The Convergence of a Collocation Method for a Class of Cauchy Singular Integral Equations*

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1. Introduction

In several recent papers we have studied the numerical solution of Cauchy singular integral equations of the first kind by a polynomial collocation method [4–6]. This technique has proved to be quite efficient for many problems, even when the kernel (but not the right-hand side) is discontinuous [4–6]. The $L_2$ convergence and stability was established in [5] and [6]. In [12] Ioakimidis extended this method to solve Cauchy singular integral equations of the second kind with constant coefficients. Although some discussion of convergence was given it appears (at least to this author) that the proof is incomplete and the conditions on the kernel and the right-hand side are somewhat more restrictive than necessary. In this paper we will show that the convergence analysis given in [5, 6] can be generalized to establish the $L_2$ convergence of collocation under rather mild restrictions on the data.

Although a variety of numerical techniques such as Galerkin's method and direct quadrature have been used extensively to solve Cauchy singular equations, they seem to be generally suited for problems with smooth kernels. For problems with discontinuous kernels collocation methods appear to be a reasonable compromise between time-consuming Galerkin methods [2] and direct quadrature methods which require the evaluation of the kernel at pairs of points where the kernel may become unbounded [3, 13]. It is well known that for Fredholm equations collocation is related to product integration quadrature methods [1]. Since such methods seem not to have been developed for the solution of Cauchy singular equations collocation appears to be a good way of dealing with fairly general integral equations of this type [4, 7].

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2. COLLOCATION—INDEX 0

We consider the Cauchy singular integral equation

\[ au(x) + \frac{b}{\pi} \int_{-1}^{1} \left[ \frac{u(\xi)}{(\xi - x)} \right] d\xi + \int_{-1}^{1} K(x, \xi) u(\xi) d\xi = f(x), \quad -1 < x < 1, \]  

(1)

where \( a \) and \( b \) are real constants with \( b \neq 0 \), and normalized so that \( a^2 + b^2 = 1 \). In this paper we will assume that \( f(x) \) and \( K(x, \xi) \) are continuous for \( 1 \leq x \leq 1 \) and \(-1 \leq \xi \leq 1 \). It is well known that

\[ v(x) = p(x) u(x), \]  

(2)

where \( p(x) \) is the weight function

\[ p(x) = (1 - x)^{a}(1 + x)^{\beta} \]  

(3)

and \( u(x) \) is "smooth." \( a \) and \( \beta \) are determined by

\[ a = \frac{1}{2\pi i} \log \left( \frac{a - ib}{a + ib} \right) + M, \quad \beta = \frac{1}{2\pi i} \log \left( \frac{a - ib}{a + ib} \right) + N, \]  

(4)

where we take \( a, \beta > -1 \) in order to have integrable solutions [2]. \( M \) and \( N \) are integers related to the index \( v \) of Eq. (1) by

\[ v = -(a + \beta) = -(M + N). \]  

(5)

In this paper we will consider only non-negative values of the index and this restricts \( v \) to the values 0 or 1 since \( a > -1 \) and \( \beta > -1 \). When \( v = 0 \) the solution to (1) is unique. On the other hand when \( v = 1 \) then (1) usually has a one-dimensional null space and to find a unique solution it has to be supplemented by an additional side condition most often of the form

\[ l(u) = c, \]  

(6)

where \( l \) is a given linear functional. For ease of exposition we will consider the two cases \( v = 0, v = 1 \), separately. The case \( a = 0, a = \frac{1}{2}, \beta = -\frac{1}{2} \) was treated in previous papers [5, 6].

To solve (1) numerically when \( v = 0 \) we use the representation (2) in (1) giving

\[ \alpha p(x) u(x) + \frac{b}{\pi} \int_{-1}^{1} \left[ \frac{\rho(\xi) u(\xi)}{(\xi - x)} \right] d\xi + \int_{-1}^{1} \rho(\xi) K(x, \xi) u(\xi) d\xi = f(x), \quad -1 < x < 1. \]  

(7)
Now let \( \{P_n^{(\alpha, \beta)}(x)\}, \ n \geq 0, \) be the Jacobi polynomials orthogonal with respect to the weight function \( \rho(x) \) and consider approximating \( u(x) \) by \( u_N(x) \) where

\[
u(x) = \sum_{n=0}^{N} a_n P_n^{(\alpha, \beta)}(x) \tag{8}\]

and \( a_n, n = 0, 1, \ldots, N, \) are unknown expansion coefficients. To determine these we form the residual

\[
\begin{align*}
    r_N &= ap(x) u_N(x) + \frac{b}{\pi} \int_{-1}^{1} \left[ \rho(\xi) \frac{u_N(\xi)}{(\xi - x)} \right] d\xi \\
    &+ \int_{-1}^{1} \rho(\xi) K(x, \xi) u_N(\xi) d\xi - f(x).
\end{align*}
\tag{9}\]

When \( v = 0 [2] \)

\[
    \begin{align*}
        ap(x) P_n^{(\alpha, \beta)}(x) &+ \frac{b}{\pi} \int_{-1}^{1} \left[ \rho(\xi) P_n^{(\alpha, \beta)}(\xi)/(\xi - x) \right] d\xi \\
        &= -b \csc(\pi \alpha) P_n^{(1-\alpha, -\beta)}(x),
    \end{align*}
\tag{10}\]

where \( \{P_n^{(-\alpha, -\beta)}(x)\} \) are the Jacobi polynomials orthogonal with respect to \( 1/\rho(x) = (1-x)^{-\alpha}(1+x)^{-\beta}. \) Then

\[
    \begin{align*}
        r_N(x) &= -b \csc(\pi \alpha) \sum_{n=0}^{N} a_n P_n^{(1-\alpha, -\beta)}(x) \\
        &+ \sum_{n=0}^{N} a_n \int_{-1}^{1} \rho(\xi) K(x, \xi) P_n^{(\alpha, \beta)}(\xi) d\xi \\
        &- f(x).
    \end{align*}
\tag{11}\]

If we let \( r_N(x_m) = 0, \ m = 0, 1, \ldots, N, \ x_m \in [-1, 1], \) then (11) gives rise to the \( N+1 \) linear equations

\[
    -b \csc(\pi \alpha) \sum_{n=0}^{N} a_n P_n^{(1-\alpha, -\beta)}(x_m) + \sum_{n=0}^{N} a_n \int_{-1}^{1} \rho(\xi) K(x, \xi) P_n^{(\alpha, \beta)}(\xi) d\xi \\
    = f(x_m), \quad m = 0, 1, 2, \ldots, N,
\tag{12}\]

for \( a_n, n = 0, 1, \ldots, N, \) and to obtain a convergent algorithm we must carefully consider the choice of collocation points \( \{x_m\}. \) Generalizing the choice for \( a = 0 [5] \) we pick \( \{x_m\} \) to be the \( N+1 \) distinct zeros of \( P_{N+1}^{(1-\alpha, -\beta)}(x) [16]. \)
In order to analyze the convergence of this scheme we begin by reformulating (1), (8) and (12) in terms of operator equations between suitable Hilbert spaces.

Let

\[ L_{2,\rho} = \left\{ f \left| \int_{-1}^{1} \rho f^2 \, d\xi < \infty \right. \right\} \]

and

\[ L_{2,1/\rho} = \left\{ f \left| \int_{-1}^{1} \left| f^2 / \rho \right| \, d\xi < \infty \right. \right\} \]

then \( \{ P_n^{(\alpha, \beta)}(x) \}_{n=0}^{\infty} \) and \( \{ P_n^{(-\alpha, -\beta)}(x) \}_{n=0}^{\infty} \) are complete orthogonal bases in \( L_{2,\rho} \) and \( L_{2,1/\rho} \), respectively. Using Eq. (10) the operator \( H \) defined by

\[ Hu = a \rho(x) u(x) + \frac{b}{\pi} \int_{-1}^{1} \left( \rho u(\xi) / (\xi - x) \right) \, d\xi \]

(13)
can be extended to be a bounded linear operator from \( L_{2,\rho} \rightarrow L_{2,1/\rho} \) by the formula

\[ Hu(x) = H \left( \sum_{n=0}^{\infty} u_n P_n^{(\alpha, \beta)}(x) \right) = \sum_{n=0}^{\infty} u_n H P_n^{(\alpha, \beta)}(x) \]

\[ = -b \csc(\pi \alpha) \sum_{n=0}^{\infty} u_n P_n^{(-\alpha, -\beta)}(x), \]

where \( u_n = \langle u, P_n^{(\alpha, \beta)} \rangle / \langle P_n^{(\alpha, \beta)}, P_n^{(\alpha, \beta)} \rangle \), and the series in (14) converge in the norms of \( L_{2,\rho} \) and \( L_{2,1/\rho} \), respectively. It follows from (14) that \( H \) has a bounded inverse \( H^{-1} \) where

\[ H^{-1}u(x) = \left[ a / \rho(x) \right] u(x) - \frac{b}{\pi} \int_{-1}^{1} \left[ u(\xi) / \rho(\xi)(\xi - x) \right] \, d\xi, \]

(15)
at least for Hölder continuous functions \( u \) [2].

Since \( K(x, \xi) \) is continuous it follows from standard arguments that

\[ Ku(x) = \int_{-1}^{1} \rho(\xi) K(x, \xi) \, u(\xi) \, d\xi \]

(16)
defines a compact operator from \( L_{2,\rho} \rightarrow L_{2,1/\rho} \). Thus if \( f(x) \in L_{2,1/\rho} \) then (1) can be written as

\[ Hu + Ku = f, \]

(17)
where \( H \) and \( K \) have the properties given above.
By the Fredholm alternative a sufficient condition for (1) to have a unique solution for each \( f \in L_{2,1/2} \) is that the homogeneous equation

\[
Hu + Ku = 0
\]

have only 0 as a solution. We will assume that this condition holds from now on.

Turning to the analysis of the collocation equations let \( C[-1, 1] \) denote the space of continuous functions on \([-1, 1]\) endowed with the sup norm

\[
\|f\|_{\infty} = \max_{x \in [-1, 1]} |f(x)|.
\]

Define the projection operators

\[
P_N : C[-1, 1] \rightarrow L_{2,1/2}
\]

by

\[
P_N g = \sum_{m=0}^{N} g(x_m) l_m(x),
\]

where \( l_m(x) \) are the standard Lagrange interpolation polynomials satisfying \( l_m(x_k) = \delta_{mk} \). \( P_N \) maps a continuous function onto the unique polynomial which interpolates to \( g \) at the points \( \{x_m\} \). Using (20) it is easily shown that \( u_N \) satisfies (8) and (12) iff

\[
P_N r_N(x) = 0.
\]

In fact if \( P_N r_N(x) = 0 \) then \( P_N r_N(x_m) = 0 \) and

\[
\sum_{k=0}^{N} r_N(x_m) l_k(x_m) = \sum_{k=0}^{N} r_N(x_m) \delta_{km} = r_N(x_m) = 0.
\]

On the other hand if \( r_N(x_m) = 0 \) then (20) shows that \( P_N r_N = 0 \). From this it follows that \( u_N \) satisfies

\[
P_N Hu_N + P_N Ku_N = P_N f.
\]

Now (10) shows that \( H \) maps the set of polynomials of degree \( \leq N \) onto itself. In addition, \( P_N \pi_n = \pi_n \), where \( \pi_n \) is a polynomial of degree \( \leq N \) so that

\[
P_N Hu_N = Hu_N
\]

showing that

\[
Hu_N + P_N Ku_N = P_N f.
\]
Using (24) we will now show that \( \{ u_N \} \) converges in \( L_{2,p} \) to \( u \), the unique solution of (1). Since \( H \) has a bounded inverse it follows that \( u_N \) satisfies

\[
\frac{1}{2} \frac{d}{dt} u_N + H^{-1} P_N K u_N = H^{-1} P_N f,
\]

and a standard theorem shows that

\[
\lim_{N \to \infty} \| u - u_N \|_{2,p} = 0
\]

if \( |1| \)

\[
\lim_{N \to \infty} \| f - P_N f \|_{1,p} = 0 \quad \text{and} \quad \lim_{N \to \infty} \| K - P_N K \|_{p,1/p} = 0.
\]  

(\( \| \cdot \|_{1/p} \) is the norm on \( L_{2,1/p} \) and \( \| \cdot \|_{p,1/p} \) denotes the induced operator norm between \( L_{2,p} \) and \( L_{2,1/p} \).) Moreover if (26) holds then [5, 6]

\[
\| u - u_N \|_{p} \leq C \| Hu - P_N Hu \|_{1/p}.
\]

To prove (26) we need two results. The first is a classical theorem on the mean square convergence of interpolation polynomials due to Erdős and Turan, and the second is an imbedding argument due to Vainikko on the uniform convergence of approximations to compact operators. (For a slightly different approach to problems of this type see the recent work of Prosdorf and Silberman [14, 15].) For the sake of completeness we state these as Theorems 2.1 and 2.2.

**Theorem 2.1 (Erdős and Turan [16]).** Let \( w(x) \) be a non-negative integrable weight function on \([-1, 1]\) \((\int_{-1}^{1} w(x) \, dx > 0)\) and let \( \{ p_n \}_{n=0}^{\infty} \) be polynomials orthogonal with respect to \( w(x) \) (deg \( p_n = n \)). That is,

\[
\int_{-1}^{1} w(x) p_n(x) p_m(x) \, dx = (n + 1) \delta_{nm}.
\]

Let \( x_k, k = 0, 1, \ldots, N \) be the distinct zeros of \( p_{N+1}(x) \) and if \( f \) is continuous on \([-1, 1]\) then \( P_N f \) is the unique polynomial which interpolates to \( f \) at the points \( \{ x_k \} \). Then

\[
\int_{-1}^{1} w(x)(f - P_N f)^2 \, dx = 0.
\]

**Theorem 2.2 (Vainikko [10]).** Let \( X, Y \) and \( Z \) be Banach spaces with \( Y \subseteq Z \) where the imbedding \( E: Y \to Z \) is continuous. Suppose that \( K \) is a compact operator from \( X \to Y \). Let \( \{ P_n \} \) be a sequence of bounded operators
from \( Y \rightarrow Z \) such that \( \| P_N \|_{X,Z} \leq M, N \geq 0 \) and assume that \( P_N \) converges pointwise to \( E \). Then \( P_N K \) converges uniformly to \( K \). That is,

\[
\lim_{N \to \infty} \| K - P_N K \|_{X,Z} = 0.
\] (29)

**Theorem 2.3.** Let \( P_N \) be the interpolation operator defined by (20) and let \( E : C[-1, 1] \rightarrow L_{2, 1/\rho} \) be the imbedding operator. Then \( P_N \) converges pointwise to \( E \) and \( \| P_N \| \leq M, N \geq 0 \).

**Proof.** The pointwise convergence of \( P_N \) to \( E \) is an immediate consequence of Theorem 2.2.

To prove the uniform boundedness of \( \| P_N \| \) observe first that the zeros of \( P_{N+1}^{(-\alpha, -\beta)}(x) \) are the nodes of the Gaussian quadrature rule for the weight \((1 - x)^{-\alpha} (1 + x)^{-\beta} \) [16]. Thus for any polynomial \( \pi_{2N+1} \) of degree \( \leq 2N + 1 \) we have

\[
\int_{-1}^{1} (1 - x)^{-\alpha} (1 + x)^{-\beta} \pi_{2N+1}(x) \, dx = \sum_{k=0}^{N} w_k \pi_{2N+1}(x_k),
\] (30)

where \( \{w_k\} \) are the weights of the rule. Now it follows from (30) that

\[
\| P_N f \|_{1/\rho}^2 = \int_{-1}^{1} (1 - x)^{-\alpha} (1 + x)^{-\beta} (P_N f)^2 \, dx
\]

\[
= \sum_{k=0}^{N} w_k [(P_N f)(x_k)]^2
\]

\[
= \sum_{k=0}^{N} w_k f^2(x_k),
\]

since \( (P_N f)^2 \) is a polynomial of degree at most \( 2N \). Thus

\[
\| P_N f \|_{1/\rho}^2 \leq \| f \|_{\infty}^2 \int_{-1}^{1} (1 - x)^{-\alpha} (1 + x)^{-\beta} \, dx
\]

\[
= \| f \|_{\infty}^2 \left[ 2^{-(\alpha + \beta) + 1} \Gamma(1 - \alpha) \Gamma(1 - \beta) / \Gamma(2 - \alpha - \beta) \right],
\]

where \( \Gamma(x) \) is the gamma function. Thus

\[
\| P_N \| \leq M,
\]

where \( M = 2^{(\alpha + \beta) + 1/2} [\Gamma(1 - \alpha) \Gamma(1 - \beta) / \Gamma(2 - \alpha - \beta)]^{1/2} \).

**Theorem 2.4.** Let \( u_N \) be the approximation to \( u \) determined by (8) and (12). Then

\[
\lim_{N \to \infty} \| u - u_N \|_{1/\rho}^2 = 0.
\]
Proof. According to our previous discussion it suffices to show that (26) holds. But this follows from Theorems 2.2 and 2.3 by taking $X = L_{2,\rho}$, $Y = C[-1, 1]$ and $Z = L_{2,1/\rho}$.

3. Collocation—Index 1

When $\nu = 1$, $H$ has a one-dimensional null space. In fact if $c$ is a constant then $Hc = 0$ [2]. Thus in order to obtain a unique solution it is customary to supplement (1) by an additional condition of the form

$$l(u) = c,$$

where $l$ is a bounded linear functional on $L_{2,\rho}$ and $c$ is a given constant. (By the Riesz–Fisher Theorem we can write $l(u) = \int_1^1 \rho(x) g(x) u(x) \, dx$, where $g \in L_{2,\rho}$. A common choice is $g = 1$ and $c = 0$ [12].) In this case we have

$$H P_n^{(a, \beta)} = \frac{-b}{2} \csc(\pi a) P_{n-1}^{(-a, -\beta)}(x), \quad n = 1, 2, \ldots,$$

and

$$H P_0^{(a, \beta)}(x) = 0,$$

so that $H$ again can be extended as bounded operator from $L_{3,\rho} \to L_{2,1/\rho}$. In this case $H$ is onto, has a one dimensional null space, and has a bounded right inverse $H'$, where for Hölder continuous functions $H'u$ is given by (15).

Using these observations it is possible to show that (1) along with (31) is equivalent to the equation of the second kind [11]

$$u + H'Ku + T_\phi u = H'f \quad l(H'f) \phi_0 + c\phi_0,$$

where $H\phi_0 = 0$, $T_\phi u = l(H'Ku) \phi_0$, and for convenience $l(\phi_0) = 1$. $T_\phi$ is a bounded finite rank operator so that $H'K - T_\phi$ is compact and the solvability of (32) can be determined by the Fredholm alternative. That is (1) and (31) have a unique solution $u \in L_{2,\rho}$ for every $f \in L_{2,1/\rho}$ iff $N(I + H'K - T_\phi) = 0$. We assume that this condition holds for the remainder of this section.

To solve (1) and (31) numerically via collocation we proceed as in Section 2 where $\nu = 0$. That is we approximate $u$ by

$$u_N = \sum_{n=0}^N a_n P_n^{(a, \beta)}(x)$$
where the coefficients \( \{a_n\} \) are now determined by

\[
-\left[ b \csc(\pi \alpha) / 2 \right] \sum_{n=1}^{N} a_n P_n^{(-\alpha, -\beta)}(x_m) + \sum_{n=0}^{N} a_n K P_n^{(\alpha, \beta)}(x_m) - f(x_m) = 0, \quad m = 0, 1, \ldots, N - 1 \quad (36)
\]

\[
l(u_N) = c \quad (37)
\]

where \( \{x_m\}, m = 0, 1, \ldots, N - 1 \) are the \( N \) distinct zeros of \( P_n^{(-\alpha, -\beta)}(x) \). (Note that in the particular case that \( l(u) = 1 \), \( u(x) \) \( dx \) so that if \( l(u) = 0 \) we get \( a_0 = 0 \) leaving only Eqs. (34) to solve. This is the situation discussed in [2] and [12].) Equations (36) and (37) can be written as

\[
\begin{align*}
&\{r_N(x_m) = 0, \quad m = 0, 1, \ldots, N - 1, \\
&l(u_N) = 0,
\end{align*} \quad (38)
\]

where \( r_N(x) = (Hu_N + Ku_N - f)(x) \).

To analyze the convergence of this method we proceed in a fashion analogous to that for \( v = 0 \).

Let \( P_{N-1}: C \rightarrow L^2_{-1/1} \) map \( u \in C \) onto the polynomial \( P_{N-1}u \) which interpolates to \( u \) at the zeros of \( P_n^{(-\alpha, -\beta)}(x) \). Then as in Lemma 2.1 it is easily shown that (36) is equivalent to

\[
\begin{align*}
&\{P_{N-1}Hu_N + P_{N-1}Ku_N = P_{N-1}f, \\
l(u_N) = c.
\end{align*} \quad (39)
\]

Since (32) shows that \( Hu_N \) is a polynomial of degree \( N - 1 \), \( P_{N-1}Hu_N = Hu_N \) showing that \( u_N \) satisfies

\[
\begin{align*}
&\{Hu_N + P_{N-1}Ku_N = P_{N-1}f, \\
l(u_N) = c.
\end{align*} \quad (40)
\]

As for (34) solving (40) is equivalent to solving

\[
u_N + H'P_{N-1}Ku_N - T_\phi u_N = H'P_{N-1}f - l(H'P_{N-1}f) + c\phi_0. \quad (41)\]

If we let \( K_N = H'P_{N-1}K - T_\phi \) and \( f_N = H'P_{N-1}f - l(H'P_{N-1}f) + c\phi_0 \) then (41) can be written as

\[
u_N + K_Nu_N = f_N \quad (42)
\]

and the convergence of \( u_N \) follows if we can show that \([1]\)

\[
\lim_{N \to \infty} \|H'f - f_N\|_\rho = 0, \quad \lim_{N \to \infty} \|H'K - K_N\|_\rho, \rho = 0. \quad (43)
\]
Since $H'$ and $l$ are bounded this follows if
\[
\lim_{N \to \infty} \| f - P_{N-1} f \|_{1,p} = 0, \quad \text{and} \quad \lim_{N \to \infty} \| K - P_{N-1} K \|_{1,p} = 0.
\]
(44)

However, these follow as for Theorem 2.4 from Theorems 2.2 and 2.3 on replacing $N$ by $N - 1$. Summarizing we have Theorem 3.1.

**Theorem 3.1.** Assume that (1) and (31) have unique solution. Let $u_N$ be the approximation to $u$ determined by (36) and (37). Then for all $N \geq N_0$, $u_N$ exists, is unique and
\[
\lim_{N \to \infty} \| u - u_N \|_{1,p} = 0.
\]

Moreover the error estimate
\[
\| u - u_N \|_p \leq C \left[ \| f - P_{N-1} f \|_{1,p} + \| K - P_{N-1} K \|_{1,p} \right]
\]
(45)
holds.

4. **An Example—$v = 1$**

As a specific example of the case where $v = 1$ we consider the equation of the first kind
\[
\frac{1}{\pi} \int_{-1}^{1} \frac{u(\xi)}{\sqrt{1 - \xi^2}} (\xi - x) d\xi + \int_{-1}^{1} K(x, \xi) u(\xi)/\sqrt{1 - \xi^2} \, d\xi = f(x), \quad -1 < x < 1.
\]
(46)

Here $\rho(\xi) = 1/\sqrt{1 - \xi^2}$, $P_n^{(-1/2,-1/2)}(\xi) - T_n(\xi) - \cos(n \cos^{-1} x)$ are the Chebyshev polynomials of the first kind and $P_n^{1/2,1/2}(x) = \sin[(n + 1) \cos^{-1}(x)]/\sin(\cos^{-1}(x))$ are the Chebyshev polynomials of the second kind. The collocation points $\{x_m\}$ are given by
\[
\sin [(n + 1) \cos^{-1} x_m] = 0
\]
(47)
so that
\[
x_m = \cos(m\pi/(n + 1)), \quad m = 1, 2, \ldots n.
\]
(48)

The convergence of the collocation method in this particular case was essentially given by Gabdulhaev and Duskov in [9] although their method was somewhat different from ours. Rather than treat (47) directly they made the
trigonometric substitution $x = \cos t$, $\xi = \cos \tau$ to convert (47) into the equation

$$\frac{1}{\pi} \int_0^\pi \frac{u(\tau)/\cos(t - \cos \tau)}{\cos t - \cos \tau} \, d\tau$$

$$+ \int_0^\pi \frac{K(\cos t, \cos \tau) u(\tau)}{\cos \tau} \, d\tau = f'(\cos t), \quad (49)$$

where $u(\tau) = u(\cos \tau)$. In this case the Chebyshev polynomials are mapped onto the trigonometric functions

$$\{\cos nt\}_{n=0}^\infty \quad \text{and} \quad \{\sin((n+1)\tau)/\sin\tau\}_{n=0}^\infty,$$

respectively. Multiplying (49) through by $\sin \tau$ it can be considered the equivalent to the equation

$$\frac{\sin \tau}{\pi} \int_{-1}^1 \frac{u(\tau)/\cos(t - \cos \tau)}{\cos t - \cos \tau} \, d\tau$$

$$+ \int_{-1}^1 \sin \tau K(\cos t, \cos \tau) u(\tau) = \sin \tau f'(\cos t).$$

Using the fact that now

$$\frac{\sin \tau}{\pi} \int_{-1}^1 \frac{\cos nt/\cos(t - \cos \tau)}{\cos t - \cos \tau} \, d\tau = \sin nt, \quad n \geq 0, \quad (50)$$

enabled them to carry out the analysis on $L_2[0, \pi]$ using results on trigonometric rather than polynomial interpolation.

5. Further Results

As we indicated in the Introduction the condition of continuity on $f(x)$ and $K(x, \xi)$ may be relaxed. Since Erdös and Turan's theorem on the mean square convergence of interpolating polynomials holds even if $f(x)$ is only Riemann integrable [16] Theorems 2.4 and 3.1 hold for certain discontinuous right-hand sides as well. Similarly the kernel $K(x, \xi)$ need only be Riemann integrable [5]. This follows since the collocation method may be regarded as a degenerate kernel method. The details are the same as those given in [5].
As a practical matter in carrying out the collocation method it is necessary to be able to evaluate the integrals

$$K P_n^{(a, \beta)}(x_m) = \int_{-1}^{1} \rho(\xi) K(x_m, \xi) P_n^{(a, \beta)}(\xi) d\xi. \quad (51)$$

In most cases (51) will have to be calculated numerically. In the case of equations of the first kind a common practice is to use the Gaussian quadrature rule for the weight \( \rho(\xi) \) with the number of nodes equal to the number of basis elements. Following this approach for the general case \((v = 0 \text{ and } v = 1)\) (51) is approximated by

$$K P_n^{(a, \beta)}(x_m) = \sum_{k=0}^{N} w_k K(x_m, \xi_k) P_n^{(a, \beta)}(\xi_k),$$

where \( \{\xi_k\} \) and \( \{w_k\} \) are the weights and nodes of the Gaussian rule for \( \rho(\xi) \). In this case it can be shown that the collocation method is numerically equivalent to the direct quadrature method introduced by Erdogan et al. [2] and Krenk [13]. In this case collocation agrees with an existing method. An advantage of this approach is that it can deal with kernels which may become unbounded when \( x = \xi \), however, in this case the convergence is slow, accuracy being limited by that of the quadrature rule [7]. For the case where \( K(x, \xi) = a(x - \xi) \log |x - \xi| + b(x - \xi), \) \( a(x) \) and \( b(x) \) are analytic and \( a = \frac{1}{2} \) and \( \beta = -\frac{1}{2} \) \((v = 0)\), a more sophisticated approach to the calculation was seen to result in a marked increase in the accuracy of the solution.

6. Conclusions

We have shown that a polynomial collocation method for solving a class of singular integral equations converges in mean square under rather mild conditions on the kernel and right-hand side. The results generalize previous ones of Gabdulhaev and Duskov [9] and Golberg and Fromme [5, 6] for equations of the first kind. Under certain conditions it is observed that the method is equivalent to the method of direct quadrature introduced by Erdogan and Krenk.

REFERENCES