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Groups of Lie type generated by long root elements in $F_4(K)$

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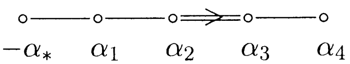
Abstract

Let K be a field and G a quasi-simple subgroup of the Chevalley group $F_4(K)$. We assume that G is generated by a class Σ of abstract root subgroups such that there are $A, C \in \Sigma$ with $[A, C] \in \Sigma$ and any $A \in \Sigma$ is contained in a long root subgroup of $F_4(K)$. We determine the possibilities for G and describe the embedding of G in $F_4(K)$.

Keywords: Group of Lie type; Chevalley group; Long root element; Root subgroup; Orthogonal group; Building; Polar space; Moufang hexagon

Introduction

For an arbitrary commutative field K, we denote by $F_4(K)$ the universal Chevalley group of type F_4 over K. This is the group generated by symbols $x_r(t)$, $t \in K$, $r \in \Phi$, with respect to the Steinberg relations; we refer to Carter [1, 12.1.1]. Here Φ is the root system of type F_4, a subset of the Euclidean space $\mathbb{R}^4$ with orthonormal basis $\{e_1, e_2, e_3, e_4\}$. In the notation of Bourbaki [2], the extended Dynkin diagram of type F_4 is



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where

$$\alpha_* = e_1 + e_2, \qquad \alpha_1 = e_2 - e_3, \qquad \alpha_2 = e_3 - e_4,$$
$$\alpha_3 = e_4, \qquad \alpha_4 = \tfrac{1}{2}(e_1 - e_2 - e_3 - e_4).$$

A long root subgroup of $F_4(K)$ is a conjugate of $X_{\alpha_*} = \{x_{\alpha_*}(t) \mid t \in K\} \simeq (K, +)$. The group $F_4(K)$ is generated by its class of long root subgroups and is simple.

In the following, we study subgroups of $F_4(K)$ which are of 'Lie type'. Here a group of Lie type is the subgroup of the automorphism group of any spherical Moufang building generated by the root subgroups (as defined by Tits [3]). These groups of Lie type are closely related to the groups generated by a class of so-called abstract root subgroups in the sense of Timmesfeld [4,5].

A conjugacy class Σ of abelian subgroups of a group G is called a class of abstract root subgroups of G, if $G = \langle \Sigma \rangle$ and for $A, B \in \Sigma$, one of the following holds:

(a) $[A, B] = 1$.
(b) $[A, B] = [a, B] = [A, b] \in \Sigma$ for $a \in A^\#$, $b \in B^\#$; moreover $[A, B]$ commutes with A and with B.
(c) $\langle A, B \rangle$ is a rank 1 group with abelian unipotent subgroups A, B. (This means that for $a \in A^\#$, there exists $b \in B^\#$ such that $A^b = B^a$ and vice versa.)

This paper is devoted to the study of the following problem:

(P) Let Σ be a class of abstract root subgroups of G such that:
(1) There are $A, B, C \in \Sigma$ such that $\langle A, B \rangle$ is a rank 1 group and $[A, C] \in \Sigma$.
(2) For $A, C \in \Sigma$, $C_\Sigma(A) = G_\Sigma(C)$ implies $A = C$.
We assume that G is a subgroup of $Y = F_4(K)$ such that any element $A \in \Sigma$ is contained in a long root subgroup $\widehat{A}$ of Y.
The problem is to determine the possible G and the embedding of G in Y. By Timmesfeld [4, (3.17)] or [5, II (2.14)], any such G is quasi-simple. For G as in (P), we say (for short) that G is a group of Lie type embedded in Y.

Problem (P) contributes to the determination of subgroups of groups of Lie type generated by long root elements. For a subsystem Ψ of Φ with fundamental root system $\{p_1, \ldots, p_r\}$, we define $M(p_1, \ldots, p_r) := \langle X_r \mid r \in \Psi \rangle$. These subgroups of $F_4(K)$ are called subsystem subgroups. In $F_4(K)$ there are the following classical subsystem subgroups:

$$M(-\alpha_*, \alpha_1, \alpha_2, \alpha_3) \simeq B_4(K),$$
$$M(\alpha_2, \alpha_3, \alpha_4) \simeq C_3(K),$$
$$M(\alpha_2, \alpha_3, \alpha_4, -e_1) \simeq C_4(K) \quad \text{in characteristic 2},$$

and the subsystem subgroups of these.

When G as in (P) already embeds in a (proper) subsystem subgroup M, then this reduces to the study of subgroups of classical groups; we refer to Steinbach [6] and Cuypers and Steinbach [7].

We prove the following theorem.

Theorem 1. *For any subgroup G of $F_4(K)$ as in* (P) *above, passing to a conjugate in $F_4(K)$, one of the following holds*:

(1) *G is contained in the classical subsystem subgroup $M(-\alpha_*, \alpha_1, \alpha_2, \alpha_3) \simeq B_4(K)$.*
(2) *G is $F_4(L)$, L a subfield of K, or a group $F_4(L, F)$ of mixed type F_4 in characteristic* 2*, where $F^2 \subseteq L \subseteq F$, F, L subfields of K.*
(3) *G arises from a Moufang hexagon.*

We remark that there is overlap between Cases (1) and (3). Below in Theorems 2–6, we give more detailed information on the possible subgroups G and their embeddings in $F_4(K)$.

In addition to the Steinberg generators and relations for $F_4(K)$ mentioned above, we use the associated building. In this building, there are four types of objects, called points, lines, planes and symplecta, and the long root subgroups of $F_4(K)$ may be identified with the points.

We use the classification of polar spaces due to Tits [3] and the classification of Moufang polygons by Tits and Weiss [8] (as stated by Van Maldeghem [9]). Another important tool is the determination of weakly embedded polar spaces by Steinbach and Van Maldeghem [10,11].

Our strategy to solve Problem (P) is as follows: let G be a subgroup of $F_4(K)$ as in (P). By the classification of groups generated by abstract root subgroups, due to Timmesfeld [4] or [5], G is on a list of groups of Lie type and of infinite-dimensional classical groups. Some of these candidates are easily eliminated. We have to deal with the cases where G arises from an orthogonal space of Witt index 3 or 4, a building of type F_4 or a Moufang hexagon. We remark that instead of using Timmesfeld's classification we could have taken the latter groups of Lie type as a starting point for the investigation of subgroups of $F_4(K)$ generated by parts of long root subgroups.

Next, we state the results obtained in the respective cases. For unexplained terminology, we refer to the section where the subgroups in question are dealt with. When G arises from an orthogonal space, we are able to reduce to a classical subsystem subgroup of $F_4(K)$ as in the next theorem.

Theorem 2. *Let G be a subgroup of $F_4(K)$ as in* (P). *We assume that there is a vector space W* (*over the field L*), *endowed with a non-degenerate quadratic* *form* $q : W \to L$ *such that* $\overline{G} := G/\mathrm{Z}(G) \simeq \mathrm{P}\Omega(W, q)$ *with* $\overline{\Sigma}$ *the class of* (*projective*) *Siegel transvection groups.*

Then a conjugate of G *in* $F_4(K)$ *is contained in* $M(-\alpha_*, \alpha_1, \alpha_2, \alpha_3) = B_4(K)$ (*with underlying orthogonal space denoted* V). *Moreover, the Witt index of* q *is* 3 *or* 4 *and the orthogonal polar space* (W, q) *is weakly embedded in* $P([V, G])$.

When G arises from a building of type F_4, we show:

Theorem 3. *Let* G *be a subgroup of* $F_4(K)$ *as in* (P). *We assume that there is a building* $\mathcal{F}$ *of type* F_4 *such that* $\overline{G} := G/\mathrm{Z}(G)$ *is isomorphic to the normal subgroup of* $\mathrm{Aut}(\mathcal{F})$ *generated by the class of long root subgroups with* $\overline{\Sigma}$ *the class of long root subgroups.*

Then, passing to a conjugate in $F_4(K)$, *one of the following holds for* G:

- *When* $\mathrm{char}(K) \neq 2$, *there is a subfield* L *of* K *such that* G *is* $F_4(L)$.
- *When* $\mathrm{char}(K) = 2$, *there are subfields* L, F *of* K *with* $F^2 \subseteq L \subseteq F$ *such that* G *is the group* $F_4(L, F)$ *of mixed type* F_4.

When G arises from a Moufang hexagon Γ, then we show that Γ is 'classical'. This means that the exceptional Moufang hexagons related to forms of E_6, ${}^2E_6$, or E_8 do not occur; but the so-called mixed hexagons in characteristic 3 do. In detail we prove the following theorem.

Theorem 4. *Let* G *be a subgroup of* $F_4(K)$ *as in* (P). *We assume that there is a Moufang hexagon* Γ, *such that* $\overline{G}$ *is isomorphic to the subgroup of* $\mathrm{Aut}(\Gamma)$ *generated by the class of long root subgroups with* $\overline{\Sigma}$ *the class of long root subgroups.*

Then Γ *is a* G_2*-hexagon, an* ${}^3D_4$*-*, ${}^6D_4$*-hexagon, or a mixed hexagon in characteristic* 3. *When* K *is algebraically closed and* Γ *is not a mixed hexagon in characteristic* 3, *then a conjugate of* G *in* $F_4(K)$ *is contained in*

$$M(e_1 - e_2, e_2 - e_3, e_3 - e_4, e_3 + e_4) = D_4(K).$$

Furthermore we show that the groups associated to a G_2- and ${}^3D_4$-, ${}^6D_4$-hexagon embed in $F_4(L)$, where L is the ground field coordinatizing the long root subgroups.

Theorem 5. *Let* $\widehat{L} : L$ *be a separable cubic field extension with Galois closure* $\overline{L}$. *Then the groups* ${}^3D_4(\widehat{L})$, ${}^6D_4(\overline{L})$ (*depending on whether* $\widehat{L} : L$ *is a Galois extension or not*) *embed in* $F_4(L)$ *such that long root subgroups are long root subgroups.*

To complete the study of groups of Lie type embedded in $F_4(K)$, we also deal with the Ree groups ${}^2F_4(L, \sigma)$. Here the centers of the long root subgroups (which are the central elation subgroups of a Moufang octagon) do not form a class of abstract root subgroups. We show the following proposition.

Proposition 6. *Let K be a field and Γ a Moufang octagon admitting central elations. Then the subgroup of* $\mathrm{Aut}(\Gamma)$ *generated by the central elation subgroups of $F_4(K)$ cannot be a subgroup of $F_4(K)$ such that the central elation subgroups are contained in long root subgroups of $F_4(K)$.*

In this paper on subgroups of $F_4(K)$ the emphasis is on arbitrary fields, including non-perfect fields in characteristic 2 (as, for example, the field of rational functions over GF(2)). The latter are involved in many interesting phenomena, in particular in the groups of mixed type F_4.

For finite groups and for algebraic groups over an algebraically closed field, results on groups of Lie type embedded in $F_4(K)$ are in the literature. Subgroups of simple algebraic groups over an algebraically closed field, which are generated by full long root subgroups have been determined by Liebeck and Seitz [12] for all classical and exceptional types. In their setting the subgroups in question arise from one of the spherical root systems. Stensholt [13] constructs embeddings among finite groups of Lie type such that long root subgroups are long root subgroups. For the exceptional types in the finite case, the embedded groups of Lie type have been determined by Cooperstein [14,15]. In [14, Part I] the results on subgroups generated by full long root subgroups are achieved geometrically; in [15], on subgroups over a subfield, classification results are used.

The paper is organized as follows: in the preliminary Section 1, we collect properties of $F_4(K)$ and we state classification results for later use. In Section 2 we begin the proof of the theorems stated above. The groups arising from an orthogonal space are dealt with in Section 2; the ones arising from a building of type F_4 in Section 3. The subgroups associated to a Moufang hexagon are investigated in Sections 4 and 5. Finally, we deal with Moufang octagons in Section 6.

Proof of Theorem 1. We use Timmesfeld's classification of groups generated by abstract root subgroups, as stated in Section 1.6. In Section 2.2 we eliminate the subgroups $E_n(L)$, $n = 6, 7, 8$. In Section 2.4 we deal with groups arising from a projective space. Now Theorem 2 on orthogonal groups (proved in Section 2) and Theorem 3 on subgroups arising from a building of type F_4 (proved in Section 3) yield Theorem 1. $\square$

This paper was taken from my Habilitationsschrift [16]. There also the subgroups G of $F_4(K)$ as in (P) but generated by a class Σ of so-called abstract transvection subgroups (where Possibility (b) never occurs) are handled; see also Steinbach [25].

1. Preliminaries

In this section we collect some properties of the Chevalley group $F_4(K)$ and its associated building. Furthermore, we state the classification of weakly embedded classical polar spaces due to Steinbach and Van Maldeghem [10,11] as well as the classification of groups generated by abstract root subgroups, due to Timmesfeld [4, Theorem 5] (see also [5, III Section 9]). Both results will be used later.

For the definition and properties of Chevalley groups and the associated root systems, we refer to Carter [1], Steinberg [17] and Bourbaki [2].

1.1. Notation for abstract root subgroups. We recall the definition of a class Σ of abstract root subgroups of the group G from the introduction. For $U \leqslant G$ and $A \in \Sigma$, we define $\Sigma \cap U := \{T \in \Sigma \mid T \leqslant U\}$, $C_\Sigma(A) := \{B \in \Sigma \mid [A, B] = 1\}$ and $M_A := \langle \Lambda_A \rangle$.

Let $A, B \in \Sigma$. When $[A, B] \in \Sigma$ as in Possibility (b), we also write $B \in \Psi_A$. Furthermore, $B \in \Lambda_A$ means that $[A, B] = 1$, $A \neq B$, and $\Sigma \cap AB$ is a partition of AB. Note that $[A, B] \in \Lambda_A$, when $B \in \Psi_A$.

1.2. Chevalley commutator relations in $F_4(K)$. Let $t, u \in K$ and $r, s \in \Phi$. When $0 \neq r + s \notin \Phi$, then $[x_r(t), x_s(u)] = 1$. When $r + s \in \Phi$, then the following holds (with signs depending on r, s, but not on t, u):

(a) If r, s are long or if $r, s, r + s$ are short, then $[x_r(t), x_s(u)] = x_{r+s}(\pm tu)$.
(b) If r, s are short and $r + s$ is long, then $[x_r(t), x_s(u)] = x_{r+s}(\pm 2tu)$.
(c) If r is long and s is short, then $[x_r(t), x_s(u)] = x_{r+s}(\pm tu)x_{r+2s}(\pm tu^2)$.

Furthermore, $\langle X_r, X_{-r} \rangle \simeq \mathrm{SL}_2(K)$.

1.3. The F_4-geometry. We consider the building associated to $F_4(K)$ (in the sense of Tits [3]) as a point-line geometry, the F_4-geometry. There are four types of objects: points, lines, planes and symplecta. For properties of symplecta, we refer to Timmesfeld [5, III Section 7], Van Maldeghem [9, p. 80], and Cooperstein [14, p. 333].

A point is a long root subgroup, the standard point being $X_{e_1+e_2}$. Two long root subgroups A, C define a line, a so-called F_4-line, precisely when any element in AC is a long root element. The standard line is $X_{e_1+e_2} X_{e_1+e_3}$. Three long root subgroups (not on a line) define a plane, when any two define a line.

The action of the subsystem subgroup $M(C_3) := M(\alpha_2, \alpha_3, \alpha_4) \simeq \mathrm{Sp}_6(K)$ on the lines of the F_4-geometry passing through $X_{e_1+e_2}$ is equivalent to the action of $\mathrm{Sp}_6(K)$ on the isotropic planes of the underlying symplectic space. Let $E = X_{e_1+e_2}$, $F = X_{-e_1-e_2}$. Any F_4-line on $X_{e_1+e_2}$ has a unique point A_i in $\Lambda_E \cap \Psi_F$. We let A_i correspond to the isotropic plane E_i in the symplectic space underlying $M(C_3)$. When $A_2 \in \Lambda_{A_1}$, then $E_1 \cap E_2$ is a line. When $[A_1, A_2] = 1$, but $A_2 \notin \Lambda_{A_1}$, then $E_1 \cap E_2$ is a point. When $A_2 \in \Psi_{A_1}$, then $E_1 \cap E_2$ is empty.

As follows from the Dynkin diagram of type F_4, all points, lines, and planes of the F_4-geometry contained in a symplecton (seen as point-line geometry) yield a polar space of type B_3. Whenever A, B are commuting long root subgroups of $F_4(K)$ which do not define an F_4-line, then A and B define a symplecton of the F_4-geometry. The standard symplecton on $X_{e_1+e_2}$ and $X_{e_1-e_2}$ is

$$S := S(X_{e_1+e_2}, X_{e_1-e_2}) = \langle X_{e_1\pm e_2}, X_{e_1\pm e_3}, X_{e_1\pm e_4}, X_{e_1}\rangle.$$

Note that $S \leqslant M(-\alpha_*, \alpha_1, \alpha_2, \alpha_3) = B_4(K)$ and that $S = \mathrm{Z}(U_J)$ in the parabolic subgroup $P_J = U_J L_J$ with Levi complement associated to the diagram $(\alpha_1, \alpha_2, \alpha_3)$ of type B_3. We may consider S as a 7-dimensional natural module for $B_3(K)$.

Let S be the symplecton on $X_{e_1+e_2}$ and $X_{e_1-e_2}$ as above. When A, B are noncollinear points in S (i.e., A, B are not on an F_4-line), then $S = S(A, B)$ and S is spanned by A, B and all T which are collinear with both A and B. For a long root subgroup E generating $\mathrm{SL}_2(K)$ with $X_{e_1+e_2}$, there is a unique long root subgroup T contained in S which commutes with E. Any point in S, which is not on an F_4-line with T, generates $\mathrm{SL}_2(k)$ with E.

1.4. Properties of $F_4(K)$. The permutation rank of $F_4(K)$ on the class of long root subgroups is five. The class of long root subgroups is a class of abstract root subgroups of $F_4(K)$ in the sense of Timmesfeld [4,5].

The center of $F_4(K)$ is trivial. Any diagonal automorphism of $F_4(K)$ is an inner automorphism. For any long root subgroup T in $Y = F_4(K)$ and $1 \neq t \in T$, we have $C_Y(t) = C_Y(T)$. Let A_i, B_i $(i = 1, \dots, 4)$ be long root subgroups of $F_4(K)$ such that $X_i := \langle A_i, B_i\rangle \simeq \mathrm{SL}_2(K)$ and $[X_i, X_j] = 1$ for $i, j = 1, \dots, 4$, $i \neq j$. Passing to a conjugate in $F_4(K)$, we may assume that $A_1, B_1, \dots, A_4, B_4$ are $X_{e_1+e_2}, X_{-e_1-e_2}, X_{e_1-e_2}, X_{-e_1+e_2}, X_{e_3-e_4}, X_{e_3+e_4}, X_{e_3+e_4}, X_{-e_3-e_4}$.

For any long root subgroup E in $F_4(K)$, we denote by M_E the unipotent radical in the parabolic subgroup $N(E)$ (see Carter [1, 8.5]). For $E = X_{e_1+e_2}$, we have $M_E = \langle X_r \mid r \in \Psi\rangle$, where $\Psi := \{e_1 + e_2, e_1, e_2, \frac{1}{2}(e_1 + e_2 \pm e_3 \pm e_4), e_1 \pm e_3, e_1 \pm e_4, e_2 \pm e_3, e_2 \pm e_4\}$. Furthermore, A^m is contained in $S(E, A)$ for $m \in M_E$ whenever E and A define a symplecton. Finally, M_E/E is a 14-dimensional symplectic space over K.

1.5. Weak embeddings of polar spaces. For polar spaces, we refer to Tits [3] and Cohen [18]. Let V be a vector space over some skew field K. We say that a polar space Γ is weakly embedded in the projective space $P(V)$, if there exists an injective map π from the set of points of Γ to the set of points of $P(V)$ such that

(a) the set $\{\pi(x) \mid x \text{ point of } \Gamma\}$ generates $P(V)$;

(b) for each line l of Γ, the subspace of $P(V)$ spanned by $\{\pi(x) \mid x \in l\}$ is a line;
(c) if x, y are points of Γ such that $\pi(y)$ is contained in the subspace of $P(V)$ generated by the set $\{\pi(z) \mid z \text{ collinear with } x\}$, then y is collinear with x.

The map π is called the weak embedding and (c) is the weak embedding axiom. We say that Γ is weakly embedded of degree > 2 in $P(V)$, if each line of $P(V)$ which is spanned by the images of two non-collinear points of Γ contains the image of a third point of Γ. Similarly, we define when the weak embedding has degree 2.

Let W be a vector space endowed with a pseudo-quadratic form or a (σ, ϵ)-hermitian form in the sense of Tits [3, Section 8]. The geometry of 1- and 2-dimensional subspaces of W where the from vanishes, yields a so-called classical polar space. Weak embeddings of classical polar spaces and of generalized quadrangles have been classified by Steinbach and Van Maldeghem [10,11]. The main result is that with known exceptions they are induced by semilinear mappings.

We close the section with a statement of the classification of groups generated by abstract root subgroups, due to Timmesfeld [4, Theorem 5] (see also [5, III Section 9]).

1.6. Timmesfeld's classification of groups generated by abstract root subgroups. Let G be a quasi-simple group generated by the class Σ of abstract root subgroups such that there are $A, B, C \in \Sigma$ with $\langle A, B\rangle$ a rank 1 group and $[A, C] \in \Sigma$. Then G arises from one of the following geometries Γ:

(A) a projective space,
(D) an orthogonal polar space,
(E) a building of type E_6, E_7 or E_8,
(F) a building of type F_4,
(G) a Moufang hexagon.

When Γ has finite rank (an assumption only in Cases (A), (D)), then $\overline{G} := G/\mathrm{Z}(G)$ is isomorphic to the normal subgroup of $\mathrm{Aut}(\Gamma)$ generated by the class of long root subgroups with $\overline{\Sigma}$ the class of long root subgroups.

Here long root subgroups are (projective) linear transvection subgroups (corresponding to incident point–hyperplane pairs) in Case (A) and (projective) Siegel transvection groups in Case (D). The latter correspond to singular lines; we refer to Timmesfeld [5, II (1.5)].

2. Subgroups of $F_4(K)$ arising from an orthogonal or a projective space

Let G be a subgroup of $F_4(K)$ as in Problem (P) of the introduction. Considering the centralizer of an SL_2, we eliminate the case that $G/\mathrm{Z}(G) \simeq E_n(L)$, $n = 6, 7, 8$. For G arising from an orthogonal space, we prove Theorem 2.

When G arises from a projective space, we show that a conjugate of G in $F_4(K)$ is contained in $M(-\alpha_*, \alpha_1, \alpha_2, \alpha_3) = B_4(K)$.

We use the notation of Section 1.1. First, we deduce the following from (P), which will be used without reference.

2.1. Let $A, B \in \Sigma$. Then $\widehat{A}$ is the unique long root subgroup of $F_4(K)$ which contains A. If $[A, B] = 1$, then $[\widehat{A}, \widehat{B}] = 1$ and $\widehat{A}$, $\widehat{B}$ define a line in the F_4-geometry, when $B \in \Lambda_A$, and they define a symplecton, when $B \notin \Lambda_A$; we refer to Section 1.3.

If $[A, B] = C \in \Sigma$, then $[\widehat{A}, \widehat{B}] = \widehat{C}$. If $\langle A, B\rangle$ is a rank 1 group, then $\langle \widehat{A}, \widehat{B}\rangle \simeq \mathrm{SL}_2(K)$. Furthermore, A is the unique element in Σ contained in $\widehat{A}$ (using the assumption in (P) that $C_\Sigma(A) = C_\Sigma(B)$ implies $A = B$).

2.2. *Let $E, F \in \Sigma$ such that $\langle E, F\rangle$ is a rank 1 group. Then there exist no $A, B \in C_\Sigma(E) \cap C_\Sigma(F)$ with $[A, B] \in \Sigma$. In particular, $G/\mathrm{Z}(G) \not\simeq E_n(L)$, $n = 6, 7, 8$.*

Proof. We may assume that $\widehat{E} = X_{e_1+e_2}$, $\widehat{F} = X_{e_1-e_2}$. Then for $A, B \in C_\Sigma(E) \cap C_\Sigma(F)$, $\widehat{A}$, $\widehat{B}$ are symplectic transvection subgroups in $M(\alpha_2, \alpha_3, \alpha_4) = C_3(K) \simeq \mathrm{Sp}_6(K)$. Hence $[\widehat{A}, \widehat{B}]$ is never a long root subgroup in $F_4(K)$. With Section 2.1 the first claim follows. In $E_n(L)$, $n = 6, 7, 8$, the centralizer of a long root SL_2 is of type A_5, D_6, E_7, respectively, and hence contains long root subgroups A, B with $[A, B]$ again a long root subgroup, a contradiction. □

Proof of Theorem 2. For Siegel transvection subgroups, we refer to Section 1.6. By Section 2.2 the Witt index of q is 3 or 4. Any element $A \in \Sigma$ may be identified with the associated Siegel transvection group T_ℓ in $\Omega(W, q)$. We fix $E, F, B, D \in \Sigma$ such that $\langle E, F\rangle$ and $\langle B, D\rangle$ are commuting rank 1 groups which have the same commutator space in W. By Section 1.4 we may pass to a conjugate a G with $\widehat{E} = X_{e_1+e_2}$, $\widehat{F} = X_{-e_1-e_2}$, $\widehat{B} = X_{e_1-e_2}$, $\widehat{D} = X_{-e_1+e_2}$. In the orthogonal group $\Omega(W, q)$ we verify $G = \langle \Lambda_E \cap \Psi_F, F\rangle$.

Let $A \in \Lambda_E \cap \Psi_F$. Considering the associated singular lines, we see that for T one of B or D, we have $C := [A, T] \in \Lambda_E$. This yields $\widehat{C} \in \Lambda_{\widehat{E}} \cap \Lambda_{\widehat{T}}$. Whence $\widehat{C}$ is contained in the symplecton $S(\widehat{E}, \widehat{T})$ by Section 1.3. We obtain $C \leqslant M(B_4)$. But A and C are conjugate in $\langle B, D\rangle$, thus also $A \leqslant M(B_4)$. This proves $G \leqslant M(B_4)$ and Theorem 2 follows with Section 2.3 below. □

2.3. *Let K be a field and V a vector space over K endowed with a non-degenerate quadratic form Q of Witt index $\geqslant 2$. Let G be a quasi-simple subgroup of $\Omega(V, Q)$ generated by the class Σ of abstract root subgroups, such that any $A \in \Sigma$ is contained in some Siegel transvection group $\widehat{A}$ of $\Omega(V, Q)$. We assume $\overline{G} := G/\mathrm{Z}(G) \simeq \mathrm{P}\Omega(W, q)$ with $\overline{\Sigma}$ the class of (projective) Siegel transvection subgroups on W.*

Let $E, F, B, D \in \Sigma$ such that $\langle E, F\rangle$ and $\langle B, D\rangle$ are commuting rank 1 groups which have the same commutator space in W. If also $[V, E]+[V, F]=[V, B]+[V, D]$, then the orthogonal space (W, q) is weakly embedded of degree 2 in $P([V, G])$.

Proof. By Γ we denote the orthogonal polar space associated to (W, q). We consider the map $\pi : \Gamma \to P(V)$ which maps the line ℓ of Γ to the singular line $[V, A]$, provided that $A \in \Sigma$ corresponds to the Siegel transvection group T_ℓ, and each singular point p of Γ is mapped to the intersection of all $[V, T]$, where $T \in \Sigma$ corresponds to a Siegel transvection group T_ℓ with $p \subseteq \ell$.

We prove that π is a weak embedding. Clearly, π maps lines to lines and is injective on lines. We fix a point p of Γ. Let A and C be elements of Σ corresponding to Siegel transvection groups T_ℓ, T_s with $p = \ell \cap s$. First, we show that $[V, A] \cap [V, C]$ is a point of V. Indeed, when $\ell + s$ is 3-dimensional singular, then also $[V, A] + [V, C]$ is 3-dimensional singular. Next assume that $\ell + s$ is 3-dimensional and non-singular. We may assume that $A = E$ and $C = B$. By assumption, $[V, A] + [V, C]$ is contained in the orthogonal sum of two hyperbolic lines in V and hence is 3-dimensional non-singular.

We deduce that π maps points to points. For non-collinear points x, y of Γ and different points z, t of Γ collinear with both x and y, the same relations hold for the images of these four points under π in the polar space associated to (V, Q). In particular, π is injective on points. Thus the weak embedding axiom holds and π is a weak embedding (of degree 2). $\square$

By Steinbach and Van Maldeghem [11], the weak embedding in Section 2.3 is induced by a semilinear mapping $\varphi : W \to V$ (with respect to an embedding $\alpha : L \to K$). Furthermore, φ commutes with the action of G.

2.4. *When G arises from a projective space, then necessarily $\overline{G} := G/\mathrm{Z}(G) \simeq \mathrm{PSL}_3(L)$, $\mathrm{PSL}_4(L)$, L a commutative field. Furthermore, a conjugate of G in $F_4(K)$ is contained in $M(-\alpha_*, \alpha_1, \alpha_2, \alpha_3) = B_4(K)$.*

Proof. For $E, F \in \Sigma$ generating a rank 1 group, we have $\langle E, F\rangle \simeq \mathrm{SL}_2(L)$, L a skew field, or G arises from the non-Desarguesian Moufang plane and $\langle E, F\rangle \simeq \mathrm{SL}(L)$, L a Cayley division algebra.

But $\langle E, F\rangle \leqslant \langle \widehat{E}, \widehat{F}\rangle \simeq \mathrm{SL}_2(K)$ with K a commutative field. We may construct an embedding $\alpha : L$ (or $\mathcal{C}$) $\to K$ as in Timmesfeld [19, (6.2)]. Whence L is necessarily a commutative field. With Section 2.2 either $\overline{G} \simeq \mathrm{PSL}_4(L) \simeq P\Omega_6^+(L)$ (and we may apply Theorem 2) or $\overline{G} \simeq \mathrm{PSL}_3(L)$. In the latter case, a conjugate of G is contained in $M(-\alpha_*, \alpha_1) = A_2(K) \simeq \mathrm{SL}_3(K)$, as $G = \langle E, F, A_1, A_2\rangle$ for $E, F \in \Sigma$ generating a rank 1 group and A_1, A_2 distinct in $\Lambda_E \cap \Psi_F$. $\square$

3. Subgroups of $F_4(K)$ arising from a building of type F_4

Let G be a subgroup of $F_4(K)$ as in (P). In this section we assume that G arises from a building of type F_4, see Section 1.6.

Using the classification of buildings of type F_4 due to Tits [3], we prove that G is isomorphic to $F_4(L)$, L a field in characteristic $\neq 2$, or to a group $F_4(L, F)$ of mixed type F_4 in characteristic 2 (as defined in Section 3.1 below). In both cases we prove that the embedding of G in $F_4(K)$ is by restriction of scalars, which proves Theorem 3.

3.1. Groups $F_4(L, F)$ of mixed type F_4. These groups were defined by Tits [3, (10.3.2)]. Let F be a field of characteristic 2 and L a subfield of F such that $F^2 \subseteq L \subseteq F$. The associated group $F_4(L, F)$ of mixed type F_4 is

$$F_4(L, F) := \big\langle x_r(t), x_s(f) \bigm| r \text{ long},\ t \in L,\ s \text{ short},\ f \in F \big\rangle \leqslant F_4(F).$$

The center of $F_4(L, F)$ is contained in the center of $F_4(F)$, whence trivial.

We have $F_4(F) = \langle B_4(F), C_3(F)\rangle$ with $B_4(F)$, $C_3(F)$ the standard subsystem subgroups of type B_4 and C_3. Similarly, $F_4(L, F) = \langle B_4(L, F), C_3(L, F)\rangle$, where the latter two groups of mixed type are classical groups, we refer to Tits [3, (10.3.2)].

3.2. Proposition. *Let G as in* (P) *arise from a building of type F_4. For $E, F \in \Sigma$ generating a rank* 1 *group, $C_\Sigma(E) \cap C_\Sigma(F)$ is the point set of a polar space of rank* 3 *which is weakly embedded in the* 6*-dimensional symplectic space underlying $C_{F_4(K)}(\langle E, F\rangle)$. Moreover, G is isomorphic to $F_4(L)$, L a field in characteristic $\neq 2$, or to a group $F_4(L, F)$ of mixed type F_4 in characteristic* 2*, with $\overline{\Sigma}$ the class of long root subgroups.*

Proof. We use the classification of buildings of type F_4 due to Tits [3, Section 10]. Let $\Delta := C_\Sigma(E) \cap C_\Sigma(F)$, where $E, F \in \Sigma$ generate a rank 1 group. Since G arises from a building of type F_4, Δ is the point set of a (thick) polar space of rank 3, we refer to Timmesfeld [5, III Section 7]. The points A, C in Δ are collinear precisely when $[A, C] = 1$. We denote the underlying symplectic space of $C_{F_4(K)}(\langle E, F\rangle)$ by V. Then $[V, A]$ is a point in V for $A \in \Delta$. For $A, B \in \Delta$, we have $[A, B] = 1$ if and only if $[V, A] \subseteq C_V(B)$. Thus $\langle [V, T] \mid T \text{ on } \ell\rangle$ is a singular line in V, for any line ℓ of Δ. (Indeed, let $\mathcal{F} := \{A_1, B_1, A_2, B_2, A_3, B_3\} \subseteq \Delta$ such that $\langle A_i, B_i\rangle$ is a rank 1 group $(i = 1, 2, 3)$ but all other pairs in $\mathcal{F}$ commute. Then $V = \bigoplus\{[V, T] \mid T \in \mathcal{F}\}$, $C_V(A_i) = \bigoplus\{[V, T] \mid T \in \mathcal{F}\backslash\{B_i\}\}$ and similarly for B_i $(i = 1, 2, 3)$. For any C on the line on A_1 and A_2, we obtain $[V, C] \subseteq \bigcap\{C_V(T) \mid T \in \mathcal{F}\backslash\{B_1, B_2\}\} = [V, A_1] + [V, A_2]$, as desired.)

This yields that Δ is weakly embedded in $P(V)$. The planes of Δ are Desarguesian (we refer to Cuypers and Steinbach [7, (3.6)]). Thus by the classification of polar spaces due to Tits [3, (8.22)], Δ arises from a vector space W over F endowed with a form. By Steinbach and Van Maldeghem [10, (5.1.1)], the weak embedding is induced by a semilinear mapping $\varphi : W \to V$ with respect to an embedding $\alpha : F \to K$. We obtain that F is commutative.

By the classification of buildings of type $F_4(K)$ in Tits [3, Section 10], either Proposition 3.2 holds or W is a 6-dimensional unitary space, endowed with a $(\sigma, -1)$-hermitian form, $\sigma \neq \mathrm{id}$. But the latter is not possible, because of the weak embedding in a symplectic space. (Indeed, let (x_1, y_1) and (x_2, y_2) be orthogonal hyperbolic pairs in W. For $0 \neq c \in F$ and $a := x_2 + cy_2$, the vector $p := x_1 - cy_1 + a$ is isotropic and $u := x_2 + c^\sigma y_1$, $v := y_2 - y_1$ and y_1 are isotropic in $p^\perp$. In the symplectic space, $a\varphi$ and $p\varphi$ are perpendicular, since $a\varphi$ is isotropic. Because of $a = u + cv + (c - c^\sigma)^\alpha y_1$, also $(c - c^\sigma(y_1\varphi)$ and $p\varphi$ are perpendicular. Whence $c^\sigma = c$ and $\sigma = \mathrm{id}$, a contradiction.) □

3.3. Theorem. *We assume that* $\overline{G} := G/\mathrm{Z}(G) \simeq S =: F_4(L, F)$ *with* $\overline{\Sigma}$ *the class of long root subgroups in* S. (*Here we define* $F_4(L, F) := F_4(F) = F_4(L)$ *in characteristic* $\neq 2$.) *Then there exists an embedding* $\alpha : F \to K$ *such that a conjugate of* G *in* $F_4(K)$ *is* $F_4(L^\alpha, F^\alpha)$.

Proof. (1) We say that $A \in \Sigma$ corresponds to the long root subgroup A_1 of $F_4(L, F)$ if the image of A in S is A_1. Passing to a conjugate of G in $Y := F_4(K)$, we achieve (see Section 1.4):

(∗) If $r \in \{\pm(e_1 + e_2), \pm(e_1 - e_2), \pm(e_3 + e_4), \pm(e_3 - e_4)\}$ and $T \in \Sigma$ corresponds to X_r (in S), then $\widehat{T} = X_r$ (in Y).

By assumption there is a central extension $\rho : G \to S$, mapping abstract root elements to long root elements. By M_1 and M_2 we denote the subgroup of G generated by all elements in Σ which correspond to a long root subgroup in the classical subgroups $B_4(L, F)$ and $C_3(L, F)$ of S, respectively. Then $G = \langle M_1, M_2\rangle$.

For any embedding $\alpha : F \to K$, K a field, let $\epsilon_\alpha : F_4(F) \to F_4(K)$ be the injective homomorphism with $x_r(t) \mapsto x_r(t^\alpha)$, for $r \in \Phi$, $t \in F$.

(2) Next, we prove that passing to a conjugate of G, we achieve that $m = m\rho\epsilon_\alpha$ for $m \in M_2$ with an embedding $\alpha : F \to K$.

By (∗), $M_2 \leqslant M(\alpha_2, \alpha_3, \alpha_4) = C_3(K)$ with underlying 6-dimensional symplectic space V. The vector space W underlying M_2 is 6-dimensional symplectic over F in characteristic $\neq 2$, and is an orthogonal space $L^{\frac{1}{2}} \oplus F^6$ over F (of dimension $6 + \dim_{F^2} L$) in characteristic 2. By Proposition 3.2, the associated polar space Δ is weakly embedded in $P(V)$. By Steinbach and Van Maldeghem [10, (5.1.1)], the weak embedding is induced by a semilinear mapping $\varphi : W \to V$ with respect to an embedding $\alpha : F \to K$. The action of M_2 commutes with φ; i.e.,

$(w(\varphi)m = (w(m\rho))\varphi$, for $w \in W$, $m \in M_2$ (we refer to Cuypers and Steinbach [7, (8.3)]).

We write the elements of $C_3(L, F)$ as 6×6 matrices on the symplectic space $W/W^\perp$ with respect to the standard hyperbolic basis $\mathcal{E}$. Similarly, we proceed for $C_3(K)$, see Carter [1, p. 186]. By J we denote the fundamental matrix of the symplectic form with respect to $\mathcal{E}$. For $m \in M_2$, the matrices of m and $m\rho$ with respect to $\mathcal{E}$ are related via $M_{\mathcal{E}}^{\mathcal{E}}(m) = D^{-1} \cdot M_{\mathcal{E}}^{\mathcal{E}}(m\rho)^\alpha \cdot D$, where D is the diagonal matrix of the base change from $\mathcal{E}\varphi$ to $\mathcal{E}$ in V. Furthermore, DJD^{T} is a scalar multiple of J, since $DJD^{\mathrm{T}} \cdot J^{-1}$ commutes with $C_3(L^\alpha, F^\alpha)$. Therefore conjugation by D is an automorphism $h(\chi)$ of $C_3(K)$, where $\chi : \{\alpha_2, \alpha_3, \alpha_4\} \to K^*$ is a character of $C_3(K)$.

We extend χ to a character χ of $F_4(K)$ by $\alpha_1 \mapsto 1$. Then $h := h(\chi)$ is an (inner) diagonal automorphism of $F_4(K)$. Passing to the conjugate $G^{h^{-1}}$ of G, we achieve that $m = m\rho\epsilon_\alpha$, for $m \in M_2$. This proves (2).

(3) Next, we prove that there exist an embedding $\beta : F \to K$ and a character $\chi : \{-a_*, \alpha_1, \alpha_2, \alpha_3\} \to K^*$ of $B_4(K)$ such that $m = m\rho\epsilon_\beta h(\chi)$ for $m \in M_1$.

By $(*)$ and Theorem 2, $M_1 \leqslant M(-\alpha_*, \alpha_1, \alpha_2, \alpha_3) = B_4(K)$ with underlying 9-dimensional orthogonal space V.

First, we assume that $\mathrm{char}(K) = 2$. The orthogonal space (W, q) associated to M_1 is $W = F \perp L^8$ over L, where L^8 is an orthogonal sum of hyperbolic lines and $q(f) = f^2 \in L$ for $f \in F$. By Section 2.3, (W, q) is weakly embedded of degree 2 in $P(V)$. By Steinbach and Van Maldeghem [11], the weak embedding is induced by an injective semilinear mapping $\varphi : W \to V$ with respect to an embedding $\beta : L \to K$. As the action of M_1 commutes with φ, we have $(w\varphi)m = (w(m\rho))\varphi$, for $w \in W$, $m \in M_1$. Furthermore, $W^\perp\varphi \subseteq V^\perp$. We write $W^\perp = Fx_0$. For $f \in F$, there exists $b_f \in K$ with $(fx_0)\varphi = b_f(x_0\varphi)$. We consider the singular vector $w := fx_0 + f^2x + y$, where (x, y) is a hyperbolic pair in W. Since $w\varphi$ is singular in V, we obtain $(f^2)^\beta = b_f^2$. Thus by $f^\beta := b_f$, we extend β to an embedding $\beta : F \to K$.

We write the elements of $B_4(L, F)$ and $B_4(K)$ as 9×9 matrices over F and K, respectively, with respect to the standard basis as in Carter [1, p. 186]. Similarly as in (2) for $M_2 \leqslant C_3(K)$, we deduce that (3) holds. For $\mathrm{char}(K) \neq 2$, we have $B_4(K)/\langle -1 \rangle = \Omega_9(K)$ and the argument is similar.

(4) Next, we compare the results obtained so far for the elements $x_{\alpha_3}(t)$, $t \in F$, contained in $C_3(L, F) \cap B_4(L, F)$. Let $m \in M_1 \cap M_2$ with $m\rho = x_{\alpha_3}(t)$. We have shown that $x_{\alpha_3}(t^\alpha) = m = x_{\alpha_3}(\chi(\alpha_3)t^\beta)$. For $t = 1$, we obtain $\chi(\alpha_3) = 1$. Hence $\alpha = \beta$. Similarly, we deduce $\chi(\alpha_2) = 1$. Next we consider $x_{-(e_1 - e_2)}(1)$, which is also in $C_3(L, F) \cap B_4(L, F)$. Since $-(e_1 - e_2) = -\alpha_* + 2\alpha_1 + 2\alpha_2 + 2\alpha_3$, necessarily $\chi(\alpha_*) = \chi(\alpha_1)^2$. We define the character $\chi_0 : \Pi(F_4) \to K^*$ by $\chi_0(\alpha_1) = \chi(\alpha_1)$, $\chi_0(\alpha_i) = 1$ $(i = 2, 3, 4)$. Then $h := h(\chi_0)$ is an (inner) diagonal automorphism of $F_4(K)$ which induces $h(\chi)$ when restricted to $B_4(K)$ and

which centralizes $C_3(K)$. Passing to the conjugate $G^{h^{-1}}$ of G, we achieve that $m = m\rho\epsilon_\alpha$, $m \in \langle M_1, M_2\rangle = G$. This proves Theorem 3.3. $\square$

Now Theorem 3 follows from Proposition 3.2 and Theorem 3.3. $\square$

4. ${}^3D_4(\widehat{L})$, ${}^6D_4(\overline{L})$ embed in $F_4(L)$

Let $\widehat{L} : L$ be a separable cubic field extension with Galois closure $\overline{L}$. First, we describe the twisted groups ${}^3D_4(\widehat{L})$ and ${}^6D_4(\overline{L})$ as fixed point groups in $D_4(\overline{L})$. Then we prove that both types of groups are subgroups of $F_4(L)$ (note L, not $\overline{L}$) such that long root subgroups are long root subgroups.

4.1. Separable cubic extensions. Let $\widehat{L} : L$ be a separable cubic field extension. For $\theta \in \widehat{L}$, $\theta \notin L$, fixed, let f be the minimal polynomial of θ over L. We denote by $\overline{L}$ the splitting field of f, the so-called Galois closure. If $\widehat{L} : L$ is a Galois extension (e.g., for $\mathrm{GF}(q^3) : \mathrm{GF}(q)$), then $\overline{L} = \widehat{L}$ and $\mathrm{Aut}(\overline{L} : L) = \langle\sigma\rangle \simeq \mathbb{Z}_3$. Otherwise (e.g., for $\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}$), $\overline{L}$ is a quadratic Galois extension of $\widehat{L}$ and $\mathrm{Aut}(\overline{L} : L) = \langle\sigma, \tau\rangle \simeq \Sigma_3$, where σ is of order 3, τ is of order 2, $\sigma\tau = \tau\sigma^2$, $\mathrm{Fix}(\tau) = \widehat{L}$.

4.2. The standard embedding of $G_2(L)$, ${}^3D_4(\widehat{L})$, ${}^6D_4(\overline{L})$. For the definition of twisted groups, see Tits [20], Carter [1], Steinberg [17].

Let L, $\widehat{L}$, $\overline{L}$ be as in Section 4.1. Let σ, τ be permutations of order 3 and 2, respectively, of the root system $\Phi(D_4)$ of type D_4, which arise from symmetries of the Dynkin diagram. We choose notation of the fundamental roots δ_i such that $\sigma = (\delta_1, \delta_2, \delta_3)$ with fixed point δ_0 and $\tau = (\delta_2, \delta_3)$ with fixed points δ_0, δ_1.

The universal Chevalley group $D_4(\overline{L})$ has automorphisms η_σ, η_τ with $\eta_\sigma : x_r(t) \mapsto x_{r\sigma}(t^\sigma)$, $\eta_\tau : x_r(t) \mapsto x_{r\tau}(t^\tau)$, where $t \in \overline{L}$, $r \in \Phi(D_4)$ (compare Section 4.4 below). We use η_τ only when $\widehat{L} : L$ is not Galois.

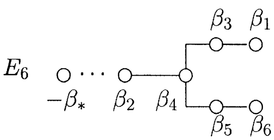
Any orbit of $\langle\sigma, \tau\rangle$ on $\Phi(D_4)$ is of the form $\{r\}$ with $r\sigma = r = r\tau$ or $\{r, r\sigma, r\sigma^2\}$ with $r\sigma \neq r = r\tau$. By definition ${}^3D_4(\widehat{L}) \leqslant \mathrm{Fix}(\eta_\sigma)$ and ${}^6D_4(\overline{L}) \leqslant \mathrm{Fix}(\langle\eta_\sigma, \eta_\tau\rangle)$ are generated by all 'long root elements' $x_r(u)$, where $r\sigma = r$ $(= r\tau)$, $u \in L$, and all 'short root elements' $x_s(t)x_{s\sigma}(t^\sigma)x_{s\sigma^2}(t^{\sigma^2})$, where $s\sigma \neq s = s\tau$, $t \in \widehat{L}$. (The superscripts 3 and 6 are the degree of the Galois closure of the extension $\widehat{L} : L$.)

Note that $G_2(L) := \langle x_r(u), x_s(t)x_{s\sigma}(t^\sigma)x_{s\sigma^2}(t^{\sigma^2}) \mid r\sigma = r,\ s\sigma \neq s = s\tau,\ u, t \in L\rangle$ is contained in ${}^3D_4(\widehat{L})$, ${}^6D_4(\overline{L})$. We call the above embedding of $G_2(L)$, ${}^3D_4(\widehat{L})$ and ${}^6D_4(\overline{L})$ in $D_4(\overline{L})$ the standard embedding.

The groups ${}^3D_4(\widehat{L})$, ${}^6D_4(\overline{L})$ are quasi-simple, as is $G_2(L)$ for $L \neq \mathrm{GF}(2)$. Let $a(c) := x_{\delta_1}(c)x_{\delta_2}(c^\sigma)x_{\delta_3}(c^{\sigma^2})$, $b(c) := x_{-\delta_1}(c)x_{-\delta_2}(c^\sigma)x_{-\delta_3}(c^{\sigma^2})$ and $n(t) := a(t)b(-t^{-1})a(t)$, where $c, t \in \widehat{L}$, $t \neq 0$. Then $a(c)^{n(t)} = b(-t^{-1}ct^{-1})$.

For the proof of Theorem 5, we construct automorphisms η_σ, η_τ of $F_4(\overline{L})$ such that the restrictions to the subsystem subgroup $D_4(\overline{L})$ are as in Section 4.2. Then we show there is an inner automorphism of $F_4(\overline{L})$ which conjugates η_σ and η_τ to the field automorphisms with respect to σ and τ, respectively; we refer to Stensholt [13] for the finite case. It is convenient to construct η_σ and η_τ in $E_6(\overline{L})$ first.

4.3. Symmetries of the root system of type E_6. We denote by $\Phi := \Phi(E_6)$ the root system of type E_6 with the following extended Dynkin diagram in the notion of Bourbaki [2]:



where

$$\begin{aligned}
\beta_1 &= \tfrac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8), \qquad \beta_2 = e_1 + e_2,\\
\beta_3 &= e_2 - e_1, \qquad \beta_4 = e_3 - e_2, \qquad \beta_5 = e_4 - e_3, \qquad \beta_6 = e_5 - e_4,\\
\beta_* &= \tfrac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 - e_6 - e_7 + e_8)\\
&= \beta_1 + 2\beta_2 + 2\beta_3 + 3\beta_4 + 2\beta_5 + \beta_6.
\end{aligned}$$

By $(\,,)$ we denote the standard scalar product on the underlying Euclidean space $\mathbb{R}^6$. The extended diagram has symmetries $(\beta_3\beta_5\beta_2)(\beta_1\beta_6 - \beta_*)$ with fixed point β_4 and $\tau := (\beta_3\beta_5)(\beta_1\beta_6)$ with fixed points β_4, β_2, $-\beta_*$, which are induced by isometrics σ and τ of $\mathbb{R}^6$ (permuting Φ).

The permutation of Φ of order 2 with $\sum_{i=1}^8 c_i e_i \mapsto \sum_{i=1}^4 c_i e_i - \sum_{i=5}^8 c_i e_i$ is induced by an isometry z of $\mathbb{R}^6$ with $z : \beta_1 \mapsto -(\beta_1 + \beta_2 + 2\beta_3 + 2\beta_4 + \beta_5)$, $\beta_6 \mapsto -(\beta_2 + \beta_3 + 2\beta_4 + 2\beta_5 + \beta_6)$, β_2, β_3, β_4, β_5 fixed. Moreover, z commutes with σ and with τ.

Let $r_1 := e_1 + e_5$, $r_2 := e_2 - e_3$, $r_3 := -\frac{1}{2}(e_1 + e_2 + e_3 - e_4 + e_5 + e_6 + e_7 - e_8)$, $r_4 := e_3 - e_4$. Then z is the diagram symmetry of order 2 with respect to the fundamental system $\{r_1, r_2, r_3, r_4, r_3 z, r_1 z\}$ of Φ (with lowest root $e_3 + e_4$). The vectors $\frac{1}{2}(r + rz)$, $r \in \Phi$, yield a root system of type F_4 with fundamental system consisting of $\delta_1 := e_2 - e_3$, $\delta_2 := e_3 - e_4$, $\delta_3 = -\frac{1}{2}(e_1 + e_2 + e_3 - e_4)$, and $\delta_4 := e_1$ (with lowest root $e_3 + e_4$).

4.4. *Let $\widehat{L} : L$ be a separable cubic field extension with Galois closure $\overline{L}$ as in Section* 4.1 *and σ, τ, z permutations of the root system of type E_6 as in Section* 4.3. *Then there exist signs $N_{r,s}$ involved in the Steinberg relations of the universal Chevalley group $E_6(\overline{L})$, such that the mappings $\eta_\sigma : x_r(t) \mapsto x_{r\sigma}(t^\sigma)$, $\eta_\tau : x_r(t) \mapsto x_{r\tau}(t^\tau)$, where $r \in \Phi$, $t \in \overline{L}$, extend to automorphisms of $E_6(\overline{L})$.* (*We consider η_τ only when $\widehat{L} : L$ is not Galois.*)

Also $\eta_z : x_r(t) \mapsto x_{rz}(t)$*, where* $r \in \Phi$*,* $t \in \overline{L}$*, extends to an automorphism of* $E_6(\overline{L})$ *with* $F_4(\overline{L}) = \langle x_r(t), x_s(t)x_{sz}(t) \mid r, s \in \Phi,\ rz = r,\ sz \neq s,\ t \in \overline{L}\rangle \subseteq \mathrm{Fix}(\eta_z)$*. Furthermore,* η_σ*,* η_τ *restrict to automorphisms of* $F_4(\overline{L})$*.*

Proof. We denote by f the bilinear form on $\mathbb{R}^6$ with fundamental matrix

$$F := \begin{pmatrix} 1 & -1 & -1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 & -2 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

with respect to the basis $\{\beta_4, \beta_2, \beta_3, \beta_5, \beta_1, \beta_6\}$. Then $(x, y) = f(x, y) + f(y, x)$, $f(x\sigma, y\sigma) = f(x, y) = f(x\tau, y\tau)$ for $x, y \in \mathbb{R}^6$. Moreover $f(r, s) \in \mathbb{Z}$ for $r, s \in \Phi$.

As did Springer [21, 10.2], we define the following structure constants $N_{r,s}$ of type E_6; i.e., signs involved in the Chevalley commutator relations: we set $\epsilon(r) := 1$, when $r \in \Phi^+$, and $\epsilon(r) := -1$, when $r \in \Phi^-$. For $r, s \in \Phi$ with $r + s \in \Phi$, we define

$$N_{r,s} := \epsilon(r)\epsilon(s)\epsilon(r+s) \cdot (-1)^{f(r,s)}$$

and we set $N_{r,s} := 0$ for $r, s \in \Phi$ with $r + s \notin \Phi$. We verify that $N_{r\sigma,s\sigma} = N_{r,s}$ and $N_{r\tau,s\tau} = N_{r,s}$, for $r, s \in \Phi$. Indeed, σ, τ respect f, and we only have to take care of the signs $\epsilon(r)$, $\epsilon(s)$. Since τ permutes the positive roots, it preserves these signs. For σ, we consider several cases according to the coefficients of β_6 in r and s. Thus the mappings η_σ, η_τ preserve the Steinberg relations and extend to automorphisms of $E_6(\overline{L})$.

Let Z be the matrix of z with respect to the same basis as for F. Then the only non-zero entries in $ZFZ^{\mathrm{T}} - F$ (modulo 2) are a right lower corner $\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$. Thus for $r = \sum_{i=1}^{6} c_i\beta_i$, $s = \sum_{i=1}^{6} d_i\beta_i \in \Phi$, we have $f(rz, sz) = f(r, s)$ modulo 2, if and only if $c_1d_6 = c_6d_1$. This yields that $N_{rz,sz} = N_{r,s}$ for $r, s \in \Phi$. Whence also η_z extends to an automorphism of $E_6(\overline{L})$. Since z commutes with σ, τ, we obtain $F_4(\overline{L})$ as a fixed point group of η_z which is invariant under η_σ, η_τ. □

4.5. *We consider* $F_4(\overline{L})$ *with automorphisms* η_σ*,* η_τ *as constructed in Section* 4.4*. Then there is an inner automorphism* ω *of* $F_4(\overline{L})$ *such that*

$$\omega\eta_\sigma\omega^{-1} = f_\sigma, \qquad \omega\eta_\tau\omega^{-1} = f_\tau,$$

where $f_\sigma : x_r(t) \mapsto x_r(t^\sigma)$*,* $f_\tau : x_r(t) \mapsto x_r(t^\tau)$ *are field automorphisms.*

Proof. The permutations σ, τ permute the roots of $\Phi(F_4)$. The action of η_σ on $F_4(\overline{L})$ is $x_r(t) \mapsto x_{r\sigma}(t^\sigma)$, for $r \in \Phi(F_4)$, $t \in \overline{L}$, and similarly for η_τ. The field automorphism f_σ of $F_4(\overline{L})$ is defined by $x_r(t) \mapsto x_r(t^\sigma)$, and similarly for f_τ.

We consider the fundamental system $\{\delta_1, \delta_2, \delta_3, \delta_4\}$ of $\Phi(F_4)$ (with highest root δ_*) of Section 4.3. The roots $-\delta_*$, δ_1 are fixed by σ and τ. Furthermore $(\delta_2, -e_1 - e_2, e_1 - e_2)$ by σ and $(\delta_2, e_1 - e_2)$ by τ.

For δ_3, δ_4, the fundamental roots of a root system of type A_2, we have $(\delta_3, \delta_4, -\delta_3 - \delta_4)$ by σ and $(\delta_3, -\delta_4)$ by τ. In the notation of Bourbaki [2] we write $\delta_3 = f_1 - f_2$, $\delta_4 = f_2 - f_3$. The above translates in $(f_1 - f_2, f_2 - f_3, f_3 - f_1)$ by σ (this means $\sigma = (123)$) and $(f_1 - f_2, f_3 - f_2)$ by τ (this means $\tau = (13)$); also $(f_1 - f_3)\tau = f_3 - f_1$. This leads to the following permutation matrices P_σ and P_τ associated to σ and τ, satisfying $M^\sigma P_\sigma = M$ and $M^\tau P_\tau = M$:

$$P_\sigma := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad P_\tau = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad M = \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix},$$

where $b \in \widehat{L}$, $b \notin L$, and $c := b^\sigma$, $a := c^\sigma$.

We write elements of $M(\delta_2, \delta_3, \delta_4) \simeq \mathrm{Sp}_6(\overline{L})$ as 6×6 matrices as in Carter [1, p. 186] (with $\begin{pmatrix} & I \\ -I & \end{pmatrix}$ as fundamental matrix of the underlying symplectic form). In the following we often consider $\eta := \eta_\sigma$ and η_τ simultaneously. We write then $\bar{t} = t^\sigma, t^\tau$, for $t \in \overline{L}$, $\bar{r} = r\sigma, r\tau$, for $r \in \Phi(F_4)$ and we omit the indices in η_σ, P_σ, f_σ. The action of η on $M(\delta_2, \delta_3, \delta_4) \simeq \mathrm{Sp}_6(\overline{L})$ is given by

$$X \mapsto \begin{pmatrix} P^{-1} & \\ & P^{\mathrm{T}} \end{pmatrix} \overline{X} \begin{pmatrix} P & \\ & P^{-\mathrm{T}} \end{pmatrix}, \quad \text{for } X \in \mathrm{Sp}_6\big(\overline{L}\big), \tag{$*$}$$

as can be checked on X_{δ_2}, X_{δ_3}, X_{δ_4}, $X_{-(e_1-e_2)}$. We define

$$g := \begin{pmatrix} M & \\ & M^{-\mathrm{T}} \end{pmatrix} \in M(\delta_2, \delta_3, \delta_4) \simeq \mathrm{Sp}_6\big(\overline{L}\big).$$

By ω we denote the inner automorphism $x \mapsto g^{-1}xg$ of $F_4(\overline{L})$. For $r \in \Phi(F_4)$, $t \in \overline{L}$, we have $x_r(t)\omega\eta\omega^{-1} = g(g^{-1}\eta)x_{\bar{r}}(t)(g\eta)g^{-1}$.

By $(*)$ we know

$$g\eta = -\begin{pmatrix} P^{-1} & \\ & P^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} \overline{M} & \\ & \overline{M}^{-\mathrm{T}} \end{pmatrix} \begin{pmatrix} P & \\ & T^{-\mathrm{T}} \end{pmatrix} = \begin{pmatrix} P^{-1} & \\ & P^{\mathrm{T}} \end{pmatrix} g,$$

since $g \in \mathrm{Sp}_6(\overline{L})$ and $\overline{M}P = M$. Thus we obtain $\omega^{-1}\eta\omega = f$ on $M(\delta_2, \delta_3, \delta_4) \simeq \mathrm{Sp}_6(\overline{L})$ with $(*)$. Moreover $X_{-\delta_*}$ commutes with $\mathrm{Sp}_6(\overline{L})$ and $-\delta_*$ is fixed by σ and τ, hence $\omega^{-1}\eta\omega = f$ on $X_{-\delta_*}$. We are left with X_{δ_1}. Since $\det P_\sigma = 1$, $\begin{pmatrix} P_\sigma & \\ & P_\sigma^{-\mathrm{T}} \end{pmatrix}$ is contained in $M(\delta_3, \delta_4) \simeq A_2(\overline{L}) \leqslant C(X_{\delta_1})$. Hence $\omega^{-1}\eta\omega = f_\sigma$ on X_{δ_1} and thus on $F_4(\overline{L})$. Since $\det P_\tau = -1$, conjugation with $\begin{pmatrix} P_\tau^{-1} & \\ & P_\tau^{\mathrm{T}} \end{pmatrix}$ maps $x_{\delta_1}(t^\tau)$ to $x_{\delta_1}(-t^\tau)$. We denote by $h := h(\chi)$ the diagonal automorphism of $F_4(\overline{L})$ with respect to the character $\chi : \delta_1 \mapsto m$, $\delta_2 \mapsto 1$, $\delta_3 \mapsto 1$, $\delta_4 \mapsto 1$, where $0 \neq m := \det M$ with $m^\sigma = m$, $m^\tau = -m$. Then $h\omega\eta\omega^{-1}h^{-1} = f$ (field automorphism) for σ and τ on $F_4(\overline{L})$. Since any diagonal automorphism of $F_4(\overline{L})$ is inner, this proves the claim. $\square$

Proof of Theorem 5. We consider $F_4(\overline{L})$ with automorphisms η_σ, η_τ as constructed in Section 4.4. By Section 4.5 there is an inner automorphism ω of $F_4(\overline{L})$ such that $\eta_\sigma = \omega^{-1} f_\sigma \omega$, $\eta_\tau = \omega^{-1} f_\tau \omega$. We obtain

$$^3D_4\big(\widehat{L}\big) \leqslant \mathrm{Fix}(\eta_\sigma) = \mathrm{Fix}(f_\sigma)\omega = F_4(L)\omega \simeq F_4(L),$$
$$^6D_4\big(\overline{L}\big) \leqslant \mathrm{Fix}\big(\langle \eta_\sigma, \eta_\tau \rangle\big) = \mathrm{Fix}\big(\langle f_\sigma, f_\tau \rangle\big)\omega = F_4(L)\omega \simeq F_4(L).$$

The long root subgroup $\{x_{\delta_*}(t) \mid t \in L\}$ of $^3D_4(\widehat{L})$, $^6D_4(\overline{L})$ is a long root subgroup of $F_4(L)\omega$. $\square$

5. Subgroups of $F_4(K)$ arising from a Moufang hexagon

Let G be a subgroup of $F_4(K)$ as in (P). In this section we assume that G arises from a Moufang hexagon, see Section 1.6.

Using the classification of Moufang hexagons due to Tits and Weiss [8], we prove that G arises from a G_2-, $^3D_4$-, or $^6D_4$-hexagon or from a mixed hexagon in characteristic 3. The further investigation of these cases proves Theorem 4.

5.1. Moufang hexagons. For the definition and properties of the Moufang hexagons, we refer to Tits and Weiss [8] and Van Maldeghem [9, 5.5]. From the classification of Moufang hexagons due to Tits and Weiss [8], we use the following facts:

Any Moufang hexagon, Γ say, belongs to the root system of type G_2. We fix an apartment of Γ together with the twelve associated root groups U_i. There exists a commutative field L such that long root subgroups (with an even index) are isomorphic to $(L, +)$ and short root subgroups are isomorphic to $(J, +)$, where J is one of the following Jordan division algebras over L:

(1) $J = L$; we say Γ is a G_2-hexagon.
(2) $J = \widehat{L}$, a separable cubic field extension of L; we say Γ is a $^3D_4$- or $^6D_4$-hexagon, depending on whether $\widehat{L} : L$ is a Galois extension or not.
(3) $J = F$ is a field extension of L, $\mathrm{char}(L) = 3$, such that $F^3 \subseteq L \subseteq F$. We say that Γ is a mixed hexagon in characteristic 3.
(4) The dimension of J over L is 9 or 27; we say Γ is an exceptional Moufang hexagon.

For the associated groups in Cases (1) and (2), we refer to Section 4.2. In Case (3), the associated group is the group $G_2(L,F)$ of mixed type G_2 in characteristic 3, which was introduced by Tits [3, (10.3.2)]. By definition

$$G_2(L,F) := \big\langle x_r(t), x_s(f) \bigm| r \text{ long},\ t \in L,\ s \text{ short},\ f \in F \big\rangle \leqslant G_2(F).$$

The only non-trivial commutator relations among $U_0, \dots, U_4$ are

$$\big[u_0(t), u_4(u)\big] = u_2(tu), \qquad \big[u_1(a), u_3(b)\big] = u_2\big(T(a,b)\big)$$

for $t, u \in L$, $a, b \in J$, where $T : J \times J \to L$ is a symmetric bilinear form, which satisfies $T(a, a^{\#}) = 3N(a)$ for $a \in J$. Here # is the adjoint map and N the (anisotropic) norm on J.

The class Σ^1 of long root subgroups of Γ is a class of abstract root subgroups of $S := \langle \Sigma^1 \rangle \leqslant \operatorname{Aut}(\Gamma)$, see Timmesfeld [5, III Section 4]. For commuting long root subgroups A^1, B^1, any element in A^1B^1 is a long root element (i.e., $B^1 \in \Lambda_{A^1}$ in the notation of Section 1.1).

5.2. *Let Γ be a Moufang hexagon as in Section* 5.1. *We set $E^1 := U_2$ and $F^1 := U_8$ and $M_{E^1} = \langle \Lambda_{E^1} \rangle$. Then $\widetilde{M_{E^1}} := M_{E^1}/E^1$ is a symplectic space over L, which is non-degenerate provided that Γ is not a mixed hexagon in characteristic* 3.

Proof. We have $M_{E^1} = U_0U_1U_2U_3U_4$, see Timmesfeld [5, III(4.10)(3)]. Thus $W := \widetilde{M_{E^1}} = \widetilde{U_0} \oplus \widetilde{U_1} \oplus \widetilde{U_3} \oplus \widetilde{U_4}$ is a vector space over L, where the scalar multiplication is given by the action of the diagonal subgroup of $\langle U_2, U_8 \rangle$ normalizing U_2 and U_8, see Timmesfeld [5, III(2.25)].

We define a symplectic form $(\,,\,) : W \times W \to L$ by $(\widetilde{m_1}, \widetilde{m_2}) := c \in L$, when $[m_1, m_2] = u_2(c)$, for $m_1, m_2 \in M_{E^1}$. Because of the commutator relation given in Section 5.1 and properties of T, we have $W^{\perp} = 0$; we refer to Tits and Weiss [8]. $\square$

5.3. Proposition. *Let G as in* (P) *arise from the Moufang hexagon Γ. Then Γ is not an exceptional Moufang hexagon.*

Proof. We may assume that Γ is not a mixed hexagon in characteristic 3. There is a central extension $\rho : G \to S$, where S is the subgroup of $\operatorname{Aut}(\Gamma)$ generated by the class Σ^1 of long root subgroups.

Let W be the non-degenerate symplectic space associated to M_{E^1}, see Section 5.2. Recall that $V := M_{\widehat{E}}/\widehat{E}$ is a 14-dimensional symplectic space over K, see Section 1.4. We define $v\varphi := \widetilde{m}$ when $v = \widetilde{m\rho}$ with $m \in M_E$. Since $M_E \cap \mathrm{Z}(G) = 1$, $\varphi : W \to V$ is a semilinear mapping, which satisfies $(v, w) = 0$ if and only if $(v\varphi, w\varphi) = 0$. Hence an orthogonal sum of hyperbolic lines in W gives rise to an orthogonal sum of hyperbolic lines in V. Thus $2(\dim_L J + 1) = \dim_L W \leqslant 14$ and $\dim_L J \leqslant 6$. $\square$

5.4. The standard apartment. Let G be a subgroup of $F_4(K)$ as in (P), arising from the (necessarily classical) Moufang hexagon Γ.

Let $E, A_1, B_1, F, B_2, A_2 \in \Sigma$ correspond to the long root subgroups of an apartment of Γ (in that ordering). Passing to a conjugate of G in $F_4(K)$, we may assume that $\widehat{E} = X_{e_1+e_2}$, $\widehat{F} = X_{-e_1-e_2}$. Note that $[A_1, A_2] = E$. By Section 1.3 we achieve $\widehat{A_1} = X_{e_2-e_3}$, $\widehat{A_2} = X_{e_1+e_3}$, since $\mathrm{Sp}_6(K)$ is transitive on pair of disjoint isotropic planes. Because of $B_1 = [A_1, F]$ and $B_2 = [A_2, F]$, we obtain that $\widehat{B_1} = X_{-e_1-e_3}$, $\widehat{B_2} = X_{-e_2+e_3}$. Whence $\langle E, A_1, B_1, F, B_2, A_2\rangle \leqslant M(-\alpha_*, \alpha_1) = A_2(K) \simeq \mathrm{SL}_3(K)$. We say that $(E, A_1, B_1, F, B_2, A_2)$ is the standard apartment in G.

5.5. Short root subgroups. We consider the standard apartment in G. Let $\rho : G \to \mathrm{Aut}(\Gamma)$ be a homomorphism with kernel $\mathrm{Z}(G)$ which maps Σ to the class of long root subgroups of Γ. For $A \in \Sigma$, $M_A = \langle \Lambda_A\rangle$ intersects $\mathrm{Z}(G)$ trivially. We denote by U_α, $U_{-\alpha}$ the short root subgroups in $\mathrm{Aut}(\Gamma)$ associated to the half apartments (E, A_2, B_2, F) and (E, A_1, B_1, F), respectively. For $u \in U_\alpha$, $v \in U_{-\alpha}$ there exist unique 'short root elements' $a \in C_G(E) \cap C_G(F) \cap M_{A_2}$ and $b \in C_G(E) \cap C_G(F) \cap M_{A_1}$ with $a\rho = u$, $b\rho = v$; we refer to Timmesfeld [5, III (4.9), (4.10)]. This defines short root subgroups A_α and $A_{-\alpha}$ in G.

We may coordinatize A_α, $A_{-\alpha}$ as follows: $A_\alpha = \{a(c) \mid c \in J\}$, $A_{-\alpha} = \{b(c) \mid c \in J\}$ such that $a(c)a(d) = a(c+d)$, $b(c)b(d) = b(c+d)$ and $a(c)^{n(t)} = b(-t^{-1}ct^{-1})$, for $n(t) := a(t)b(-t^{-1})a(t)$ and $c, d, t \in J$, $t \neq 0$. Indeed, this is possible in the groups $G_2(F)$ and ${}^3D_4(\widehat{L})$, ${}^6D_4(\overline{L})$ by Section 4.2.

Moreover, the following relations hold for $c, d, t \in J$, $t \neq 0$: $n(t)^{-1} = n(-t)$, $b(c)^{n(t)} = a(-tct)$, $a(t)^{b(t^1)} = b(-t^{-1})^{a(t)}$ and $a(c)^{h(t)} = a(t^{-1}ct^{-1})$, $b(c)^{h(t)} = b(tct)$, where $h(t) := n(t)n(-1)$.

5.6. *The short root subgroup A_α in G with respect to the standard apartment is contained in the unipotent radical of the stabilizer of an isotropic plane* (*the one corresponding to A_2*) *in $M(\alpha_2, \alpha_3, \alpha_4) = C_3(K) \simeq \mathrm{Sp}_6(K)$. This means that $A_\alpha \leqslant \langle X_r \mid r \in \Psi\rangle$, where $\Psi := \{e_1 - e_2, e_3 + e_4, e_3 - e_4, \frac{1}{2}(e_1 - e_2 + e_3 + e_4), \frac{1}{2}(e_1 - e_2 + e_3 - e_4), e_3\}$. Similarly, $A_{-\alpha} \leqslant \langle X_{-r} \mid r \in \Psi\rangle$.*

Proof. Since A_α centralizes E and F, we have $A_\alpha \leqslant M(\alpha_2, \alpha_3, \alpha_4) = C_3(K) \cong \mathrm{Sp}_6(K)$. Furthermore, A_α stabilizes the isotropic plane corresponding to A_2 (see Section 1.3). Hence A_α is contained in the parabolic subgroup $U_J L_J$ with Levi complement L_J of type A_2 with diagram (α_3, α_4) and unipotent radical $U_J = \langle X_r \mid r \in \Psi\rangle \leqslant M_{\widehat{A_2}}$ (the unipotent radical of the point stabilizer $N(\widehat{A_2})$ in the F_4-geometry). We obtain $A_\alpha \leqslant M_{\widehat{A_2}} \cap U_J L_J = (M_{\widehat{A_2}} \cap L_J)U_J = U_J$. $\square$

5.7. Notation. We write each element of $M(\alpha_2, \alpha_3, \alpha_4) = C_3(K) \simeq \mathrm{Sp}_6(K)$ as a 6×6 matrix as in Carter [1, p. 186]. The first/last three basis vectors span the isotropic plane corresponding to A_2 and A_1, respectively. The fundamental matrix of the underlying symplectic form is $\begin{pmatrix} & I \\ -I & \end{pmatrix}$. By Section 5.6 the short root elements $a(c) \in A_\alpha$ and $b(c) \in A_{-\alpha}$, written as 6×6 matrices, are

$$a(c) = \begin{pmatrix} I & \\ M(c) & I \end{pmatrix}, \qquad b(c) = \begin{pmatrix} I & N(c) \\ & I \end{pmatrix}, \qquad c \in J.$$

Any $M(c)$ is a symmetric matrix, since $a(c)$ respects the symplectic form. The matrix $M(t)$ is invertible for $t \neq 0$, since then $A_1^{a(t)}$ corresponds to an isotropic plane which is disjoint from the planes corresponding to A_1 and A_2.

The element $x_{e_1-e_2}(a)x_{e_3+e_4}(d)x_{e_3-e_4}(f)x_{\frac{1}{2}(e_1-e_2+e_3+e_4)}(b)x_{\frac{1}{2}(e_1-e_2+e_3-e_4)}(c) \times x_{e_3}(e)$ as 6×6 matrix is $\begin{pmatrix} I & \\ M & I \end{pmatrix}$ with

$$M = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} \quad \text{and}$$

$$\begin{pmatrix} R^{-1} & \\ & R^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} I & \\ M & I \end{pmatrix} \begin{pmatrix} R & \\ & R^{-\mathrm{T}} \end{pmatrix} = \begin{pmatrix} I & \\ R^{\mathrm{T}} M R & I \end{pmatrix}$$

for $R \in \mathrm{GL}_3(K)$. The latter describes the action of $M(\alpha_3, \alpha_4) = A_2(K) \simeq \mathrm{SL}_3(K)$ on the unipotent radical of the plane stabilizer.

5.8. *For $c \in J$, let $M(c)$ be the* 3×3 *matrix defined in Section* 5.7. *Suppose that $M(1) = I$. Then the mapping $M : c \mapsto M(c)$ has the following properties*: *for $c, d, t \in J$, $t \neq 0$,*

$$M(c+d) = M(c) + M(d), \qquad M(tct) = M(t)M(c)M(t),$$
$$M\left(t^{-1}\right) = M(t)^{-1}, \quad t \neq 0.$$

Moreover, M is injective and any $M(c)$ is symmetric.

Proof. Let $c, d, t \in J$, $t \neq 0$. We use the relations between the short root elements in A_α, $A_{-\alpha}$ given in Section 5.5. With $a(c+d) = a(c)a(d)$, we see $M(c+d) = M(c) + M(d)$ (and similarly for N). Since any $M(t)$ is invertible, M is injective. Because of $a(t)^{b(t^{-1})} = b(-t^{-1})^{a(t)}$, we have $N(t^{-1}) = M(t)^{-1}$ and $n(1) = \begin{pmatrix} & -I \\ I & \end{pmatrix}$. Since $b(t) = a(-t)^{n(1)}$, we obtain $N(t) = M(t)$ and $M(t^{-1}) = M(t)^{-1}$. We deduce $h(t) := n(t)n(-1) = \begin{pmatrix} M(t^{-1}) & \\ & M(t) \end{pmatrix}$. Now $b(c)^{h(t)} = b(tct)$ implies that $M(tct) = M(t)M(c)M(t)$. □

5.9. A subgroup of type G_2 in G. Let Γ_0 be a G_2-subhexagon of Γ obtained by restricting the short root subgroups from J to L. (The existence of a G_2-subhexagon in any Moufang hexagon is due to Ronan, see Van Maldeghem [9, (5.5.12)].) Let G_0 be the subgroup of G generated by all $T \in \Sigma$ which correspond to a long root subgroup of Γ_0.

5.10. Theorem. *Passing to a conjugate in $F_4(K)$, the subgroup G_0 of type G_2 is contained in the standard subsystem subgroup $M(D) := M(e_1 - e_2, e_2 - e_3, e_3 - e_4, e_3 + e_4) = D_4(K)$ of $F_4(K)$. When K is quadratically closed, there exists an embedding $\alpha : L \to K$ such that a conjugate of G_0 in $F_4(K)$ is $G_2(L^\alpha)$.*

Proof. We pass to a conjugate of G as in Section 5.4. For $a(1) = \left(\begin{smallmatrix} & I \\ M(1) & I \end{smallmatrix}\right)$ as in Section 5.7, there exists a matrix R with $\det R = 1$ such that $R^{\mathrm{T}} M(1) R$ is a diagonal matrix (also in characteristic 2).

We identify R with $\left(\begin{smallmatrix} R & \\ & R \end{smallmatrix}\right) \in M(\alpha_2, \alpha_3, \alpha_4) \simeq \mathrm{Sp}_6(K)$. Then R centralizes $M(-\alpha_*, \alpha_1) \simeq \mathrm{SL}_3(K)$, since $R \in M(\alpha_3, \alpha_4) \simeq \mathrm{SL}_3(K)$. After conjugation with R we may thus assume that $M(1)$ is a diagonal matrix. Hence $a(1)$ is contained in $M(D_4)$. Since $G_0 = \langle F, A_1, A_2, a(1)\rangle$, the first claim follows.

By Steinbach [22], the embedding of G_0 in the 8-dimensional orthogonal group $M(D_4)/\langle -1\rangle$ is induced by a semilinear mapping. Similarly as in Step (3) of the proof of Theorem 3.3, we obtain that there exist an embedding $\alpha : L \to K$ and a diagonal automorphism h of $D_4(K)$ such that $g = g\rho\epsilon_\alpha h$, for $g \in G$. (Here $\epsilon_\alpha : D_4(L) \to D_4(K)$ with $x_r(t) \mapsto x_r(t^\alpha)$, for $r \in \Phi(D_4)$, $t \in L$.) Since K is quadratically closed, there is a character $\chi : \Pi(F_4) \to K^*$ such that the restriction of $h(\chi)$ to $M(e_1 - e_2, e_2 - e_3, e_3 - e_4, e_3 + e_4) = D_4(K)$ is h. Any diagonal automorphism of $F_4(K)$ is inner by Section 1.4. This proves the theorem. □

5.11. We remark that when a conjugate of G_0 is contained in the standard $G_2(K)$ in $F_4(K)$, then there exist $R \in \mathrm{GL}_3(K)$ and $0 \neq t \in K$ such that $R^{\mathrm{T}} M(1) R = tI$. This means that a form similar to the one defined by $M(1)$ admits an orthonormal basis. For arbitrary fields this is not true for all non-degenerate symmetric bilinear forms in dimension 3.

5.12. *We assume that the subgroup $G_0 \simeq G_2(L)$ of G is embedded via the standard embedding in $G_2(K) \leqslant F_4(K)$ (with respect to the embedding $\alpha : L \to K$). For any short root element $a(x)$, $x \in J$, of G, let $M(x)$ be as in Section* 5.7. *Then $M(cx) = c^\alpha M(x)$, for $x \in J$, $c \in L$. Furthermore, when $J = \widehat{L}$, a separable cubic extension of L, then M respects multiplication.*

Proof. Let $t \in K^*$, $h := h_{e_1+e_2}(t)$. For $s \in \{e_1 - e_2, e_3 + e_4, e_3 - e_4, \frac{1}{2}(e_1 - e_2 + e_3 + e_4), \frac{1}{2}(e_1 - e_2 + e_3 - e_4), e_3\}$, the standard scalar product $(-\alpha_1, s)$ is 1. Hence $x_s(u)^h = x_s(tu)$.

In G we have $a(x)^h = a(cx)$, for $x \in J$, $c \in L$, where h is a diagonal element in $\langle A_2, B_1\rangle$, where $\widehat{A_2} = X_{e_1+e_3}$, $\widehat{B_1} = X_{-e_1-e_3}$. Via the standard embedding $h = h_{e_1+e_3}(c^{-\alpha}) \in F_4(K)$ and $a(cx) = \left(\begin{smallmatrix} & I \\ c^\alpha M(x) & I \end{smallmatrix}\right)$; i.e., $M(cx) = c^\alpha M(x)$. This proves the first claim. For the second, we write $\widehat{L} = L(\theta)$. Since $M(1) = 1$ and M respects addition and products tct by Section 5.8, we obtain $M(\theta^n) = M(\theta)^n$ for $n \in \mathbb{N}$. Whence M respects multiplication. □

Next, we investigate the embeddings of the groups associated to the ${}^3D_4$- and ${}^6D_4$-hexagons, see Section 4.2. Our aim is to show that a conjugate of G in $F_4(K)$ is contained in a subsystem subgroup of type D_4, when K is algebraically closed. We refer to Theorem 5, for an embedding of ${}^3D_4(\widehat{L})$, ${}^6D_4(\overline{L})$ in $F_4(L)$.

5.13. Theorem. *Let G arise from a ${}^3D_4$- or ${}^6D_4$-hexagon and assume that K is algebraically closed. Then a conjugate of G in $F_4(K)$ is contained in the subsystem subgroup $M(D_4) := M(e_1 - e_2,\, e_2 - e_3,\, e_3 - e_4,\, e_3 + e_4) = D_4(K)$ of $F_4(K)$. Furthermore, the embedding is either the standard embedding* (*with respect to an embedding $\beta : \overline{L} \to K$*) *or the embedding is in a subgroup of type B_3 in $M(D_4)$.*

Proof. By Theorem 5.10, we may assume that the embedding of $G_0 \simeq G_2(L)$ in $F_4(K)$ is the standard embedding, with respect to an embedding $\alpha : L \to K$. We use that $G = \langle F, A_2, A_1, a(t) \mid t \in \widehat{L}\rangle$, since $G = \langle \Lambda_E \cap \Psi_F, F\rangle$ by Section 1.6.

We write $\widehat{L} = L(\theta)$ and we denote by f the minimal polynomial of θ over L. Since $\widehat{L}$ is a separable cubic extension of L, the polynomial f^α has three different roots in K. There is an embedding $\beta : \overline{L} \to K$ with $\beta|_L = \alpha$ which maps the set of roots of f to the set of roots of f^α.

Let $M := M(\theta)$, a symmetric invertible 3×3 matrix, see Section 5.7. Since $f^\alpha(M) = 0$ by Section 5.12, the Jordan normal form of M is diagonal. Thus one of the following holds:

(1) There exists R with $R^{\mathrm{T}}R = I$ and $\det R = 1$ such that $R^{\mathrm{T}}MR$ is a diagonal matrix (with eigenvalues on the diagonal).

(2) $\mathrm{char}(K) = 2$ and there exists R with

$$R^{\mathrm{T}}R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{such that} \quad R^{\mathrm{T}}MR = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b \\ 0 & b & 0 \end{pmatrix}$$

with a, b different eigenvalues of M.

We identify R with $\begin{pmatrix} R & \\ & R \end{pmatrix} \in M(\alpha_2, \alpha_3, \alpha_4) \simeq \mathrm{Sp}_6(K)$. Then R centralizes $M(-\alpha_*, \alpha_1) \simeq \mathrm{SL}_3(K)$, since $R \in M(\alpha_3, \alpha_4) \simeq \mathrm{SL}_3(K)$.

In Possibility (1), R centralizes also $a(1)$. Thus after conjugation with R we may assume that M is $\mathrm{diag}(\theta^\beta, \theta^{\sigma\beta}, \theta^{\sigma^2\beta})$ or $\mathrm{diag}(a, b, b)$, where possibly a is b. Using Sections 5.7 and 5.12, the first case leads to the standard embedding of G in $M(D_4)$; in the second case G embeds in the subgroup $B_3(K)$, which is obtained from $M(D_4)$ as a fixed point group under the graph automorphism of order 2 interchanging $e_3 - e_4$ and $e_3 + e_4$.

Similarly, Possibility (2) yields that G embeds in the subgroup $B_3(K)$ with diagram $(e_1 - e_2, e_2 - e_3, e_3)$; in particular $G \leqslant M(D_4)$. $\square$

We remark that we expect that a reduction to $B_3(K)$ is not possible. Due to results in representation theory, there should be no 7-dimensional representation for ${}^3D_4(\widehat{L})$, ${}^6D_4(\overline{L})$.

5.14. When G arises from a mixed hexagon in characteristic 3, see Section 5.1, we remark the following: let $f \in F\backslash L$ and set $M := M(f)$, see Section 5.7. Because of $f^3 \in L$, the matrix M^3 is a scalar multiple of the identity matrix and M has only one eigenvalue. When M has an eigenvector v such that $vv^{\mathrm{T}} \neq 0$, then there exists a 3×3 matrix R over K with $RR^{\mathrm{T}} = I$ such that $R^{\mathrm{T}}MR$ is a diagonal block matrix with an upper left 1×1 block. Thus the associated long root element $a(f)$ is contained in $M(-\alpha_*, \alpha_1, \alpha_2, \alpha_3) = B_4(K)$, see Section 5.7. This reduces to the study of embeddings of $G_2(L, L(f))$ in $B_4(K)$. But the eigenspace of M might be spanned by v with $vv^{\mathrm{T}} = 0$; for example,

$$M := \begin{pmatrix} f - t & \mathrm{i}t & \lambda \\ \mathrm{i}t & f + t & \mathrm{i}\lambda \\ -\lambda & \mathrm{i}\lambda & f \end{pmatrix},$$

where $\mathrm{i}, t, \lambda \in K$ and $\mathrm{i}^2 = -1$, satisfies $M^3 = f^3 I$ and the eigenspace for f is spanned by $(-1, \mathrm{i}, 0)$.

The results obtained in Section 5.3, Theorems 5.10, and 5.13 yield Theorem 4. □

6. Moufang octagons

In this section, we prove Proposition 6. For the definition and properties of Moufang octagons and the Ree groups ${}^2F_4(L, \sigma)$, we refer to Van Maldeghem [9], Tits [23].

Let Γ be a Moufang octagon admitting central elations. The root subgroups of Γ corresponding to a half apartment with a line in the middle are isomorphic to $(L, +)$, where L is a field of characteristic 2 admitting an endomorphism $\sigma : L \to L$ with $c^{\sigma^2} = c^2$ for $c \in L$. Let A, B be opposite root subgroups of Γ of the other kind with A_0, B_0 the set of involutions in A and B, respectively. Then A, B are isomorphic to $L \times L$ with addition $(x, y) \oplus (x', y') = (x + x', y + y' + x^\sigma x')$ for $x, x', y, y' \in L$. Moreover A_0, B_0 are central elation subgroups of Γ, isomorphic to $(L, +)$.

The group $X := \langle A, B\rangle$ is isomorphic to the Suzuki group ${}^2B_2(L, \sigma)$. The description of X as a 2-transitive group on $(L \times L) \dot\cup \infty$ in Tits [24] yields that $X_0 := \langle A_0, B_0\rangle$ is normalized by X.

Proof of Proposition 6. We use the above notation and assume that Proposition 6 is false. Then the long root subgroups $\widehat{A}$, $\widehat{B}$ of $F_4(K)$ containing A_0 and B_0, respectively, generate $\mathrm{SL}_2(K)$ with $\mathrm{char}(K) = 2$; up to conjugation $\widehat{A} = X_{\alpha_*}$, $\widehat{B} = X_{-\alpha_*}$. Now A normalizes $\langle A_0, B_0 \rangle$ and $X_{\pm\alpha_*}$. The normalizer of $X_{\pm\alpha_*}$ in $F_4(K)$ is known; it is $H_{\alpha_1} X_{\pm\alpha_*} \mathrm{Sp}_6(K)$. We obtain that it is impossible that a central elation is a square in A, a contradiction. $\square$

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