf{1}\binom{v}{w}$. Write $\Delta(\mathbf{1}\binom{v}{w}) = \sum_{z\in\Gamma^{(0)}} \mathbf{1}\binom{v}{z} \otimes \mathbf{1}\binom{z}{w}$ and $\Delta(u_{e,f}) = \sum_{g\in\Gamma^{(1)}} u_{e,g} \otimes u_{g,f}$, which makes sense in $M(B(\Gamma) \otimes B(\Gamma))$ as the degree of Γ is finite. Then an easy computation shows that $\sum_{t(g)=w} \Delta(u_{g,e})^*\Delta(u_{g,f}) = \delta_{e,f}\Delta(\mathbf{1}\binom{w}{t(e)})$. Similarly, the analogue of (5.2) holds for $\Delta(u_{e,f})$. As also the relation for the adjoint holds trivially for $\Delta(u_{e,f})$, it follows that we can define a *-algebra homomorphism $\Delta: B(\Gamma) \to M(B(\Gamma) \otimes B(\Gamma))$ sending $u_{e,f}$ to $\Delta(u_{e,f})$ and $\mathbf{1}\binom{v}{w}$ to $\Delta(\mathbf{1}\binom{v}{w})$. Cutting down, we obtain maps $\Delta_{vw}: {}^r_t B(\Gamma)^s_z \to {}^r_v B(\Gamma)^s_w \otimes {}^v_t B(\Gamma)^w_z$ which are easily seen to satisfy Definition 1.5. Moreover, the Δ_{vw} are coassociative as they are coassociative on generators.

Let now $E_{v,w}$ be the matrix units for $l^2(\Gamma^{(0)})$. Then one verifies again directly from the defining relations of $B(\Gamma)$ that one can define a *-homomorphism $\widetilde{\epsilon}: B(\Gamma) \to B(l^2(\Gamma^{(0)}))$ sending $\mathbf{1}\binom{v}{w}$ to $\delta_{v,w}\, e_{v,v}$ and $u_{e,f}$ to $\delta_{e,f}\, e_{s(e),t(e)}$. We can hence define a map $\epsilon: B(\Gamma) \to \mathbb{C}$ such that $\widetilde{\epsilon}(x) = \epsilon(x)e_{v,w}$ for all $x \in {}^v_v B(\Gamma)^w_w$, and which is zero elsewhere. Clearly it defines a morphism on the partial algebra $\mathscr{B}(\Gamma)$. As ϵ satisfies the counit condition on generators, it follows by partial multiplicativity that it satisfies the counit condition on the whole of $B(\Gamma)$, i.e. $B(\Gamma)$ is a partial *-bialgebra.

It is clear now that the $u_{e,f}$ define a unitary corepresentation U of $B(\Gamma)$ on $\mathcal{H}^\Gamma$. Moreover, from (5.1) and the formula for $u_{e,f}^*$ we can deduce that $R_\Gamma: \mathbb{C}_{\Gamma^{(0)}} \to \mathcal{H}^\Gamma \otimes_{\Gamma^{(0)}} \mathcal{H}^\Gamma$ is a morphism from $\mathbb{C}^{(\Gamma^{(0)})}$ to $U \circledT_{\Gamma^{(0)}} U$ in $\mathrm{Corep}_{u,\mathrm{rcfd}}(\mathscr{B}(\Gamma))$, cf. (5.3). From the universal property of $\mathcal{T}_q$, it then follows that we have a tensor $*$-functor $G^\Gamma: \mathcal{T}_q \to \mathrm{Corep}_{u,\mathrm{rcfd}}(\mathscr{B}(\Gamma))$ with $G^\Gamma(X) = U$. On the other hand, as we have a Δ-preserving *-homomorphism $B(\Gamma) \to A(\Gamma)$ by the universal property of $\mathscr{B}(\Gamma)$, we have a strongly monoidal *-functor $H^\Gamma: \mathrm{Corep}_{\mathrm{rcfd},u}(\mathscr{B}(\Gamma)) \to \mathrm{Corep}_u(\mathscr{A}(\Gamma)) = \mathcal{T}_q$ which is inverse to G^Γ. Since the commutation relations of $\mathscr{A}(\Gamma)$ are completely determined by the morphism spaces of $\mathcal{T}_q$, it follows that we have a *-homomorphism $\mathscr{A}(\Gamma) \to \mathscr{B}(\Gamma)$ sending $v_{e,f}$ to $u_{e,f}$. This proves the theorem. □

We remark that for finite graphs with their canonical weights coming from the Perron–Frobenius eigenvalues ([8, Section 3.1]), these partial compact quantum groups were considered in [14, Section 6].

5.3. Partial compact quantum groups from homogeneous reciprocal random walks

Let us now consider a particular class of 'homogeneous' $-(q+q^{-1})$-reciprocal random walks. Namely, assume that there exists a finite set T partitioning $\Gamma^{(1)} = \cup_a \Gamma^{(1)}_a$ such that for each $a \in T$ and $v \in \Gamma^{(0)}$, there exists a unique $e_a(v) \in \Gamma^{(1)}_a$ with source v. Write av for the range of $e_a(v)$. Assume moreover that T has an involution $a \mapsto \bar{a}$ such that

$\overline{e_a(v)} = e_{\bar{a}}(av)$. Then for each a, the map $v \mapsto av$ is a bijection on $\Gamma^{(0)}$ with inverse $v \mapsto \bar{a}v$. In particular, also for each $w \in \Gamma^{(0)}$ there exists a unique $f_w(a) \in \Gamma_a^{(1)}$ with target w.

Let us further denote $w_a(v) = w(e_a(v))$ and $\mathrm{sgn}_a(v) = \mathrm{sgn}(e_a(v))$. Let again $A(\Gamma)$ be the total *-algebra of the associated partial compact quantum group. Using Theorem 5.2, we have the following presentation of $A(\Gamma)$: it is generated by self-adjoint mutually orthogonal idempotents $\mathbf{1}\binom{v}{w}$ and elements $(u_{a,b})_{v,w} := u_{e_a(v),e_b(v)}$ for $a,b \in T$ and $v,w \in \Gamma^{(0)}$ with defining relations $(u_{a,b})^*_{v,w} = \frac{\mathrm{sgn}_b(w)\sqrt{w_b(w)}}{\mathrm{sgn}_a(v)\sqrt{w_a(v)}}(u_{\bar{a},\bar{b}})_{av,bw}$ and

$$\sum_{a\in T}(u_{a,b})^*_{\bar{a}v,w}(u_{a,c})_{\bar{a}v,z} = \delta_{w,z}\delta_{b,c}\mathbf{1}\binom{v}{bw}, \quad \sum_{a\in T}(u_{b,a})_{w,v}(u_{c,a})^*_{z,v} = \delta_{b,c}\delta_{w,z}\mathbf{1}\binom{w}{v}.$$

The element $(u_{a,b})_{v,w}$ lives inside the component ${}^{v}_{w}A(\Gamma)^{av}_{bw}$.

Let us now consider $M(A(\Gamma))$, the multiplier algebra of $A(\Gamma)$. For a function f on $\Gamma^{(0)} \times \Gamma^{(0)}$, write $f(\lambda,\rho) = \sum_{v,w} f(v,w)\mathbf{1}\binom{v}{w} \in M(A(\Gamma))$. Similarly, for a function f on $\Gamma^{(0)}$ we write $f(\lambda) = \sum_{v,w} f(v)\mathbf{1}\binom{v}{w}$ and $f(\rho) = \sum_{v,w} f(w)\mathbf{1}\binom{v}{w}$. We then write for example $f(a\lambda,\rho)$ for the element corresponding to the function $(v,w) \mapsto f(av,w)$.

We can further form in $M(A(\Gamma))$ the elements $u_{a,b} = \sum_{v,w}(u_{a,b})_{v,w}$. Then $u = (u_{a,b})$ is a unitary $m \times m$ matrix for $m = \#T$. Moreover,

$$u^*_{a,b} = u_{\bar{a},\bar{b}}\frac{\gamma_b(\rho)}{\gamma_a(\lambda)}, \tag{5.4}$$

where $\gamma_a(v) = \mathrm{sgn}_a(v)\sqrt{w_a(v)}$. We then have the following commutation relations between functions on $\Gamma^{(0)} \times \Gamma^{(0)}$ and the entries of u:

$$f(\lambda,\rho)u_{a,b} = u_{a,b}f(\bar{a}\lambda,\bar{b}\rho), \tag{5.5}$$

where $f(\bar{a}\lambda,\bar{b}\rho)$ is given by $(v,w) \mapsto f(\bar{a}v,\bar{b}w)$. Further, $\Delta(u_{a,b}) = \Delta(1)\sum_c(u_{a,c} \otimes u_{c,b})$. Note that the *-algebra generated by the $u_{a,b}$ is no longer a weak Hopf *-algebra when $\Gamma^{(0)}$ is infinite, but rather one can turn it into a Hopf *-algebroid.

Remark 5.3. The weak multiplier Hopf algebra $A(\Gamma)$ is related to the free orthogonal dynamical quantum groups introduced in [41] as follows. Denote by G the free group generated by the elements of T subject to the relation $\bar{a} = a^{-1}$ for all $a \in T$. By assumption on Γ, the formula $(af)(v) := f(\bar{a}v)$ defines a left action of G on $\mathrm{Fun}(\Gamma^{(0)})$. Denote by $C \subseteq \mathrm{Fun}(\Gamma^{(0)})$ the unital subalgebra generated by all γ_a and their inverses and translates under G, write the elements of $T \subseteq G$ as a tuple in the form $\nabla = (a_1, \bar{a_1}, \ldots, a_n, \bar{a_n})$, and define a $\nabla \times \nabla$ matrix F with values in C by $F_{a,b} := \delta_{b,\bar{a}}\gamma_a$. Then the free orthogonal dynamical quantum group $A_{\mathrm{o}}^C(\nabla, F, F)$ introduced in [41] is the universal unital $*$-algebra generated by a copy of $C \otimes C$ and the entries of a unitary $\nabla \times \nabla$-matrix $v = (v_{a,b})$ satisfying

$$v_{a,b}(f \otimes g) = (af \otimes bg)v_{a,b}, \qquad (aF_{a,\bar{a}} \otimes 1)v^*_{\bar{a},\bar{b}} = v_{a,b}(1 \otimes F_{b,\bar{b}})$$

for all $f, g \in C$ and $a, b \in \nabla$. The second equation can be rewritten as $v^*_{\bar{a},\bar{b}} = v_{a,b}(\gamma_a^{-1} \otimes \gamma_b)$. Comparing with (5.4) and (5.5), we see that there exists a $*$-homomorphism

$$A_o^C(\nabla, F, F) \to M(A(\Gamma)), \quad \begin{cases} f \otimes g & \mapsto f(\lambda)g(\rho), \\ v_{a,b} & \mapsto u_{\bar{a},\bar{b}}. \end{cases}$$

The two quantum groupoids are related by an analogue of the unital base changes considered for dynamical quantum groups in [41, Proposition 2.1.12]. Indeed, Theorem 5.2 shows that $A(\Gamma)$ is the image of $A_o^C(\nabla, F, F)$ under a non-unital base change from C to $\mathrm{Fun}_f(\Gamma^{(0)})$ along the natural map $C \to M(\mathrm{Fun}_f(\Gamma^{(0)}))$.

5.4. Dynamical quantum $SU(2)$ as a partial compact quantum group

As a particular example, take $0 < |q| < 1$ and $x > 0$, and consider the graph with $\Gamma_x^{(0)} = x|q|^{\mathbb{Z}}$ and $\Gamma_x^{(1)} = \{(y, z) \mid y/z \in \{|q|, |q|^{-1}\}\}$. Endow Γ with the involution $(y, z) \mapsto (z, y)$, the weight $w(y, z) = \frac{z+z^{-1}}{y+y^{-1}}$ and the sign $\mathrm{sgn}(y, z) = \sigma_\mu$ if $y/z = |q|^\mu$, where $\sigma_+ = 1$ and $\sigma_- = -\mathrm{sgn}(q)$. Consider further the set $T = \{+, -\}$ with the non-trivial involution, and label the edges (y, z) with μ if $y/z = |q|^\mu$. Write $F(y) = |q|^{-1}\frac{|q|y+|q|^{-1}y^{-1}}{y+y^{-1}}$, and put $\alpha = \frac{F^{1/2}(\rho-1)}{F^{1/2}(\lambda-1)}u_{--}$ and $\beta = \frac{1}{F^{1/2}(\lambda-1)}u_{-+}$. Then our relations for the $u_{\epsilon,\nu}$ are equivalent to the commutation relations

$$\alpha\beta = qF(\rho - 1)\beta\alpha \qquad \alpha\beta^* = qF(\lambda)\beta^*\alpha \tag{5.6}$$

$$\alpha\alpha^* + F(\lambda)\beta^*\beta = 1, \qquad \alpha^*\alpha + q^{-2}F(\rho - 1)^{-1}\beta^*\beta = 1, \tag{5.7}$$

$$F(\rho - 1)^{-1}\alpha\alpha^* + \beta\beta^* = F(\lambda - 1)^{-1}, \qquad F(\lambda)\alpha^*\alpha + q^{-2}\beta\beta^* = F(\rho), \tag{5.8}$$

$$f(\lambda)g(\rho)\alpha = \alpha f(\lambda + 1)g(\rho + 1), \qquad f(\lambda)g(\rho)\beta = \beta f(\lambda + 1)g(\rho - 1). \tag{5.9}$$

These are precisely the commutation relations for the dynamical quantum $SU(2)$-group as in [24, Definition 2.6], except that the precise value of F has been changed by a shift in the parameter domain. The (total) coproduct on A_x also agrees with the one on the dynamical quantum $SU(2)$-group. Note that the case of q a root of unity case was considered in [20, Section 5], see also [18,10] for generalizations to higher rank (in resp. the unitary and non-unitary case).

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