Characters and Complexity of Finite Semigroups*

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Abstract

Let $S$ be a finite semigroup and let $K$ be an algebraically closed field of characteristic zero. Herein, we derive a formula for the congruence $\equiv$ induced on $S$ by the direct sum of all the irreducible representations of $S$ over $K$. This congruence $\equiv$ is proved to be the same as the congruence induced by the minimal homomorphic image of $S$, which is one-to-one on the subgroups of $S$ and such that two distinct principal ideals of $S$, each generated by an idempotent, have distinct images.

Assume $\varphi$ is a faithful finite dimensional representation of $S$. Let $R(\varphi)$ be the associated completely reducible representation having the same character as $\varphi$. That is, $R(\varphi)$ is the direct sum of the Jordan-Hölder factors of $\varphi$. Then by applying the Burnside-Steinberg theorem we prove that the congruence induced by $R(\varphi)$ on $S$ equals $\equiv$. That is, the operator $R$ preserves "one-to-oneness as much as possible."

We next apply these results to compute the complexity, $\#c(S)$, of $S$ when $S$ is a union of groups. A linear transformation $T = B(S)$ is defined on the character ring of $S$ into itself. Then by a theorem of Krohn-Rhodes (which determines the complexity of $S$ in terms of its homomorphic images), together with the previous character results, we prove that $T$ is nilpotent and $\text{index}(T) = \#c(S)$.

Finally the character results proved here imply that, if the Fundamental Lemma on Complexity is valid, then the complexity of $S$ is the maximum of the images of all its irreducible representations. This is known to be the case for all regular semigroups.

In the following all semigroups $S$ are assumed to have finite order and $K$ denotes an algebraically closed field of characteristic zero. We assume the reader is familiar with the following material although this paper is reasonably self-contained:

1. Standard theorems from the representation theory of finite dimensional $K$ algebras. See [2], [7], and [9]. Standard finite dimensional
representation theory of $S$ by matrices with coefficients in $K$. See Chapter 5 of [1].

(2) The definition and elementary properties of the (group) complexity of $S$, $\#_g(S)$. See Chapter 6 of [8] and the introduction of [5]. Statement of the main theorem of complexity for semigroups $S$ which are union of groups. See Theorem 9.2.5 of [8] and Theorem A of [5].

(3) Standard theorems for finite semigroups, e.g., Rees Theorem, the Green relations, the Schützenberger representation, etc. See [1] or [8]. The calculus of homomorphisms on $S$ including the definition and existence of the minimal homomorphomorphic image of $S$ which is one-to-one on the subgroups of $S$, denoted $S \rightarrow S'$, definition and elementary properties of group mapping semigroups, etc. See Chapter 8 of [8].

In the following all undefined notation is given in the previously cited references.

In this paper we derive a formula for the congruence $\equiv$ induced on $S$ by the direct sum of all the irreducible representations of $S$ over $K$. We show that $\equiv$ is the same as the congruence induced on $S$ by $S \rightarrow S^{\equiv}$, the minimal homomorphomorphic image of $S$ which is one-to-one on each subgroup of $S$ and such that two distinct regular $\mathcal{J}$-classes of $S$ have disjoint images. (See Chapter 8 of [8].)

Assume $\varphi$ is a faithful finite dimensional representation of $S$. Let $R(\varphi)$ be the associated completely reducible representation having the same character as $\varphi$. That is, $R(\varphi)$ is the direct sum of the Jordan-Hölder factors of $\varphi$. Then by applying the Burnside-Steinberg theorem [7] we prove that the congruence induced by $R(\varphi)$ on $S$ equals $\equiv$.

We next apply these results to compute the complexity, $\#_g(S)$, of $S$ when $S$ is a union of groups. A linear transformation $T = B(S)$ is defined on the character ring of $S$ into itself. Then by a theorem of Krohn-Rhodes [5, 8] (which determines the complexity of $S$ in terms of its homomorphic images), together with the previous character result, we prove that $T$ is nilpotent and index $(T) = \#_g(S)$. For a detailed exposition see Chapter 9 of [8], which assumes the character theory results proved here.

1. CONGRUENCES INDUCED BY IRREDUCIBLE REPRESENTATIONS

NOTATION 1.1. In the following, all semigroups are of finite order. $R$, $S$, $T$, $U$, and $V$ denote semigroups. $K$ denotes an algebraically closed field of characteristic zero. All representations $\mathcal{R}$ considered will be finite dimensional right $K[S]$-modules. $K[S]$ denotes the semigroup algebra of $S$. 
over $K$ (which need not have an identity). We will speak interchangeably about $\rho$ as being a representation and a module. See [1, Ch. 5], [2], and [9].

In this paper epimorphism means onto homomorphism.

$L, R, H, D = J$ denote the Green relations (see [1] or [8]). Let $\alpha$ be one of the Green relations and let $\varphi : S \rightarrow T$ be an epimorphism (the double arrow will signify that the mapping is surjective). Then $\varphi$ is an $\alpha$-homomorphism iff $[\varphi(s_1) \alpha \varphi(s_2)$ iff $s_1 \alpha s_2]$ for all $s_1, s_2 \in S$. $\varphi$ is an $\alpha'$-homomorphism iff $\varphi(s_1) \alpha \varphi(s_2)$ implies $s_1 \alpha s_2$ for all regular elements $s_1, s_2$ of $S$. See Chapter 8 of [8]. Notice $\alpha'$ and $\alpha$ epimorphisms coincide if $S$ is regular.

Let $\psi : S \rightarrow T$ (be an epimorphism), then $\psi$ is a $\gamma$-homomorphism iff $\psi$ is one-to-one when restricted to each subgroup of $S$. Let $\varphi_i : S \rightarrow T_i$ be homomorphisms for $1 \leq i \leq n$. Then $\Pi \varphi_i : S \rightarrow T$ is the induced epimorphism defined by $\Pi \varphi_i(s) = (\varphi_1(s), \ldots, \varphi_n(s))$ for $s \in S$ and

$$T = \Pi \varphi_i(S) \triangleleft T_1 \times \cdots \times T_n,$$

where $\triangleleft$ denotes subdirect product. See Chapter 8 of [8].

Let $\mathcal{R}_1, \ldots, \mathcal{R}_n$ be a complete set of inequivalent irreducible representations (IRR) of $S$. The number $n$ is finite by the Wedderburn theory (see [9] or [2]).

**Definition 1.1.** Let $S \rightarrow S^{\text{GGM}}$ denote the epimorphism

$$\Pi \mathcal{R}_i : S \rightarrow (\Pi \mathcal{R}_i)(S).$$

**Notation 1.2.** Let $A$ be a non-empty set. $F_R(A)$ denotes the semigroup of all function of $A$ into $A$ under the multiplication $f \cdot g = h$, $h(a) = g(f(a))$, $f, g \in F_R(A)$.

$F_L(A)$ denotes the reverse semigroup of $F_R(A)$. Let $I$ be a left ideal of $S$, then $M_I^L : S \rightarrow F_L(I)$ is the homomorphism defined by $(M_I^L(s))(x) = sx$ for $s \in S, x \in I$. $M_I^R$ for $J$ a right ideal is defined dually.

The following definition is fundamental in investigations concerning both the complexity of $S$ (see [5] or [8]) and the irreducible representations of $S$ (see Theorem 1.1(a) below). This definition provides the critical link between the concepts of characters and complexity.

**Definition 1.2.** $S$ is a **generalized group mapping (GGM) semigroup** iff $S = \{1\}$ or $S$ has a minimal or 0-minimal two-sided ideal $I$ so that both $M_I^L$ and $M_I^R$ are one-to-one homomorphisms.

If $S \neq \{1\}$, and $S$ is a GGM semigroup then the ideal $I$ is necessarily regular, non-zero, and uniquely determined. See 8.2.15 of [8]. As in [8] let $S \rightarrow S^{\text{GGM}}$ denote $\Pi\{\varphi_i \cdot \varphi_i : S \rightarrow T_i, \ T_i\text{GGM and } T_i \neq T_j \text{ if } i \neq j\}$. 
Notation 1.3. Let \( q_k : S \to T_k \) be epimorphisms for \( k = 1, 2 \). Then \( q_k \) is equivalent to \( q_2 \) iff there exists an isomorphism \( \alpha : T_1 \to T_2 \) so that \( \alpha q_1 = q_2 \). (This is not to be confused with equivalent of two matrix representations.) For an epimorphism \( \varphi : S \to T \) let \( \text{mod}(\varphi) \) be the congruence induced on \( S \) by \( \varphi \), i.e., \( s_1 \equiv s_2 \text{ mod}(\varphi) \) iff \( \varphi(s_1) = \varphi(s_2) \). Clearly \( q_1 \) is equivalent to \( q_2 \) iff \( \text{mod}(q_1) = \text{mod}(q_2) \).

Let \( q_k : S \to T_k \) be epimorphisms for \( k = 1, 2 \). We write \( q_1 \succeq q_2 \) iff there exists an epimorphism \( \psi : T_1 \to T_2 \) so that \( \psi q_1 = q_2 \). Clearly \( q_1 \succeq q_2 \) iff \( s_1 \equiv s_2 \text{ (mod } q_1 \text{) implies } s_1 \equiv s_2 \text{ (mod } q_2 \text{).}

Let \( S \) be a fixed semigroup. Let \( \mathcal{L} \) be a collection of epimorphisms of \( S \) closed under equivalence. An epimorphism \( \varphi : S \to T \) is functorially minimal with respect to \( \mathcal{L} \) iff \( \varphi \in \mathcal{L} \) and for any \( \psi \in \mathcal{L}, \psi \succeq \varphi \). See Chapter 8 of [8].

Recall Notation 1.1. In this paper \( S \to S^{+\mathcal{F}'} \) denotes the functorially minimal \( \gamma \) and \( \mathcal{F}' \)-homomorphism of \( S \). It will follow from Lemma 1.7 that \( S \to S^{+\mathcal{F}'} \) exists and is clearly unique up to equivalence.

Theorem 1.1. (a) Let \( \mathcal{R} \) be an irreducible representation of \( S \). Then \( \mathcal{R}(S) \) is a GGM semigroup.

(b) \( S \to S^{\ominus\mathcal{RR}} \) is equivalent to \( S \to S^{\ominus\mathcal{GGM}} \).

(c) \( S \to S^{\ominus\mathcal{GGM}} \) is equivalent to \( S \to S^{+\mathcal{F}'} \) so \( S \to S^{\ominus\mathcal{RR}}, S \to S^{\ominus\mathcal{GGM}} \) and \( S \to S^{+\mathcal{F}'} \) are all equivalent.

Proof: The proof will proceed via several lemmas. First, some notation.

Notation 1.4. Let \( S \neq \{ \emptyset \} \) be a 0-simple semigroup. Let \( \mathcal{P}(S) \) be the collection of all pairs \((\varphi, T)\) such that \( \varphi : T \to T, \varphi \) is a \( \gamma \)-homomorphism and \( T \neq \{0\} \). We partially order \( \mathcal{P}(S) \), as in Notation 1.3, under \( \succeq \). Let \( \equiv \) be the congruence on \( S \) given by \( s_1 \equiv s_2 \) iff \( x_1 x_1 x_2 = x_1 x_2 x_2 \) for all \( x_1, x_2 \in S \). Let \( S \to S/\equiv \) denote the canonical epimorphism.

\( M_S^R \) is the right regular representation of \( S \). See Notation 1.2. Let \( \text{mod}(M_S^R) \), the congruence induced by \( M_S^R \), be denoted by \( \equiv(R) \). Thus \( s_1(\equiv(R)) s_2 \) iff \( x_1 s_1 = x_1 s_2 \) for all \( x_1 \in S \). The congruence \( \equiv(L) \) is defined dually.

Lemma 1.1. Let \( S \) be a 0-simple semigroup \( S \neq \emptyset \), then

(a) \( S \to S/\equiv \) is a \( \succeq \)-minimal element of \( \mathcal{P}(S) \). Any \( \succeq \)-minimal element of \( \mathcal{P}(S) \) is equivalent to \( S \to S/\equiv \).

(b) \( S \to S/\equiv \) is equivalent to \( S \to S^{\ominus\mathcal{GGM}} \).

(c) The congruence \( \equiv \) is the transitive closure of \( \equiv(L) \) and \( \equiv(R) \).
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Proof: We first prove (a). By the Rees theorem we may assume \( S = \mathcal{M}(G, A, B, C) \) a regular Rees matrix semigroup of \( A \times B \) matrices, with structure group \( G \) and regular structure matrix \( C \). See [1] or [8]. Clearly \( S \to S/\equiv \) is a \( \gamma \)-homomorphism with \( S/\equiv \neq \{0\} \), thus \( S \to S/\equiv \) is an element of \( \mathcal{P}(S) \). Let \( \varphi \in \mathcal{P}(S) \), then to prove (a), it will suffice to show that \( \varphi(s_1) = \varphi(s_2) \) implies \( s_1 \equiv s_2 \). Let \( x_1, x_2 \in S \), then

\[ \varphi(x_1s_1x_2) = \varphi(x_1s_1x_2). \]

Now, \( \varphi^{-1}(0) = \{0\} \), since \( S \) is 0-simple and \( \varphi(S) \neq \{0\} \). Thus either \( x_1s_1x_2 = x_1s_2x_2 = 0 \) or both \( x_1s_1x_2 \) and \( x_1s_2x_2 \) lie in \( S - \{0\} \). In the latter case, \( x_1s_1x_2 \equiv x_1s_1x_2 \) and there are \( a_1, a_2, b_1, b_2 \in S \) so that

\[ \alpha_1 = a_1x_1s_1x_2a_2^\mathcal{H}a_1x_1s_2x_2a_2 = \alpha_2, \]

\( \alpha_1 \) and \( \alpha_2 \) belong to the same (maximal) subgroup \( G \neq \{0\} \) of \( S \), and \( b_1x_1b_2 = x_1s_1x_2, b_1x_2b_2 = x_1s_2x_2 \). But \( \varphi \) being 1-1 on \( G \), \( \varphi(\alpha_1) = \varphi(\alpha_2) \) implies \( \alpha_1 = \alpha_2 \) and hence \( x_1s_1x_2 = b_1x_2b_2 = x_1s_2x_2 \) and (a) is proved.

We now prove (b). Since \( S^* = S \), it is easy to verify that \( S/\equiv \) is a GGM semigroup. Let \( \varphi : S \to T \) be any epimorphism with \( T \) a GGM semigroup. It will suffice to prove that \( s_1 \equiv s_2 \) implies \( \varphi(s_1) = \varphi(s_2) \). This is trivial if \( T = \{0\} \). Otherwise \( T \) is a 0-simple GGM semigroup and thus for \( t_1, t_2 \in T, y_1t_1 = y_1t_2 \) for all \( y_1 \in T \) or \( t_1y_2 = t_2y_2 \) for all \( y_2 \in T \) implies \( t_1 = t_2 \). Thus \( x_1s_1x_2 = x_1s_2x_2 \) for all \( x_1, x_2 \in S \) implies

\[ \varphi(x_1)(\varphi(s_1)(\varphi(x_2)) = \varphi(x_2)(\varphi(s_2)(\varphi(x_2)) \]

for all \( x_1, x_2 \in S \), hence \( y_1(\varphi(s_1)\varphi(x_2)) = y_1(\varphi(s_2)\varphi(x_2)) \) for all \( y_1 \in T \), thus \( \varphi(s_1) = \varphi(s_2) \) and (b) proved.

We now prove (c). Let \( \equiv^* \) denote the transitive closure of \( \equiv(R) \) and \( \equiv(L) \), i.e., the lub of \( \equiv(R) \) and \( \equiv(L) \) in the lattice of congruences on \( S \). Clearly \( s_1 \equiv^* s_2 \) implies \( s_1 \equiv s_2 \). Also, \( s_1 \equiv^* 0 \) iff \( s_1 = 0 \). Now, let \( S = \mathcal{M}(G, A, B, C), s_1 = (g)_{ab}, s_2 = (g')_{a'b'} \) and assume \( x_1s_1x_2 = x_1s_2x_2 \) for all \( x_1, x_2 \in S \). Now \( (x_1s_1)x_2 = (x_1s_2)x_2 \) for all \( x_1, x_2 \in S \) implies \( C(b, a) = kC(b', a') \) for some \( k = k(b, b') \in G \), all \( a \in A \). Similarly \( C(b, a) = C(b, a')h \) for some \( h = h(a, a') \in G \), all \( b \in B \). Thus, for all \( a \in A \) and \( x_2 \in S \), \( (g)_{ab}x_2 = (gk)_{a'b'}x_2 \) and, for all \( b \in B \) and \( x_1 \in S \), \( x_1(g)_{ab} = x_1(hg)_{a'b'} \), i.e., \( (g)_{ab} = (L)(hg)_{a'b'} \) and \( (g)_{ab} = (R)(hk)_{a'b'} \) for all \( a \in A \) and \( b \in B \). Hence \( s_1 = (g)_{ab} \equiv^* (h)_{a'b'} = s_3 \), thus \( s_1 \equiv s_3 \) and so \( s_2 \equiv s_3 \). Now we may choose \( a_1, a_2, b_1, b_2 \in S \) so that \( a_1s_3a_2 \equiv a_1s_3a_2 \), \( a_1s_3a_2 \), and \( a_1s_3a_2 \) lie in the same subgroup \( G \neq \{0\} \) of \( S \), and \( b_1a_1s_3a_2b_2 = s_2 \), \( b_1a_1s_3a_2b_2 = s_3 \). Thus \( s_2 \equiv s_3 \equiv s_2 \) and (c) is proved.
**Lemma 1.2.** Let \( S \) be a 0-simple semigroup with \( S \neq \{0\} \). Let \( \psi : S \rightarrow S^{\oplus_{\text{IRR}}} \). Then \( \psi(S) \neq \{0\} \) and \( \psi \) is a \( \gamma \)-homomorphism, i.e., \( \psi \in \mathcal{P}(S) \). Furthermore, either \( s_1 = (L)s_2 \) or \( s_1 = (R)s_2 \) implies \( \psi(s_1) = \psi(s_2) \).

**Proof:** By Maschke's theorem, \( G \rightarrow G^{\oplus_{\text{IRR}}} \) is an isomorphism when \( G \) is a group. By the results of Clifford-Suschkewitsch ([1, Sect. 5.4]), we deduce immediately from the above that \( \psi \in \mathcal{P}(S) \).

The last assertion of the lemma is also immediate from the Clifford-Suschkewitsch results. Alternatively, we may argue directly as follows: Let \( L \) be the left regular representation of \( S \), \( R \) the right regular representation of \( S \), and \( L^* \), the dual of \( L \), i.e., the second right regular representation of \( S \). Clearly \( \text{mod } L^* = \text{mod}(L) = (\equiv (L)) \) and \( \text{mod } R = (\equiv (R)) \).

Now, dividing out the radical and observing that every irreducible representation vanishes on the radical, it follows from the Wedderburn theory that every (right) irreducible representation is a constituent of both \( R \) and \( L^* \), thus \( R \supseteq \psi \) and \( L^* \supseteq \psi \) and the lemma is proved.

**Lemma 1.3.** Let \( S \) be a 0-simple semigroup. Then (b) of Theorem 1.1 holds for \( S \).

**Proof:** We may assume \( S \neq \{0\} \), since otherwise the lemma is trivial. Then, by Lemma 1.2, \( \psi : S \rightarrow S^{\oplus_{\text{IRR}}} \) belongs to \( \mathcal{P}(S) \). Then Lemma 1.1(a) and (b) implies \( (S \rightarrow S^{\oplus_{\text{IRR}}}) \supseteq S \rightarrow S^{\oplus_{\text{GMM}}} \). However, by Lemma 1.2

\[
(S \rightarrow S/\equiv(R)) \supseteq (S \rightarrow S^{\oplus_{\text{IRR}}}) \quad \text{and} \quad ((S \rightarrow S/\equiv(L)) \supseteq (S \rightarrow S^{\oplus_{\text{IRR}}}).
\]

Then by Lemma 1.1(c)

\[
(S \rightarrow S/\equiv) \supseteq (S \rightarrow S^{\oplus_{\text{IRR}}}).
\]

Thus by Lemma 1.1(b)

\[
(S \rightarrow S^{\oplus_{\text{GMM}}}) \supseteq (S \rightarrow S^{\oplus_{\text{IRR}}}).
\]

So \( S \rightarrow S^{\oplus_{\text{GMM}}} \) and \( S \rightarrow S^{\oplus_{\text{IRR}}} \) are equivalent and the lemma is proved.

**Lemma 1.4.** Let \( I \) be a (two-sided) ideal of \( S \).

(a) Let \( R \) be a non-zero irreducible representation of \( I \). Then \( R \) has a unique extension \( R \) to \( S \). \( R \) is irreducible.

(b) Let \( T \) be any representation of \( S \) and let \( R \) be \( T \) restricted to \( I \). Let \( R_1, \ldots, R_k \) be the non-zero irreducible constituents of \( R \). Then \( R_1, \ldots, R_k \) are among the irreducible constituents of \( T \).

**Proof:** This lemma is easily proved by the techniques of Munn-
Hewitt-Zuckermann (see [1, Theorem 5.33]). To prove (a), choose \( e \in K[1] \) so that \( \mathcal{A}(e) = I \), the identity matrix. Then let

\[
\mathcal{A}'(s) = \mathcal{A}(e \cdot s) = \mathcal{A}(e \cdot s \cdot e) = \mathcal{A}(s \cdot e)
\]

where \( \cdot \) is the multiplication in \( K[S] \) and \( e \cdot s, e \cdot s \cdot e, s \cdot e \in K[I] \) since \( I \) is an ideal.

To prove (b), let \( M \) be the \( K[S] \)-module associated with \( \mathcal{A} \) and let \( M = M_0 \supset M_1 \supset \cdots \supset M_n = 0 \) be a composition series for \( M \). Then \( M_j/M_{j+1} \) for \( j = 0, 1, \ldots, n-1 \) is a simple \( K[S] \)-module or a 1-dimensional zero action \( K[S] \)-module. In the first case the assumptions about \( K \) and the Wedderburn theory imply that every vector space endomorphism of \( M_j/M_{j+1} \) is a right multiplication by an element of \( K[S] \). Now, since \( I \) is an ideal of \( S \), the right multiplication by elements of \( K[I] \) form an ideal in \( \text{hom}_K(M_j/M_{j+1}, M_j/M_{j+1}) \). But \( \text{hom}_K(M_j/M_{j+1}, M_j/M_{j+1}) \) being a simple algebra, it follows that \( M_j/M_{j+1} \) for \( j = 0, \ldots, n-1 \) is either a simple (i.e., irreducible) \( K[I] \)-module or a zero action \( K[I] \)-module. Let \( M_j/M_{j+1}, M_{j+1}/M_{j+1} \) be the simple non-zero action \( K[I] \)-modules among the \( M_j/M_{j+1} \)'s. Clearly we may assume \( l = k \) and \( \mathcal{A}_r = M_r/M_{r+1} \) for \( 1 \leq r \leq l = k \). Let \( \mathcal{A}_r \) be \( M_j/M_{j+1} \) considered as a \( K[S] \)-module. \( \mathcal{A}_r \) is simple since \( \mathcal{A}_r \) is simple. Now \( \mathcal{A}_r \) considered as a representation of \( I \) is \( \mathcal{A}_r \). Thus by (a), \( \mathcal{A}_r = \mathcal{A}'_r \). This proves (b) and hence Lemma 1.4.

**Notation 1.5.** (see Section 2 of Chapter 8 of [8]). Let \( S \) be a semi-group. For \( s \in S \), let \( s^\# \) denote the \( J \)-class of \( S \) containing \( s \). We write \( s_1^\# \leq s_2^\# \) iff \( S^1 s_2^1 s \subset S^1 s_2^1 S \). Let \( F(s) = F(s^\#) \) be the ideal

\[
\bigcup\{s^\# : s^\# \leq s^\# \text{ is false}\}
\]

Let \( \eta_s : S \to S/F(s) \) be the natural homomorphism with \( \eta_s(s_1) = s_1 \) when \( s_1 \in S - F(s) \) and \( \eta_s(s_1) = 0 \) otherwise. Let \( s \) be a regular element of \( S \). \( \text{GGM}_s(S) = (S/F(s^\#))/\equiv \) where, for \( r_1, r_2 \in S/F(s) \), \( r_1 \equiv r_2 \) iff \( x_1 r_1 x_2 = x_1 r_2 x_2 \) in \( S/F(s^\#) \) for all \( x_1, x_2 \in s^\# \). Let \( (\eta \equiv) (\eta_s) = H_s : S \to \text{GGM}_s(S) \), where \( (\eta \equiv) : S/F(s^\#) \to (S/F(s^\#))/\equiv \) is the natural homomorphism. For extensive background see 8.2.11 ff. of [8].

We say \( T \) is a basic \( \text{GGM(BGGM)} \) of \( S \) iff \( T = \text{GGM}_s(S) \) for some regular element \( s \in S \). Since \( (s^\#)^2 = s^\# \), it is easy to verify that \( \text{GGM}_s(S) \) is a \( \text{GGM} \) semi-group. \( S \to S^{\oplus \text{BGGM}} \) denotes the epimorphism

\[
\Pi H_s : S \to \Pi H_s(S),
\]

where \( s \) runs through the regular elements of \( S \). If \( \varphi : S \to T \) and \( A \subseteq S \varphi \) \( \varphi \) restricted to \( A \).

The following lemma justifies the introduction of \( \text{GGM} \) semi-groups.
LEMMA 1.5. Let $\mathcal{R}$ be an irreducible representation of $S$. Then $\mathcal{R}(S)$ is a GGM semigroup.

Proof: Since $\{0\}$ is a GGM semigroup, we can assume $\mathcal{R}$ is not the null representation. Let $J$ be the apex of $\mathcal{R}$ (see [1, Ch. 5]), that is, $J$ is the unique $\leq$-minimal member of $\{s^\# : \mathcal{R}(s^\#) \neq 0\}$. Let $j \in J$, then $\mathcal{R}(F(j)) \subseteq \{0\}$. Now, $J$ is regular (see [1, Ch. 5]), hence $J^0[=J \cup F(j)/F(j)]$ is a 0-simple semigroup. $\mathcal{R}$ induces the non-null irreducible representation $\mathcal{R}^{(1)}$ on $J^0$ with $\mathcal{R}^{(1)}(j) = \mathcal{R}(j)$ for all $j \in J$ and $\mathcal{R}^{(1)}(0) = 0$. Let $\text{mod}(\mathcal{R}^{(1)})$ be denoted by $\equiv^{(1)}$. Then $\mathcal{R}^{(1)}$ induces a faithful (in particular non-null) irreducible representation $R^{(2)}$ of $J^0(\equiv^{(1)})$ by $R^{(2)}([x]) = R^{(1)}(x)$, where $x \in J^0$ and $[x]$ is the equivalence class of $\equiv^{(1)}$ containing $x$. Now, $T = J^0(\equiv^{(1)})$ is a 0-simple semigroup having a faithful (non-null) irreducible representation. Thus, by Lemma 1.3, the identity map: $T \rightarrow T$ is equivalent to $T \rightarrow T \bigodot_{\text{GGM}}$. But, by Lemma 1.1, $T \rightarrow T \bigodot_{\text{GGM}}$ is equivalent to $T \rightarrow T/\equiv$ ($\equiv$ as defined in Notation 1.4). Thus $(T/\equiv) = T$.

Let $X = S - (J \cup F(j))$ and $V = X \cup T$. Let $\theta : S/F(j) \rightarrow V$ with $\theta(x) = x$ for $x \in X$ and $\theta(j) = [J]$ for $j \in J$, where $[J]$ is the equivalence class of $\equiv^{(1)}$ containing $J$. Finally $\theta(0) = 0 \in T$. Now it is very easy to verify that mod $\theta$ is a congruence on $S/F(j)$ and thus there is a unique way to define a multiplication in $V$ so that $\theta$ is an epimorphism. Thus

$$V \cong (S/F(j))/(\text{mod } \theta).$$

Now consider $\beta = H_v \theta_\eta$, where $v \in T - \{0\}$:

$$\beta : S \rightarrow S/F(j) \rightarrow V \rightarrow \text{GGM}_v(V) = U.$$ 

By construction, $\mathcal{R}$ and $\theta_\eta$ induce the same congruence on the ideal $J \cup F(j)$. Moreover, $H_v$ is one-to-one on $T$, since $(T/\equiv) = T$. Thus $\beta$ and $\mathcal{R}$ induce the same congruence on the ideal $J \cup F(j)$.

Let $I = H_v(T) = \beta(J \cup F(j))$. Then $I$ is 0-simple, being isomorphic to $T$, so $I^2 = I$. Thus it is easy to verify that $U = \text{GGM}_v(V)$ is a GGM semigroup with respect to the 0-minimal ideal $I$. See 8.2 of [8].

Now, let $\psi$ be the right Schützenberger representation of $\text{GGM}_v(V) = U$ with respect to $I$ (see Sect. 3.5 of [1] and 8.2 of [8]). Since $U$ is GGM with respect to $I$, $\psi$ is one-to-one on $U$. Furthermore, $\psi$ takes values in row-monomial matrices with coefficients in $G^0$, where $G$ is a maximal subgroup of $I$, not equal to 0 in $I$ so $\psi : U \rightarrow \text{RM}(m, G)$. Let $R$ be the right regular representation of $K[G]$. Let $R^\#$ be the homomorphism which assigns to the $m \times m$ row-monomial matrix $(x_{ij})$ over $G^0$ the $mn \times mn$ matrix $(R(x_{ij}))$ over $K$. $R^\#$ is one-to-one since $R$ is one-to-one. Let $\varphi = R^\# \psi$, then $\varphi$ is one-to-one on $U$, hence also on $T$, so $\alpha = \varphi \beta$ is a
representation of $S$ which induces the same congruence on the ideal $J \cup F(j)$ as $\mathcal{R}$. Moreover $\mathcal{R}(F(j)) = 0$ and $\alpha(F(j)) = 0$. Finally, since $\psi$ and $\varphi$ are one-to-one on $U$, mod $\alpha = \mod \beta$ and $\beta(S) = U$, a GGM semigroup. Thus, to complete the proof, it suffices to show that $\mathcal{R} \geq \beta$ and $\alpha \geq \alpha \mathcal{R}$.

We first show $\mathcal{R} \geq \beta$. Suppose $\beta(s_1) \neq \beta(s_2)$, then since $\beta(S) = U$ is a GGM, there exists $x \in I$ such that $x\beta(s_1) \neq x\beta(s_2)$, $x\beta(s_1)$, $x\beta(s_2) \in I$. Now, $I = \beta(J \cup F(j))$. Pick $x' \in J \cup F(j)$ so that $\beta(x') = x$. Since $J \cup F(j)$ is an ideal, $x's_1$, $x's_2 \in J \cup F(j)$ and $\beta(x's_1) \neq \beta(x's_2)$. But $\mathcal{R}$ induces the same congruence as $\beta$ on $J \cup F(j)$, hence $\mathcal{R}(x's_1) \neq \mathcal{R}(x's_2)$. Thus $\mathcal{R}(s_1) \neq \mathcal{R}(s_2)$ proving $\mathcal{R} \geq \beta$.

To show $\alpha \geq \mathcal{R}$, it will suffice to prove that $\mathcal{R}$ is an irreducible constituent of $\alpha$. Let $N$ be the kernel of $\alpha$ restricted to $H$, $H$ a maximal subgroup of $S$ contained in $J$. By Maschke's and Wedderburn's theorem, $\varphi$ restricted to $\beta(H) = G$ contains all irreducible representations of $G$ as constituents since the right regular representation of $G$ does. Further, $G$ is a maximal subgroup of $I - \{0\}$. (See 7.2.5(c) of [8].) Now, since $G \cong H/N$, every irreducible representation $\gamma$ of $H$ whose kernel contains $N$ induces an irreducible representation $\gamma'$ of $G$ where $\gamma = \gamma' \beta | H$. But $\gamma'$ is a constituent of $\varphi | G$, hence $\gamma$ is a constituent of $\alpha$ restricted to $H$.

In particular, $\mathcal{R}$ restricted to $H$ has kernel $N$. Let $\mathcal{R}^* = R \mid H$. Then $\mathcal{R}^*$ is an irreducible plus (perhaps) null component by the Clifford-Suschkewitsch results. Redefine $\mathcal{R}^*$ by taking only the irreducible part. Then $\mathcal{R}^*$ has $H$ in its kernel so $\mathcal{R}^*$ is a constituent of $\alpha^* = \alpha \mid H$. Now, $\alpha(F(j)) = \mathcal{R}(F(j)) = 0$, so $\alpha$ and $\mathcal{R}$ induce representations $\alpha^{(1)}$, $\mathcal{R}^{(1)}$, respectively on $J^0 = J \cup F(j)/F(j)$, a 0-simple semigroup with $\alpha^{(1)}$ and $\mathcal{R}^{(1)}$ irreducible. Since $\mathcal{R}^*$ is a constituent of $\alpha^*$, again by Clifford-Suschlewitsch $\mathcal{R}^{(1)}$ is a constituent of $\alpha^{(1)}$. Thus $\mathcal{R}^* = \mathcal{R} | J \cup F(j)$ is a constituent of $\alpha^* = \alpha | J \cup F(j)$. But Lemma 1.4 proves $\mathcal{R}$ is a constituent of $\alpha$ and the proof is complete.

In the following let $\bar{s} = (s^\# \cup F(s))/F(s)(=s^\#^0)$.

**Lemma 1.6.** Let $H_s : S \rightarrow GGM_s(S)$ be given for some regular $s \in S$.

Then $S \rightarrow S \otimes \text{IRR} \geq H_s$.

**Proof:** Let $\psi_s = \Pi \mathcal{R} : \mathcal{R}$ is an irreducible representation with apex $s^\#$. By Lemmas 1.4(a) and 1.1, $\psi_s$ induces the congruence $\equiv$ on $\bar{s}$. But $H_s$ also induces the congruence $\equiv$ on $s$. Moreover,

$$\psi_s(F(s)) = H_s(F(s)) = 0.$$
Now, \( H_s(S) \) is a GGM semigroup, so it follows easily (as in the preceding proof of \( \mathcal{R} \geq \beta \)) that \( \psi_s \geq H_s \).

**Lemma 1.7.** \( S \rightarrow S^{\circ \text{BGGM}} \) is equivalent to \( S \rightarrow S^{\circ \text{GGM}} \). Further, \( S \rightarrow S^{\gamma + J'} \) exists and is equivalent to \( S \rightarrow S^{\circ \text{GGM}} \) and \( S \rightarrow S^{\circ \text{BGGM}} \).

**Proof:** (See 8.3.15 of [8].) The first assertion follows immediately from Fact 8.3.4 of [8]. Let \( s \) be a regular element of \( S \). Then it is very easy to verify that \( H_s \) is one-to-one on the subgroups of \( S \) contained in \( s^\# \), while \( H_s \) is never zero on \( s^\# \). Thus \( S \rightarrow S^{\circ \text{BGGM}} \) is a \( \gamma + J' \) epimorphism. Now suppose \( \varphi : S \rightarrow \varphi(S) \) is a \( \gamma + J' \) epimorphism. To complete the proof it is sufficient to show that, for \( s \in S \) a regular element, \( H_s(s_1) \neq H_s(s_2) \) implies \( \varphi(s_1) \neq \varphi(s_2) \).

Now \( H_s(s_1) \neq H_s(s_2) \) iff there exists \( x_1, x_2 \in s^\# \) so that either (1) \( x_1s_1x_2 \in s^\# \) and \( x_1s_2x_2 \notin s^\# \) or (2) both \( x_1s_1x_2 \) and \( x_1s_2x_2 \) lie in \( s^\# \) and \( x_1s_1x_2 \neq x_1s_2x_2 \). In case (1), since \( \varphi \) is a \( J' \) homomorphism, \( s \) regular and Fact 8.3.9(b) of [8], we have \( \varphi(x_1s_1x_2) \neq \varphi(x_1s_2x_2) \) and so \( \varphi(s_1) \neq \varphi(s_2) \). In case (2), \( x_1s_1x_2 \notin \mathcal{H} \) and so \( \varphi(x_1s_1x_2) \neq \varphi(x_1s_2x_2) \), since \( \varphi \) is one-to-one on the \( H \)-classes of \( S \) contained in the regular \( J \)-class \( s^\# \) by Remark 8.3.13(b) of [8]. Thus again \( \varphi(s_1) \neq \varphi(s_2) \). This proves Lemma 1.7.

**Proof of Theorem 1.1:** The statement of (a) is Lemma 1.5. Then (a) implies \( (S \rightarrow S^{\circ \text{GGM}}) \geq (S \rightarrow S^{\circ \text{IRR}}) \). Lemma 1.6 implies

\[
(S \rightarrow S^{\circ \text{IRR}}) \geq (S \rightarrow S^{\circ \text{BGGM}}).
\]

Lemma 1.7 implies \( (S \rightarrow S^{\circ \text{BGGM}}) \) and \( (S \rightarrow S^{\circ \text{GGM}}) \) are equivalent. This proves (b).

Finally, (c) follows from (b) and Lemma 1.7. This proves Theorem 1.1.

**Notation 1.6.** Let \( \mathcal{A} \) be a representation of \( S \). Then \( R(\mathcal{A}) \) is the completely reducible module with the same character as \( \mathcal{A} \). That is, \( R(\mathcal{A}) \) is the direct sum \( \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_k \) of the Jordan-Hölder factors of \( \mathcal{A} \). Thus

\[
\mathcal{A}(s) = \begin{pmatrix}
\mathcal{A}_1(s) & 0 \\
& \ddots \& \\
& & \mathcal{A}_k(s)
\end{pmatrix}
\]
and

\[
R(\mathcal{R})(s) = \begin{pmatrix}
R_1(s) & 0 \\
& \ddots & \ddots \\
& & R_k(s)
\end{pmatrix}.
\]

Clearly, \( R_j \) is an irreducible representation of \( S \) for \( 1 \leq j \leq k \).

The next theorem states that the operator \( R \) "preserves one-to-oneness as much as possible."

**Theorem 1.2.** (a) Let \( \mathcal{R} \) be a representation of \( S \). Let \( \mathcal{R}(S) = T \). Let \( \psi : T \to T^{\oplus \text{IRR}} \), then \( R(\mathcal{R}) : S \to R(\mathcal{R})(S) \) is equivalent (in the sense of Notation 1.3) to \( \psi R : S \to T^{\oplus \text{IRR}} \).

(b) Thus \( R(\mathcal{R}) \) is equivalent (in the sense of Notation 1.3) to \( \varphi \mathcal{R} \), where \( \varphi : T \to T^{r+\text{IRR}} \).

**Proof:** Theorem 1.2 follows from the Burnside-Steinberg theorem [7] and Theorem 1.1. The details are as follows: To prove (a) we may assume \( \mathcal{R} \) is a faithful representation of \( T \) and then we must show

\[
R(\mathcal{R}) : T \to R(\mathcal{R})(T)
\]

is equivalent to \( T \to T^{\oplus \text{IRR}} \).

Let \( U \) be a representation and let \( U^n = U \otimes \cdots \otimes U \) (\( n \) terms) for \( n = 0, 1, 2, \ldots \), where \( \otimes \) denotes the tensor product and \( U \) denotes the representation always taking the value \( (1) \), the one by one matrix with entry \( 1 \in K \). Let \( \chi(U) \) denote the character of \( U \).

Let \( \chi_1 = \chi(\mathcal{R}_1), \ldots, \chi_q = \chi(\mathcal{R}_q) \) be the non-zero irreducible characters of \( S \). Let \( \chi(\mathcal{R}^n) = \sum_{j=1}^{q} a_{nj} \chi_j \). Then the Burnside-Steinberg theorem asserts that, for each \( j \) with \( 1 \leq j \leq q \), there exists an \( m(j) = m \geq 0 \) so that \( a_{mj} \neq 0 \).

\[
\chi(\mathcal{R}^n) = [\chi(\mathcal{R})]^n = [\chi(R[\mathcal{R}])]^n = \chi[(R(\mathcal{R})^n] = \sum_{j=1}^{q} a_{nj} \chi_j.
\]

It is well known that, if \( K \) has characteristic zero, every completely reducible module is uniquely determined by its character (see [9]). Let \( R(\mathcal{R})^{(n)} = \bigoplus_{p=0}^{n} R(\mathcal{R})^p \). Then it is easy to check that \( R(\mathcal{R})^{(n)} \) is a completely reducible module and the irreducible representation \( \mathcal{R}_j \) occurs as a constituent of \( R(\mathcal{R})^{(m)} \) where \( m = m(j), a_{mj} \neq 0 \). Thus \( R(\mathcal{R})^{(m)} \supseteq \mathcal{R}_j \) (in the sense of Notation 1.3). But clearly \( R(\mathcal{R})^{(1)}, R(\mathcal{R})^{(2)}, \ldots \) all induce
congruences equal to mod $R(\mathcal{R})$. Thus $R(\mathcal{R}) \geq R(\mathcal{R})^{(m)} \geq \mathcal{R}_j$ for $1 \leq j \leq q$, and $m = m(j)$. So

$$R(\mathcal{R}) \geq \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_q : S \rightarrow S^{\otimes \mathcal{IRR}}.$$ 

But the reverse inequality is immediate since $R(\mathcal{R})$ is completely reducible. This proves (a).

The assertions of (b) follow immediately from (a) and Theorem 1.1. This proves Theorem 1.2.

2. Characters and Complexity

**Notation 2.1.** (See Chapters 1 and 5-9 of [8] for extensive background and exposition. For additional references see [3], [4], [5], and [6].)

We recall the definition of the (group) complexity of a finite semigroup $S$.

If $S_1$ and $S_2$ are semigroups and $Y$ is a homomorphism of $S_1$ into $\text{endo}(S_2)$, the semigroup of endomorphisms of $S_2$, the semidirect product of $S_2$ by $S_1$ with connecting homomorphism $Y$, denoted by $S_2 \rtimes Y S_1$, is the semigroup with elements $S_2 \times S_1$ and product defined by

$$(s_2, s_1)(s_2', s_1') = (s_2 Y(s_1)(s_2'), s_1 s_1').$$

We write

$$S_n \times Y_{n-1} S_{n-1} \times Y_{n-2} \cdots \times Y_1 S_1 = T_n$$

for the semigroup

$$\cdots (S_n \times Y_{n-1} S_{n-1}) \times Y_{n-2} S_{n-2} \cdots \times Y_1 S_1,$$

where

$$Y_{n-1} : S_{n-1} \rightarrow \text{endo}(S_n), \ldots, Y_j : S_j \rightarrow \text{endo}(S_n \times Y_{n-1} \cdots \times Y_{j+1} S_{j+1}), \ldots,$$

$$Y_1 : S_1 \rightarrow \text{endo}(S_n \times Y_{n-1} \cdots \times Y_2 S_2)$$

are homomorphisms.

$S \mid T$, read $S$ divides $T$, iff there exists a subsemigroup $T' \subseteq T$ and an epimorphism $\varphi : T' \rightarrow S$.

**Definition 2.1.** $S$ is a combinatorial semigroup iff each subgroup of $S$ has order 1. Let $S$ be a semigroup. Then $\#(S)$, read the complexity number of $S$, is the smallest positive integer $n$ so that

$$S \mid T_n \times Y_{n-1} T_{n-1} \times Y_{n-2} \cdots \times Y_2 T_2 \times Y_1 T_1$$

(2.1)
where either

(a) \(T_1, T_3, T_5, \ldots\) are groups and \(T_2, T_4, T_6, \ldots\) are combinatorial semigroups or

(b) \(T_1, T_3, T_5, \ldots\) are combinatorial semigroups and \(T_2, T_4, T_6, \ldots\) are groups. See Chapters 6 and 9 of [8].

**Notation 2.2.** We define \(C(S) = (n, G)\) iff (a) above holds with \(n = \#(S)\), but (b) never holds with \(n = \#(S)\). Similarly, we define \(C(S) = (n, C)\) iff (b) holds with \(n = \#(S)\) but (a) never holds with \(n = \#(S)\). Finally, \(C(S) = (n, C \lor G)\) iff either (a) or (b) can hold with \(n = \#(S)\). \(C(S)\) is called the complexity of \(S\). That \(C(S)\) is well defined for every finite semigroup follows from [8].

**Notation 2.3.** (See [8].) Let \(\mathcal{C}\), the set of all complexities, equal

\[
\{1, 2, 3, \ldots\} \times \{C, G, C \lor G\}.
\]

Let \(\#\) be the function from \(\mathcal{C}\) to \(\{1, 2, 3, \ldots\}\) with \(\#(n, \alpha) = n\). We note that \(\#(S) = \#(C(S))\). We order \(\mathcal{C}\) by \(\leq\) where \(C_1 \leq C_2\) iff

(a) \(C_1 = C_2\), or

(b) \(\#(C_1) < \#(C_2)\), or

(c) \(\#(C_1) = \#(C_2) = n\) and \(C_1 = (n, C, \lor G)\).

Then \((\mathcal{C}, \leq)\) is a lattice with minimal element \((1, C, \lor G)\).

Let \((C, \lor G, n) = (n, C \lor G)\) for all \(n \geq 1\). Let \((C, 2n) = (2n, G)\) and \((G, 2n) = (2n, C)\) for \(n = 1, 2, 3, \ldots\). Let \((C, 2n + 1) = (2n + 1, C)\) and \((G, 2n + 1) = (2n + 1, G)\) for \(n = 0, 1, 2, \ldots\).

Let \(\#_G(2n, C) = \#_G(2n + 1, C) = \#_G(2n, G) = n\) for \(n \geq 1\). Let \(\#_G(2n + 1, G) = n + 1\) for \(n \geq 0\) and let \(\#_G(1, C) = 0\). Let

\[
\#_G(k, C \lor G) = \#_G(k, C) \quad \text{for} \quad k \geq 0.
\]

Then \(\#_G(C(S))\) is the smallest number of groups appearing in the solutions of equation (2.1). Let \(\#_G(S) = \#_G(C(S))\).

Finally we introduce the following notation. Let

\[
(C, 1) \oplus (C, n) = (C, 1) \oplus (C, \lor G, n) = (C, n).
\]

Let \((C, 1) \oplus (G, n) = (C, n + 1)\). We notice that

\[
(C, 1) \oplus \text{lub}(X) = \text{lub}(((C, 1) \oplus \alpha : \alpha \in X)) \geq \text{lub}(X),
\]

for \(X\) any finite set of complexities.
Recall Notations 1.1 and 1.3. $S ightarrow S^v, S ightarrow S^g$ denote the functorially minimal $\gamma$ and $\alpha$-homomorphisms, respectively, where $\alpha$ is any of the Green relations. See Chapter 8 of [8]. Notice if $S$ is regular (e.g., $S$ is a union of groups), then $\alpha$ and $\alpha'$ epimorphisms coincide.

**Theorem A** (See Chapter 9 of [8], especially Theorem 9.2.5). Let $S$ be a semigroup which is a union of groups, then:

(a) $\#_\sigma(S) = \#_\sigma(S')$.

(b) If $S' \neq \{1\}$, then $\#_\sigma(S^{\gamma+L}) + 1 = \#_\sigma(S')$.

(c) Consider

$$S \rightarrow S^v \rightarrow S^{\gamma+L} \rightarrow S^{\gamma+L} \rightarrow \cdots \rightarrow \{1\}, \quad (2.2)$$

then $\#_\sigma(S)$ equals the number of $L$ operators in (2.2), i.e., the number of non one-to-one $L$ epimorphisms.

(a') $S^\sigma$ is combinatorial and $\#_\sigma(S) = \#_\sigma(S^{\gamma+L})$.

(b') If $S' \neq \{1\}$, then $\#_\sigma(S^{\gamma+L}) + 1 = \#_\sigma(S^{\gamma+L})$.

(c') Consider

$$S \rightarrow S^{\gamma+L} \rightarrow S^{(\gamma+L)^2} \rightarrow S^{(\gamma+L)^3} \rightarrow \cdots \rightarrow \{1\}, \quad (2.3)$$

then $\#_\sigma(S)$ equals the number of $L$ operators in (2.3), i.e., the number of non one-to-one $L$ epimorphisms.

**Proof:** The theorem is proved by applying Theorem A of [5] together with its corollaries as developed in [6]. For a detailed exposition see Chapter 9 of [8], especially Definition 9.24 and Theorem 9.2.5.

The assertion of (a) follows by Proposition 6.10 of [6] or Theorem 9.2.15 and Corollary 9.3.4 of [8]. To prove (b) first assume $S^{\gamma+L} = \{1\}$. Then $S'$ is right simple. Thus by the well-known structure theorem for right simple semigroups (see [1] or [8]), $\#_\sigma(S') = 1$, $\#_\sigma(S^{\gamma+L}) = 0$, so (b) is true in this case. Now, assume $S^{\gamma+L} \neq \{1\}$, then Remark 6.5 of [6] (see also [5] and [8]) yields

$$C(S) \leq (C, 1) \oplus (G, 1) \oplus C(S^{\gamma+L}), \quad (2.4)$$

Thus

$$(G, n) = C(S') \leq (C, 1) \oplus (G, 1) \oplus C(S^{\gamma+L}), \quad (2.5)$$

Thus

$$C(S^{\gamma+L}) = (C, k).$$
But (2.5) implies \((G, k) \leq (G, n)\), thus \(k < n\). Also (2.5) implies \((G, n) \leq (G, k + 2)\), thus \((G, n) \leq (G, k + 1)\), so \(n \leq k + 1\). Hence \(n = k + 1\) and (b) is proved.

Now (c) follows from (a) and (b). The series (2.2) reaches \([1]\) by [5] or Chapters 8 and 9 of [8].

That \(S^f\) is combinatorial follows because \(s \rightarrow S^1sS^1\) is a homomorphism of \(S\) into the semigroup of subsets of \(S\) under intersection. See [1, Ch. 4] or [8, Proposition 7.24]. In fact, \(S^f\) is a semilattice or commutative band. Now, \(S^{v+j} \leq S^v \times S^f\), by Proposition 8.3.15 of [8]. Thus

\[
C(S^{v+f}) = \text{lub}(C(S^v), C(S^f)).
\]

Now \(C(S^f) \leq (C, 1)\) so \#_c(S^{v+j}) = \#_c(S^v)\) and (a') follows from (a).

We next prove (b'). Assume \(S^v\) is not a group. Then

\[
C(S^{v+f}) = \text{lub}(C(S^v), C(S^f)) = \text{lub}(C(S^v), (C, 1)) = C(S^v).
\]

Further, assume \(S^{(v+j)}, \varnothing \neq \{1\}\). Then (2.4) implies

\[
(G, n) = C(S^v) = C(S^{v+f}) \leq (C, 1) \oplus (G, 1) \oplus C(S^{(v+f)}, \varnothing),
\]

Then as before, we find \(n = k + 1\). Now suppose \(S^v\) is a group \(\neq \{1\}\). Then \((G, 1) \leq C(S^{v+f}) \leq (C \vee G, 2)\). Further \(S^{v+f}\) divides \(S^v \times S^f\).

Now, by Proposition 6.7 of [6] the projection map \(S^v \times S^f \rightarrow S^f\) is an \(L\)-homomorphism and since \(S^f\) is a commutative band, it is clearly minimal so \((S^v \times S^f, \varnothing) = S^f\). Thus \#_c(S^{(v+f)}, \varnothing) = 0 and \#_c(S^{v+f}) = 1. So (b') holds in this case. Next, assume \(S^{v+f}, \varnothing = \{1\}\), then \(S^{v+f}\) is left simple, so again \#_c(S^{v+f}) = 1 and \#_c(S^{v+f}, \varnothing) = 0. This proves (b').

Now (c') follows from (a') and (b'). The series (2.3) reaches \(S^f\) by Chapters 8 and 9 of [8]. This proves Theorem A.

**NOTATION 2.5.** (See Definition 9.2.4(j), pp. 238–239 of [8].) Let \(S\) be a finite semigroup. Let \(\mathcal{R}_1, \ldots, \mathcal{R}_k, \ldots, \mathcal{R}_n\) be a complete set of inequivalent non-zero irreducible representations of \(S\). Let \(\chi_j = \chi(\mathcal{R}_j)\) be the associated characters. \(\chi_1, \ldots, \chi_k\) are those characters taking only the values zero and one.

Let \(s\) be a regular element of \(S\). Let \(s^#\) be the \(\mathcal{J}\)-class containing \(s \in S\). Let \(\mathcal{L}_b : b \in B_s\) be the \(\mathcal{L}\) classes of \(S\) contained in \(s^#\). The semigroup \(F_R(A)\) was defined in Notation 1.2. Let \(R_s : S \rightarrow F_R(B_s, 0)\) be the homomorphism given by \(R_s(s')(b) = b'\) where \(b' = 0\) if \(b = 0\), or \((\mathcal{L}_b s') \cap L_b = \phi\) for all \(b \in B\). Otherwise \(b'\) is the (unique) \(b' \in B_s\) satisfying \((\mathcal{L}_b s') \cap L_b \neq \phi\). We denote \(R_s(S)\) by \(\text{RLM}_s(S)\). See Definition 8.2.8 ff. of [8]. Now, by
identifying $0 \in B_0$ with the zero of $K[S]$ and considering the elements of $R_\lambda(S)$ as matrices (having entries 0 and 1) we obtain a right $K$-representation $R_\lambda(S)$ of $S$, with character $\chi = \chi(R_\lambda(S))$, written $\chi(RLM_\lambda(S))$, where $\chi(s) = |\{b \in B : L_0s \cap L_b \neq \emptyset\}|$. Here $|X|$ denotes the cardinality of $X$.

When $S$ has a unique minimal or 0-minimal ideal $I$ (e.g., $S$ a GGM semigroup $\neq \{0\}$) we write $RLM(S)$ for $RLM_\lambda(S) \neq 0, s \in I$.

$$C(S) = \{\sum a_i x_i : a_i \text{ an integer}\}$$

denotes the character ring of $S$, the operations being pointwise addition and multiplication. As is well known, $\chi_1, \ldots, \chi_n$ are linearly independent over $K$, see [2], and hence over the integers.

The following definition is fundamental. (See Definition 9.2.4(j), pp. 238–239 of [8].)

**DEFINITION 2.2.** $A(S): C(S) \rightarrow C(S)$ is the linear transformation given by

$$A(S)(\chi_i) = \chi(RLM_\lambda(S)))$$

$$A(S)(\sum a_i \chi_i) = \sum a_i A(S)(\chi_i). \quad (2.7)$$

Note that the matrix $(\alpha_{ij})$ of $A(S)$ has non-negative integer coefficients.

We recall from notation 2.5 that $\chi_1, \ldots, \chi_k$ are those characters taking only the values of zero and one. $A'$ denotes the semigroup with elements $A$ and multiplication $aa' = a'$. $B'$ denotes the semigroup with elements $B$ and multiplication $b'b = b'$.

**LEMMA 2.1.** Let $S$ be a semigroup which is a union of groups. Then $A(S)(\chi_i) = \chi_j$ iff $1 \leq j \leq k$ iff $\chi_j(S) \subseteq \{0, 1\} \subset K$.

**PROOF:** Let $J_i$ be the apex of $R_\lambda$. Since $S$ is a union of groups, $J_i$ is a simple subsemigroup of $S$. Let $I_j = R_\lambda(J_i)$. Then $I_j$ is a simple semigroup which has a faithful irreducible representation (induced by $R_\lambda$).

Thus, by Lemma 1.1, $I_j \equiv \simeq J_i$.

Now, suppose that each subgroup of $I_j$ has order one, i.e., $I_j$ is combinatorial. Then it is well known (see [1] or [8]) that $I_j \simeq A_j \times B_j$, but then $I_j \equiv$ has order one. Thus $I_j$ combinatorial implies

$$I_j = R_\lambda(J_i) = \{x\}.$$ 

Now since $x$ is a non-zero idempotent and $R_\lambda$ is irreducible it follows easily that $x = 1 \in K$.

Now define an irreducible representation $T_\lambda$ of $S$ by $T_\lambda(s) = 1$ when $J_0s \subseteq J_i$ and zero otherwise. $T_\lambda$ agrees with $R_\lambda$ on the ideal $J_i \cup I(J_i)$. 

Thus, by Lemma 1.4, $\mathcal{F}_j = \mathcal{R}_j$, so $1 \leq j \leq k$. Now, from the definition of $\mathcal{F}_j = \mathcal{R}_j$, it follows that $\mathcal{A}(\mathcal{S})(\chi_j) = \chi_j$, $\chi_j = \chi(\mathcal{R}_j) = \mathcal{R}_j$.

Assume now that $I_j$ is non-combinatorial. Then there exists a maximal subgroup $G$ of $I_j$ so that $\mathcal{R}_j \mid G$ has a constituent $\mathcal{R}$ which is an irreducible representation of $G$ whose kernel is properly contained in $G$. Now, by the character relations for groups $\sum \{\chi(\mathcal{R})(g) : g \in G\} = 0$ and $\chi(\mathcal{R})$ is not identically zero (in fact,

$$\sum \{\chi(\mathcal{R})(g) \chi(\mathcal{R})(g^{-1}) : g \in G\} = |G|.$$  

Thus $\chi(\mathcal{R})$ cannot assume only the values zero and one, and the same is true of $\chi(\mathcal{R}_j) = \chi_j$. Let $s \in J_j$, then $\chi(\mathcal{R}_j, (\mathcal{R}_j(s))) \subseteq \{0, 1\}$, thus $\mathcal{A}(\mathcal{S})(\chi_j) \neq \chi_j$. This proves Lemma 2.1.

DEFINITION 2.3. Since, by Lemma 2.1, $\mathcal{A}(\mathcal{S})$ is the identity transformation when restricted to $W = \{\sum_{i=1}^k a_i \chi_i\}$, we can define the linear transformation $B(\mathcal{S}) : \mathcal{C}(\mathcal{S})/W \rightarrow \mathcal{C}(\mathcal{S})/W$ to be the transformation induced by $\mathcal{A}(\mathcal{S})$.

NOTATION 2.6. Let index $(B(\mathcal{S})) = 0$ iff $\mathcal{C}(\mathcal{S})/W = \{0\}$. Let index $(B(\mathcal{S})) = n \geq 1$ iff $\mathcal{C}(\mathcal{S})/W \neq \{0\}$ and $n$ is the smallest positive integer such that $(B(\mathcal{S}))^n$ equals the zero operator. In all other cases let index $(B(\mathcal{S})) = +\infty$.

THEOREM B. Let $\mathcal{S}$ be a semigroup which is a union of groups. Then

(a) $B(\mathcal{S})$ is nilpotent, i.e., index $(B(\mathcal{S})) < +\infty$,
(b) index $(B(\mathcal{S})) = \#_\mathcal{C}(\mathcal{S})$.

PROOF: (For a detailed exposition of the proof of Theorem B assuming the lemmas of this paper proved earlier, see Lemma 9.2.32 of [8].) We introduce the following notation. Let $X$ be a non-empty subset of $\mathcal{C}(\mathcal{S})$. Then $H(X) : \mathcal{S} \rightarrow H(X)(\mathcal{S})$ is the epimorphism

$$H(X) = \prod \left\{ \mathcal{R}_j : \text{there exists } x \in X, x = \sum_{i=1}^n a_i \chi_i, a_i > 0, 1 \leq i \leq n \right\}.$$  

By Theorem 1.1 and Lemma 1.7 we have

$$H[\mathcal{C}(\mathcal{S})] : \mathcal{S} \rightarrow \mathcal{S}^{\oplus GGM} = \mathcal{S}^{\oplus GGM} = \mathcal{S}^{\oplus BGM}.$$  

By Theorem A,

$$\#_\mathcal{C}(\mathcal{S}) = \#_\mathcal{C}(\mathcal{S})^{\oplus GGM} = \#_\mathcal{C}[H[\mathcal{C}(\mathcal{S})]()] = (2.8)$$  

Recall Notation 1.5. Let \( \psi_1 = \prod H_i : S \rightarrow \text{GGM}_s(S) : s^# \text{ is combinatorial} \). Now, when the simple subsemigroup \( s^# \) is combinatorial, \( s^# \) is isomorphic to \( A_r \times B^t \) and thus \( s^/# = \{1\} \). Now, since \( s = S^{1} \cdot s^{1} \) is a homomorphism of \( S \) into the set of subset of \( S \) under intersection, \( H_\psi(x) = 1 \), when \( xs^# \subseteq s^# \) and \( H_\psi(x) = 0 \) otherwise, i.e., \( H_\psi = \chi \), for some \( 1 \leq j \leq k \). Thus \( \psi_1 \) is equivalent to \( S \rightarrow S^{#} \) and to \( \prod \{ \mathcal{A}_i : 1 \leq i \leq k \} \).

Now consider the epimorphism

\[
R_\psi \cdot H_\psi : S \rightarrow \text{GGM}_s(S) \rightarrow \text{RLM}_s'(\text{GGM}_s(S)),
\]

where \( s^# \) is non-combinatorial and \( s' = H_\psi(s) \). Now it is easily seen that \( [\text{GGM}_s(S)]^* = \text{GGM}_s(S) \) when \( s^# \) is non-combinatorial,

\[
\text{RLM}_s'(\text{GGM}_s(S))^* = \text{RLM}_s'(\text{GGM}_s(S))
\]

and \( R_\psi \) is an \( \mathcal{L} \)-homomorphism. See 8.3.25 of [8]. Thus

\[
\text{RLM}_s'(\text{GGM}_s(S)) \neq \{1\}
\]

and equation (2.4) implies \( C(\text{GGM}_s(S)) = (G, n) \) and

\[
C(\text{RLM}_s'(\text{GGM}_s(S))) = (C, n - 1).
\]

For more details see [5] and Chapters 8 and 9 of [8].

Now consider the epimorphism

\[
\theta_\psi \cdot R_\psi \cdot H_\psi : S \rightarrow \text{GGM}_s(S) \rightarrow \text{RLM}_s'(\text{GGM}_s(S))
\]

\[
\rightarrow \text{RLM}_s'(\text{GGM}_s(S)) \otimes^{\text{BGGM}}.
\]

Then assuming \( \text{RLM}_s'(\text{GGM}_s(S)) \neq \{1\} \) we have

\[
\#_s(\text{GGM}_s(S)) - 1 = \#_G(\text{RLM}_s'(\text{GGM}_s(S)))
\]

\[
= \#_G((\text{RLM}_s'(\text{GGM}_s(S))) \otimes^{\text{BGGM}}).
\]  

(2.9)

Note that, by the Proof of Lemma 2.1, all irreducible characters belong to \( W \) iff \( S \) is combinatorial. In the trivial case in which \( S \) is combinatorial, \( \#_G(S) = 0 \), and by our convention index \( B(S) = 0 \), thus the theorem holds in this case. Hence we may assume \( k < n \), i.e., \( \#_G(S) \geq 1 \). For \( k < i \leq n \), we have by (2.9) and Theorems 1.2 and 1.1 that

\[
\#_s(\mathcal{A}_i(S)) - 1 = \#_G(\chi(\text{RLM}[\mathcal{A}_i(S)])(S)).
\]
Hence
\[
\#\sigma(S) - 1 = \#\sigma(S^{\otimes 1}^{\text{IRR}}) - 1 = \max_{k < i \leq n} (\#\sigma(R_i(S)) - 1) \\
= \max_{k < i \leq n} \#\sigma(H(A(S)(X_i))(S)) \\
= \#\sigma[H(A(S)[\xi(S)])(S)].
\]

Thus
\[
\#\sigma(S) - 1 = \#\sigma[H(A(S)[\xi(S)])(S)].
\]

Replacing $S$ by $H(A(S)[\xi(S)])(S)$ and repeating the argument $k$ times, we have
\[
\#\sigma(S) - k = \#\sigma[H(A(S)^k[\xi(S)])(S)]
\]
as long as the right-hand side is positive. When $H(A(S)^k[\xi(S)])(S)$ is combinatorial, $H(A(S)^k[\xi(S)])$ contains only irreducible representations with range $\mathbb{C}\{0, 1\}$, thus $B(S)^k$ is the zero operator. Now (a) and (b) follow from Theorem A.

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**REFERENCES**